Random velocity fields with known Lagrangian law

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Articles and these slides are available at the URL above.

Title pages of two articles are available from the presenter.

Brief introduction

Consider a single particle moving in a random velocity field. From the statistical description of the velocity field, I would like to derive such things as the effective diffusivity, the Lagrangian autocorrelation function, the Lagrangian decorrelation time, and the evolution of the probability density function for particle location.

Outline

- Qualitative results for continuous velocity fields
- Tracking particles to obtain numerical values
- Velocity fields in discrete space and time
- Examples and illustrations
- The probability law of the Lagrangian velocity

Continuous velocity fields – three classes:

(i) Physical flows (oceans, atmosphere, laboratory)

(ii) Numerical solution of Navier–Stokes with forcing

(iii) Outcomes of a random velocity field with given law

The particle follows the trajectory equation

$$\frac{dX_t}{dt} = U(X_t, t) \qquad (+\text{molecular diffusion})$$

Qualitative results

concerning the trajectory X_t and the Lagrangian velocity $U(X_t, t)$:

- If U is homogeneous, stationary, and non–divergent, then $U(X_t, t)$, $t \geq 0$ is strictly stationary (Lumley (1957), ..., Zirbel (2000))

- Rescaled probability density function of X_t converges to a solution of an effective diffusion equation (review: Isichenko (1992))

- Rescaled trajectory $\varepsilon X_{t/\varepsilon^2},\ t \geq 0$ converges to Brownian motion as $\varepsilon \to 0$ (Fannjiang and Komorowski (1999), several others)

- If U is homogeneous and Markov in time, then $U(X_t + \bullet, t),\ t \geq 0$ is Markov as well, with known generator (Komorowski (2000), Zirbel (2000))

Obtaining quantitative results

We would like to know the numerical values of such things as the effective diffusivity matrix

$$\mathbf{D} = \lim_{t\to\infty} \frac{\mathrm{Var}(X_t)}{t}$$

The current technique:

- Drop a particle into the velocity field
- Track it
- Repeat many times
- Analyze the data statistically

Examples

(i) Physical flows: remotely–tracked floats and submersibles (Davis (1991))

(ii) Navier–Stokes: direct numerical simulation (Yeung and Pope (1989))

(iii) Random velocity fields: simulation and tracking (Elliott and Majda (1996))

These are certainly useful and important techniques, but **we ought to be able to compute numbers directly from the statistical description of the velocity field.**

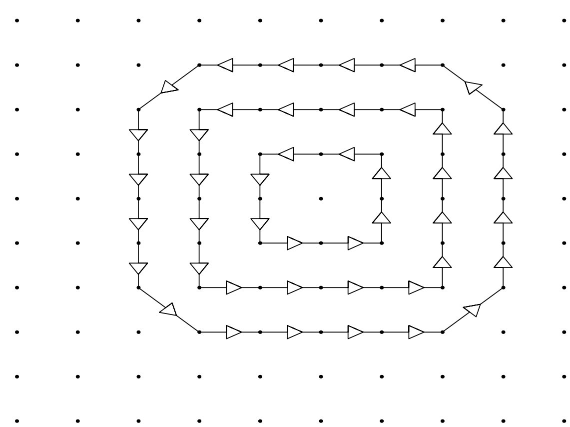
Velocity fields in discrete space and time

Spatial grid repeats periodically

A 10 by 10 grid is shown here, but it can be as fine as you like

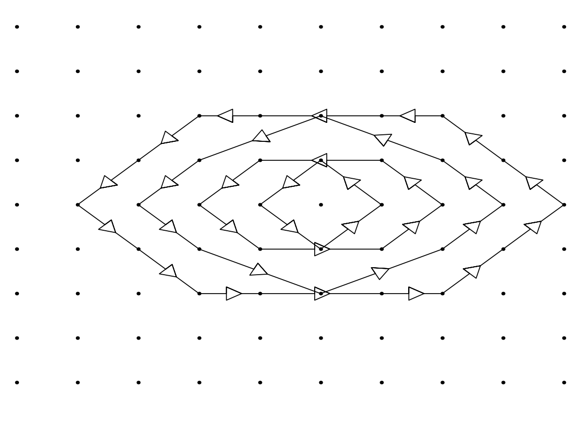
Particles follow the arrow at each time step

Snapshot:



Time evolution I will consider:

- This vortex can translate according to an additive random walk
- The shape of the vortex can change according to a Markov transition matrix, perhaps to



Generality of this model

When the velocity field is discrete, homogeneous, and Markov in time, it can be decomposed into m different vortices with Markov transitions between vortices; the location of the current vortex performs an additive random walk according to the current and next vortex.

A much wider range of velocity fields may be constructed by allowing additional hidden variables controlling the translation of each vortex; in this way, a vortex may be given momentum in a particular direction.

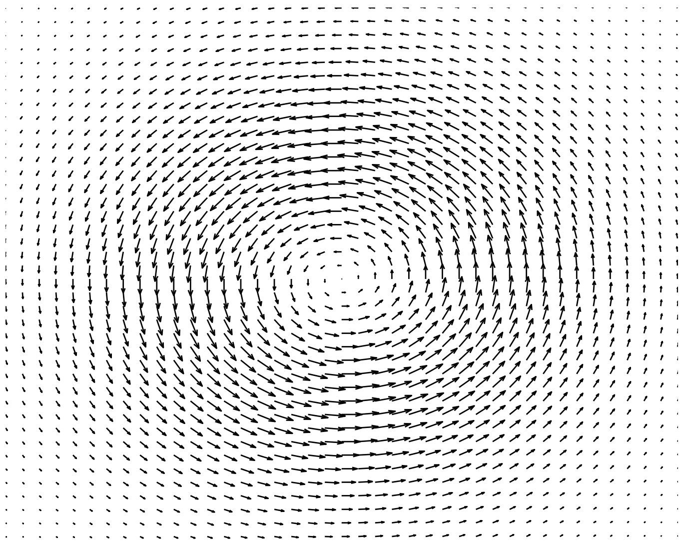
Constructing discrete vortices

This can be done in the incompressible case with various resolutions by solving a bipartite matching problem. Examples follow.

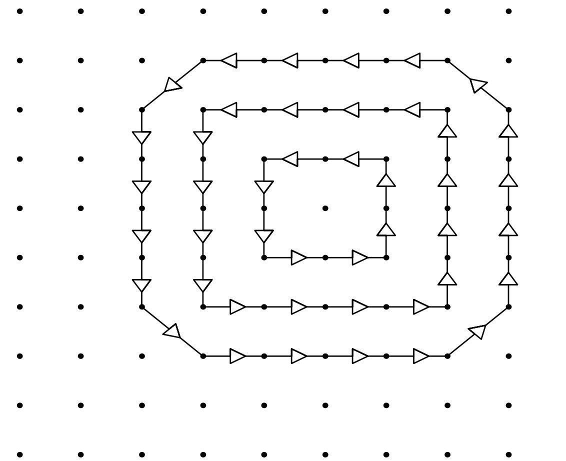
Constructing discrete dynamics

Right now, only the simplest "dynamics" are considered.
In the future, perhaps 50 to 100 vortex shapes may be selected to represent all possible velocity field shapes. Transitions between these should approximate the desired dynamics and forcing.

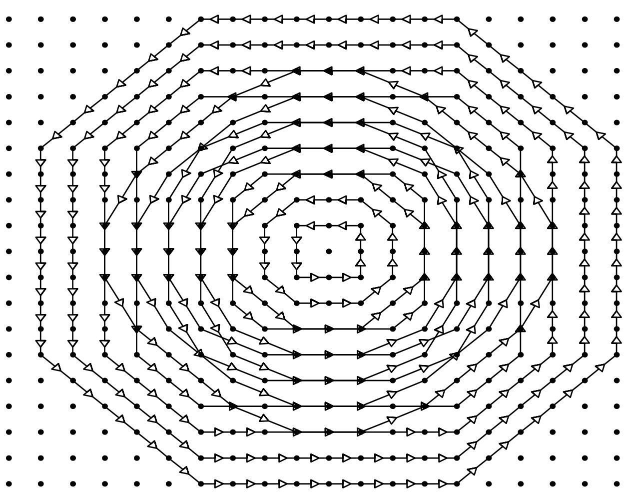
Original velocity field



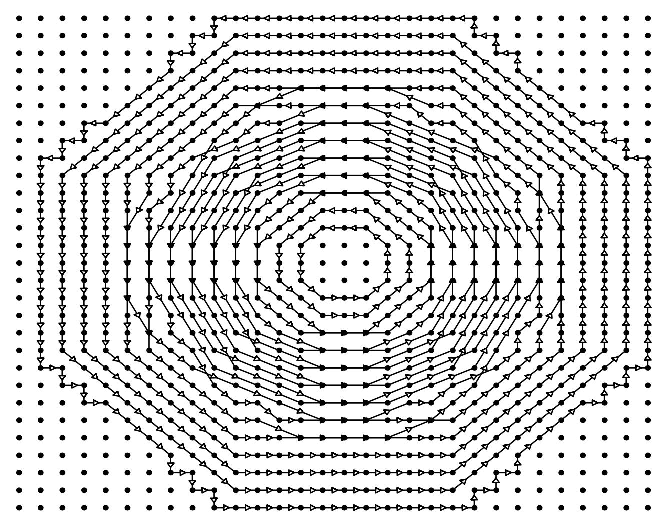
10 by 10 approximation



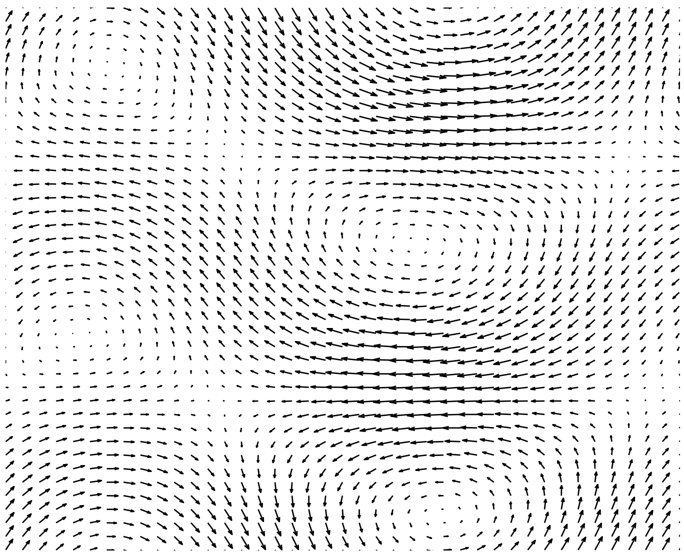
20 by 20 approximation



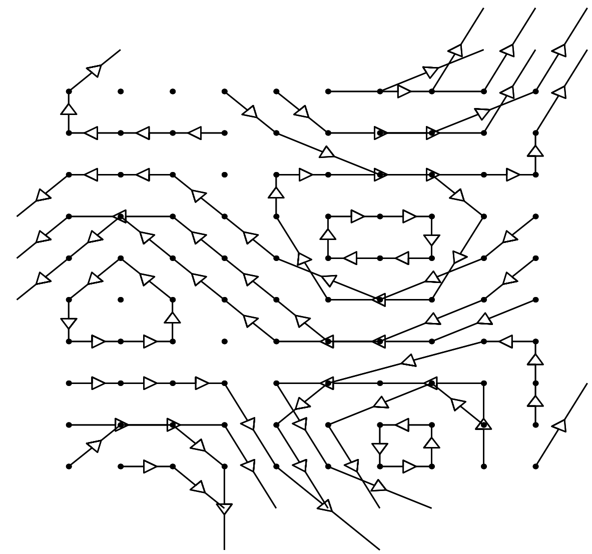
30 by 30 approximation



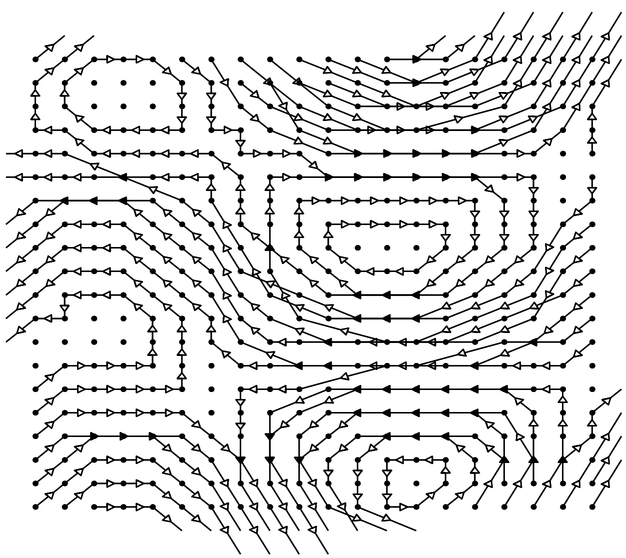
Original velocity field



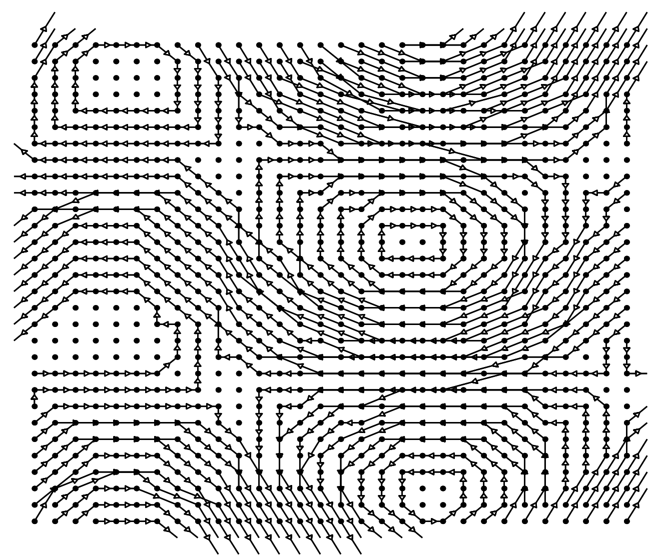
10 by 10 approximation



20 by 20 approximation

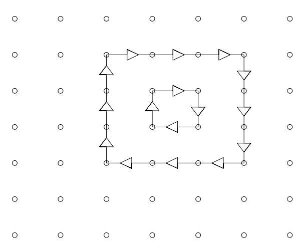


30 by 30 approximation



One vortex examples

The velocity field has only one vortex type, which is repeated periodically throughout $\mathbb{Z}^2$. The vortex moves up and to the right, with probabilities 0.3 (right) 0.3 (up) 0.2 (up and right) 0.2 (don't move). The vortices are standardized to have the same "energy" $\mathbb{E}|U(0,0)|^2$. The Lagrangian drift is $\mu = \lim_{t\to\infty} \mathbb{E}X_t/t$.
The diffusivity matrix is $\beta = \lim_{t\to\infty} \mathrm{Cov}(X_t)/t$.

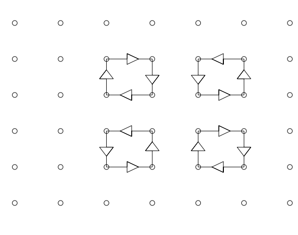


Eulerian spectral gap 0.0990,

Lagrangian 0.1456 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.2499 & 0.0945 \\ 0.0945 & 0.2499 \end{bmatrix}$.

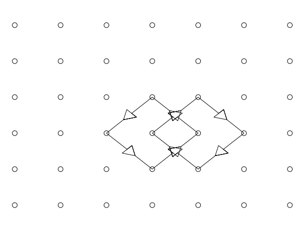


Eulerian spectral gap 0.0990,

Lagrangian 0.1565 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.1841 & 0.0484 \\ 0.0484 & 0.1841 \end{bmatrix}$.

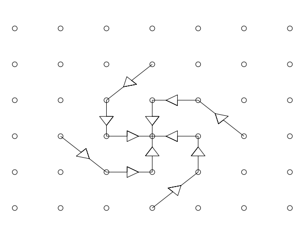


Eulerian spectral gap 0.0990,

Lagrangian 0.1281 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.0929 & 0.0545 \\ 0.0545 & 0.1724 \end{bmatrix}$.



Eulerian spectral gap 0.0990,

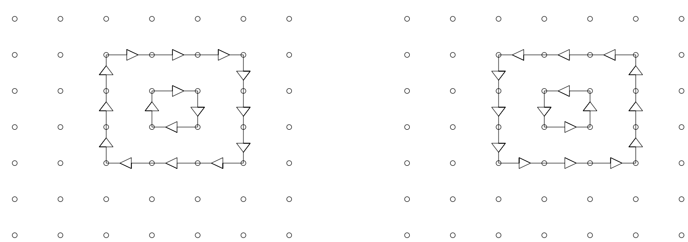
Lagrangian 0.1820 (mixes faster).

Lagrangian drift $\mu = [0.2415 \;\; 0.1934]$.

Diffusivity matrix $\beta = \begin{bmatrix} 0.6925 & 0.5639 \\ 0.5639 & 0.8110 \end{bmatrix}$.

Two–type examples

The velocity field has two vortex types, which alternate according to the transition matrix $\begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$. The current vortex is repeated periodically throughout $\mathbb{Z}^2$. Vortex 1 moves up and to the right, with probabilities 0.3 (right) 0.3 (up) 0.2 (up and right) 0.2 (don't move), while vortex 2 moves down and to the right, with probabilities 0.3 (right) 0.3 (down) 0.2 (down and right) 0.2 (don't move).

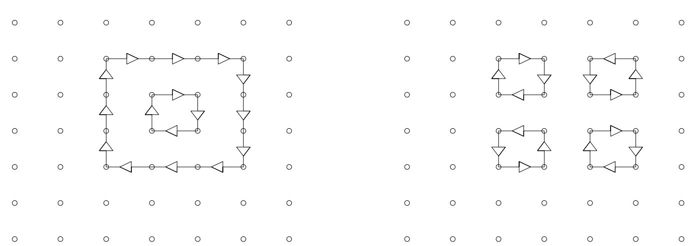


Eulerian spectral gap 0.0949,

Lagrangian 0.1468 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.1473 & 0.0000 \\ 0.0000 & 0.1550 \end{bmatrix}$.

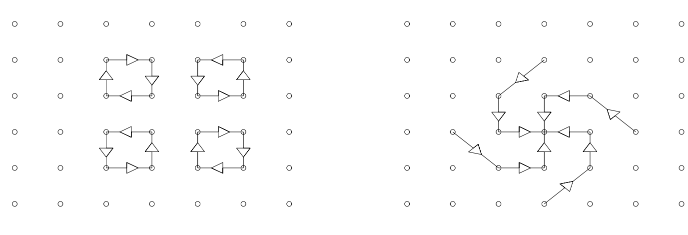


Eulerian spectral gap 0.0949,

Lagrangian 0.1583 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.2177 & -0.0089 \\ -0.0089 & 0.1885 \end{bmatrix}$.

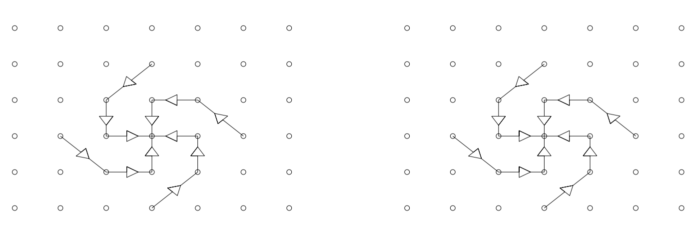


Eulerian spectral gap 0.0949,

Lagrangian 0.1613 (mixes faster).

Lagrangian drift μ is $[0.0394 \quad -0.0610]$.

Diffusivity matrix $\beta = \begin{bmatrix} 0.3209 & -0.1360 \\ -0.1360 & 0.3077 \end{bmatrix}$.



Eulerian spectral gap 0.0949,

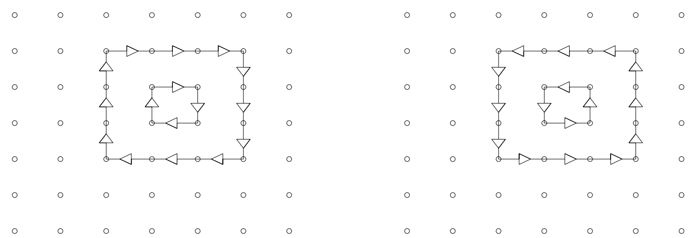
Lagrangian 0.1442 (mixes faster).

Lagrangian drift μ is $[0.2109 \quad -0.0169]$.

Diffusivity matrix $\beta = \begin{bmatrix} 0.6871 & 0.0595 \\ 0.0595 & 0.5622 \end{bmatrix}$.

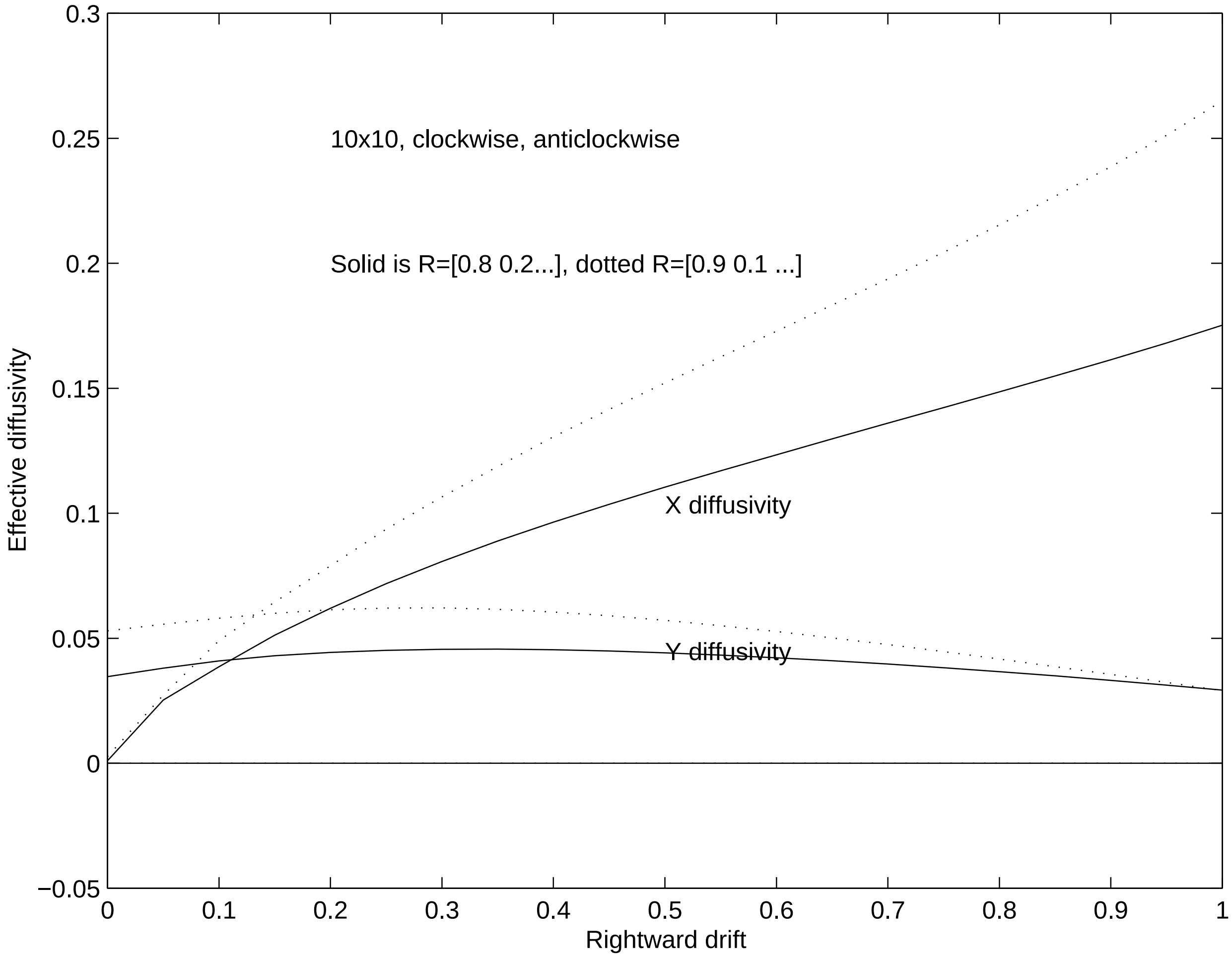
Illustration: the effect of drift

Consider these vortices:



Allow the first to move up and right, the second down and right.

Graphed below is effective diffusivity versus rightward drift speed.



Discrete Markov velocity fields

Proposition If U is discrete, homogeneous, and Markov, then U satisfies $U(x, n) = u(I_n, x - L_n), n = 0, 1, 2, ...$ for some *vortex types* $u(1, \cdot), u(2, \cdot), \ldots$, a *type process* I and a *location process* L.

The velocity field consists of **vortices** which change **type** and **location**.

The type process I is Markov, and the location process L is an additive random walk.

Discrete hidden Markov velocity fields

Arbitrary vortex types $u(1, \cdot), u(2, \cdot), ...$

Type changes according to a Markov chain I.

Location process L is an additive random walk whose characteristics depend on the current vortex type:

$$L_{n+1} = L_n + A(I_n, n)$$

The random velocity field is $U(x, n) = u(I_n, x - L_n)$.

The vortex is repeated periodically throughout $\mathbb{Z}^2$.

The particle trajectory obeys

$$X_{n+1} = X_n + U(X_n, n) + \Delta_n,$$

where Δ_n represents molecular diffusion.

Lagrangian velocity process

Let $M_n = L_n - X_n$. Then M is the location of the vortex from the point of view of the moving particle.

Note that $U(X_n, n) = u(I_n, X_n - L_n) = u(I_n, -M_n)$ and that M evolves according to

$$M_{n+1} = \sigma(I_n, M_n) + A(I_n, n) + \Delta_n,$$

where σ is a mapping on the grid generated by u. This is almost the same as how L evolves.

The point is this: the Lagrangian velocity depends on a Markov chain (I, M), where I is Markov and M evolves in a simple way. The state space of (I, M) is the same size as that of (I, L), the chain which determines the velocity field. The transition matrix Q of (I, M) is easy to get from the transition matrix P of (I, L), in this way:

$$Q = \Sigma D P$$

where Σ is a matrix built from the vortex types and D corresponds to molecular diffusion. When the vortices are incompressible, Σ will be a permutation matrix.

For two–particle motion, there are two Lagrangian location processes M^1 and M^2. They evolve in the same way as above, but with different diffusion:

$$\begin{aligned} M^1_{n+1} &= \sigma(I_n, M^1_n) + A(I_n, n) + \Delta^1_n \\ M^2_{n+1} &= \sigma(I_n, M^2_n) + A(I_n, n) + \Delta^2_n \end{aligned}$$

Then $(U(X^1_n, n), U(X^2_n, n)) = (u(I_n, -M^1_n), u(I_n, -M^2_n))$.