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Random measures carried by Brownian flows on $\mathbb{R}^d$

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Abstract. We study the measure-valued random process associated with a mass distribution carried by a Brownian flow on $\mathbb{R}^d$. We give conditions under which the integral of a test function with respect to this random measure forms a well-defined random process. For sufficiently smooth test functions, we show that this process is a semimartingale and compute its quadratic variation. We illustrate with detailed computations for the center of mass and dispersion matrix of the mass distribution.

Key words. Stochastic flows; Brownian flows; Mass transport

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Contents

1 Introduction

Brownian motion and other diffusion processes have long provided models for the random motion of one or several particles in physical space. Since roughly 1980, Brownian flows have been developed as one logical extension of these models. A Brownian flow on $\mathbb{R}^d$ is a collection $F_t,\ t \geq 0$, of random homeomorphisms on $\mathbb{R}^d$ satisfying certain continuity and independence properties (see the end of this section). The map F_t describes the motion of particles from time 0 to time t. The joint trajectory of any n particles, namely the process

$$(F_t x_1, \ldots, F_t x_n),\ t \geq 0, \tag{1.1}$$

is a diffusion on $\mathbb{R}^{d \times n}$ (the initial locations $x_1, \ldots, x_n$ are fixed). For specific choices of F, the one-particle motion $F_t x,\ t \geq 0$, will be Brownian motion.

What sets Brownian flows apart from the earlier models is that they model the simultaneous motion of *all* particles in the space. Furthermore, the map F_t is one to one and onto, and the mapping $(x, t) \mapsto F_t x$ is continuous. This makes Brownian flows suitable for modeling fluid motion rather than, for example, molecular diffusion. They retain the temporal independence features of diffusions, but in order to maintain spatial continuity, they require that the particle motions be correlated in certain ways depending on their relative locations.

One way to understand the effect of a Brownian flow is to measure the size of the image $F_t(A)$ of a set $A \subseteq \mathbb{R}^d$ as $t \to \infty$. Following KUNITA (1990), Section 4.3, we may fix a measure Π_0 on $\mathbb{R}^d$ and define a random measure Π_t by

$$\Pi_t(A) = \Pi_0(F_t(A)), \qquad \text{Borel } A \subseteq \mathbb{R}^d \tag{1.2}$$

The case of an isotropic flow is instructive. Let Π_0 be the Lebesgue measure, so that $\Pi_t(A)$ corresponds to the volume of $F_t(A)$. Then, unless the flow preserves Lebesgue measure (i.e., $\Pi_t = \Pi_0$ almost surely), Π_t converges vaguely to a random measure which is singular with respect to the Lebesgue measure. Moreover, if all d Lyapunov exponents of the flow are negative, then the limiting measure is almost surely the zero measure. In this case, the flow compresses sets of finite Lebesgue measure onto sets of zero measure as $t \to \infty$. See LE JAN (1985), DARLING and LE JAN (1988), and the review article DARLING (1991) for details.

In this paper, we are interested in the transport and mixing properties of Brownian flows. To this end, we fix a finite Borel measure M_0 on $\mathbb{R}^d$ and examine the image measure M_t given by

$$M_t(A) = M_0(F_t^{-1}(A)) \qquad \text{Borel } A \subseteq \mathbb{R}^d \tag{1.3}$$

or, more briefly, $M_t = M_0 \circ F_t^{-1}$. We keep in mind the following interpretation: the measure M_0 represents the initial location of a mass distribution (discrete, continuous, or otherwise) and the random variable $M_t(A)$ is the amount of mass in the set A at time t. We would like to understand how the flow moves this mass around in $\mathbb{R}^d$.

We often understand a random measure M_t through integrals of the form

$$M_t f = \int M_t(dx) f(x),$$

where the test function f is a measurable map from $\mathbb{R}^d$ to $\mathbb{R}$. This paper gives some general results concerning the random variable $M_t f$ and the process $M_t f,\ t \geq 0$. We also give computations for some particularly interesting choices of the function f.

If f is bounded, then $M_t f$ is well defined and finite. This may not be the case, however, if f is unbounded. In Section 2 we give conditions on M_0 and f which guarantee that not only $M_t f$ but the whole process $M_t f,\ t \geq 0$, is well defined and finite. In Section 3 we show that if f is sufficiently smooth, then the process $M_t f,\ t \geq 0$, is a semimartingale, and we compute the joint quadratic variation of $M_t f,\ t \geq 0$, and $M_t g,\ t \geq 0$. This is the main result of the paper.

Finally, in Section 4 we apply these results to the centroid and dispersion processes, defined by

$$C_t = \int M_t(dx) x \tag{1.4}$$

$$D_t = \int M_t(dx)(x - C_t)(x - C_t)^T \tag{1.5}$$

(C_t is vector-valued and D_t is matrix-valued.) The centroid is the center of mass of M_t, while the dispersion matrix measures how spread out M_t is relative to its center of mass.

The original motivation for this work comes from statistical fluid mechanics, where a longstanding problem is to characterize the transport and mixing properties of a given turbulent fluid flow. Brownian flows have been proposed (ZIRBEL and ÇINLAR (1997)) as a model for such physical flows because they mimic their spatial structure but have much simpler temporal dependences. The present paper provides computational results which facilitate working with the mass location process $M_t,\ t \geq 0$. Using these results, ZIRBEL and ÇINLAR (1996) and ZIRBEL (1997) investigate the long-time behavior of the centroid and dispersion processes in isotropic Brownian flows.

Brownian flows

We conclude this section with a list of facts about Brownian flows which will be needed in this article. KUNITA (1990) is a complete reference for Brownian flows; a shorter, heuristic introduction may be found in ZIRBEL and ÇINLAR (1997).

Let U be a Gaussian random vector field on $\mathbb{R}^d \times \mathbb{R}$ such that

$$\mathbb{E}U(x,t) = u(x)t$$

$$\operatorname{Cov}(U^i(x,s), U^j(y,t)) = a^{ij}(x,y)\min(s,t)$$

for some functions u and a, called the *drift* and *covariance*, respectively.

We assume throughout this paper that for some finite constant R, u and a satisfy

$$|u(x,t)| \leq R(1+|x|) \tag{1.6}$$

$$|u(x,t) - u(y,t)| \leq R|x-y| \tag{1.7}$$

$$|a^{ij}(x,y)| \leq R(1+|x|)(1+|y|) \tag{1.8}$$

$$|a^{ij}(x,y) - a^{ij}(x',y) - a^{ij}(x,y') + a^{ij}(x',y')| \leq R|x-x'||y-y'| \tag{1.9}$$

for all x, x', y, y' in $\mathbb{R}^d$ and t in $\mathbb{R}_+$. Then U is continuous and by Theorem 4.2.5 of KUNITA (1990), there exists a flow $F = \{F_t;\ 0 \leq t < \infty\}$ of homeomorphisms satisfying the stochastic equation:

$$F_t x = x + \int_0^t U(F_s x, ds) \tag{1.10}$$

for all x in $\mathbb{R}^d$ and t in $\mathbb{R}_+$ simultaneously.

The mapping $(x,t) \mapsto F_t x$ is continuous. The map from time s to a later time t, namely $F_t \circ F_s^{-1}$, is independent of the history to time s, namely, the σ-algebra generated by F_r for $r \leq s$.

The one-point motion under F, namely the process $F_t x,\ t \geq 0$, for fixed x in $\mathbb{R}^d$, is a diffusion on $\mathbb{R}^d$ with generator $\mathcal{A}$ given by

$$\mathcal{A}f(x) = \tfrac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x,x)\partial_i\partial_j f(x) + \sum_{i=1}^{d} u^i(x)\partial_i f(x) \tag{1.11}$$

In fact, the n-point motion $(F_t x_1, \ldots, F_t x_n),\ t \geq 0$, for fixed n in $\mathbb{N}$ and $x_1, \ldots, x_n$ in $\mathbb{R}^d$ is a diffusion on $\mathbb{R}^{d\times n}$. The joint quadratic variation between two components is given by the formula

$$\langle F^i x, F^j y\rangle_t = \int_0^t ds\ a^{ij}(F_s x, F_s y) \tag{1.12}$$

See KUNITA (1990), Theorem 3.2.4.

2 Regularity of the process $M_t f,\ t \geq 0$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be Borel measurable. The definition (1.3) of M_t immediately yields

(2.1)
$$M_t f = \int M_t(dx) f(x) = \int M_0(dx) f(F_t x).$$

For bounded f, the integral $M_t f$ will always exist and be finite. This is not necessarily the case for unbounded f. We show in this section that the process $M_t f,\ t \geq 0$, is well defined if M_0 and f satisfy the following condition:

(2.2) **Condition.** For some $p > 2$ and $C < \infty$, we have

$$\int M_0(dx)|x|^p < \infty$$

$$|f(x)| \leq C(1 + |x|)^p, \qquad x \in \mathbb{R}^d.$$

If M_0 and f satisfy this condition, it is not difficult to show that for each fixed $t \geq 0$, the random variable $M_t|f|$ is finite almost surely. Indeed, KUNITA (1990), Lemma 4.5.3 shows that for each $p \geq 0$, there exists a constant K_p such that

(2.3)
$$\mathbb{E}(1 + |F_t x|^2)^p \leq e^{K_p t}(1 + |x|^2)^p$$

for all x in $\mathbb{R}^d$ and $t \geq 0$. So,

$$\begin{aligned} \mathbb{E} M_t|f| &= \mathbb{E} \int M_0(dx)|f(F_t x)| \\ &\leq \int M_0(dx) \mathbb{E} C(1 + |F_t x|)^p \\ &\leq C e^{K_{p/2} t} 2^{p/2} \int M_0(dx)(1 + |x|)^p, \end{aligned}$$

which is finite by Condition (2.2). Thus, $M_t|f|$ is finite almost surely, and so $M_t f$ is well defined and finite almost surely.

One can do this for all rational t simultaneously. The problem, however, is showing that $M_t f$ is well defined for all $t \geq 0$ simultaneously. What we need is an L^p bound on the supremum of $(1 + |F_t x|^2)^p$. The following lemma does this.

(2.4) **Lemma.** Let $p > 1$. Then there exist finite constants C_1 and C_2 (depending on p) such that, for all $T > 0$ and x in $\mathbb{R}^d$,

$$\mathbb{E} \sup_{t \leq T} (1 + |F_t x|^2)^p \leq C_1 e^{C_2 T}(1 + |x|^2)^p.$$

Proof: Let $g(x) = 1 + |x|^2$. By Lemma 4.5.2 of KUNITA (1990), the process

$$L_t = g(F_t x) - g(x) - \int_0^t ds\ \mathcal{A}g(F_s x), \qquad t \geq 0$$

is a mean-zero martingale. Also, from the proof of Lemma 4.5.3 of the same book, we have $|\mathcal{A}g(x)| \leq K_1 g(x)$ for all x in $\mathbb{R}^d$.

We proceed to bound $\mathbb{E}\sup_{t\leq T} g(F_t x)^p$ as follows:

$$\begin{aligned}
\mathbb{E}\sup_{t\leq T} g(F_t x)^p &= \mathbb{E}\sup_{t\leq T}(L_t + g(x) + \int_0^t ds\ \mathcal{A}g(F_s x))^p \\
&\leq 3^{p-1}\mathbb{E}(\sup_{t\leq T}|L_t|^p + g(x)^p + \mathbb{E}\sup_{t\leq T}|\int_0^t ds\ \mathcal{A}g(F_s x)|^p)
\end{aligned}$$

Consider the first term on the right side. By Doob's inequality, we have

$$\mathbb{E}\sup_{t\leq T}|L_t|^p \leq C_3 \sup_{t\leq T}\mathbb{E}|L_t|^p$$

where $C_3 = (p/(p-1))^p$. Now, using the definition of L,

$$\begin{aligned}
\mathbb{E}|L_t|^p &= \mathbb{E}|g(F_t x) - g(x) - \int_0^t ds\ \mathcal{A}g(F_s x)|^p \\
&\leq 3^{p-1}\mathbb{E}(g(F_t x)^p + g(x)^p + \mathbb{E}|\int_0^t ds\ \mathcal{A}g(F_s x)|^p)
\end{aligned}$$

Thus, $\mathbb{E}\sup_{t\leq T} g(F_t x)^p$ is bounded by

$$3^{p-1}(C_3 3^{p-1} + 1)(\sup_{t\leq T}\mathbb{E}g(F_t x)^p + g(x)^p + \mathbb{E}\sup_{t\leq T}|\int_0^t ds\ \mathcal{A}g(F_s x)|^p)$$

Now by (2.3), we have

$$\mathbb{E}g(F_t x)^p = \mathbb{E}(1 + |F_t x|^2)^p \leq e^{K_p T}(1 + |x|^2)^p = e^{K_p T} g(x)^p$$

Moreover, recalling that $|\mathcal{A}g| \leq K_1 g$,

$$\begin{aligned}
\mathbb{E}\sup_{t\leq T}|\int_0^t ds\ \mathcal{A}g(F_s x)|^p &\leq \mathbb{E}\sup_{t\leq T} t^p \int_0^t ds\ |\mathcal{A}g(F_s x)|^p \\
&\leq T^p \mathbb{E}\int_0^T ds\ K_1^p g(F_s x)^p \\
&\leq T^p K_1^p \int_0^T ds\ e^{K_p T} g(x)^p \\
&\leq T^{p+1} K_1^p e^{K_p T} g(x)^p
\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\sup_{t\le T} g(F_t x)^p &\le 3^{p-1}(C_3 3^{p-1}+1)(e^{K_pT}+1+T^{p+1}K_1^p e^{K_pT})g(x)^p \\ &\le C_1 e^{C_2T} g(x)^p,\end{aligned}$$

which completes the proof. □

(2.5) **Proposition.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be Borel measurable. Suppose that M_0 and f satisfy Condition (2.2). Then the processes

$$M_t f,\ t\ge 0, \qquad \int_0^t ds\ M_s f,\ t\ge 0,$$

are well defined and finite almost surely. Moreover, for each $t \ge 0$, $M_t f$ and $\int_0^t ds M_s f$ have finite expectations. Finally, if f is continuous, then the process $M_t f,\ t\ge 0$, is continuous.

Proof: The integral $M_t|f|$ is always defined, but may be infinite. Fix $T > 0$. Then,

$$\begin{aligned}\mathbb{E}\sup_{t\le T} M_t|f| &= \mathbb{E}\sup_{t\le T}\int M_0(dx)|f(F_t x)| \\ &\le \mathbb{E}\int M_0(dx)\sup_{t\le T}|f(F_t x)| \\ &\le \int M_0(dx)\mathbb{E}\sup_{t\le T} C(1+|F_t x|)^p,\end{aligned}$$

by the second part of Condition (2.2). The preceding lemma yields

$$\mathbb{E}\sup_{t\le T}(1+|F_t x|)^p \le C_1 2^{p/2} e^{C_2 T}(1+|x|)^p, \qquad x\in\mathbb{R}^d,$$

for some finite constants C_1 and C_2. This last function is integrable with respect to M_0 by the first part of Condition (2.2). Thus $\mathbb{E}\sup_{t\le T} M_t|f| < \infty$. Noting that

$$\sup_{t\le T}\int_0^t ds\ M_s|f| \le T\sup_{t\le T} M_t|f|$$

shows that $\mathbb{E}\sup_{t\le T}\int_0^t ds\ M_s|f| < \infty$ also.

Let T_n be a deterministic sequence increasing to ∞. For each n,

$$M_t|f| < \infty, \qquad \int_0^t ds\ M_s|f| < \infty, \qquad 0\le t\le T_n,$$

almost surely. Thus, $M_t f$ and $\int_0^t ds M_s f$ are well defined on $[0, T_n]$, almost surely. Taking the limit as $n\to\infty$ yields the first claim. The second claim has also been shown.

Continuity follows from the dominated convergence theorem, since $\sup_{t\le T}|f(F_t x)|$ dominates $|f(F_t x)|$ for $t \le T$, and the first has a finite integral with respect to M_0. □

3 Stochastic calculus of M_t

We have seen conditions under which the process $M_tf;\ t \geq 0$, is well defined. If f is continuous, the process is continuous. Is it also a semimartingale? The following theorem offers an answer. It first appeared in ZIRBEL (1993) and was stated in the review article ZIRBEL and ÇINLAR (1997).

Let us decompose U into its mean and martingale parts:

$$U(x,t) = u(x)t + U_0(x,t) \tag{3.1}$$

Then U_0 is a Gaussian random field with zero drift and covariance a. We will impose the following condition on M_0 and f. In particular, by the results of the last section, it insures that the process $M_tf,\ t \geq 0$, is well defined, finite, and continuous.

(3.2) **Condition.** The function $f : \mathbb{R}^d \to \mathbb{R}$ is twice continuously differentiable; for some constants $p > 2$ and $C < \infty$,

$$\int M_0(dx)|x|^p < \infty$$

$$|f(x)| \leq C(1+|x|)^p, \qquad x \in \mathbb{R}^d$$

$$|\mathcal{A}f(x)| \leq C(1+|x|)^p, \qquad x \in \mathbb{R}^d;$$

and for each $i = 1, \ldots, d$, the function $\partial_i f$ is of polynomial growth. □

(3.3) **Theorem.** *Suppose that f and M_0 satisfy Condition (3.2).*

a) The process $M_tf,\ t \geq 0$, satisfies the following equation:

$$M_tf = M_0f + \int_0^t ds M_s(\mathcal{A}f) + L_t, \tag{3.4}$$

where L is a continuous martingale with mean 0, given by

$$L_t = \int M_0(dx)J_t(x). \tag{3.5}$$

For each x in $\mathbb{R}^d$, $J_t(x),\ t \geq 0$ is a martingale on $\mathbb{R}$, defined by

$$J_t(x) = \sum_{i=1}^d \int_0^t \partial_i f(F_s x) U_0^i(F_s x, ds). \tag{3.6}$$

b) Let N_0 be a finite deterministic measure on $\mathbb{R}^d$ and define N_t by $N_t = N_0 \circ F_t^{-1}$. Suppose that N_0 and a function g satisfy Condition (3.2), with $\bar{p}$ in place of p. In addition, suppose that for x, y in $\mathbb{R}^d$ and $i, j = 1, \ldots, d$,

$$|\partial_i f(x) \partial_j g(y) a^{ij}(x,y)| \leq C(1+|x|)^p(1+|y|)^{\bar{p}}. \tag{3.7}$$

Then the joint quadratic variation of $M_t f$ and $N_t g$ satisfies

$$\langle M_t f, N_t g\rangle = \sum_{i,j=1}^{d} \int_0^t ds \int M_s(dx) \int N_s(dy)\ \partial_i f(x) \partial_j g(y) a^{ij}(x,y) \tag{3.8}$$

(3.9) **Remark.** The martingale L in part (a) will be square integrable if f satisfies

$$|\partial_i f(x) \partial_j f(y) a^{ij}(x,y)| \le C(1+|x|)^p (1+|y|)^p$$

using the same value of p as in Condition (3.2). Square integrability follows from $\mathbb{E}L_t^2 = \mathbb{E}\langle L, L\rangle_t$ and part (b) via some simple estimates.

Proof: Fix t in $\mathbb{R}_+$.

a) First, we use Itô's Lemma to expand $f(F_t x)$. The proof of Lemma 4.5.2 of KUNITA (1990) shows that

$$f(F_t x) = f(x) + \int_0^t ds\ (\mathcal{A}f)(F_s x) + J_t(x), \tag{3.10}$$

the last term of which is the martingale given by (3.6). The precise definition of the integral in (3.6) is:

$$\int_0^t \partial_i f(F_s x) U_0^i(F_s x, ds) = \int_0^t V^i(F_s x, ds), \tag{3.11}$$

where V is the Gaussian random field defined by $V^i(x,t) = \partial_i f(x) U_0^i(x,t)$. This is discussed after Theorem 3.3.3 of KUNITA (1990).

By Lemma (2.5) and the conditions placed on M_0, f, and $\mathcal{A}f$, the integrals of the first three terms in (3.10) with respect to M_0 are well defined, and so the integral of the last term with respect to M_0 is also well defined. Integrate over M_0 and interchange integrals in the middle term on the right to obtain

$$M_t f = M_0 f + \int_0^t ds\ M_s(\mathcal{A}f) + M_0 J_t \tag{3.12}$$

The last term is L_t, and all that remains is to show that it is a martingale. Let r be less than t. In (3.10), the absolute value of the last term is bounded by the sum of the absolute values of the other terms. Integrating each such term with respect to M_0 and then taking the expectation gives a finite value, according to Lemma (2.5). Hence L_t is integrable and we can interchange integrations in the following computation:

$$\begin{aligned} \mathbb{E}_{\mathcal{H}_r} L_t &= \mathbb{E}_{\mathcal{H}_r} \int M_0(dx) J_t(x) \\ &= \int M_0(dx) \mathbb{E}_{\mathcal{H}_r} J_t(x) \\ &= \int M_0(dx) J_r(x) = L_r \end{aligned} \tag{3.13}$$

Thus L is a martingale, which finishes the proof of part (a).

b) Recall equation (3.10). Rewrite it for $g(F_t y)$, denoting the last term by $K_t(y)$. Then (3.4) becomes, for $N_t g$,

$$N_t g = N_0 g + \int_0^t ds\ N_s(\mathcal{A}g) + N_0 K_t$$

This allows us to compute the joint quadratic variation of $M_t f$ and $N_t g$:

$$\langle M_t f, N_t g\rangle = \langle M_0 J_t, N_0 K_t\rangle \tag{3.14}$$

We want to rewrite this as $\int M_0(dx) \int N_0(dy)\langle J_t(x), K_t(y)\rangle$.

It is well known that for two continuous martingales Z and Z', the joint quadratic variation $\langle Z, Z'\rangle_t$ is the unique continuous process of bounded variation such that $Z_t Z'_t - \langle Z, Z'\rangle_t$ is a martingale. The following calculation makes use of this uniqueness property. We will justify interchanging integrals at the end.

$$\begin{aligned}
&\mathbb{E}_{\mathcal{H}_r}[(M_0 J_t)(N_0 K_t) - \int M_0(dx) \int N_0(dy)\langle J_t(x), K_t(y)\rangle] \\
&= \mathbb{E}_{\mathcal{H}_r} \int M_0(dx) \int N_0(dy)[J_t(x)K_t(y) - \langle J_t(x), K_t(y)\rangle] \\
&= \int M_0(dx) \int N_0(dy)\mathbb{E}_{\mathcal{H}_r}[J_t(x)K_t(y) - \langle J_t(x), K_t(y)\rangle] \\
&= \int M_0(dx) \int N_0(dy)[J_r(x)K_r(y) - \langle J_r(x), K_r(y)\rangle] \\
&= (M_0 J_r)(N_0 K_r) - \int M_0(dx) \int N_0(dy)\langle J_r(x), K_r(y)\rangle
\end{aligned}$$

This shows that

$$\langle M_0 J_t, N_0 K_t\rangle = \int M_0(dx) \int N_0(dy)\langle J_t(x), K_t(y)\rangle, \tag{3.15}$$

since both are the joint quadratic variation of $M_0 J_t$ and $N_0 K_t$.

Next, we use (3.11) and Theorem 3.2.4 of KUNITA (1990) to compute $\langle J_t(x), K_t(y)\rangle$ as follows

$$\begin{aligned}
\langle J_t(x), K_t(y)\rangle &= \left\langle \sum_{i,j=1}^d \int_0^t \partial_i f(F_s x) U_0^i(F_s x, ds),\ \int_0^t \partial_j g(F_s y) U_0^j(F_s y, ds)\right\rangle \\
&= \sum_{i,j=1}^d \int_0^t ds\ \partial_i f(F_s x)\partial_j g(F_s y) a^{ij}(F_s x, F_s y),
\end{aligned}$$

Combining this with (3.14) and (3.15), we obtain

$$\begin{aligned}
\langle M_t f, N_t g\rangle &= \int M_0(dx) \int N_0(dy)\langle J_t(x), K_t(y)\rangle \\
&= \int M_0(dx) \int N_0(dy) \sum_{i,j=1}^d \int_0^t ds\ \partial_i f(F_s x)\partial_i g(F_s y) a^{ij}(F_s x, F_s y)
\end{aligned}$$

Interchanging integrations and recalling (2.1) yields (3.8).

In order to justify the interchanges of integrals used above, consider the following bound:

$$\begin{aligned}
\mathbb{E}|\partial_i f(F_t x)\partial_j g(F_t y) a^{ij}(F_t x, F_t y)| &\le C\mathbb{E}(1+|F_t x|)^p(1+|F_t y|)^{\bar{p}} \\
&\le C\left[\mathbb{E}(1+|F_t x|)^{2p}\mathbb{E}(1+|F_t y|)^{2\bar{p}}\right]^{\frac{1}{2}} \\
&\le C2^{\tilde{p}}e^{K_{\tilde{p}}t}(1+|x|)^p(1+|y|)^{\bar{p}},
\end{aligned}$$

using (3.7), Hölder's inequality, and (2.3). We have set $\tilde{p} = \max(p, \bar{p})$. This justifies the following interchange of integrals

$$\mathbb{E}_{\mathcal{H}_r}\int M_0(dx)\int N_0(dy)\langle J_t(x), K_t(y)\rangle = \int M_0(dx)\int N_0(dx)\mathbb{E}_{\mathcal{H}_r}\langle J_t(x), K_t(y)\rangle$$

We also need to justify the interchange of integrals in

$$\mathbb{E}_{\mathcal{H}_r}\int M_0(dx)\int N_0(dy) J_t(x) K_t(y) = \int M_0(dx)\int N_0(dy)\mathbb{E}_{\mathcal{H}_r} J_t(x) K_t(y)$$

Solve (3.10) for $J_t(x)$ and the analogue for $K_t(y)$, then substitute into $\mathbb{E}|J_t(x)K_t(y)|$. This is bounded by a sum of terms of the form $\mathbb{E}|h(F_t x)\ell(F_t y)|$ or time integrals of these (at the worst), where h and ℓ satisfy inequality (3.7). Proceeding as above, each such term is bounded above by $C^2 2^{\tilde{p}} e^{K_{\tilde{p}}t}(1+|x|)^p(1+|y|)^{\bar{p}}$ or time integrals of these, which are integrable with respect to M_0 and N_0. This justifies interchanging integrals. □

4 Centroid and dispersion formulas

This section is devoted to finding formulas for the means, second moments, and quadratic variations of the centroid process C and dispersion process D which were defined in Section 1. The results of Sections 2 and 3 and standard properties of semimartingales are used extensively.

Formulas (4.5) and (4.8) appeared in ZIRBEL (1993) and ZIRBEL and ÇINLAR (1996), and a sketch of these calculations appeared in ZIRBEL and ÇINLAR (1997).

Recall the definitions (1.4) and (1.5) of C_t and D_t. Let $f^i(x) = x^i$. Then $C_t^i = M_t f^i$ for each $i = 1, \ldots, d$. For D_t, note that by its definition and the fact that $M_t(\mathbb{R}^d) = 1$,

$$D_t^{ij} = \int M_t(dx)(x^i - C_t^i)(x^j - C_t^j) = \int M_t(dx)x^i x^j - C_t^i C_t^j.$$

Thus, if we set $g^{ij}(x) = x^i x^j$, we have $D_t^{ij} = M_t g^{ij} - C_t^i C_t^j$. Now we may appeal to Theorem (3.3).

For the rest of the section we assume that for some $p > 2$, M_0 is such that $\int M_0(dx)|x|^p$ is finite. We will make further assumptions later as necessary. By Proposition (2.5), the processes $C_t,\ t \geq 0$, and $D_t,\ t \geq 0$, are well defined and continuous.

Formulas for the centroid

By Theorem (3.3) and the fact that $\mathcal{A}f^i = u^i$, the centroid satisfies

$$C_t = C_0 + \int_0^t ds\ M_s u + K_t, \tag{4.1}$$

The process K is a mean-zero square-integrable martingale on $\mathbb{R}^d$ by Remark (3.9). By part (b) of Theorem (3.3),

$$\langle C^i, C^j \rangle_t = \int_0^t ds\ \int M_s(dx) \int M_s(dy)\ a^{ij}(x,y) \tag{4.2}$$

for all $i, j = 1, \ldots, d$ and $t \geq 0$.

The moment $\mathbb{E}C_t^i C_t^j$ is computed as follows. We assume that $\int M_0(dx)|x|^4$ is finite. By the integration by parts formula,

$$C_t^i C_t^j = C_0^i C_0^j + \int_0^t C_s^i dC_s^j + \int_0^t C_s^j dC_s^i + \langle C^i, C^j \rangle_t$$

But notice that, by (4.1) and (4.2), this equals

$$\begin{aligned} C_0^i C_0^j \quad &+ \quad \int_0^t ds\ [C_s^i M_s u^j + C_s^j M_s u^i + \int M_s(dx) \int M_s(dy)\ a^{ij}(x,y)] \\ &+ \quad \int_0^t C_s^i dK_s^j + \int_0^t C_s^j dK_s^i \end{aligned}$$

Let us define a matrix A by

$$A^{ij}(x,y) = x^i u^j(y) + u^i(x) y^j + a^{ij}(x,y) \tag{4.3}$$

Then the quantity in brackets equals $\int M_s(dx) \int M_s(dy)\ A^{ij}(x,y)$, and thus

$$\begin{aligned} C_t^i C_t^j = C_0^i C_0^j \quad &+ \quad \int_0^t ds\ \int M_s(dx) \int M_s(dy)\ A^{ij}(x,y) \\ &+ \quad \int_0^t C_s^i dK_s^j + \int_0^t C_s^j dK_s^i \end{aligned} \tag{4.4}$$

The last two terms are square-integrable martingales: K^j is square integrable by Remark (3.9), and we will show now that $\mathbb{E} \int_0^t (C_s^i)^2 d\langle K^j, K^j \rangle_s$ is finite. First, by (4.1) and (4.2),

$$d\langle K^j, K^j \rangle_t = d\langle C^j, C^j \rangle_t = \int M_t(dx) \int M_t(dy) a^{jj}(x,y) dt,$$

and by (1.8), $a^{jj}(x,y) \le R(1+|x|)(1+|y|)$. Also, by the definition (1.4) of C_t,

$$|C_t^i| \le \int M_t(dx)|x^i| \le \int M_t(dx)(1+|x|),$$

so that

$$\begin{aligned} \mathbb{E}\int_0^t (C_s^i)^2 d\langle K^j, K^j\rangle_s &\le \mathbb{E}\int_0^t ds\ R(\int M_s(dx)(1+|x|))^4 \\ &\le R\mathbb{E}\int_0^t ds \int M_s(dx)(1+|x|)^4, \end{aligned}$$

which is finite by Proposition (2.5) since we have assumed that $\int M_0(dx)|x|^4$ is finite.

Taking the mean of (4.4),

$$\mathbb{E}C_t^iC_t^j = C_0^iC_0^j + \mathbb{E}\int_0^t ds \int M_s(dx)\int M_s(dy)\ A^{ij}(x,y). \tag{4.5}$$

This bears formal similarity to (4.2) due to our definition of A. When $u \equiv 0$, C is a martingale and A reduces to a. The covariance of C_t^i and C_t^j may now be computed using (4.5) and the mean of (4.1).

Formulas for the dispersion

Above, we set $g^{ij}(x) = x^ix^j$ and saw that $D_t^{ij} = M_tg^{ij} - C_t^iC_t^j$. Now by Theorem (3.3) and the fact that $\mathcal{A}g^{ij}(x) = A^{ij}(x,x)$, we have

$$M_tg^{ij} = M_0g^{ij} + \int_0^t ds \int M_s(dx)A^{ij}(x,x) + L_t^{ij}, \tag{4.6}$$

the last term of which is a square-integrable martingale. Combining this with (4.4) and recalling that $M_s(\mathbb{R}^d) = 1$, we obtain

$$\begin{aligned} D_t^{ij} = D_0^{ij} &+ \int_0^t ds \int M_s(dx)\int M_s(dy)\ [A^{ij}(x,x) - A^{ij}(x,y)] \\ &+ L_t^{ij} - \int_0^t C_s^i dK_s^j - \int_0^t C_s^j dK_s^i \end{aligned} \tag{4.7}$$

The last three terms here are square-integrable martingales, thus,

$$\mathbb{E}D_t^{ij} = D_0^{ij} + \mathbb{E}\int_0^t ds \int M_s(dx)\int M_s(dy)\ [A^{ij}(x,x) - A^{ij}(x,y)]. \tag{4.8}$$

The calculation of $\langle D^{ij}, D^{k\ell}\rangle_t$ begins by substituting in from (4.7) and using identities of the form $\langle X, \int Y_s dZ_s\rangle = \int Y_s d\langle X, Z\rangle_s$ repeatedly.

$$\begin{aligned}
\langle D^{ij}, D^{k\ell}\rangle_t &= \langle L^{ij}, L^{k\ell}\rangle_t - \int_0^t C_s^k d\langle L^{ij}, C^\ell\rangle_s - \int_0^t C_s^\ell d\langle L^{ij}, C^k\rangle_s \\
&- \int_0^t C_s^i d\langle C^j, L^{k\ell}\rangle_s + \int_0^t C_s^i C_s^k d\langle C^j, C^\ell\rangle_s + \int_0^t C_s^i C_s^\ell d\langle C^j, C^k\rangle_s \\
&- \int_0^t C_s^j d\langle C^i, L^{k\ell}\rangle_s + \int_0^t C_s^j C_s^k d\langle C^i, C^\ell\rangle_s + \int_0^t C_s^j C_s^\ell d\langle C^i, C^k\rangle_s
\end{aligned}$$

Now L^{ij} is the martingale part of $M_t g^{ij}$, and of course $C_t^i = M_t f^i$, so the required quadratic variations can be found from Theorem (3.3):

$$\begin{aligned}
\langle L^{ij}, L^{k\ell}\rangle_t &= \int_0^t ds \int M_s(dx) \int M_s(dy)\ [x^i y^k a^{j\ell} + x^i y^\ell a^{jk} + x^j y^k a^{i\ell} + x^j y^\ell a^{ik}] \\
\langle L^{ij}, C^k\rangle_t &= \int_0^t ds \int M_s(dx) \int M_s(dy)\ [x^i a^{jk} + x^j a^{ik}] \\
\langle C^i, L^{k\ell}\rangle_t &= \int_0^t ds \int M_s(dx) \int M_s(dy)\ [y^k a^{i\ell} + y^\ell a^{ik}]
\end{aligned}$$

We have suppressed the argument (x, y) of the covariance a. The third formula simply restates the second in a form which will be convenient. Of course, $\langle C^i, C^j\rangle_t$ is given in (4.2).

Combining carefully into the expression above for $\langle D^{ij}, D^{k\ell}\rangle_t$, the integrands we obtain are the four permutations $(i \leftrightarrow j, k \leftrightarrow \ell)$ of the general term

$$a^{j\ell}(x, y)[x^i y^k - x^i C_s^k - C_s^i y^k + C_s^i C_s^k]$$

which is the same as $(x^i - C_s^i) a^{j\ell}(x, y)(y^k - C_s^k)$. We can rewrite the sum of these terms as the quadratic form $(x - C_s)^T B(x, y)(y - C_s)$, where B is the matrix

$$B^{mn} = \delta_{im}\delta_{kn} a^{j\ell} + \delta_{im}\delta_{\ell n} a^{jk} + \delta_{jm}\delta_{kn} a^{i\ell} + \delta_{jm}\delta_{\ell n} a^{ik} \tag{4.9}$$

with the argument (x, y) suppressed. With this notation,

$$\langle D^{ij}, D^{k\ell}\rangle_t = \int_0^t ds \int M_s(dx) \int M_s(dy)\ (x - C_s)^T B(x, y)(y - C_s), \tag{4.10}$$

which is as succinct a formula as one could desire.

We will now compute $\mathbb{E} D_t^{ij} D_t^{k\ell}$. Use the integration by parts formula:

$$D_t^{ij} D_t^{k\ell} = D_0^{ij} D_0^{k\ell} + \int_0^t D_s^{ij} dD_s^{k\ell} + \int_0^t D_s^{k\ell} dD_s^{ij} + \langle D^{ij}, D^{k\ell}\rangle_t$$

The middle two terms are treated by substituting in from (4.7). For example,

$$\begin{aligned}\int_0^t D_s^{ij} dD_s^{k\ell} &= \int_0^t ds\ D_s^{ij} \int M_s(dx) \int M_s(dy)\ [A^{k\ell}(x,x) - A^{k\ell}(x,y)] \\ &+ \int_0^t D_s^{ij} dL_s^{k\ell} - \int_0^t D_s^{ij} C_s^k dK_s^\ell - \int_0^t D_s^{ij} C_s^\ell dK_s^k\end{aligned}$$

One may verify that, if $\int M_0(dx)|x|^8$ is finite, the last three terms satisfy integrability conditions guaranteeing that they are square-integrable martingales. Thus,

$$\begin{aligned}\mathbb{E} D_t^{ij} D_t^{k\ell} &= D_0^{ij} D_0^{k\ell} \\ &+ \mathbb{E} \int_0^t ds \int M_s(dx) \int M_s(dy)\ [D_s^{ij}(A^{k\ell}(x,x) - A^{k\ell}(x,y)) \\ &+ D_s^{k\ell}(A^{ij}(x,x) - A^{ij}(x,y)) + (x - C_s)^T B(x,y)(y - C_s)]\end{aligned} \tag{4.11}$$

The covariance of D_t^{ij} and $D_t^{k\ell}$ may now be computed.

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