A class of velocity fields with known Lagrangian law

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Abstract. We introduce a large class of random velocity fields on the periodic lattice and in discrete time having a certain hidden Markov structure. The generalized Lagrangian velocity (the velocity field as viewed from the location of a single moving particle) has similar hidden Markov structure, and its law is found exactly. As a result, the law of the trajectory of an individual particle is known in principle. The rate of convergence to equilibrium of the generalized Lagrangian velocity is studied in small numerical examples and in rigorous results giving absolute and relative bounds on the size of the spectral gap. The effect of molecular diffusion on the rate of convergence is also investigated. After repeating the velocity field periodically throughout the integer lattice, it is shown that, with the usual diffusive rescaling, the single-particle motion converges to Brownian motion in both compressible and incompressible cases. An exact formula for the effective diffusivity is given and numerical examples are shown.

Key words and phrases. Lagrangian velocity, Lagrangian observations, hidden Markov model, spectral gap, homogeneous turbulence.


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1 Background and introduction

Let $U = \{U(x,t), x \in \mathbb{R}^d, t \geq 0\}$ be a stochastic process taking values in $\mathbb{R}^d$. We think of $U$ as a random velocity field and consider the motion of a particle whose position follows the velocity field according to

\[
\frac{dX_t}{dt} = U(X_t, t), \quad t > 0,
\]

assuming sufficient smoothness of $U$. In some cases we also make the particle subject to molecular diffusion; then the trajectory equation is

\[
dX_t = U(X_t, t)dt + \sigma(X_t, t)dW_t, \quad t > 0,
\]

where $\sigma > 0$ and $W$ is a Wiener process independent of $U$.

One major goal of statistical fluid mechanics is to determine the law of the particle’s trajectory $X_t, t \geq 0$, from knowledge of the law of the velocity field $U$. A slightly simpler goal is to determine the law of the Lagrangian velocity process $U(X_t, t), t \geq 0$, which is the particle’s velocity under (1.1) or its drift under (1.2). (By contrast, $U$ is called the Eulerian velocity field.) These goals are rarely stated so plainly, perhaps because they have proven so far beyond our ability for so long.

The motivation for this problem is to understand the motion of fluid parcels and passive tracers in turbulent and randomly-forced fluid flows. It is nontrivial even to determine the appropriate law for $U$, but working forward from a given law is an important part of the larger problem. Besides, one must deal with this part in order to use particle trajectory data to estimate the law of $U$, even the low-order moments, as is now done in oceanography; see Davis (1991).

Here is a brief overview of the work done in this area. In oceanography, various drifting instruments have been tracked remotely to obtain approximate particle trajectories which can be used to estimate the laws of $X_t, t \geq 0$ and $U(X_t, t), t \geq 0$ in the ocean; see the review Davis (1991) and references therein. The same sort of analysis has been carried out for particle motion in numerical simulations of forced Navier–Stokes turbulence by Yeung and Pope (1989) and Gotoh et al. (1993) among others. More in line with our original problem, Avellaneda et al. (1993), Carmona and Wang (1996), Elliott and Majda (1996), Elliott et al. (1997), Carmona and Cerci (1999), and others have examined particle motion in numerical simulations of model velocity fields with given laws.

There is a long tradition of deriving approximate evolution equations for the mean concentration of a passive tracer carried by the velocity field using various closure methods. See, for example, Monin and Yaglom (1971), Section 10.3 and the references below. The
mean concentration at time $t$ can be written in terms of the distribution of $X_t$, so it is natural that similar methods can be used to make approximations of the distributions of $X_t$ and $U(X_t, t)$, as is done, for example, by Weinstock (1976), Davis (1982) and A. F. Bennett (1987). These closures rely on assumptions which are difficult (if not impossible) to verify theoretically, but there have been attempts to verify the approximations experimentally.

There are several rigorous results concerning the convergence of the rescaled processes $\varepsilon X_{t/\varepsilon^2}$, $t \geq 0$ to Brownian motion as $\varepsilon \to 0$; among these are Papanicolaou and Varadhan (1982), Bhattacharya (1985), Molchanov (1996), Carmona and Xu (1997), Komorowski and Papanicolaou (1997), Fannjiang and Komorowski (1997), Landim et al. (1998), and Fannjiang and Komorowski (1999). These are known as homogenization results from the original connection with partial differential equations; indeed the mean concentration in these cases evolves according to an effective diffusion equation under a similar rescaling of space and time. The review Isichenko (1992) has a discussion of more limited results on the effective diffusivity via partial differential equation techniques.

A key ingredient in the last six homogenization results cited is that $U$ is homogeneous, stationary, and divergence free. These conditions are helpful because then $U(X_t, t)$, $t \geq 0$ is strictly stationary for both (1.1) and (1.2), as was shown progressively by Lumley (1962), Port and Stone (1976), and Zirbel (1993). Moreover, the generalized Lagrangian velocity $V$ defined by

$$V(x, t) = U(x + X_t, t), \quad x \in \mathbb{R}^d, \; t \geq 0,$$

is strictly stationary, as shown by Osada (1982) for (1.2) and Zirbel (1993) for (1.1) and (1.2). The stationarity of $V$ implies the stationarity of $\frac{\partial U}{\partial x}(X_t, t)$, $t \geq 0$, which was used by Carmona et al. (1997) and Çağlar (1997) in studies of Lyapunov exponents. Once stationarity is assured, the main issue in homogenization and Lyapunov exponents has been to show that $V$ loses memory of its previous values quickly enough.

When the Eulerian velocity field $U$ is homogeneous and Markov in time, the generalized Lagrangian velocity $V$ is Markov, even when $U$ is divergent, cf. Zirbel (2000a). Carmona and Xu (1997) showed this in a case in which $U$ is, in addition, stationary, Gaussian, and divergence free. They and Fannjiang and Komorowski (1999) have computed the generator of $V$ and have obtained $L^2$ rate of convergence results for functionals of $V$ in terms of the spectral gap of the Eulerian field.

We conclude with an outline of the main ideas of the paper.

We introduce a large class of Eulerian velocity fields $U$ on the periodic lattice and in discrete time for which the law of the generalized Lagrangian velocity $V$ can be found explicitly in both incompressible and compressible cases. This opens the way to a theoretical approach to the problems listed above as an alternative to simulation, approximation, and
asymptotics. The models are especially well suited to studying in detail the phenomenology of particle motion due to simple fluid motifs such as localized vortices which move over time, as the examples will show. The method and results suggest several extensions to continuous space–time.

Discrete space–time has not been used often in studies of the Lagrangian velocity, but it is an honest simplification of the problem. The fundamental difficulty in continuous space and time is the non–linear relationship between $U$ and $X$, and this is retained in the discrete setting with analogues of (1.1) and (1.2) (Section 2). We believe that working in discrete space and time provides valuable insight into the continuous space–time case. For instance, the exact nature of the computations allows one to quickly check (or reject) conjectures numerically. Also, note that numerical approximations of particle motion in random or Navier–Stokes velocity fields are, at heart, discrete in space and time.

When $U$ is homogeneous and Markov in time, it can be decomposed nicely into two parameters whose evolution is easy to describe (Section 3). This observation forms the basis for the class of Eulerian velocity fields introduced in Section 4. We define $U$ to be a simple functional of a Markov chain with two components, the velocity field type $I$ and the location $L$. By choosing the state space of $I$ to be large enough, a very large class of Eulerian laws is allowed, as is usual with hidden Markov models. In particular, numerical simulations of Navier–Stokes turbulence can be included here.

The generalized Lagrangian velocity has the same hidden Markov structure, except that its location parameter $M$ evolves differently than $L$ (Section 5). The transition matrix of $(I, M)$ can be written in terms of that of $(I, L)$, and so the law of the generalized Lagrangian velocity is found explicitly. Moreover, the particle trajectory $X$ is an additive functional of $V$, and so its law is known in principle.

The model we develop is an ideal testbed for investigating the effect of the space–time structure of $U$ on the law of the generalized Lagrangian velocity without resort to approximations or numerical simulations of $U$. A few illustrative examples are given in Section 6.

The transition structure of the Markov chain $(I, M)$ makes it possible to prove results about the eigenvectors (Section 7) and eigenvalues of its transition matrix when the Eulerian velocity field is incompressible. The moduli of the eigenvalues can be used to understand the rate of convergence to equilibrium for the generalized Lagrangian velocity $V$. Results include an absolute bound (Section 8) and two relative bounds: when $I$ and $L$ are independent (Section 9) and when $(I, L)$ is reversible (Section 10). The results of Section 9 can be interpreted as a statement about enhanced diffusivity in incompressible velocity fields. The effect of adding molecular diffusion as in (1.2) is studied in Section 11.
Finally, homogenization of single particle motion in \( \mathbb{Z}^d \) is studied by periodically repeating \( U \) throughout \( \mathbb{Z}^d \) (Section 12). Under the usual diffusive scaling, this motion converges to Brownian motion. More importantly, we give an exact, computable formula for the effective diffusivity and provide some numerical examples.

2 Setup for discrete space and time

Particle motion on the lattice has become very familiar with the various studies of Markov motion in a homogeneous random environment. Our situation differs primarily in that the environment (the velocity field) changes over time (it is not “quenched”) and that it models particles carried by a fluid rather than the molecular motion of systems of particles. The closest known relative to our situation is the early paper Lumley and Corrsin (1959). The recent papers on card shuffling (for example, Bayer and Diaconis (1992)) also bear some similarity to our situation, with each step moving all particles (cards) incompressibly. The main difference is that successive shuffles are independent, whereas we seek models in which the velocity field exhibits strong dependence in time. Similar comments apply to random transposition models; see Diaconis (1988).

The spatial domain in what follows will be the set
\[
\mathcal{D} = \{0, 1, \ldots, n_{1} - 1\} \times \cdots \times \{0, 1, \ldots, n_{d} - 1\},
\]
which we may think of as a subset of \( \mathbb{Z}^d \) with \( n = n_{1}n_{2} \cdots n_{d} \) points. Addition of elements of \( \mathcal{D} \) is done componentwise modulo the numbers \( n_{1}, \ldots, n_{d} \), which we call addition modulo \( \mathcal{D} \). We may think of \( \mathcal{D} \) as a lattice with “periodic boundary conditions” or as a finite Abelian group.

A velocity field \( u \) on \( \mathcal{D} \) is simply a mapping from \( \mathcal{D} \) to \( \mathbb{D} \), thought of as a vector field on \( \mathcal{D} \). Figure 2.1(a) shows a velocity field in one dimension \( (d = 1) \) with \( n_{1} = 7 \) where \( \mathcal{D} \) is drawn as a circle. We write \( u(0) = 1, u(1) = 1, u(2) = -2 \equiv 5 \pmod{7}, u(3) = 0, u(4) = 0, u(5) = 1, u(6) = -1 \equiv 6 \pmod{7} \). Note that it is sometimes more natural to specify the values of \( u \) as numbers in \( \mathbb{Z} \). Figure 2.1(b) shows an example in two dimensions. We denote by \( \mathcal{U} \) the set of all velocity fields on \( \mathbb{D} \). A random velocity field \( U \) is a stochastic process \( U_{t}, \ t = 0, 1, \ldots \) taking values in \( \mathbb{U} \).

The trajectory equation in discrete time corresponding to (1.1) is
\[
X_{t+1} = X_{t} + U_{t}(X_{t}), \quad t = 0, 1, \ldots,
\]
with \( X_{0} \) fixed. The analogue of motion under \( U \) subject to molecular diffusion is
\[
X_{t+1} = X_{t} + U_{t}(X_{t}) + \Delta_{t}, \quad t = 0, 1, \ldots,
\]
where $\Delta_0, \Delta_1, \ldots$ are independent, identically distributed random variables taking values in $\mathbb{D}$, and independent of $U$. Note that (2.3) is identical to the numerical implementation of the Euler scheme for simulating the solution of a stochastic differential equation with Brownian noise. We defer consideration of this case until Section 11.

The generalized Lagrangian velocity $V$ is defined by

$$ V_t(x) = U_t(x + X_t), \quad x \in \mathbb{D}, \ t = 0, 1, \ldots, $$

which is the same as (1.3). Note that in case (2.2), $X_t = X_0 + \sum_{s=0}^{t-1} V_s(0)$ for $t = 0, 1, \ldots$.

A velocity field $u$ in $\mathcal{U}$ generates a mapping $\alpha : \mathbb{D} \to \mathbb{D}$ by

$$ \alpha(x) = x + u(x), \quad x \in \mathbb{D}. $$

If $\alpha$ is a permutation on $\mathbb{D}$, we say that $u$ is *incompressible*. This is the discrete-space analogue of a divergence-free vector field. We denote by $\mathcal{U}_0$ the set of all incompressible vector fields. A random velocity field $U$ taking values in $\mathcal{U}_0$ is said to be *incompressible*, otherwise it is called *compressible*.

The velocity field $U$ is said to be *homogeneous* if, for all $z$ in $\mathbb{D}$, the random velocity field $\tilde{U}$ defined by

$$ \tilde{U}_t(x) = U_t(x + z), \quad x \in \mathbb{D}, \ t = 0, 1, \ldots $$

has the same distribution as $U$. In other words, the law of $U$ is invariant under spatial translation. The definition of stationarity is similar. Note that even in the discrete setting, if $U$ is homogeneous, stationary, and incompressible, then $V$ is strictly stationary, cf. Zirbel (2000b).
3 The case of homogeneous Markov Eulerian velocity

The special case considered in this section will motivate the general class of velocity fields introduced in Section 4. The idea is to identify circumstances in which the generalized Lagrangian velocity \( V_t, \ t = 0, 1, \ldots \) is Markov. It is not enough to just assume that \( U_t, \ t = 0, 1, \ldots \) is Markov, for knowledge of, say, \( V_0 \) does not tell the value of \( X_6 \), and so one cannot recover \( U_6 \) to make the transition to \( U_7 \) and then \( V_7 \). These steps would be necessary if the law of \( U \) were inhomogeneous.

It can be shown in a general setting that if \( U \) is homogeneous and Markov, then \( V \) is also Markov, cf. Zirbel (2000a). We need not depend on that result here, for in the discrete case one can get much more detailed information on the transition structure of \( U \) and \( V \). In fact, homogeneity of \( U \) is a little stronger than what is really needed.

In the finite spatial domain we are considering here, the Eulerian velocity field \( U \) is best dealt with in terms of the “shape” of \( U \) and the absolute “location” of this shape. To make this idea more precise define an equivalence relation \( \sim \) on \( \mathcal{U} \) by

\[
(3.1) \quad u \sim v \text{ if } u = v(\cdot - z) \text{ for some } z \text{ in } \mathbb{D}.
\]

(The meaning of \( u = v(\cdot - z) \) is that \( u(x) = v(x - z) \) for all \( x \) in \( \mathbb{D} \), but we will often use the abbreviation.) This relation partitions \( \mathcal{U} \) into equivalence classes \( \mathcal{U}_1, \ldots, \mathcal{U}_M \). From each class \( \mathcal{U}_i \) we select a representative \( u_i \). We refer to \( i \) as the shape of the elements of \( \mathcal{U}_i \), and if \( v = u_i(\cdot - z) \) we say that \( v \) has shape \( i \) and location \( z \).

Given an element \( u \) of \( \mathcal{U} \), its shape is well defined but its location may not be, for there are shapes \( i \) for which \( u_i = u_i(\cdot - z) \) for nontrivial \( z \) in \( \mathbb{D} \), for example, the velocity field of all zeros. To avoid this degeneracy, we will assume for the remainder of this section that \( U \) takes values in the set of velocity fields with uniquely-defined locations. This restriction will be lifted in Section 4.

Now we may write \( U \) as

\[
(3.2) \quad U_t = u_t(\cdot - L_t), \quad t = 0, 1, \ldots,
\]

which implicitly defines the shape process \( I \) and the location process \( L \). When \( U_t, \ t = 0, 1, \ldots \) has the Markov property, the process \( (I_t, L_t), \ t = 0, 1, \ldots \) inherits the Markov property because the relation (3.2) between the two is deterministic and invertible. The result below shows that, if we assume in addition that \( U \) is homogeneous, then the shape process \( I \) is Markov relative to the filtration generated by \( (I, L) \) and the location process behaves like an additive random walk given \( I \). This structure will be mimicked in Section 4. The proof that \( V \) is Markov is deferred until Section 5.
(3.3) Proposition. Suppose that $U$ is homogeneous and Markov. Then for all types $i,j$, locations $y,z$ in $\mathbb{D}$, and times $t = 0,1,\ldots$, we have

\begin{align}
\mathbb{P}(I_{t+1} = j, L_{t+1} = z | I_t = i, L_t = y) & = \mathbb{P}(L_{t+1} - L_t = z - y | I_{t+1} = j, I_t = i) \mathbb{P}(I_{t+1} = j | I_t = i) \\
\end{align}

Proof. We begin by writing the left-hand side of (3.4) in terms of $U$ and using homogeneity:

$$
\mathbb{P}(U_{t+1} = u_j(\cdot - z) | U_t = u_i(\cdot - y)) = \mathbb{P}(U_{t+1} = u_j(\cdot - (z - y + x)) | U_t = u_i(\cdot - x)),
$$

for all $x$ in $\mathbb{D}$. In terms of $I$ and $L$, this is equal to

$$
\mathbb{P}(I_{t+1} = j, L_{t+1} = z - y + x | I_t = i, L_t = x) = \mathbb{P}(I_{t+1} = j, L_{t+1} - L_t = z - y | I_t = i, L_t = x).
$$

As this probability does not depend on $x$, it is equal to

$$
\mathbb{P}(I_{t+1} = j, L_{t+1} - L_t = z - y | I_t = i)
$$

which can be written as the right-hand side of (3.4). \qed

4 Eulerian velocity fields with hidden Markov structure

Motivated by the results of Section 3, we now introduce the class of Eulerian velocity fields with hidden Markov structure that will be considered in the remainder of the paper. Using hidden Markov structure allows a much richer variety of velocity field models than the Markov models of Section 3. For example, it allows model velocity fields in which a vortex such as is depicted in Figure 2.1(b) has, on occasion, a continued eastward motion or, on other occasions, a continued westward motion. In other words, the vortex has a kind of momentum. See Example (6.15).

Generalizing the notion of the shape of a velocity field is what we will call type, which may include additional information such as the preferred direction of motion of the velocity field. Let $T = \{1,\ldots,m\}$ be the set of types. Each type has a shape in the sense that there is a mapping $u : T \rightarrow U$ associating to each type $i$ a velocity field $u_i$. In contrast to Section 3, we now allow $u_i \sim u_j$ for $i \neq j$.

The construction of the velocity field $U$ proceeds as follows. Let $I_t$, $t = 0,1,\ldots$ be a Markov chain on $T$ with transition matrix $R$. We assume that this type process is irreducible and aperiodic. The distribution of $I_0$ is immaterial at this point.
For each \( i \) and \( j \) in \( \mathcal{I} \), let \( c_{ij} : \mathbb{D} \to [0, 1] \) be a function for which \( \sum_{x \in \mathbb{D}} c_{ij}(x) = 1 \). For all types \( i, j \) and times \( t = 0, 1, \ldots \), let \( A_t(i, j) \) be a random variable taking values in \( \mathbb{D} \) with \( \mathbb{P}(A_t(i, j) = x) = c_{ij}(x) \), and, moreover, let the collection \( A = \{ A_t(i, j), i, j \in \mathcal{I}, \ t = 0, 1, \ldots \} \) be mutually independent and independent of \( I \). Let \( L_0 \) be a random variable taking values in \( \mathbb{D} \) and independent of \( A \) and \( I \). Define the location process \( L_t, t = 0, 1, \ldots \) recursively by

\[
L_{t+1} = L_t + A_t(I_t, I_{t+1}), \quad t = 0, 1, \ldots
\]

We must impose a condition on the \( c_{ij} \) to guarantee that for all \( t \) large enough, the distribution of \( L_t \) is supported on all of \( \mathbb{D} \). This can be done by requiring that there be a sequence \( i_1, i_2, \ldots, i_N \) of types for which \( R_{i_1i_2}, \ldots, R_{i_{N-1}i_N} > 0 \) and the convolution of \( c_{i_1i_2}, \ldots, c_{i_{N-1}i_N} \) is supported on all of \( \mathbb{D} \), since by irreducibility of \( R \), the type process \( I \) will make all of these transitions at some point.

It is clear from this construction that the process \( (I_t, L_t), t = 0, 1, \ldots \) is an aperiodic, irreducible Markov chain and that

\[
\mathbb{P}(I_{t+1} = j, L_{t+1} = z | I_t = i, L_t = y) \\
= \mathbb{P}(L_{t+1} - L_t = z - y | I_{t+1} = j, I_t = i) \mathbb{P}(I_{t+1} = j | I_t = i) \\
= c_{ij}(z - y) R_{ij},
\]

in complete accord with (3.4).

Finally, define the velocity field \( U \) by

\[
U_t = u_{I_t}(- L_t), \quad t = 0, 1, \ldots
\]

The velocity field is thus determined by the type–location process \( (I_t, L_t), t = 0, 1, \ldots \). Note that \( U \) is not Markov in general, but will be Markov if knowledge of \( U_t \) allows one to uniquely determine \( I_t \) and \( L_t \), which will be the case if \( u_i \sim u_j \) implies \( i = j \) and if \( u \) takes values in the space of velocity fields with well-defined locations (cf. the discussion above (3.2)). The process \( U_t, t = 0, 1, \ldots \) will be homogeneous if the initial location \( L_0 \) is uniformly distributed on \( \mathbb{D} \). It will be stationary if, in addition, \( I_0 \) has the invariant distribution for \( R \). Thus, the construction does generalize the strictly Markov case of Section 3.

The transition matrix \( P \) of \( (I, L) \) will be needed in later sections. Here we indicate the block and circulant structure of \( P \). Recall (4.2), which shows that

\[
P((i, y); (j, z)) = c_{ij}(z - y) R_{ij}.
\]

For each \( i, j \) in \( \mathcal{I} \), define a matrix \( C_{ij} \) by

\[
C_{ij}(y, z) = c_{ij}(z - y), \quad y, z \in \mathbb{D}.
\]
Note that $C_{ij}$ is doubly indexed by $\mathbb{D}$. When $\mathbb{D}$ is one-dimensional this is quite natural and (4.5) means that $C_{ij}$ is a circulant matrix. In higher-dimensional cases $C_{ij}$ is more awkward to write out as a conventional matrix. However, property (4.5) is the natural analogue of the circulant property and will be used in Section 7. The circulant property is closely related to homogeneity.

Now we may write $P$ as a block matrix:

\begin{equation}
\begin{bmatrix}
R_{11}C_{11} & \cdots & R_{1m}C_{1m} \\
\vdots & \ddots & \vdots \\
R_{m1}C_{m1} & \cdots & R_{mm}C_{mm}
\end{bmatrix}.
\end{equation}

Note that each matrix $C_{ij}$ is doubly stochastic. We think of the blocks as being indexed by type, with type transitions given by the $R_{ij}$. Once the type transition is made, the appropriate circulant matrix $C_{ij}$ is used to determine the transition of the location.

5 The law of the generalized Lagrangian velocity

The generalized Lagrangian velocity is subject to two influences, the change in type and location of the Eulerian velocity field, and the change in perspective brought about by the movement of the particle. In this section we show that the generalized Lagrangian velocity can be thought of in terms of the type process $I$ and a new location process $M$, and that the paired process $(I, M)$ is Markov. We compute its transition matrix explicitly, thus finding the law of the generalized Lagrangian velocity exactly.

Recall from (2.4) that $V_t(x) = U_t(x + X_t)$ and from (4.3) that $U_t(z) = u_L(z - L_t)$. Combining these, we obtain

\[ V_t(x) = u_L(x - (L_t - X_t)). \]

The generalized Lagrangian velocity has the same type process as the Eulerian field. Its location process $M$ is defined by

\begin{equation}
M_t = L_t - X_t, \quad t = 0, 1, \ldots,
\end{equation}

and thus $V_t = u_L(\cdot - M_t)$.

The construction of $L$ in Section 4 now leads naturally to a construction of $M$. With $X_0 = 0$ we have $M_0 = L_0$. The evolution of $M$ goes according to:

\begin{equation}
M_t = L_{t+1} - X_{t+1}
\end{equation}
\[ L_t + A_t(I_t, I_{t+1}) - (X_t + U_t(X_t)) \]
\[ = L_t - X_t - u_{I_t}(X_t - L_t) + A_t(I_t, I_{t+1}) \]
\[ = M_t - u_{I_t}(-M_t) + A_t(I_t, I_{t+1}). \]

This bears striking resemblance to (4.1), the only difference being the term \(-u_{I_t}(-M_t)\), which accounts for the motion of the particle. We collect some obvious conclusions into a theorem.

(5.3) **Theorem.** For the Eulerian velocity field \( U \) of Section 4, the generalized Lagrangian velocity \( V \) can be written as

\[ V_t = u_{I_t}(-M_t), \quad t = 0, 1, \ldots, \]

where \( I \) is a Markov chain on \( \mathcal{I} \) with transition matrix \( R \) and \( M \) is defined recursively by (5.2). The paired process \((I, M)\) is Markov.

Let us rewrite (5.2) in a simpler way. For each \( i \) in \( \mathcal{I} \), define a mapping \( \sigma_i : \mathbb{D} \rightarrow \mathbb{D} \) by

\[ \sigma_i(x) = x - u_i(-x), \quad x \in \mathbb{D}. \]

This differs from the mapping defined in (2.5), but note that \( \sigma_i \) is incompressible if and only if the corresponding \( \alpha_i \) is. Now we rewrite (5.2) side by side with (4.1) for comparison:

\[ L_{t+1} = L_t + A_t(I_t, I_{t+1}), \quad t = 0, 1, \ldots \]
\[ M_{t+1} = \sigma_{I_t}(M_t) + A_t(I_t, I_{t+1}), \quad t = 0, 1, \ldots \]

The only difference, then, between the Eulerian and generalized Lagrangian velocity is that the location process \( M \) for the generalized Lagrangian velocity undergoes a kind of shuffle \( \sigma_{I_t} \) before it is shifted by \( A \), whereas the Eulerian location process is simply shifted.

The transition probabilities for \((I, M)\) are computed as in (4.2):

\[ P(I_{t+1} = j, M_{t+1} = z | I_t = i, M_t = y) \]
\[ = P(M_{t+1} - \sigma_{I_t}(M_t) = z - \sigma_i(y) | I_{t+1} = j, I_t = i) \]
\[ = c_{ij}(z - \sigma_i(y)) R_{ij} \]

Thus, the transition matrix \( Q \) of \((I, M)\) is given by

\[ Q((i, y); (j, z)) = c_{ij}(z - \sigma_i(y)) R_{ij}, \]

similar to (4.4).
However, it is easier to obtain $Q$ from $P$ in the following way. From (5.6) and (5.7), the only difference between $(I, L)$ and $(I, M)$ is that $M$ is first shuffled by $\sigma_I$. We can represent this deterministic transition of $(I, M)$ by a transition matrix

\[
\Sigma = \begin{bmatrix}
\Sigma_1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \Sigma_i & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \Sigma_m
\end{bmatrix},
\]

where $\Sigma_i$ is the transition matrix corresponding to $\sigma_i$. That is, $\Sigma_i(x, y)$ equals 1 if $y = \sigma_i(x)$ and is 0 otherwise. The off-diagonal blocks of $\Sigma$ are zero because only the location changes at this step. The transition given by $\Sigma$ is followed by a transition of $I$ and the addition of $A_i(I_t, I_{t+1})$, which is accomplished by $P$. Thus, we have

\[
Q = \Sigma P = \begin{bmatrix}
R_{11}\Sigma_1 C_{11} & \cdots & R_{1m}\Sigma_1 C_{1m} \\
\vdots & \ddots & \vdots \\
R_{m1}\Sigma_m C_{m1} & \cdots & R_{mm}\Sigma_m C_{mm}
\end{bmatrix}.
\]

We have found the exact law of the generalized Lagrangian velocity $V$ in terms of the law of the Eulerian velocity field $U$. Moreover, since the particle position $X$ satisfies $X_t = X_0 + \sum_{s=0}^{t-1} V_s(0)$, it is an additive functional of the Markov process $(I, M)$, and so its law is known in principle.

When the Eulerian velocity field $U$ is incompressible, the matrices $\Sigma_i$ and $\Sigma$ are permutation matrices which have the effect of permuting the rows of $P$ within each block according to the velocity field corresponding to that block. These matrices are doubly stochastic and invertible. When $U$ is compressible, at least one $\Sigma_i$ is not a permutation matrix, and at least one row of $P$ appears twice in $Q$. As we shall see, this has many consequences for the law of $V$ and makes this case more difficult than the incompressible case.

6 Examples

One very promising feature of the discrete model is that one can study simple examples exactly. The examples in this section are chosen to illustrate the construction given so far and to see what can happen with the rate of convergence to equilibrium for the chains $(I, L)$ and $(I, M)$. One-dimensional examples were chosen because it is easier to decide what is sensible there; in two dimensions there are too many free parameters to choose.
(6.1) Example. *One velocity field type.* Let \( d = 1 \) and \( \mathbb{D} = \{0, 1, 2, 3, 4, 5, 6\} \). Consider only one type of velocity field represented by \( u_1 = [1 -2 0 0 0 0 1] \), as shown in Figure 6.1(a). The type process is trivial, \( I_t = 1 \) for all \( t \). The location process performs a random walk on \( \mathbb{D} \). Suppose the distribution \( c_{11} \) of the increments is given by \( c_{11} = [0.2 \ 0.75 \ 0 \ 0 \ 0 \ 0 \ 0.05]^T \). Thus, when the vortex moves, it moves preferentially clockwise around \( \mathbb{D} \). Figure 6.1(b) shows the vortex with location 3, i.e., \( u_1(\cdot - 3) \).

![Figure 6.1: Velocity field \( u_1 \) with (a) location 0 and (b) location 3.](image)

It is clear that the Eulerian and generalized Lagrangian velocities are Markov in this case. The transition matrix \( P \) of the Eulerian velocity is

\[
P = \begin{bmatrix}
0.2 & 0.75 & 0 & 0 & 0 & 0 & 0.05 \\
0.05 & 0.2 & 0.75 & 0 & 0 & 0 & 0 \\
0 & 0.05 & 0.2 & 0.75 & 0 & 0 & 0 \\
0 & 0 & 0.05 & 0.2 & 0.75 & 0 & 0 \\
0 & 0 & 0 & 0.05 & 0.2 & 0.75 & 0 \\
0.75 & 0 & 0 & 0 & 0 & 0.05 & 0.2
\end{bmatrix}
\]

(6.2)

The permutation \( \sigma_1 \) defined by \( \sigma_1(x) = x - u_1(-x) \) has transition matrix

\[
\Sigma_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(6.3)
Thus, the transition matrix $Q$ of the generalized Lagrangian velocity is given by

\[
Q = \Sigma P = \begin{bmatrix}
0.75 & 0 & 0 & 0 & 0 & 0.05 & 0.2 \\
0.2 & 0.75 & 0 & 0 & 0 & 0 & 0.05 \\
0 & 0.05 & 0.2 & 0.75 & 0 & 0 & 0 \\
0 & 0 & 0.05 & 0.2 & 0.75 & 0 & 0 \\
0 & 0 & 0 & 0.05 & 0.2 & 0.75 & 0 \\
0.05 & 0.2 & 0.75 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(6.4)

For example, when $M_0 = 0$, we see from Figure 6.1(a) that the particle is moved to 1 by the vortex. If the vortex doesn’t move, the value of $M_1$ will be $-1 \equiv 6 \pmod{7}$ since the location of the vortex relative to the particle is $-1$; this happens with probability 0.2 as reflected by $Q_{06} = 0.2$.

The moduli of the eigenvalues of $P$ are given by

\[
(6.5) \quad 1.0000, 0.8876, 0.8876, 0.6828, 0.6828, 0.6029, 0.6029
\]

while the corresponding values for $Q$ are

\[
(6.6) \quad 1.0000, 0.7845, 0.7845, 0.7319, 0.7319, 0.6363, 0.6363
\]

Note that the second-largest modulus for $P$ is larger than that for $Q$. We interpret this to mean that the generalized Lagrangian velocity converges to equilibrium faster than the Eulerian velocity field.

This is a general result for velocity fields with only one vortex type as will be shown in Proposition (9.1). In particular, if the vortex is made to move preferentially counterclockwise by setting $c_1 = [0.2 \ 0.05 \ 0 \ 0 \ 0 \ 0.75]^T$, then the moduli of the eigenvalues of $P$ are unchanged while those of $Q$ become

\[
(6.7) \quad 1.0000, 0.8421, 0.8421, 0.6620, 0.6620, 0.6554, 0.6554
\]

One way to think of this result is that a single vortex undergoing a random walk is not able to "hold onto" a particle, otherwise $V$ would converge to equilibrium more slowly than $U$.

(6.8) **Example.** Two velocity field types. Let $\mathcal{D}$ be the same as in Example (6.1), but now let the Eulerian velocity field change between two vortex types, $u_1 = [1 \ -2 \ 0 \ 0 \ 0 \ 0 \ 1]$ and $u_2 = [-1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 2]$, as shown in Figure 6.2. The type $I$ makes transitions according to

\[
(6.9) \quad R = \begin{bmatrix}
0.8 & 0.2 \\
0.2 & 0.8
\end{bmatrix}
\]
When type 1 is followed by type 1, the additive increment has distribution
\[(6.10) \quad c_{11} = [0.2 \ 0.75 \ 0 \ 0 \ 0 \ 0.05]^T,\]
so that \(u_1\) moves preferentially clockwise around \(\mathbb{D}\). Let
\[(6.11) \quad c_{12} = c_{21} = [1 \ 0 \ 0 \ 0 \ 0 \ 0]^T,\]
so that when type changes, location doesn’t, and finally let
\[(6.12) \quad c_{22} = [0.2 \ 0.05 \ 0 \ 0 \ 0 \ 0.75]^T,\]
so that \(u_2\) moves predominately counterclockwise. This completes the specification of the Eulerian velocity field \(U\).

![Figure 6.2: Velocity fields \(u_1\) and \(u_2\) for Example (6.8).](image)

Forming the matrices \(P, \Sigma,\) and \(Q\), we find that the second-largest eigenvalue moduli (denoted \(\text{eig}_2\)) are
\[(6.13) \quad \text{eig}_2(P) = 0.6813, \quad \text{eig}_2(Q) = 0.8455,\]
so that the generalized Lagrangian velocity \(V\) converges to equilibrium more slowly than \(U\). In the words of the previous example, this pair of velocity fields is able to hold onto a particle. Similar examples can be constructed for two-dimensional velocity fields.

On the other hand, if we reverse the movements of \(u_1\) and \(u_2\) by interchanging the values of \(c_{11}\) and \(c_{22}\), then we have
\[(6.14) \quad \text{eig}_2(P) = 0.6813, \quad \text{eig}_2(Q) = 0.6516,\]
so that the same velocity field types are no longer able to hold onto a particle when they move in this way.

(6.15) **Example.** *Velocity field with momentum.* Consider the velocity field \(u_1\) from Example (6.8). Give it two characteristic directions of motion as follows. Set
\[(6.16) \quad u_2 = u_1 = [1 \ -2 \ 0 \ 0 \ 0 \ 0 \ 1]\]
and define $R, c_{11}, c_{12}, c_{21}, c_{22}$ as in (6.9) – (6.12). Then $U$ always has the same shape, but when the type is 1, the vortex tends to move clockwise around $\mathbb{D}$, whereas when the type is 2, it tends to move counterclockwise. Thus, the vortex has a kind of momentum that makes Markov transitions between clockwise and counterclockwise.

Here we find that

\begin{equation}
\text{eig}_2(P) = 0.6813, \quad \text{eig}_2(Q) = 0.6388,
\end{equation}

so that the generalized Lagrangian velocity converges to equilibrium more quickly than the Eulerian. It is not known whether this is a general result, however.

\begin{equation}
\text{(6.18) Example. Compressible velocity field.} \text{ Consider again Example (6.8) but with one modification: Let}
\end{equation}

\begin{equation}
\text{u}_2 = [0 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0],
\end{equation}

which is a compressible velocity field since it maps both 0 and 1 to 0. See Figure 6.3. The second eigenvalues become

\begin{equation}
\text{eig}_2(P) = 0.6813, \quad \text{eig}_2(Q) = 0.7718.
\end{equation}

This pair of velocity fields is able to hold onto a particle, but this can be foiled by making small changes to the matrix $R$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6.3}
\caption{Velocity fields $u_1$ and $u_2$ for Example (6.18).}
\end{figure}

In incompressible cases, the type $I$ and location $M$ of the generalized Lagrangian velocity are asymptotically independent with $M$ being uniform over $D$, as will be shown in Section 7. However, in the compressible case, they converge to a nontrivial limiting distribution, which we now describe for this example. Given that the type is 1, the distribution of the location is

\begin{equation}
\text{(6.21)} \quad [0.4162 \ 0.1129 \ 0.1049 \ 0.0905 \ 0.0769 \ 0.0797 \ 0.1189]
\end{equation}

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and given that the type is 2, the distribution is

\[
\begin{bmatrix}
0.2033 & 0.1096 & 0.0659 & 0.0500 & 0.0354 & 0.0207 & 0.5151
\end{bmatrix}
\]

The marginal distribution of $M$ is just the arithmetic average of these, since the types occur with equal probability in the limit from the form (6.9) of $R$. It is not uniform on $\mathbb{D}$.

## 7 Eigenvectors of $P$ and $Q$

We begin the study of the eigenvectors and eigenvalues of the transition matrices $P$ and $Q$ underlying $U$ and $V$ in this section. We identify a class of eigenvectors common to both, clarify the structure of the remaining eigenvectors of $Q$ in the incompressible case, and provide more detailed calculations of the eigenvectors of $P$.

We think of $P, \Sigma$, and $Q$ as block matrices, as in (4.6), (5.10), and (5.11). The corresponding vectors are elements of $\mathbb{C}^m \times \mathbb{C}^\mathbb{D}$; we write them in block form as well. When necessary, we think of such a vector $f$ as a function $(i, x) \mapsto f_i(x)$ from $I \times \mathbb{D}$ to $\mathbb{R}$. Here $i$ denotes the block and $x$ the position within the block.

For $b$ in $\mathbb{C}^m$ and $c$ in $\mathbb{C}^\mathbb{D}$, the Kronecker product $b \otimes c$ is a vector $f$ in $\mathbb{C}^m \times \mathbb{C}^\mathbb{D}$ given in block form as

\[
f = b \otimes c = \begin{bmatrix}
b_1c \\
\vdots \\
b_mC
\end{bmatrix},
\]

where $b_1, \ldots, b_m$ are the components of $b$. We may also write $f_i(x) = b_i c(x)$. We denote by $\bar{1}$ the vector in $\mathbb{C}^\mathbb{D}$ with all components equal to 1.

Let $\pi$ denote the invariant distribution of the type transition matrix $R$. Using the fact that the $C_{ij}$ are doubly stochastic, we see that

\[
(\pi^T \otimes \bar{1})^T P = (\pi^T \otimes \bar{1})^T,
\]

so that under the invariant distribution $\frac{1}{n}(\pi^T \otimes \bar{1})^T$, the type $I_t$ is independent of the Eulerian location $L_t$, which is uniformly distributed on $\mathbb{D}$.

Moreover, in the incompressible case, the matrix $\Sigma$ is a permutation matrix, and so

\[
(\pi^T \otimes \bar{1})^T Q = (\pi^T \otimes \bar{1})^T
\]

as well, so the same comments apply to $(I, M)$. In particular, when the Eulerian parameters $(I, L)$ are started in the invariant distribution, both the Eulerian and generalized Lagrangian velocity are stationary with the same invariant distribution.
However, in the compressible case, (7.3) fails because $\Sigma$ is no longer doubly stochastic. The invariant type distribution is still $\pi$, but the Lagrangian location parameter $M$ is no longer independent of the type under the invariant distribution. See Example (6.18).

Vectors of the form $b \otimes \mathbf{1}$ also play an important role for (right) eigenvectors, for we easily compute
\begin{equation}
(7.4) \quad P(b \otimes \mathbf{1}) = Q(b \otimes \mathbf{1}) = (Rb) \otimes \mathbf{1},
\end{equation}
in both the incompressible and compressible cases. Thus, if $b$ is an eigenvector of $R$, then $b \otimes \mathbf{1}$ is an eigenvector of both $P$ and $Q$ with the same eigenvalue as $b$ has for $R$.

Set $\mathcal{G} = \{b \otimes \mathbf{1} : b \in \mathbb{C}^m\}$. The following result shows that in the incompressible case, the eigenvectors of $Q$ split naturally over $\mathcal{G}$ and its orthogonal complement $\mathcal{G}^\perp = \{h \in \mathbb{C}^m \times \mathbb{C}_D : g^*h = 0 \text{ for all } g \in \mathcal{G}\}$, where $*$ denotes conjugate transpose. The result may also be applied to $P$ by taking $\Sigma = I$.

(7.5) **Proposition.** Suppose that $\Sigma$ is a permutation matrix and that $Q = \Sigma P$ is diagonalizable. Then a basis of eigenvectors of $Q$ can be chosen so that each is of one of the following forms:

(i) $b \otimes \mathbf{1}$ where $b$ is a right eigenvector of $R$

(ii) an element of $\mathcal{G}^\perp$

Moreover, $Q$ can be written as the direct sum of its restrictions to $\mathcal{G}$ and $\mathcal{G}^\perp$.

**Proof:** Equation (7.4) shows that $Q \mathcal{G} \subseteq \mathcal{G}$. We claim that $Q \mathcal{G}^\perp \subseteq \mathcal{G}^\perp$ as well. Let $h \in \mathcal{G}^\perp$ and $g = b \otimes \mathbf{1} \in \mathcal{G}$. We must show that $g^*Qh = 0$. But $g^*Qh = g^*\Sigma Ph$, and $g^*\Sigma = g^*$ because $\Sigma$ is a permutation matrix of the form (5.10). Moreover, $g^*P = ((b^*R)^* \otimes \mathbf{1})^*$, and so $g^*Ph = 0$. Thus, $Q \mathcal{G}^\perp \subseteq \mathcal{G}^\perp$. This establishes the last claim of the proposition.

Now let \{ $f^{(j)}$ \} be a basis for $\mathbb{C}^m \times \mathbb{C}_D$ consisting of eigenvectors of $Q$ with eigenvalues $\mu^{(j)}$. We may write $f^{(j)} = g^{(j)} + h^{(j)}$ where $g^{(j)} \in \mathcal{G}$ and $h^{(j)} \in \mathcal{G}^\perp$. Because $Q \mathcal{G} \subseteq \mathcal{G}$ and $Q \mathcal{G}^\perp \subseteq \mathcal{G}^\perp$, we have that $Qg^{(j)} = \mu^{(j)}g^{(j)}$ and $Qh^{(j)} = \mu^{(j)}h^{(j)}$. From among the $g^{(j)}$ we may choose a basis for $\mathcal{G}$, and from the $h^{(j)}$ a basis for $\mathcal{G}^\perp$, which gives the basis claimed in the proposition.

Finally, each $g^{(j)}$ chosen may be written in the form $b \otimes \mathbf{1}$. The equation $Qg = \lambda g$ becomes $(Rb) \otimes \mathbf{1} = (\lambda b) \otimes \mathbf{1}$ by (7.4), and so $b$ must be an eigenvector of $R$, as claimed. \(\square\)

The eigenvectors of $P$ will be treated succinctly after some preliminaries on the eigenvectors of circulant matrices.
Let $c$ be a vector in $\mathbb{C}^d$. Define a matrix $C$ doubly-indexed by $\mathbb{D}$ by

$$C(y, z) = c(z - y), \quad y, z \in \mathbb{D}. \quad (7.6)$$

We say that $C$ is circulant by extension of the case $d = 1$. For each $k$ in $\mathbb{D}$ define a vector $\phi^{(k)}$ in $\mathbb{C}^d$ by

$$\phi^{(k)}(x) = \frac{1}{\sqrt{n}} \exp(2\pi ik \cdot x), \quad x \in \mathbb{D}, \quad (7.7)$$

where $n = n_1 n_2 \cdots n_d$ and the inner product is defined by

$$k \cdot x = \frac{k_1 x_1}{n_1} + \cdots + \frac{k_d x_d}{n_d} \quad (7.8)$$

(it is immaterial whether we regard the multiplication as being modulo $n_1, \ldots, n_d$ or not). The vectors $\phi^{(k)}$ are the Fourier basis. They are orthonormal,

$$\phi^{(k)*} \phi^{(\ell)} = \delta_{k\ell}, \quad k, \ell \in \mathbb{D}, \quad (7.9)$$

and for each $k$ in $\mathbb{D}$, $\phi^{(k)}$ is an eigenvector of $C$, for

$$(C \phi^{(k)})(x) = \sum_{y \in \mathbb{D}} c(y - x) \phi^{(k)}(y) \quad (7.10)$$

$$= \sum_{z \in \mathbb{D}} c(z) \phi^{(k)}(x + z)$$

$$= (\sqrt{n} c^T \phi^{(k)}) \phi^{(k)}(x).$$

The associated eigenvalue is $\lambda^{(k)} = \sqrt{n} c^T \phi^{(k)}$.

The eigenvectors of $P$ split over a large number of mutually orthogonal subspaces, as the next result will show. It is an extension of Proposition (7.5) for the case $\Sigma = I$. Set

$$G^{(k)} = \{ b \otimes \phi^{(k)} : b \in \mathbb{C}^m \}. \quad (7.11)$$

Then $G^0 = G$ from above. Let $\lambda^{(k)}_{ij}$ denote the eigenvalue of $C_{ij}$ corresponding to $\phi^{(k)}$. For each $k$ in $\mathbb{D}$, let $R^{(k)}$ be the matrix

$$R^{(k)} = [R_{ij} \lambda^{(k)}_{ij}]. \quad (7.12)$$

Note that $R^{(0)} = nR$.

(7.13) **Proposition.** Suppose that $P$ is diagonalizable. Then the eigenvectors of $P$ are $b^{(i,k)} \otimes \phi^{(k)}$, $i \in \mathcal{I}$, $k \in \mathbb{D}$ where $b^{(i,k)}$, $i \in \mathcal{I}$ are the eigenvectors of $R^{(k)}$. 
Proof: A simple computation similar to (7.4) shows that

$$P(b \otimes \phi^{(k)}) = (R^{(k)}b) \otimes \phi^{(k)}.$$  

Thus, for each $k$ in $\mathbb{D}$, $PG^{(k)} \subseteq G^{(k)}$. The subspaces $G^{(k)}$ are clearly mutually orthogonal by (7.9). We now proceed as in the proof of Proposition (7.5) to select a basis $b^{(i,k)} \otimes \phi^{(k)}$, $i \in I$ of eigenvectors of $P$ for each subspace $G^{(k)}$ and recognize that the $b^{(i,k)}$ must be eigenvectors of $R^{(k)}$. 

8 Absolute bound on eigenvalues

The theorem below gives an absolute bound on the moduli of the eigenvalues of $P$ and $Q$ in terms of the type transition matrix $R$ and the eigenvalues of the matrices $C_{ij}$. This bound gives an estimate of the rate of convergence to equilibrium of the Lagrangian velocity and will play a role in the relative bound of Section 9.

Let $B$ be a diagonalizable matrix. Denote by $\text{eig}_1(B)$ the largest of the moduli of the eigenvalues of $B$, by $\text{eig}_2(B)$ the second largest, and so on. Similarly, we will write $\text{eig}_1(B, \mathcal{H})$ for the largest eigenvalue modulus corresponding to eigenvectors in the subspace $\mathcal{H}$.

If $B$ is an irreducible diagonalizable Markov matrix, then $\text{eig}_1(B) = 1$ and $\text{eig}_2(B) < 1$. The size of $\text{eig}_2(B)$ bounds the rate at which powers of $B$ converge to their limiting value; smaller $\text{eig}_2(B)$ means faster convergence to equilibrium for the Markov chain. The number $1 - \text{eig}_2(B)$ is called the spectral gap.

As discussed after (7.4), the matrices $P$ and $Q$ share eigenvectors of the form $b \otimes \mathbf{1}$ where $b$ is an eigenvector of $R$. Thus all three matrices have $m$ eigenvalues in common. As these eigenvalues depend only on type transitions and not on the velocity fields encoded in $\Sigma$, they are of relatively little interest.

We will henceforth restrict attention to the incompressible case because, by Proposition (7.5), the remaining eigenvalues correspond to eigenvectors which are known to lie entirely in $G^{\perp}$.

(8.1) **Theorem.** Suppose that $U$ is incompressible and that $Q = \Sigma P$ is diagonalizable. Then

$$\text{eig}_1(Q, G^{\perp}) \leq \text{eig}_1(T),$$  

where $T$ is the $m \times m$ matrix

$$T = [R_{ij}\text{eig}_2(C_{ij})].$$  

(Take $\Sigma = I$ to obtain the same bound for $P$.)
**Proof.** Let \( h \in \mathcal{G}^\perp \) be an eigenvector of \( Q \) with eigenvalue \( \mu \). Writing \( h_i \) for the blocks of \( h \), the equation \( Qh = \mu h \) becomes

\[
(8.4) \quad \sum_{j=1}^{m} R_{ij} \Sigma_i C_{ij} h_j = \mu h_i, \quad i = 1, \ldots, m.
\]

Fix \( i \). Taking the \( \mathbb{C}^D \) vector norm and using the fact that it is invariant under the permutation matrix \( \Sigma_i \), we obtain

\[
(8.5) \quad \| \mu \| \| h_i \| = \left\| \sum_{j=1}^{m} R_{ij} C_{ij} h_j \right\| \leq \sum_{j=1}^{m} R_{ij} \| C_{ij} h_j \|,
\]

by the triangle inequality.

Now \( h \) is in \( \mathcal{G}^\perp \) and so, in particular, \( h \) is orthogonal to \( e_j \otimes \bar{1} \) where \( e_j \) is the \( j \)th standard basis vector for \( \mathbb{C}^m \). Thus, \( \bar{1}^T h_j = 0 \), and so \( h_j \) is orthogonal to \( \phi^{(k)} \). Writing \( h_j \) as a linear combination of the \( \phi^{(k)} \), \( k \neq 0 \), we see that

\[
(8.6) \quad \| C_{ij} h_j \|^2 = \sum_{k \neq 0} |\phi^{(k)*} h_j|^2 |\lambda_{ij}^{(k)}|^2 \leq \text{eig}_2(C_{ij})^2 \| h_j \|^2,
\]

and thus \( \| C_{ij} h_j \| \leq \text{eig}_2(C_{ij}) \| h_j \| \). Using this in (8.5) yields

\[
(8.7) \quad |\mu| \| h_i \| \leq \sum_{j=1}^{m} R_{ij} \text{eig}_2(C_{ij}) \| h_j \| = \sum_{j=1}^{m} T_{ijj} \| h_j \|,
\]

using (8.3).

Now \( T \) has non-negative entries, so by a corollary of the Perron–Frobenius Theorem (cf. Theorem 15.5.1 of Lancaster and Tismenetsky (1985)), \( T \) has a left eigenvector \( a \) with non-negative entries and a positive real eigenvalue equal to \( \text{eig}_1(T) \). Multiplying (8.7) by \( a_i \) and summing over \( i \) yields

\[
(8.8) \quad |\mu| \sum_{i=1}^{m} a_i \| h_i \| \leq \sum_{j=1}^{m} \sum_{i=1}^{m} a_i T_{ijj} \| h_j \| = \sum_{j=1}^{m} \text{eig}_1(T) a_j \| h_j \|,
\]

from which we conclude \( |\mu| \leq \text{eig}_1(T) \).

\[
(8.9) \quad \text{Remarks.} \quad \text{a) The number } \text{eig}_1(Q, \mathcal{G}^\perp) \text{ pertains to the rate at which the Lagrangian location parameter converges to equilibrium, which is the uniform distribution on } \mathbb{D} \text{ in the incompressible case (cf. (7.3)). It is natural that it should be bounded in terms of } \text{eig}_2(C_{ij}), \text{ since these numbers measure rate of convergence to uniform for motion in } \mathbb{D} \text{ under the transition matrices } C_{ij}.
\]
b) What is still lacking is an understanding of the effect of the permutation matrix $\Sigma$ on the eigenvalues of $Q$. Theorem (8.1) makes no condition on $\Sigma$ other than that it be a permutation matrix, so it applies just as well to the case $\Sigma = I$.

c) The values of $\text{eig}_1(T)$ in Examples (6.1), (6.8), and (6.15) are, respectively, 0.8876, 0.9101, and 0.9101.

9 Relative bound on eigenvalues I: $I$ and $L$ independent

In this section and the next we identify conditions on the Eulerian velocity field which are sufficient to guarantee that $\text{eig}_1(Q, G^\perp) \leq \text{eig}_1(P, G^\perp)$, meaning that the generalized Lagrangian velocity $V$ converges to equilibrium at least as fast as $U$. These conditions are rather restrictive as we shall see. And yet relative bounds appear to be crucial in continuous space and time, where useful absolute bounds on eigenvalues may not be available. So it is good to develop them in the discrete setting first.

(9.1) Proposition. Suppose that $U$ is incompressible, $C_{ij} = C$ for all $i, j$ in $I$ and that $R$ is diagonalizable. Then $\text{eig}_1(Q, G^\perp) \leq \text{eig}_1(P, G^\perp)$.

Proof: First we compute $\text{eig}_1(P, G^\perp)$. Let $b^{(i)}, i = 1, \ldots, m$ denote the eigenvectors of $R$ and $\rho^{(i)}$ the corresponding eigenvalues. Recalling (7.12), we have

\begin{equation}
R^{(k)} = [R_{ij}^k \lambda^{(k)}] = \lambda^{(k)}R,
\end{equation}

and so the $b^{(i)}$ are also the eigenvectors of $R^{(k)}$. The eigenvectors of $P$ are thus of the form $b^{(i)} \otimes \phi^{(k)}$ and the corresponding eigenvalues are $\rho^{(i)} \lambda^{(k)}$. The eigenvalues over $G^\perp$ correspond to $k \neq 0$, and the largest of these occurs when $\rho^{(i)} = 1$ and $|\lambda^{(k)}| = \text{eig}_2(C)$. Thus $\text{eig}_1(P, G^\perp) = \text{eig}_2(C)$.

Now recall (8.3). In the current case,

\begin{equation}
T = [R_{ij} \text{eig}_2(C)] = \text{eig}_2(C)R,
\end{equation}

and so $\text{eig}_1(T) = \text{eig}_2(C)$. Theorem (8.1) gives

\begin{equation}
\text{eig}_2(Q, G^\perp) \leq \text{eig}_1(T) = \text{eig}_2(C) = \text{eig}_1(P, G^\perp),
\end{equation}

which is the desired result. $\square$

Recall that $P$ and $Q$ are the transition matrices of $(I, L)$ and $(I, M)$, respectively. The eigenvalues due to the common element $I$ are the eigenvalues over $G$, which are the same
by Equation (7.4). The remaining eigenvalues concern the rate at which $L$ and $M$ converge to their limiting distributions on $\mathbb{D}$, which are uniform due to incompressibility.

The inequality $\text{eig}_1(Q, G^\perp) \leq \text{eig}_1(P, G^\perp)$ can be interpreted to mean that $M$ converges to uniform faster than $L$. Now $M$ is the location of the velocity field relative to the moving particle $X$, while $L$ is the location relative to the origin. Turning this around, $M$ is the location of the particle relative to the velocity field, while $L$ is the location of the origin. That the relative location $M$ converges to uniform faster than $L$ means that the velocity field is unable to "hold" the particle near it, but rather loses track of the particle faster than it loses track of the origin.

This result applies, in particular, when there is only one velocity field type. It is a bit surprising because this situation so closely resembles so-called ring currents in the ocean which preserve their shape and carry distinct water masses even as they move great distances in the ocean (Robinson (1983)). We do not understand the discrepancy.

It is also interesting to interpret this result in terms of the turbulent diffusion of a particle. Equations (5.6) and (5.7) become, in this case,

\begin{align}
L_{t+1} &= L_t + A_t \\
M_{t+1} &= \sigma_t(M_t) + A_t.
\end{align}

We have written $A_t$ in place of $A_t(I_t, I_{t+1})$ because the law of $A_t(i, j)$ does not depend on $i$ and $j$ when $C_{i,j} = C$. It is clear that the processes $I$ and $L$ are independent.

These equations can be thought of as describing the motion of a particle $L$ undergoing diffusion and a particle $M$ subject to advection ($\sigma_t$) and diffusion. The advection is determined by the Markov process $I$. Note that the advecting velocity field $W_t(x) = \sigma_t(x) - x$ need not be homogeneous, and can be made to have virtually any law by suitably enlarging the state space of $I$. It need only be incompressible.

Now $L$ alone is Markov but $M$ alone is not. For a proper comparison of their rates of convergence to equilibrium we compare the Markov chains $(I, L)$ and $(I, M)$. The eigenvalues over $G^\perp$ are germaine to the rate of convergence of the distributions of $L$ and $M$ to the uniform distribution, and Proposition (9.1) concludes that $\text{eig}_1(Q, G^\perp) \leq \text{eig}_1(P, G^\perp)$, which we interpret to mean that diffusion plus incompressible advection makes particle location converge to uniform more quickly than diffusion alone. This is a perfectly natural result. The analogue for motion on $\mathbb{Z}^d$ or $\mathbb{R}^d$ is that the effective diffusivity exceeds the molecular diffusivity; see Section 12 and Isichenko (1992), Equation (4.16).
10 Relative bound on eigenvalues II: Reversible \((I, L)\)

When the Eulerian velocity field is incompressible and reversible, we will see that \(\text{eig}_1(Q, G^\perp)\) is less than or equal to \(\text{eig}_1(P, G^\perp)\), as in Section 9. The inspiration for this case is Section 2 of Carmona and Xu (1997), where the Eulerian velocity field is incompressible, Markov, and reversible (its generator is self-adjoint with respect to the invariant measure). These facts are used to bound the spectral gap of the generalized Lagrangian velocity away from 0 in terms of the spectral gap of the Eulerian velocity, which is the continuous-time analogue of what we do in this section. Our result suggests that it may be possible to carry this program through in some generality in continuous space and time.

Reversible Markov processes and homogenization were also considered by Kipnis and Varadhan (1986) and De Masi et al. (1989). The object of study was an additive functional of a reversible Markov process. However, in our case, the particle position \(X\) is not an additive functional of \(U\). Rather, it is an additive functional of \(V\) through the equation 
\[
X_t = X_0 + \sum_{s=0}^{t-1} V_s(0),
\]
but it is highly unlikely that \(V\) will be reversible, as one may check by considering its transition matrix \(Q = \Sigma P\).

As in Section 7, let \(\pi\) be the invariant distribution of the type process \(I\), so that \(\frac{1}{n}(\pi \otimes \mathbf{1})\) is the invariant distribution of \(U\). Note that the components of \(\pi\) are non-zero. Set
\[
(10.1) \quad \Pi = \text{diag}(\pi \otimes \mathbf{1}).
\]
The Markov chain \((I_t, L_t), t = 0, 1, \ldots\) is reversible if \(\Pi P\) is symmetric; then the law of \(U\) is invariant under time reversal.

Reversibility puts rather severe restrictions on \(U\). Consider that
\[
(10.2) \quad \Pi P = \begin{bmatrix}
\pi_1 R_{11} C_{11} & \cdots & \pi_1 R_{1m} C_{1m} \\
\vdots & & \vdots \\
\pi_m R_{m1} C_{m1} & \cdots & \pi_m R_{mm} C_{mm}
\end{bmatrix}.
\]
Looking at the \(ij\) block, symmetry of \(\Pi P\) requires \(\pi_i R_{ij} C_{ij} = \pi_j R_{ji} C_{ji}^T\). This is a matrix equation. Summing across the first row yields
\[
(10.3) \quad \pi_i R_{ij} = \pi_j R_{ji},
\]
because \(C_{ij}\) and \(C_{ji}\) are doubly stochastic. With this said, we must also have
\[
(10.4) \quad C_{ij} = C_{ji}^T.
\]
Thus, when \((I, L)\) is reversible, the type process \(I\) must be reversible, and the motion of the velocity fields must satisfy (10.4). In particular, when type \(i\) is followed by type \(i\),
the location must make a transition according to a symmetric circulant matrix \( C_{ii} \). This prevents preferential drift in any direction for type \( i \).

(10.5) **Proposition.** Suppose \( U \) is incompressible and that \((I,L)\) is reversible. Then \( \text{eig}_1(Q, G^\perp) \leq \text{eig}_1(P, G^\perp) \).

**Proof:** For vectors \( f \) and \( g \) in \( \mathbb{C}^n \times \mathbb{C}^\mathbb{D} \), define an inner product by

\[
(f, g)_\Pi = f^* \Pi g,
\]

where \( * \) denotes conjugate transpose. The inner product induces a norm by

\[
\|f\|_\Pi^2 = (f, f)_\Pi = \sum_{i=1}^m \pi_i \sum_{x \in \mathbb{D}} |f_i(x)|^2 = \sum_{i=1}^m \pi_i \|f_i\|^2.
\]

Because \( \Sigma \) is a permutation matrix with block diagonal form (5.10), it does not change the norms of the blocks \( f_i \), and so \( \|\Sigma f\|_\Pi = \|f\|_\Pi \) for all \( f \) in \( \mathbb{C}^n \times \mathbb{C}^\mathbb{D} \).

Let \( h \in G^\perp \) be an eigenvector of \( Q \) with eigenvalue \( \mu \). Then

\[
|\mu| \|h\|_\Pi = \|\mu h\|_\Pi = \|Qh\|_\Pi = \|\Sigma P h\|_\Pi = \|P h\|_\Pi
\]

by the preceding paragraph.

An easy computation shows that \( P \) is self-adjoint with respect to the inner product (10.6). Also, by the proof of Proposition (7.5), \( P \) preserves the subspace \( G^\perp \), and so its restriction to \( G^\perp \) is self-adjoint. Thus, by the spectral theorem, there exists a basis for \( G^\perp \) consisting of eigenvectors of \( P \), and these eigenvectors are orthonormal with respect to the inner product (10.6). A standard argument (cf. (8.6)) shows that \( \|P h\|_\Pi \leq \text{eig}_1(P, G^\perp) \|h\|_\Pi \). Combining this with (10.8) yields \( |\mu| \leq \text{eig}_1(P, G^\perp) \), which was to be shown.

\[\square\]

### 11 Molecular diffusion

To this date, most results on the convergence (under certain scalings) of the trajectory of a single particle in a random or periodic velocity field have relied heavily on the presence of a diffusion term in addition to the velocity field. (Exceptions are Molchanov (1996), Carmona and Xu (1997) with \( \kappa = 0 \), Komorowski and Papanicolaou (1997), and Fanjiqian and Komorowski (1999).) One may speculate that diffusion is helpful because it insures a certain rate of convergence to equilibrium of the (Lagrangian) velocity of the particle.
Let us investigate the effect of an analogue of molecular diffusion in the discrete setting. Recall Equation (2.3) for motion with diffusion:

\[
X_{t+1} = X_t + U_t(X_t) + \Delta_t, \quad t = 0, 1, \ldots
\]

The common distribution of \( \Delta_t \) is given by a function \( d_0 \) on \( \mathbb{D} \). The strength of the diffusion can be controlled by concentrating \( d_0 \) around 0 or not.

The diffusion does not affect the Eulerian velocity field \( U \), of course. But it does affect the generalized Lagrangian velocity: Equation (5.7) for the Lagrangian location parameter becomes

\[
M_{t+1} = \sigma I_t(M_t) + \Delta_t + A_t(I_t, I_{t+1}), \quad t = 0, 1, \ldots
\]

It is easy to see that \((I, M)\) is still Markov and that its transition matrix \( Q_\Delta \) can be written as

\[
Q_\Delta = \Sigma D P,
\]

where \( D \) is a block diagonal matrix with each diagonal block equal to the circulant matrix \( D_0 \) having \( d_0 \) as its 0th row. In the degenerate case in which \( \Delta_t = 0 \) with probability 1, \( D \) reduces to the identity matrix and (11.3) reduces to (5.11).

The presence of diffusion modifies the absolute bound on eigenvalues for incompressible velocity fields given in Theorem (8.1) in the following way. The product \( DP \) above is a block matrix of the form \([R_{ij}D_0C_{ij}]\), and so in Equation (8.3) we may replace \( C_{ij} \) by \( D_0C_{ij} \). As \( D_0 \) and \( C_{ij} \) are both circulant, they share eigenvectors \( \delta^{(k)}, k \in \mathbb{D} \). Denoting the eigenvalues of \( D_0 \) and \( C_{ij} \) by \( \delta^{(k)} \) and \( \lambda_{ij}^{(k)} \), the eigenvalues of \( D_0C_{ij} \) are equal to \( \delta^{(k)}\lambda_{ij}^{(k)} , k \in \mathbb{D} \). Then (8.3) can be replaced by

\[
(T_\Delta)_{ij} = R_{ij} \max(|\delta^{(k)}\lambda_{ij}^{(k)}| , k \neq 0),
\]

and Equation (8.2) reads \( \text{eig}_1(Q_\Delta, G^+) \leq \text{eig}_1(T_\Delta) \).

In particular, we have

\[
(T_\Delta)_{ij} \leq R_{ij}\text{eig}_2(D_0)\text{eig}_2(C_{ij}),
\]

from which,

\[
\text{eig}_1(Q_\Delta, G^+) \leq \text{eig}_1(T_\Delta) \leq \text{eig}_2(D_0)\text{eig}_1(T) \leq \text{eig}_2(D_0).
\]

The first inequality shows a reduction in the absolute bound on eigenvalues due to the presence of diffusion, and the second shows an absolute bound which depends solely on the molecular diffusion.

While the presence of diffusion lowers the eigenvalue bound \( \text{eig}_1(T_\Delta) \), in specific instances it can be shown to \textit{increase} \( \text{eig}_1(Q_\Delta, G^+) \). That is, in some cases we may have

\[
\text{eig}_1(\Sigma P, G^+) < \text{eig}_1(\Sigma DP, G^+).
\]
In such cases, the addition of diffusion slows the convergence of the Lagrangian location parameter $M$ to its equilibrium distribution.

(11.8) **Example.** **Diffusion slows convergence.** Consider again the velocity fields of Example (6.8), but with the following definitions:

$$R = \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{bmatrix}$$

$$c_{11} = [0.262 \ 0.143 \ 0.037 \ 0.098 \ 0.162 \ 0.298]^T$$
$$c_{12} = [0.275 \ 0.184 \ 0.042 \ 0.035 \ 0.189 \ 0.275]^T$$
$$c_{21} = [0.099 \ 0.211 \ 0.104 \ 0.106 \ 0.132 \ 0.348]^T$$
$$c_{22} = [0.166 \ 0.237 \ 0.113 \ 0.147 \ 0.158 \ 0.179]^T$$

$$d_0 = [0.5 \ 0.25 \ 0 \ 0 \ 0 \ 0.25]^T.$$ Then we find

(11.9) $$\text{eig}_1 (\Sigma P, G) = 0.1623, \quad \text{eig}_1 (\Sigma DP, G) = 0.2201$$

so that the addition of diffusion slows convergence to equilibrium.

Finally, let us consider the results of Sections 9 and 10 in the presence of diffusion. The case $C_{ij} = C$ is already closely connected with diffusion; we simply replace $C$ by $D_0 C$. The only change is that now Proposition (9.1) may conclude with the stronger inequality

(11.10) $$\text{eig}_1 (Q_\Delta, G) \leq \text{eig}_2 (D_0) \text{eig}_1 (P, G)$$

using (11.6).

The reversible case is similar to the proof of Proposition (10.5). Letting $h \in G$ be an eigenvector of $Q_\Delta = \Sigma DP$ and returning to (10.7), we find

(11.11) $$\| DPh \|_H^2 = \sum_{i=1}^m \pi_i \| D_0 (Ph)_i \|^2 \leq \text{eig}_2 (D_0)^2 \sum_{i=1}^m \pi_i \| (Ph)_i \|^2$$

$$= \text{eig}_2 (D_0)^2 \| Ph \|_H^2$$

$$= \text{eig}_2 (D_0)^2 \text{eig}_1 (P, G) \| h \|_H^2.$$ We have written $(Ph)_i$ for the $i$th block of $Ph$ and have used the fact from the proof of Proposition (7.5) that $Ph \in G$ and a bound similar to (8.6). The upshot for the reversible case is that

(11.12) $$\text{eig}_1 (Q_\Delta, G) \leq \text{eig}_2 (D_0) \text{eig}_1 (P, G),$$

and so again the presence of diffusion lowers the bound.
12 Homogenization

In this section we consider a particle moving in \( \mathbb{Z}^d \) according to a time-varying, spatially periodic velocity field together with molecular diffusion. We will see that the one-particle motion converges to Brownian motion upon rescaling space and time in the usual way, and we will compute the limiting diffusion coefficient exactly in terms of the law of the Eulerian velocity field and the distribution of the diffusion. The result holds for compressible and incompressible cases.

Let \( U \) be an Eulerian velocity field as in Section 4 but now regard \( U \) as taking values in \( \mathbb{Z}^d \), so that we may distinguish between, say, motion one unit to the right versus \( n_1 - 1 \) units to the left. If molecular diffusion is incorporated (as in Section 11), regard \( \Delta t \), \( t = 0, 1, \ldots \) as taking values in \( \mathbb{Z}^d \) as well.

Repeat \( U \) periodically throughout \( \mathbb{Z}^d \) and let \( Y_t \) be the trajectory of a particle moving in this velocity field with \( Y_0 = 0 \). Then \( Y \) satisfies

\[
Y_{t+1} = Y_t + U(Y_t \text{ mod } \mathbb{Z}) + \Delta t, \quad t = 0, 1, \ldots.
\]

If we set \( X_t = Y_t \text{ mod } \mathbb{Z} \), then \( X \) evolves as in Section 11, as does the type–location process \((I, M)\). Then,

\[
Y_{t+1} = Y_t + u_{I_t}(-M_t) + \Delta t, \quad t = 0, 1, \ldots.
\]

The process \((I, M, \Delta)\) is Markov and \( Y \) is an additive functional of it. As such, we expect that \( Y \) will converge to Brownian motion when properly scaled, and this is shown in the following theorem. More importantly, the limiting diffusivity is expressed to the extent possible in terms of the transition matrix \( Q_\Delta \) of \((I, M)\) and the covariance of \( \Delta \), rather than using the much larger transition matrix of \((I, M, \Delta)\).

In order to state the theorem we need to set some notation. Let \( \pi \) be the invariant distribution corresponding to the transition matrix \( Q_\Delta \) of (11.3). Let \( \pi u = \sum_{i,m} \pi(i, m) u_i(-m) \) be the mean drift due to \( U \) and \( \mathbb{E} \Delta = \sum_{a \in \mathbb{Z}^d} d_0(a) a \) be the mean drift due to the diffusion. For integer values of \( t \) let \( Z_t = Y_t - (\pi u + \mathbb{E} \Delta) t \) and for the remaining values of \( t \) define \( Z_t \) by linear interpolation. Finally, define an inner product for functions from \( \mathcal{I} \times \mathbb{Z} \) to \( \mathbb{R} \) by

\[
\langle f, g \rangle_\pi = \sum_{i,m} \pi(i, m) f(i, m) g(i, m).
\]

(12.3) **Theorem.** Suppose that \( Q_\Delta \) is irreducible and aperiodic. Then as \( \varepsilon \to 0 \), the processes \( \varepsilon Z_{t/\varepsilon^2} \), \( t \geq 0 \) converge in distribution to \( d \)-dimensional Brownian motion with zero drift and covariance matrix \( \beta \) given by

\[
\beta^{k\ell} = -\langle f^k, f^\ell \rangle_\pi - \langle f^k, g^\ell \rangle_\pi - \langle g^k, f^\ell \rangle_\pi + \text{Cov}(\Delta^k, \Delta^\ell) - \langle 1, \Sigma D_k P g^\ell + \Sigma D_\ell P g^k \rangle_\pi,
\]

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where $f$ is the $\mathbb{Z}^d$-valued function given by $f(i, m) = u_i(-m) - \pi u$ and $g^k$, $k = 1, \ldots, d$ is the solution of
\begin{equation}
(I - Q_\Delta) g^k = -f^k,
\end{equation}
and $D_k$ is the matrix
\begin{equation}
D_k = \sum_{a \in \mathbb{Z}^d} d_0(a) (a^k - IE \Delta^k) \Sigma_a
\end{equation}
where for each $a$ in $\mathbb{Z}^d$, $\Sigma_a$ is the transition matrix on $\mathcal{I} \times \mathbb{D}$ corresponding to the addition of $a$ to the second component modulo $\mathbb{D}$.

(12.7) **Remark.** The first three terms in (12.4) reflect the limiting covariance of
$\sum_{s=0}^{t-1} u_{I_s}(-M_s)$, the fourth comes straight from molecular diffusion, and the last term reflects an interaction between diffusion and advection. In the absence of molecular diffusion,
\begin{equation}
\beta^{k\ell} = -\langle f^k, f^\ell \rangle_\pi - \langle f^k, g^\ell \rangle_\pi - \langle g^k, f^\ell \rangle_\pi,
\end{equation}
where $g^k$ satisfies $(I - Q) g^k = -f^k$.

**Proof:** By (12.2),
\begin{equation}
Y_t - (\pi u + IE \Delta)t = \sum_{s=0}^{t-1} \hat{f}(M_s, \Delta_s),
\end{equation}
where
\begin{equation}
\hat{f}(i, m, a) = (u_i(-m) - \pi u) + (a - IE \Delta) = f(i, m) + \mu(a),
\end{equation}
with $\mu(a) = a - IE \Delta$. The process $(I, M, \Delta)$ is a Markov chain with transition matrix $\hat{Q}$ given by
\begin{equation}
\hat{Q}(i, m, a; j, n, b) = R_{ij} C_{ij}(n - \sigma_i(m) - a)d_0(b).
\end{equation}
This chain has state space $\mathcal{I} \times \mathbb{D} \times \text{support}(d_0)$ and is irreducible aperiodic because $Q_\Delta$ is irreducible aperiodic and the value of $\Delta$ is chosen independently at each step. The invariant distribution $\hat{\pi}$ of $(I, M, \Delta)$ has the form $\pi(i, m)d_0(a)$, for
\begin{equation}
\sum_{i, m, a} \pi(i, m) d_0(a) R_{ij} C_{ij}(n - \sigma_i(m) - a)d_0(b) = \sum_{i, m} \pi(i, m) Q_\Delta(i, m; j, n)d_0(b) = \pi(j, n)d_0(b),
\end{equation}
the first equality by (11.3) and the discussion after it, the second by $\pi Q_\Delta = \pi$.

Note that the mean of $\hat{f}$ under $\hat{\pi}$ is zero. Thus, for each $k = 1, \ldots, d$, the equation
$(I - \hat{Q}) g^k = -\hat{f}^k$ has a solution, as will be explained below. By Theorem VII.3.74 of Jacod
and Shiryaev (1987), the processes \( \varepsilon Z_{t/\varepsilon^2} \), \( t \geq 0 \) converge in distribution as \( \varepsilon \to 0 \) to a Brownian motion with zero drift and covariance matrix \( \beta \) given by

\[
\beta^{kt} = -(\hat{f}^k, \hat{f}^\ell)_x - (\hat{f}^k, \hat{g}^\ell)_x - (\hat{g}^k, \hat{f}^\ell)_x.
\]

The inner product here is analogous to \( \langle \cdot, \cdot \rangle_x \) defined before the theorem. To be precise, Jacod and Shiryaev (1987) establish (12.12) for \( k = \ell \), but it may be checked for \( k \neq \ell \) by consideration of the limiting variance of the additive functional based on \( \hat{f}^k + \hat{f}^\ell \) and polarization.

We now compute \( \hat{g} \) and \( \beta \) in terms of the smaller matrices \( P, \Sigma, Q_\Delta \), and \( \Sigma_a \). The solution of \( (I - \hat{Q}) \hat{g}^k = -\hat{f}^k \) is

\[
\hat{g}^k = -\sum_{n=0}^{\infty} \hat{Q}^n \hat{f}^k,
\]

which converges because \( \hat{\pi} \hat{f} = 0 \) and \( \hat{Q} \) is irreducible aperiodic. Now \( \hat{Q}^n \mu = 0 \) for \( n = 1, 2, \ldots \), so we have

\[
\hat{g}^k = -\mu^k - \sum_{n=0}^{\infty} \hat{Q}^n f^k,
\]

where we are thinking of \( f \) as a function of \( i, m \), and \( a \), although it does not really depend on \( a \).

The infinite sum in (12.13) may be written

\[
\sum_{n=0}^{\infty} (\hat{Q}^n f^k)(i, m, a) = f(i, m) + \sum_{n=1}^{\infty} \mathbb{E}[f^k(I_n, M_n)|I_0 = i, M_0 = m, \Delta_0 = a].
\]

The influence of \( \Delta_0 \) does not last long. It only affects the value of \( M_1 \), since \( \Delta_1 \) is independent of \( (I_0, M_0, \Delta_0) \). After time 1, \( (I, M) \) evolves exactly as in Section 11, with transition matrix \( Q_\Delta \), and we may ignore the value of \( \Delta \). For the first step, note that from (11.2),

\[
M_1 = \sigma_1(m) + a + A_0(i, I_1),
\]

so that the first transition of \( (I, M) \) is according to the matrix \( \Sigma \Sigma_a P \), where \( \Sigma_a \) corresponds to the deterministic addition of \( a \) modulo \( \mathbb{Z} \). Thus,

\[
\mathbb{E}[f^k(I_n, M_n)|I_0 = i, M_0 = m, \Delta_0 = a] = (\Sigma \Sigma_a PQ^{-1}_{\Delta} f^k)(i, m),
\]

where we have returned to regarding \( f \) as a function of \( i \) and \( m \) alone. By changing the index of summation, (12.13) becomes

\[
\hat{g}^k(\cdot, \cdot, a) = -\mu^k - f^k - \Sigma \Sigma_a P \sum_{n=0}^{\infty} Q^n f^k
\]

\[
= -\mu^k - f^k + \Sigma \Sigma_a P g^k,
\]

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where \( g^k \) satisfies (12.5).

Finally, we compute the terms in (12.12). Because \( \sum_{a \in \mathbb{Z}_d} d_0(a) \mu(a) = 0 \),

\[
\langle \hat{f}^k, \hat{f}^\ell \rangle \pi = \sum_{i,m,a} \pi(i,m)d_0(a)(f^k(i,m) + \mu^k(a))(f^\ell(i,m) + \mu^\ell(a))
\]

\[
= \langle f^k, f^\ell \rangle \pi + \text{Cov}(\Delta^k, \Delta^\ell).
\]

Next, from (12.14), \( \langle \hat{f}^k, \hat{g}^\ell \rangle \pi \) equals

\[
\sum_{i,m,a} \pi(i,m)d_0(a)(f^k(i,m) + \mu^k(a))(-\mu^\ell(a) - f^\ell(i,m) + (\Sigma \sum a P g^\ell)(i,m))
\]

\[
= -\langle f^k, f^\ell \rangle \pi + \sum_{i,m} \pi(i,m)f^k(i,m)(Q_\Delta g^\ell)(i,m)
\]

\[
- \text{Cov}(\Delta^k, \Delta^\ell) + \sum_{i,m} \pi(i,m)(\Sigma D_k P g^\ell)(i,m)
\]

\[
= \langle f^k, g^\ell \rangle \pi - \text{Cov}(\Delta^k, \Delta^\ell) + \langle 1, \Sigma D_k P g^\ell \rangle \pi,
\]

using \( Q_\Delta g^\ell = f^\ell + g^\ell \) from (12.5). Combining this and (12.15) into (12.12) yields (12.4).

(12.16) **Example.** We may now compute the effective diffusivity in periodic velocity fields based on Examples (6.8) and (6.18).

Recall that Example (6.8) has two cases, one in which \( u_1 \) moves clockwise and \( V \) mixes more slowly than \( U \), the other in which \( u_1 \) moves counterclockwise and \( V \) mixes more quickly than \( U \). The effective diffusivity \( \beta \) is found to be 0.7389 and 0.1388, respectively. Apparently \( V \) mixing more slowly allows the particle to be transported farther.

Example (6.18) contains a compressible velocity field. The mean drift \( \pi u \) is found to be -0.1119 while the effective diffusivity \( \beta \) is 1.3165, even larger than the two–type velocity fields.

We illustrate the effect of adding molecular diffusion in these examples by taking \( d_0(a) = c(p)p^{10} \) for \( a = 0, \pm 1, \pm 2, \pm 3 \) where \( c(p) \) is chosen to normalize \( d_0 \) and \( p \) ranges from 0 (no diffusion) to about 0.5. We graph \( \beta \) against the variance of \( \Delta \) in Figure 12.1. For comparison, the variance of \( \Delta \) is graphed against itself to show what the effective diffusivity would be with diffusion alone. It is clear that the effective diffusivity is larger than for molecular diffusion alone. For the incompressible cases, the effective diffusivity increases as a function of the variance of \( \Delta \). However, for the compressible case, effective diffusivity first decreases when molecular diffusivity is added.
Figure 12.1: — — — compressible case of Example (6.18), · · · · u_1 clockwise case of Example (6.8), · · · · u_1 counterclockwise case of Example (6.8), solid line is variance of Δ.

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References


