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MEAN OCCUPATION TIMES OF CONTINUOUS ONE-DIMENSIONAL MARKOV PROCESSES

Craig L. Zirbel

Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403-0221
zirbel@math.bgsu.edu

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Abstract. We give a general method for finding the long-time asymptotic growth rate of mean occupation times of one-dimensional continuous strong Markov processes. The method uses a well-known decomposition of the resolvent, previous work of Y. Kasahara, and some new comparison results. Particular attention is paid to occupation times measured according to a function which is supported on the whole range of the process. We give an extended example concerning isotropic Brownian flows. A companion paper gives several other examples.

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1 Introduction

Let X be a continuous strong Markov process on an interval I on the real line and let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Borel function. In this paper we discuss a general method for finding the growth rate of the expected occupation time

$$L(t) = \mathrm{E}^x \int_0^t ds\, f(X_s) \tag{1.1}$$

as $t \to \infty$. This problem is well understood when X is positive Harris recurrent – in this case $L(t)/t \to \int_I M(dx) f(x)$, where M is the invariant probability measure for X. We will focus on the null recurrent and transient cases. We are particularly interested in situations in which f is supported near the endpoints of I.

We have encountered such problems in connection with isotropic Brownian flows. The companion paper ZIRBEL (1997) makes extensive use of the method described here. A further complication is also introduced: X has an initial distribution which places mass near the endpoints of I. Section 6 of the present article contains a simpler, but still instructive, example.

The method we propose is to first take the Laplace-Stieltjes transform of L:

$$\hat{L}(\alpha) = \mathrm{E}^x \int_0^\infty dt\, e^{-\alpha t} f(X_t) \tag{1.2}$$

(This is $U_\alpha f(x)$, where U_α is the α-potential or resolvent of X.) Second, find the asymptotic behavior of $\hat{L}(\alpha)$ as $\alpha \to 0$, and third, use a Tauberian theorem to find the asymptotic behavior of $L(t)$ as $t \to \infty$. The second step is the focus of the paper. For completeness, we quote the Tauberian theorem of BINGHAM, GOLDIE and TEUGELS (1986) in this context:

(1.3) **Theorem.** Suppose L is non-decreasing and $c \geq 0$. If g varies regularly at ∞ with exponent $\delta \in [0, \infty)$ (see Definition (3.6)), then the following are equivalent:

$$L(t) \sim cg(t)/\Gamma(1+\delta), \qquad t \to \infty$$

$$\hat{L}(\alpha) \sim cg(1/\alpha), \qquad \alpha \downarrow 0$$

When $c = 0$, we interpret these as $L(t) = o(g(t))$ and $\hat{L}(\alpha) = o(g(1/\alpha))$. □

The paper is organized as follows. We begin in Section 2 with various definitions concerning the following decomposition of $\hat{L}(\alpha)$:

(1.4)
$$\hat{L}(\alpha) = h(\alpha) \int_I v_1(x \wedge y, \alpha) v_2(x \vee y, \alpha) f(y) M(dy)$$

The asymptotic behavior of $h(\alpha)$ has been treated by Y. Kasahara and co-authors for a certain class of Markov processes. In Section 3, we restate some of their results for our case and then give a new version which is considerably easier to apply (Theorem (3.9)).

The functions v_1 and v_2 are positive and are monotone as functions of their first arguments. When f is supported near an endpoint of I, we need tight bounds on v_1 and v_2 near that endpoint in order to examine the integral in (1.4) as $\alpha \to 0$. In Sections 4 and 5, we show how to obtain such bounds by comparing X to another Markov process having similar behavior in the region of interest.

In Section 6 we carry out an analysis of occupation times of the two-point distance process in a two-dimensional isotropic Brownian flow with top Lyapunov exponent equal to 0. This example illustrates the use of the method and resolves an unsettled question concerning this process. Further examples may be found in the companion article ZIRBEL (1997).

Finally, let us mention that knowing the asymptotic behavior of the mean occupation time in (1.1) can sometimes be parlayed into statements concerning the law of the occupation time process $\int_0^t ds f(X_s)$, $t \geq 0$. The original result of DARLING and KAC (1957) has been greatly improved and generalized by STONE (1963), BINGHAM (1971), KASAHARA (1975, 1977, 1982), and BINGHAM, GOLDIE and TEUGELS (1986), Section 8.11. These results require some uniformity in the asymptotics of L as x ranges over the support of f, which will often fail to hold when f is supported near the endpoints of I. We are unaware of any general results on the law of the occupation time process in this case.

2 Decomposition of Laplace transform

We assume that X is a strong Markov process with continuous paths, taking values in an interval I in $(-\infty, +\infty)$. Denote by ℓ the left endpoint of I and by r the right endpoint. These may or may not lie in I. We assume that X is *regular* on (ℓ, r), that is, for each x in (ℓ, r) and y in I,

$$\mathrm{P}^x\{T_y < +\infty\} > 0. \tag{2.1}$$

Here T_y is the first hitting time of y by X and P^x is a probability measure under which $X_0 = x$ almost surely. With this assumption, X is a Feller process.

Denote by $\mathcal{A}$ the infinitesimal generator of X, by s its scale function, and by M its speed measure. The function s is continuous and strictly increasing (BREIMAN (1968), Theorem 16.27). We extend s to $[\ell, r]$ (if necessary) by setting $s(r) = \lim_{x\uparrow r} s(x)$ and $s(\ell) = \lim_{x\downarrow \ell} s(x)$. When M is absolutely continuous with respect to Lebesgue measure, we denote its density by m.

Let $f : I \to \mathbb{R}$ be a bounded Borel function. The following decomposition is proven in DYNKIN (1965), Theorem 17.9 and ITÔ and MCKEAN (1965), Section 4.11, but see BREIMAN (1968), Theorem 16.75 for an introduction.

$$\mathrm{E}^x \int_0^\infty dt\, e^{-\alpha t} f(X_t) = h(\alpha) \int_I v_1(x \wedge y, \alpha) v_2(x \vee y, \alpha) f(y) M(dy) \tag{2.2}$$

Here E^x denotes expectation using the probability measure P^x. The left hand side is the resolvent of X. (Note that the definition of the speed measure varies by a factor of 2 in DYNKIN (1965), while BREIMAN (1968) assumes the process is already in natural scale.)

The functions v_1 and v_2 are defined in terms of the hitting times of X:

$$v_1(x, \alpha) = \begin{cases} \mathrm{E}^x e^{-\alpha T_c}, & \text{if } x \le c \\ \frac{1}{\mathrm{E}^c e^{-\alpha T_x}}, & \text{if } x > c \end{cases} \tag{2.3}$$

$$v_2(x, \alpha) = \begin{cases} \frac{1}{\mathrm{E}^c e^{-\alpha T_x}}, & \text{if } x \le c \\ \mathrm{E}^x e^{-\alpha T_c}, & \text{if } x > c \end{cases} \tag{2.4}$$

Here $c \in (\ell, r)$ is arbitrary but fixed. In essence, v_1 has to do with motion to the right, v_2 with motion to the left. The function $v_1(\cdot, \alpha)$ is increasing, $v_2(\cdot, \alpha)$ is decreasing, and both are positive. They are the unique such solutions of

$$\mathcal{A}v = \alpha v, \qquad v(c, \alpha) = 1. \tag{2.5}$$

See BREIMAN (1968), Theorem 16.69. Let D_s^+ be the operator defined by

$$D_s^+ v(x) = \lim_{y \downarrow x} \frac{v(y) - v(x)}{s(y) - s(x)}$$

The function h is the reciprocal of the Wronskian of v_1 and v_2:

(2.6) $$h(\alpha) = (D_s^+ v_1(x,\alpha) v_2(x,\alpha) - v_1(x,\alpha) D_s^+ v_2(x,\alpha))^{-1}$$

The right side is constant as a function of x, although it depends on the choice of c.

(2.7) **Example. BES^2 process.** Let $X_t,\ t \geq 0$ be a BES^2 process, that is, the norm of a planar Brownian motion. Its generator $\mathcal{A}$ is given by

$$(\mathcal{A}f)(x) = \frac{1}{2} f''(x) + \frac{1}{2x} f'(x), \qquad x > 0$$

The scale function of X is $s(x) = \log x$ and the density m of the speed measure satisfies $m(x) = 2x$. The equation $\mathcal{A}v = \alpha v$ becomes

$$\frac{1}{2} v''(x) + \frac{1}{2x} v'(x) - \alpha v(x) = 0, \qquad v(c) = 1$$

or

$$x^2 v''(x) + x v'(x) - 2\alpha x^2 v(x) = 0, \qquad v(c) = 1$$

This is the modified Bessel equation of order 0. The general solutions are

$$I_0(\sqrt{2\alpha} x), \qquad K_0(\sqrt{2\alpha} x),$$

which are respectively increasing and decreasing. The functions v_1 and v_2 are thus given by

(2.8) $$v_1(x,\alpha) = \frac{I_0(\sqrt{2\alpha} x)}{I_0(\sqrt{2\alpha} c)}, \qquad v_2(x,\alpha) = \frac{K_0(\sqrt{2\alpha} x)}{K_0(\sqrt{2\alpha} c)}$$

where $c \in (0, \infty)$ is fixed. A direct calculation gives

(2.9) $$h(\alpha) = I_0(\sqrt{2\alpha} c) K_0(\sqrt{2\alpha} c).$$

Limit of v_1 and v_2 as $\alpha \to 0$

For the limit of $\hat{L}(\alpha)$ as $\alpha \to 0$, we will need to know $\lim_{\alpha \downarrow 0} v_1(x,\alpha)$ and $\lim_{\alpha \downarrow 0} v_2(x,\alpha)$. The limits are not equal to 1 in the transient case since, for example, T_c may be infinite with positive probability. Indeed, for v_1, we obtain

$$\lim_{\alpha \downarrow 0} v_1(x,\alpha) = \begin{cases} \mathbf{P}^x\{T_c < \infty\}, & \text{if } x \leq c \\ \frac{1}{\mathbf{P}^c\{T_x < \infty\}}, & \text{if } x > c \end{cases}$$

Standard properties of the scale function yield, for $a < b$,

$$\mathbf{P}^a\{T_b < \infty\} = \frac{s(a) - s(\ell)}{s(b) - s(\ell)},$$

which we interpret to be 1 if $s(\ell) = -\infty$. Making this substitution,

$$\lim_{\alpha \downarrow 0} v_1(x, \alpha) = \frac{s(x) - s(\ell)}{s(c) - s(\ell)} \tag{2.10}$$

with the same interpretation if $s(\ell) = -\infty$. Similarly,

$$\lim_{\alpha \downarrow 0} v_2(x, \alpha) = \frac{s(r) - s(x)}{s(r) - s(c)}, \tag{2.11}$$

which equals 1 if $s(r) = \infty$.

Change of variable

It is often useful to rescale the real line in order to obtain a process with nicer properties. Let us see how this affects the functions h, v_1, and v_2 in the decomposition (2.2).

Let $\phi : I \to \mathbb{R}$ be continuous and strictly increasing. Define a process Y by $Y_t = \phi(X_t)$. Then Y is a regular, strong Markov process with continuous paths, taking values in the interval $\phi(I)$. Its scale function equals $s \circ \phi^{-1}$ and its speed measure N is the image of M under ϕ, that is $N(A) = M(\phi^{-1}(A))$ for each Borel $A \subseteq \phi(I)$. Occupation times of Y have a decomposition

$$\mathbf{E}^y \int_0^\infty dt\, e^{-\alpha t} g(Y_t) = \tilde{h}(\alpha) \int_{\phi(I)} u_1(y \wedge z, \alpha) u_2(y \vee z, \alpha) g(z) N(dz) \tag{2.12}$$

where $\tilde{h}, u_1$, and u_2 are defined as before, but with respect to Y. We take the reference location $\tilde{c}$ equal to $\phi(c)$.

Suppose that X starts at a and first hits b at time T_b. Then Y starts at $\phi(a)$ and first hits $\phi(b)$ at time T_b. Looking at (2.3) and (2.4), we see that

$$v_1(x, \alpha) = u_1(\phi(x), \alpha), \qquad v_2(x, \alpha) = u_2(\phi(x), \alpha). \tag{2.13}$$

The relationship between h and $\tilde{h}$ is even easier: they are equal. First, note that for positive $f : I \to \mathbb{R}$,

$$\mathbf{E}^x \int_0^\infty dt\, e^{-\alpha t} f(X_t) = \mathbf{E}^{\phi(x)} \int_0^\infty dt\, e^{-\alpha t} f(\phi^{-1}(Y_t)),$$

where $\mathbf{E}^{\phi(x)}$ means that $Y_0 = \phi(x)$. By the decomposition (2.12), this equals

$$\tilde{h}(\alpha) \int_{\phi(I)} u_1(\phi(x) \wedge z, \alpha) u_2(\phi(x) \vee z, \alpha) f(\phi^{-1}(z)) N(dz)$$

Changing variables via $z = \phi(y)$ makes this

$$\tilde{h}(\alpha) \int_I u_1(\phi(x) \wedge \phi(y), \alpha) u_2(\phi(x) \vee \phi(y), \alpha) f(y) M(dy)$$

But $\phi(x) \wedge \phi(y) = \phi(x \wedge y)$ and $\phi(x) \vee \phi(y) = \phi(x \vee y)$, so this is

$$\tilde{h}(\alpha) \int_I v_1(x \wedge y, \alpha) v_2(x \vee y, \alpha) f(y) M(dy)$$

which is identical to (2.2) if and only if $\tilde{h} = h$.

3 Asymptotic behavior of h

In this section we discuss methods to determine the asymptotic behavior of $h(\alpha)$ as $\alpha \to 0$. When X is *transient*, meaning that for some a and b in (ℓ, r) we have $\mathbf{P}^a\{T_b = \infty\} > 0$, Problem 4.11.8 of ITÔ and MCKEAN (1965) shows that

$$\lim_{\alpha \downarrow 0} h(\alpha) < +\infty. \tag{3.1}$$

Conversely, this condition implies that X is transient. In particular, if I is the open interval (ℓ, r), then X is transient if and only if $s(\ell)$ and $s(r)$ are not both infinite (REVUZ and YOR (1991), Chapter VII, Exercise (3.21)).

We will quote two results of Y. Kasahara and others, one concerning the limit of h when X is transient and the other for the asymptotic behavior of $h(\alpha)$ as $\alpha \to 0$ when X is recurrent. Kasahara's work concerns one-dimensional martingale diffusions (i.e., diffusions on natural scale) which are closely related to the spectral theory of strings. See KOTANI and WATANABE (1982) for a review of this connection.

Consider the martingale Y defined by $Y_t = s(X_t)$ for $t \geq 0$. Without loss of generality, we assume that $s(c) = 0$. As we saw in Section 2, Y takes values in the interval with endpoints $s(\ell)$ and $s(r)$. Its scale function is the identity map and its speed measure N is the image of M under s. It has associated functions h, u_1, and u_2; the function h is the same as for X, while u_1 and u_2 are given by (2.13).

In order to relate Kasahara's results, first note that, by Theorem 16.36 of BREIMAN (1968), $M[a, b] < \infty$ when $\ell < a \leq b < r$. Second, we assume that the speed measure satisfies

$$M\{\ell\} = +\infty, \qquad M\{r\} = +\infty. \tag{3.2}$$

A measure satisfying these conditions is said to be *inextensible* in the terminology of KASAHARA, KOTANI, and WATANABE (1980); see also pp. 246-247 of KOTANI and WATANABE (1982). Note that the measure N is also inextensible. Condition (3.2) makes no difference at a natural or entrance boundary, since such points are inaccessible from (ℓ, r). It makes no difference at an exit boundary, since the process stays there once it hits there and the decomposition (2.2) is unaffected by the values of $M\{\ell\}$ and $M\{r\}$ (see (5.1) and (5.2)). However, regular endpoints must be absorbing, not reflecting.

On pages 246-247 of KOTANI and WATANABE (1982) and pp. 178-179 of KASAHARA, KOTANI, and WATANABE (1980), we see the same situation as we have here for Y. The resolvent of Y is written in terms of functions h, u_-, u_+, and m corresponding to our h, u_1, u_2, and N, respectively. The function h is further broken up as

$$\frac{1}{h(\alpha)} = \frac{1}{h_+(\alpha)} + \frac{1}{h_-(\alpha)} \tag{3.3}$$

The definition of h_+ depends only on the restriction of the speed measure m (our N) to $[0, \infty]$; similarly h_- depends on the restriction to $[-\infty, 0)$. This relationship between h_+ and the restriction of N to $[0, \infty)$ is called *Krein's correspondence.*

KASAHARA (1975) is the first of several articles concerning Krein's correspondence and its implications for one-dimensional Markov processes. From Equation (9) of that article,

$$\lim_{\alpha \downarrow 0} h_+(\alpha) = s(r) \tag{3.4}$$

$$\lim_{\alpha \downarrow 0} h_-(\alpha) = -s(\ell) \tag{3.5}$$

The first is exactly Kasahara's Equation (9), and the second follows by reflecting Y about 0. This allows us to compute $\lim_{\alpha \downarrow 0} h(\alpha)$ when X is transient.

When X is recurrent, we will have $h_\pm(\alpha) \to \infty$ as $\alpha \to 0$. We wish to quote a comparison theorem concerning the asymptotic behavior of $h_+(\alpha)$ as $\alpha \to 0$. First we need to mention regular variation.

(3.6) **Definition.** A Borel measurable function $g : (0, \infty) \to (0, \infty)$ is said to be *regularly varying with exponent* δ if, for all $\lambda > 0$,

$$\lim_{x \to \infty} \frac{g(\lambda x)}{g(x)} = \lambda^\delta$$

If $\delta = 0$, we say that g is *slowly varying.* If g is regularly varying with exponent δ, then we can write $g(x) = x^\delta f(x)$ for a slowly varying function f. □

The following result involves the restrictions to $[0, \infty]$ of two inextensible measures N_1 and N_2 and their corresponding functions h_1 and h_2 (these play the role of h_+). It first appeared as Theorem 2 of KASAHARA, KOTANI, and WATANABE (1980); we use the version given as Theorem 2.1 of KOTANI and WATANABE (1982).

(3.7) **Theorem.** *Define* $Q_i(y) = yN_i[0,y]$ *for* $i = 1,2$ *and let* Q_i^{-1} *be the generalized inverse of* Q_i: $Q_i^{-1}(z) = \inf\{y : Q_i(y) > z\}$. *Suppose* f *is slowly varying. Then the following statements are equivalent.*

$$Q_1^{-1}(z) \sim Q_2^{-1}(z)f(z), \qquad z \to \infty$$

and

$$h_1(\alpha) \sim h_2(\alpha)f(\frac{1}{\alpha}), \qquad \alpha \to 0.$$

To use this result, we need some reference cases of Krein's correspondence:

(3.8) **Example.** (KASAHARA (1975)) Suppose $\delta \in (0,1]$ and let $N[0,y] = y^{1/\delta - 1}$ for $y \geq 0$ and $N\{\infty\} = +\infty$. Then the corresponding function h_+ is given by $h_+(\alpha) = D_\delta \alpha^{-\delta}$, where $D_1 = 1$ and $D_\delta = \Gamma(1+\delta)\delta^{-\delta}(1-\delta)^{-1-\delta}$ for $\delta \in (0,1)$.

We may extend this to the case $\delta = 0$ by interpreting N as $N[0,y] = 0$ for $0 \leq y < 1$ and $N[0,y] = +\infty$ for $y \geq 1$. We have $D_0 = 1$ and $h_+(\alpha) = 1$ in this case. In this case N is the restriction to $[0,\infty]$ of an inextensible measure but is not the speed measure of a continuous strong Markov process. Still, we may still consider Krein's correspondence and use Theorem (3.7); see pp. 235-236 of KOTANI and WATANABE (1982). □

Finally, we come to our interpretation of Theorem (3.7). It bears resemblance to Theorem 2 of KASAHARA (1975) and the corollary to Theorem 2.1 of KOTANI and WATANABE (1982), which are both combinations of Theorem (3.7) and Example (3.8). However, we have found our version easier to apply because it does not require us to look at the inverse of a slowly varying function. A version for h_- is stated afterward as a corollary. Recall that condition (3.2) is still in force.

(3.9) **Theorem.** *Suppose that* $s(r) = +\infty$, M *has no atoms in* $[c,r)$, *and* g *varies regularly with exponent* $\delta \in [0,1]$. *Then the following are equivalent:*

$$\lim_{x \uparrow r} \frac{g(s(x)M[c,x])}{s(x)} = 1 \tag{3.10}$$

and

$$\lim_{\alpha \downarrow 0} \frac{h_+(\alpha)}{g(\frac{1}{\alpha})} = D_\delta \tag{3.11}$$

where D_δ *is defined in Example (3.8).*

Proof: Let N_1 be the restriction of the speed measure of Y to $[0,\infty]$. Then $h_1 = h_+$. Let $N_2[0,y] = y^{1/\delta-1}$ so that $h_2(\alpha) = D_\delta \alpha^{-\delta}$ by Example (3.8). Let $Q_i(y) = yN_i[0,y]$ as in Theorem (3.7). Also, write $g(z) = z^\delta f(z)$, where f is slowly varying.

First note that $M[c,x] = N_1[0,s(x)]$ by the definition of N. Equation (3.10) can be rewritten as

$$1 = \lim_{y\to\infty} \frac{g(yN_1[0,y])}{y} = \lim_{y\to\infty} \frac{g(Q_1(y))}{y} = \lim_{z\to\infty} \frac{g(z)}{Q_1^{-1}(z)}$$

by setting $y = s(x)$ and then $z = Q_1(y)$. (Q_1 is invertible since it is strictly increasing by definition and is continuous because M has no atoms.) But $g(z) = z^\delta f(z) = Q_2^{-1}(z)f(z)$, so this is equivalent to $Q_1^{-1}(z) \sim Q_2^{-1}(z)f(z)$ as $z\to\infty$. By Theorem (3.7), this is equivalent to $h_1(\alpha) \sim h_2(\alpha)f(\frac{1}{\alpha})$. But $h_2(\alpha)f(\frac{1}{\alpha}) = D_\delta g(\frac{1}{\alpha})$, so this is equivalent to condition (3.11), completing the proof. □

(3.12) **Remark.** We can prove the same result if M has an atomic component, provided we impose an additional requirement to insure that

$$\lim_{y\to\infty} \frac{g(Q_1(y))}{y} = 1 \text{ implies } \lim_{z\to\infty} \frac{g(z)}{Q_1^{-1}(z)} = 1$$

(The other direction is not problematic.)

One possibility is to require Q_1 to be regularly varying. Supposing the first limit equals 1, we let $y = Q_1^{-1}(z)$ to find

$$\lim_{z\to\infty} \frac{g(z)}{Q_1^{-1}(z)} = \lim_{z\to\infty} \frac{g(z)}{g(Q_1(Q_1^{-1}(z)))} \lim_{y\to\infty} \frac{g(Q_1(y))}{y}$$

If Q_1 is regularly varying, then $Q_1(Q_1^{-1}(z)) \sim z$ as $z\to\infty$ (BINGHAM, GOLDIE and TEUGELS (1986), Theorem 1.5.12) and so $g(Q_1(Q_1^{-1}(z))) \sim g(z)$ as $z\to\infty$ (BINGHAM, GOLDIE and TEUGELS (1986), Proposition 1.5.7ii), and we are done. □

(3.13) **Corollary.** *Suppose that $s(\ell) = -\infty$, M has no atoms in (ℓ, c), and g varies regularly with exponent $\delta \in [0,1]$. Then the following are equivalent:*

$$\lim_{x\downarrow\ell} \frac{g(-s(x)M[x,c))}{-s(x)} = 1 \qquad \textit{and} \qquad \lim_{\alpha\downarrow 0} \frac{h_-(\alpha)}{g(\frac{1}{\alpha})} = D_\delta$$

where D_δ is defined in Example (3.8). □

4 Growth of v_1 near r and v_2 near ℓ

Consider $v_1(x,\alpha)$ for x near r. If $r \notin I$, then $v_1(x,\alpha) \to \infty$ as $x \uparrow r$ (BREIMAN (1968), Theorem 16.69). On the other hand, for fixed $x > c$, $v_1(x,\alpha)$ decreases as $\alpha \downarrow 0$. Similar comments apply to $v_2(x,\alpha)$ for x near ℓ. We want to find upper and lower bounds on v_1 and v_2 in these regions so that we can sort out these competing effects.

We note that if the initial position x of the process X is a single, fixed point, then these bounds will not be necessary, as one can see in the decomposition (2.2). The bounds are useful when one is interested in quantifying the dependence of occupation times on initial position. For example, this arises in the companion paper ZIRBEL (1997), where the initial state is random.

The idea is to find another diffusion $\tilde{X}$ which behaves much like X in the region of interest, but for which we can find the corresponding $\tilde{v}_1$ and $\tilde{v}_2$ explicitly. For example, the separation process Z considered in Section 6 is essentially a BES^2 process when it is far away from the origin, and the BES^2 process is simple to analyze, as we have seen in Example (2.7).

The starting point is an explicit construction of solutions of $\mathcal{A}v = \alpha v$ by successive approximations.

(4.1) **Theorem.** *Fix k_1 and k_2 in $\mathbb{R}$. Define functions $\xi_0, \xi_1, \ldots$ and ξ by*

$$\begin{aligned} \xi_0(x) &= k_1 + k_2(s(x) - s(c)) \\ \xi_{n+1}(x) &= \int_c^x (s(x) - s(y))\xi_n(y)M(dy), \qquad n = 0, 1, \ldots \\ \xi(x,\alpha) &= \sum_{n=0}^{\infty} \xi_n(x)\alpha^n \end{aligned}$$

Then this series converges uniformly on each closed subinterval of (ℓ, r). For each $\alpha > 0$, ξ is a solution of

$$\mathcal{A}\xi = \alpha\xi, \qquad \xi(c) = k_1, \qquad D_s^+\xi(c) = k_2.$$

Proof: Let $\lambda = \alpha/2$. The differential equation for ξ can be written as

$$\tfrac{1}{2} dD_s^+\xi - \lambda\xi dM = 0,$$

in the terminology of DYNKIN (1965), Section 17.7. Theorem 17.7 of the same book can be applied to show that the series $\sum_{n=0}^{\infty} \xi_n(x)\alpha^n$ does indeed converge uniformly to a function ξ satisfying the conditions at c. □

Suppose now that $\tilde{X}$ is a regular, continuous, strong Markov process on I with scale function $\tilde{s}$, speed measure $\tilde{M}$, and generator $\tilde{\mathcal{A}}$. We have the following comparison result for solutions of $\mathcal{A}\xi = \alpha\xi$ and $\tilde{\mathcal{A}}\tilde{\xi} = \alpha\tilde{\xi}$.

(4.2) **Theorem.** *Let J be either the interval $(\ell, c]$ or $[c, r)$. Suppose that for some $K > 0$ we have*

$$s(b) - s(a) \leq K(\tilde{s}(b) - \tilde{s}(a)), \qquad M(a,b] \leq K\tilde{M}(a,b]$$

for all a, b in J with $a \leq b$. Consider solutions $\xi(x,\alpha)$ and $\tilde{\xi}(x,\tilde{\alpha})$ of the equations

$$\mathcal{A}\xi = \alpha\xi, \qquad \xi(c) = k_1, \qquad D_s^+\xi(c) = k_2$$

$$\tilde{\mathcal{A}}\tilde{\xi} = \tilde{\alpha}\tilde{\xi}, \qquad \tilde{\xi}(c) = k_1, \qquad D_{\tilde{s}}^+\tilde{\xi}(c) = \tilde{k}_2$$

a) If $k_1 > 0$ and $k_2 = \tilde{k}_2 = 0$, then for all x in J and $\alpha > 0$, $\xi(x,\alpha) \leq \tilde{\xi}(x, K^2\alpha)$.

b) If $k_1 \geq 0$, J equals $[c, r)$, and $k_2, \tilde{k}_2 > 0$, then there exists a constant B (depending on K, k_2, and $\tilde{k}_2$) such that $\xi(x,\alpha) \leq B\tilde{\xi}(x, K^2\alpha)$ for all $x \geq c$ and all $\alpha > 0$.

c) If $k_1 \geq 0$, J equals $(\ell, c]$, and $k_2, \tilde{k}_2 < 0$, then there exists a constant B (depending on K, k_2, and $\tilde{k}_2$) such that $\xi(x,\alpha) \leq B\tilde{\xi}(x, K^2\alpha)$ for all $x \leq c$ and all $\alpha > 0$.

Proof: Let ξ_n and $\tilde{\xi}_n$ be the functions defined in Theorem (4.1) for X and $\tilde{X}$, respectively. Consider first the case $k_1 > 0$ and $k_2 = \tilde{k}_2 = 0$. We claim that for all $n \geq 0$,

$$0 \leq \xi_n(x) \leq K^{2n}\tilde{\xi}_n(x), \qquad x \in J. \tag{4.3}$$

The proof is by induction on n. First, note that $0 < \xi_0(x) = k_1 = \tilde{\xi}_0(x)$, so the claim is true for $n = 0$. Next, suppose that we have shown that $0 \leq \xi_n(x) \leq K^{2n}\tilde{\xi}_n(x)$ for x in J. Then,

$$\begin{aligned} \xi_{n+1}(x) &= \int_c^x (s(x) - s(y))\xi_n(y)M(dy) \\ &\leq \int_c^x K(\tilde{s}(x) - \tilde{s}(y))K^{2n}\tilde{\xi}_n(y)K\tilde{M}(dy) \\ &= K^{2(n+1)}\tilde{\xi}_{n+1}(x) \end{aligned}$$

The first line also shows that $\xi_{n+1}(x) \geq 0$. We have proven (4.3) for all $n \geq 0$ and x in J.

Now it follows easily that

$$\xi(x,\alpha) = \sum_{n=0}^{\infty} \xi_n(x)\alpha^n \leq \sum_{n=0}^{\infty} \tilde{\xi}_n(x)(K^2\alpha)^n = \tilde{\xi}(x, K^2\alpha),$$

which completes the proof of case (a).

Next, consider the case $k_1 \geq 0$, $J = [c, r)$, and $k_2, \tilde{k}_2 > 0$. We claim that, for some constant $B > 0$, we have

$$0 \leq \xi_n(x) \leq BK^{2n}\tilde{\xi}_n(x), \qquad x \geq c. \tag{4.4}$$

for all $n \geq 0$. The base step is more difficult this time. Let $x \geq c$. Then,

$$\begin{aligned} 0 \leq \xi_0(x) &= k_1 + k_2(s(x) - s(c)) \\ &\leq k_1 + k_2 K(\tilde{s}(x) - \tilde{s}(c)) \\ &= k_1 + \frac{k_2 K}{\tilde{k}_2}(\tilde{\xi}_0(x) - k_1) \\ &= k_1(1 - \frac{k_2 K}{\tilde{k}_2}) + \frac{k_2 K}{\tilde{k}_2}\tilde{\xi}_0(x) \\ &\leq B\tilde{\xi}_0(x), \end{aligned}$$

Note that if $k_1 = 0$, we can take $B = \frac{k_2 K}{\tilde{k}_2}$; otherwise we set

$$B = |1 - \frac{k_2 K}{\tilde{k}_2}| + \frac{k_2 K}{\tilde{k}_2} = \max(1, \frac{2k_2 K}{\tilde{k}_2} - 1)$$

This establishes (4.4) for $n = 0$. The induction step is the same as above, and the inequality in case (b) follows readily. The proof for case (c) is similar. □

This theorem applies, in particular, to the functions v_1 and $\tilde{v}_1$, and also v_2 and $\tilde{v}_2$, where $\tilde{v}_1$ and $\tilde{v}_2$ have the obvious meanings. Note that k_2 and $\tilde{k}_2$ are now functions of α, so B depends on α. Under the hypotheses of the proposition, using $J = [c, r)$ yields

$$v_1(x, \alpha) \leq B_1(\alpha)\tilde{v}_1(x, K^2\alpha), \qquad x \geq c. \tag{4.5}$$

On the other hand, using $J = (\ell, c]$ gives

$$v_2(x, \alpha) \leq B_2(\alpha)\tilde{v}_2(x, K^2\alpha), \qquad x \leq c. \tag{4.6}$$

The following corollary casts the conditions of Theorem (4.2) in terms of the asymptotic behavior of the derivatives of $s, \tilde{s}, M$, and $\tilde{M}$, and yields upper *and* lower bounds on v_1 and v_2.

We assume that s and $\tilde{s}$ are continuously differentiable and that M and $\tilde{M}$ have continuous densities m and $\tilde{m}$ with respect to Lebesgue measure. In addition, $s', \tilde{s}', m$, and $\tilde{m}$ must be strictly positive.

(4.7) **Corollary.** *a) Suppose the functions $s'(x)/\tilde{s}'(x)$ and $m(x)/\tilde{m}(x)$ have limits in $(0, \infty)$ as $x \uparrow r$. Then there exist constants $R_1, R_2 > 0$ and functions $B_1(\alpha), B_2(\alpha) > 0$ such that*

$$B_1(\alpha)\tilde{v}_1(x, R_1\alpha) \leq v_1(x, \alpha) \leq B_2(\alpha)\tilde{v}_1(x, R_2\alpha).$$

for all $x \in [c, r)$ *and* $\alpha > 0$.

b) Suppose the functions $s'(x)/\tilde{s}'(x)$ *and* $m(x)/\tilde{m}(x)$ *have limits in* $(0, \infty)$ *as* $x \downarrow \ell$. *Then there exist constants* $R_1, R_2 > 0$ *and functions* $B_1(\alpha), B_2(\alpha) > 0$ *such that*

$$B_1(\alpha)\tilde{v}_2(x, R_1\alpha) \le v_2(x, \alpha) \le B_2(\alpha)\tilde{v}_2(x, R_2\alpha).$$

for all $x \in (\ell, c]$ *and* $\alpha > 0$.

Proof: For part (a), let $e = r$; for (b), let $e = \ell$. Then the functions $s'/\tilde{s}'$ and $m/\tilde{m}$ are continuous and strictly positive, with finite, non-zero limits at e. Thus, they are bounded above and below by constants C_1 and C_2 with $0 < C_1 \le C_2 < \infty$:

$$C_1 \le \frac{s'(x)}{\tilde{s}'(x)} \le C_2, \qquad C_1 \le \frac{m(x)}{\tilde{m}(x)} \le C_2,$$

for x between c and e. The upper bounds imply that the conditions of Theorem (4.2) hold with $K = C_2$. Equations (4.5) and (4.6) give the rightmost inequalities in the conclusions of parts (a) and (b) with $R_2 = C_2^2$.

On the other hand, the lower bounds above show that the conditions of Theorem (4.2) are satisfied for $K = 1/C_1$ and with the tildes reversed on s and m. Then equations (4.5) and (4.6) yield the leftmost inequalities in parts (a) and (b) with $R_1 = C_1^2$. □

5 Decay of v_1 near ℓ and v_2 near r

It is more difficult to get bounds on the decay of $v_1(x, \alpha)$ as $x \to \ell$ and $v_2(x, \alpha)$ as $x \to r$ than it is to bound their growth at the opposite endpoint. The inductive procedure of Theorem (4.2) will not work: for example, in the construction of Theorem (4.1), if we have $k_2 < 0$, the functions ξ_n change sign on $[c, r)$, which ruins the inequalities of the induction step of Theorem (4.2).

There is a simple condition on the speed measure which guarantees that v_1 and v_2 decay to zero:

(5.1) $$\lim_{x \downarrow \ell} M[x, c] = +\infty \text{ implies } \lim_{x \downarrow \ell} v_1(x, \alpha) = 0$$

(5.2) $$\lim_{x \uparrow r} M[c, x] = +\infty \text{ implies } \lim_{x \uparrow r} v_2(x, \alpha) = 0$$

See ITÔ and MCKEAN (1965), Section 4.6 for the proof.

We can sometimes obtain upper and lower bounds on v_1 and v_2 if the process X is a strong solution of a stochastic differential equation of the form

(5.3) $$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt.$$

We wish to compare X with another process $\tilde{X}$ satisfying the same equation but with a different drift $\tilde{\mu}$ replacing μ. (It is immaterial whether they solve (5.3) with the same Wiener process W.)

We state our result for v_2; the analogous result holds for v_1. The lower bound we give can be traded for an upper bound by reversing the roles of X and $\tilde{X}$.

We assume that μ and $\tilde{\mu}$ are bounded and at least one of them is Lipschitz, and that $|\sigma(x) - \sigma(y)|^2 \leq f(|x-y|)$ for some f satisfying $\int_{(0,1)} \frac{dz}{f(z)} = \infty$.

(5.4) **Proposition.** *Let X and $\tilde{X}$ be as above. Suppose that, for some $x^* \geq c$ we have*

$$\mu(x) \leq \tilde{\mu}(x), \qquad x \geq x^*.$$

Then,

$$\frac{\tilde{v}_2(x,\alpha)}{\tilde{v}_2(x^*,\alpha)} \leq \frac{v_2(x,\alpha)}{v_2(x^*,\alpha)}, \qquad x \geq x^*.$$

Proof: Fix $x \geq x^*$. We will start both X and $\tilde{X}$ at x; the expectation $\mathbf{E}^x$ will be used for this. By the definition of v_2 and the strong Markov property,

$$\begin{aligned} v_2(x,\alpha) &= \mathbf{E}^x e^{-\alpha T_c} \\ &= \mathbf{E}^x e^{-\alpha T_{x^*}} \mathbf{E}^{x^*} e^{-\alpha T_c} \\ &= v_2(x^*,\alpha) \mathbf{E}^x e^{-\alpha T_{x^*}}. \end{aligned}$$

Similarly, $\tilde{v}_2(x,\alpha) = \tilde{v}_2(x^*,\alpha)\mathbf{E}^x e^{-\alpha \tilde{T}_{x^*}}$, where $\tilde{T}_{x^*}$ is the first time $\tilde{X}$ hits x^*.

Let X and $\tilde{X}$ be solutions of their respective stochastic differential equations but with the same Wiener process in each. Theorem 3.7 of Chapter IX of REVUZ and YOR (1991) and our assumption that $\mu \leq \tilde{\mu}$ give

$$X_t \leq \tilde{X}_t, \qquad 0 \leq t < \infty,$$

almost surely. Thus $T_{x^*} \leq \tilde{T}_{x^*}$, so

$$\mathbf{E}^x e^{-\alpha \tilde{T}_{x^*}} \leq \mathbf{E}^x e^{-\alpha T_{x^*}}.$$

This completes the proof. □

6 Example

We close with an extended example concerning two-dimensional isotropic Brownian flows. We intend this to be instructive, showing how to resolve some of the difficulties that may be encountered. Our analysis also clears up a minor issue concerning Brownian flows.

Isotropic Brownian flows are treated in detail by LE JAN (1985) and BAXENDALE and HARRIS (1986); see also ZIRBEL and ÇINLAR (1996b) and the companion article ZIRBEL (1997). We assume that the covariance tensor of the flow is C^4 and not constant, and is normalized so that the one-point motion is the standard Brownian motion on $\mathbb{R}^2$. We are interested in the distance between two points in the flow at time t. This is a continuous, regular, strong Markov process $Z_t,\ t \geq 0$, with Z_0 being the initial separation.

The behavior of Z_t as $t \to \infty$ depends on the largest Lyapunov exponent λ of the flow, as is shown in LE JAN (1985), pp. 617-8. In two dimensions,

if $\lambda < 0$, then $Z_t \to 0$ almost surely;

if $\lambda \geq 0$, then Z is null recurrent on $(0, \infty)$;

if $\lambda > 0$, then $Z_t \to \infty$ in probability.

When $\lambda = 0$, the invariant measure for Z puts infinite mass on both $(0,1)$ and $(1,\infty)$, so we cannot tell immediately how Z_t behaves as $t \to \infty$. We wish to remedy this here by finding how the mean occupation time of various intervals grows as $t \to \infty$. We will show the following:

(6.1) **Proposition.** *Let $d = 2$ and $\lambda = 0$. Then there exist a constant K and a speed measure M such that for all $0 < a < b < \infty$ and all $c > 0$,*

$$\begin{array}{rcll} \mathbf{E}^c \int_0^t ds 1_{(0,a]}(Z_s) & \sim & K\sqrt{\frac{2}{\beta_L \pi}}\sqrt{t}\log t, & t \to \infty \\ \mathbf{E}^c \int_0^t ds 1_{[a,b]}(Z_s) & \sim & \frac{1}{2} K M[a,b] \log t, & t \to \infty \\ \mathbf{E}^c \int_0^t ds 1_{[a,\infty)}(Z_s) & \sim & t, & t \to \infty \end{array}$$

The values of K and M are given in the proof.

The third statement follows from the first two. For those, we need some preliminaries about the speed and scale of Z.

Under our assumptions on the covariance b, Z is a diffusion on $(0, \infty)$. From the form of its generator, it has the same law as the strong solution of the stochastic differential equation

$$dZ_t = \sigma(Z_t) dW_t + \mu(Z_t) dt$$

where W is the Wiener process and

$$\sigma(z) = \sqrt{2(1 - b_L(z))}, \qquad \mu(z) = \frac{1 - b_N(z)}{z}.$$

Here b_L and b_N are deterministic functions related to the covariance of the flow. What is important is that b_L and b_N are bounded and continuous, they have a common limit $\mu_0 \in [0, 1)$ as $z \to \infty$, and, as $z \to 0$,

$$b_L(z) = 1 - \tfrac{1}{2}\beta_L z^2 + \gamma_L z^4 + o(z^4) \tag{6.2}$$

$$b_N(z) = 1 - \tfrac{1}{2}\beta_N z^2 + \gamma_N z^4 + o(z^4) \tag{6.3}$$

The top Lyapunov exponent of the flow equals $\frac{1}{2}(\beta_N - \beta_L)$, so the case $\lambda = 0$ corresponds to $\beta_L = \beta_N$.

The scale function s and the density m of the speed measure satisfy

$$s'(z) = \exp(-\int_{z_0}^{z} dy\, \frac{2\mu(y)}{\sigma^2(y)}), \qquad m(z) = \frac{2}{\sigma^2(z)s'(z)}$$

where z_0 is arbitrary, but will be fixed below for our convenience. Write the derivative s' as

$$s'(z) = \frac{z_0}{z}\exp(-\int_{z_0}^{z} dy\, \frac{\delta(y)}{y}),$$

where $\delta(y) = (b_L(y) - b_N(y))/(1 - b_L(y))$. BAXENDALE and HARRIS (1986), p. 1163 show that $\int_{z_0}^{z} dy\, \frac{\delta(y)}{y}$ has a finite limit as $z \to \infty$, so we have $s'(z) \sim K/z$ as $z \to \infty$, and moreover,

$$s(z) \sim K \log z, \qquad m(z) \sim \frac{2z}{K(1-\mu_0)}, \qquad M[c, z] \sim \frac{z^2}{K(1-\mu_0)}$$

where $K = z_0 \exp(-\int_{z_0}^{\infty} dy\, \frac{\delta(y)}{y})$.

We will need more precise information as $z \to 0$, however. Begin by noting that, as $y \to 0$,

$$\frac{\delta(y)}{y} \sim \frac{2(\gamma_L - \gamma_N)}{\beta_L} y.$$

Thus, $R(z_0) = \exp(\int_{z_0}^{0} dy\, \frac{\delta(y)}{y})$ is finite, and

$$\exp(\int_{z_0}^{z} dy\, \frac{\delta(y)}{y}) - R(z_0) = O(z^2), \qquad z \to 0$$

In other words,

$$\frac{1}{s'(z)}\frac{z_0}{z} - R(z_0) = O(z^2), \qquad z \to 0, \tag{6.4}$$

Now $R(z_0)$ is continuous, $R(0) = 1$, and $R(z_0)$ has a finite limit as $z_0 \to \infty$ (as mentioned above), so we may choose z_0 so that $R(z_0) = z_0$. Note that then $s'(z) \sim 1/z$ and $s(z) \sim \log z$ as $z \to 0$. We also have $K = \exp(-\int_0^\infty dy\, \frac{\delta(y)}{y})$.

Moreover, by multiplying through (6.4) by $\frac{2z}{\sigma^2}$ and rearranging, we find that $m(z) - \frac{2z}{\sigma^2} = O(z)$ as $z \to 0$. Using (6.2) yields $\frac{2z}{\sigma^2} - \frac{2}{\beta_L z} = O(z^3)$, and so

$$m(z) - \frac{2}{\beta_L z} = O(z), \qquad z \to 0, \tag{6.5}$$

which is the detailed information we will need later. From this, we have $M[z,c) \sim -\frac{2}{\beta_L}\log z$ as $z \to 0$.

Theorem (3.9) and its corollary allow us to conclude

$$h_+(\alpha) \sim \frac{K}{2}\log\frac{1}{\alpha} \qquad \text{and} \qquad h_-(\alpha) \sim D_{\frac{1}{2}}\sqrt{\frac{\beta_L}{2\alpha}}, \qquad \alpha \to 0$$

by letting $g(z) = \frac{K}{2}\log z$ for h_+ and $g(z) = \sqrt{\beta_L z/2}$ for h_-. Thus, $h(\alpha) \sim \frac{K}{2}\log\frac{1}{\alpha}$ as $\alpha \to 0$.

Consider the occupation time $\mathbf{E}^c \int_0^t ds 1_{[a,b]}(Z_s)$. Its Laplace transform is, by (2.2),

$$\hat{L}(\alpha) = h(\alpha)\int_a^b v_1(c \wedge y, \alpha) v_2(c \vee y, \alpha) M(dy)$$

The integrand is bounded and converges to 1 as $\alpha \to 0$, so $\hat{L}(\alpha)$ is asymptotic to $\frac{K}{2}M[a,b]\log\frac{1}{\alpha}$ as $\alpha \to 0$. The Tauberian Theorem (1.3) establishes the second statement of Proposition (6.1).

Next, consider the first claim of Proposition (6.1). We will show below that for all c small enough, $\mathbf{E}^c \int_0^t ds 1_{(0,c]}(Z_s) \sim C\sqrt{t}\log t$ as $t \to \infty$. Fix one such c and consider $L(t,d) = \mathbf{E}^d \int_0^t ds 1_{(0,c]}(Z_s)$ for $d > c$. By (1.2) we have $\hat{L}(\alpha, d) = \mathbf{E}^d e^{-\alpha T_c}\hat{L}(\alpha, c)$. But $\mathbf{E}^d e^{-\alpha T_c} \to 1$ as $\alpha \to 0$ since Z is recurrent, so we have $L(t,d) \sim C\sqrt{t}\log t$ by the Tauberian Theorem. In fact, this is true for all $d > 0$. Finally, using the result of the preceding paragraph, we obtain the first conclusion of Proposition (6.1).

Consider then $L(t) = \mathbf{E}^c \int_0^t ds 1_{(0,c]}(Z_s)$, where $c \in (0, c_{\max})$ with $c_{\max}$ to be specified below. By (2.2),

$$\hat{L}(\alpha) = h(\alpha)\int_0^c dz\, v_1(z,\alpha) m(z) \tag{6.6}$$

We cannot solve for v_1 directly from the differential equation $\mathcal{A}v = \alpha v$ since σ and μ are too unwieldy. Nor can we use the comparison method of Section 5 directly, again because of σ. Our approach is to change variables to simplify the diffusion coefficient and *then* use the comparison method. The fact that σ is linear near 0 guides our choice of rescaling.

Define a positive, increasing function ϕ by

$$\phi(z) = \exp(\int_1^z dy\ /\sigma(y)), \qquad z > 0 \tag{6.7}$$

and set $Y_t = \phi(Z_t)$ for $t \geq 0$ when $Z_0 > 0$; set $Y_t = 0$ for all t when $Z_0 = 0$. Then Y is a continuous, strong Markov process on $[0, \infty)$ which satisfies the stochastic differential equation

$$dY_t = Y_t dW_t + Y_t \pi(Y_t) dt$$

where π is defined implicitly by $\pi(\phi(z)) = \frac{\mu(z)}{\sigma(z)} + \frac{1}{2}(1 - \sigma'(z))$. It is readily verified that, as $z \to 0$,

$$\pi(\phi(z)) \sim \tfrac{1}{2} - \frac{\gamma_N - 2\gamma_L}{\sqrt{\beta_L}} z^2 \quad \text{and} \quad \phi(z) \sim \nu z^{1/\sqrt{\beta_L}} \tag{6.8}$$

so that $\pi(y) \sim \frac{1}{2} - Ay^\delta$ as $y \to 0$, for some constant A and $\delta = 2\sqrt{\beta_L}$. Thus, π can be bounded above and below on some interval $(0, \phi(c_{\max}))$ by functions of the form $\frac{1}{2} - \gamma y^\delta$. This defines $c_{\max}$.

Let $\tilde{Y}$ solve the stochastic differential equation

$$d\tilde{Y}_t = \tilde{Y}_t dW_t + \tilde{Y}_t(\tfrac{1}{2} - \gamma \tilde{Y}_t^\delta) dt$$

(the drift is Lipschitz on $(0, \phi(c))$, which is all we need). Let u_1 and $\tilde{u}_1$ denote the increasing functions of (2.3) for Y and $\tilde{Y}$. By Proposition (5.4), by choosing different values for γ we can make $\tilde{u}_1$ lie completely above or completely below u_1. By the results of Section 2, $v_1(z, \alpha) = u_1(\phi(z), \alpha)$, so the Laplace transform $\hat{L}$ of (6.6) is bounded above and below by expressions of the form

$$h(\alpha) \int_0^c dz\ \tilde{u}_1(\phi(z), \alpha) m(z)$$

for different values of γ. We will find the asymptotic behavior of this expression and show that it does not depend on γ.

The function $\tilde{u}_1$ is the unique increasing solution of $\tilde{\mathcal{A}}u = \alpha u$, or

$$y^2 u'' + y(1 - 2\gamma y^\delta) u' - 2\alpha u = 0 \tag{6.9}$$

normalized to equal 1 at $\phi(c)$. We will make a change of variables suggested by KAMKE (1959), Equation 2.215. This is an excellent source for solutions of such equations. Let $u(y) = y^{\sqrt{2\alpha}} F(2\gamma y^\delta/\delta)$. Then F satisfies

$$xF'' + (b - x)F' - aF = 0,$$

with $a = \sqrt{2\alpha}/\delta$ and $b = 1 + 2a$. This is the confluent hypergeometric equation (KAMKE (1959), Equation 2.113).

The solution we want is

$$F(a,b,x) = 1 + \sum_{k=1}^{\infty} \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)}\frac{x^k}{k!}$$

Thus, a solution of (6.9) is $u(y) = y^{\sqrt{2\alpha}} F(a,b,2\gamma y^\delta/\delta)$. The identity $F(a,b,x)$ $= e^x F(b-a,b,-x)$ (KAMKE (1959), Section 2.113) shows that u is positive regardless of the sign of γ. Moreover, term-by-term differentiation shows that

$$u'(y) = a\delta y^{\sqrt{2\alpha}-1} F(a+1,b,2\gamma y^\delta/\delta).$$

This is clearly positive for $\gamma \geq 0$, and the identity above shows that it is positive for $\gamma < 0$ as well. Thus, $\tilde{u}_1(y,\alpha) = u(y)/u(\phi(c))$.

We have argued above that $\hat{L}(\alpha)$ is bounded above and below by expressions of the form

$$h(\alpha)\int_0^c dz\; \tilde{u}_1(\phi(z),\alpha)m(z) = h(\alpha)\int_0^c dz\; \frac{u(\phi(z))}{u(\phi(c))}m(z)$$

As $\alpha \to 0$, $u(\phi(c)) \to 1$. If we substitute in the series for u and the asymptotic form (6.5) of m, we see that only the first term is infinite as $\alpha \to 0$. That term is

$$h(\alpha)\int_0^c dz\; \phi(z)^{\sqrt{2\alpha}} \frac{2}{\beta_L z}$$

Now $\phi(z) \sim \nu z^{1/\sqrt{\beta_L}}$ as $z \to 0$ from (6.8), so we can bound it above and below by a function of the form $\bar{\nu} z^{1/\sqrt{\beta_L}}$. But we have

$$\frac{2}{\beta_L} h(\alpha)\int_0^c dz\; \bar{\nu}^{\sqrt{2\alpha}} z^{\sqrt{2\alpha/\beta_L}-1} = \frac{2}{\beta_L} h(\alpha)\bar{\nu}^{\sqrt{2\alpha}} \sqrt{\frac{\beta_L}{2\alpha}}\, c^{\sqrt{2\alpha/\beta_L}} \sim \frac{K}{\sqrt{2\alpha\beta_L}} \log\frac{1}{\alpha}$$

as $\alpha \to 0$, regardless of the value of $\gamma, \bar{\nu}$, or c. Thus $\hat{L}(\alpha)$ from (6.6) shares this asymptotic behavior, and the first conclusion of Proposition (6.1) holds with $C = K\sqrt{\frac{2}{\beta_L \pi}}$.

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