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MASS TRANSPORT BY BROWNIAN FLOWS

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Abstract. We consider the motion of a mass distribution in a random velocity field which is "δ-correlated" in time but which has arbitrary spatial correlations. We discuss the rigorous formulation of this problem in terms of Brownian flows and stochastic calculus. We use numerical simulations to illustrate the effect of the flow on the mass. We present results concerning the evolution of the mass distribution and, in particular, the long-time asymptotics of the center of mass and relative dispersion.

Key words. Stochastic flows; Brownian flows; Mass transport

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1. Introduction. In statistical fluid mechanics, an important problem is the behavior of a passive tracer carried by a random velocity field. The problem is generally difficult, because of the statistical dependence of particle motions on each other. In this paper we consider this problem with velocity fields that are "δ-correlated" in time but have arbitrary spatial correlations. The spatial correlations make joint particle motions non-trivial, while the short temporal memory makes possible various explicit calculations.

We begin with a general description of the mass transport problem, introducing our notation along the way. Let v be a time-dependent random velocity field on $\mathbb{R}^d$. Let $F_{st}(x)$ be the position at time t of a particle that was at x at an earlier time s. Then the motion of the particle in the velocity field is described by

$$F_{ss}(x) = x; \qquad \frac{d}{dt}F_{st}(x) = v(F_{st}(x), t), \qquad t > s \tag{1.1}$$

For $s \le t$ fixed, F_{st} is a random transformation from $\mathbb{R}^d$ into $\mathbb{R}^d$. The family F of transformations F_{st}, $0 \le s \le t < \infty$, is called a *random flow*, since F satisfies the flow equations

$$F_{ss} = \text{identity} \qquad \text{for each } s \ge 0, \tag{1.2}$$

$$F_{st} \circ F_{rs} = F_{rt} \qquad \text{for } 0 \le r \le s \le t. \tag{1.3}$$

Mass transport is concerned with the evolution of a mass distribution carried by the flow. The mass distribution may be continuous or discrete (a

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collection of individual particles). We use the unified formalism of measures to deal with both cases. Let $M_t(R)$ be the amount of mass in region $R \subseteq \mathbb{R}^d$ at time t. In terms of M_0, the equation for conservation of mass is

$$M_t(R) = M_0(\{x \in \mathbb{R}^d \ : \ F_{0t}(x) \in R\}) \tag{1.4}$$

For t fixed, M_t is a *random measure* on $\mathbb{R}^d$, that is, for each realization of the velocity field (or, equivalently, of the flow F), it defines a measure on $\mathbb{R}^d$. If M_0 has a smooth density $\rho_0(x)$, then M_t admits a random density $\rho_t(x)$, and (1.4) is equivalent to the familiar conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{1.5}$$

The mass transport problem is to describe the evolution of M_t in time, using the probability law of the velocity field v.

An important, much-studied quantity is the *mean measure* of M_t, namely the measure μ_t defined by $\mu_t(R) = \mathrm{E} M_t(R)$, where E denotes expectation (average over the probability space). Commonly, one describes μ_t in terms of its density m_t with respect to the Lebesgue measure on $\mathbb{R}^d$. The density $m_t(x)$ is the mean tracer concentration. There has been and continues to be much interest in obtaining partial differential equations for m_t.

A second order concern is the statistical dependence between concentrations at different points in space, for this begins to describe the spatial structure of M_t. Assuming that M_t has the (random) density $\rho_t(x)$ at x, one is interested in the joint distribution of $\rho_t(x)$ and $\rho_t(y)$ for $x \neq y$, or with less ambition, their covariance. Of course, $\mathrm{E}\rho_t(x) = m_t(x)$.

For many applications, it is important to have descriptive measures of where the mass is concentrated and how spread out it is likely to be. As we shall see in Sections 4 and 5, mean concentration provides little information in this regard. Instead, we will emphasize random descriptors like the *centroid* $C_t = (C_t^i)$ and *dispersion matrix* $D_t = (D_t^{ij})$. These are defined by

$$C_t^i = \frac{1}{M_0(\mathbb{R}^d)} \int_{\mathbb{R}^d} M_t(dx) x^i, \tag{1.6}$$

$$D_t^{ij} = \frac{1}{M_0(\mathbb{R}^d)} \int_{\mathbb{R}^d} M_t(dx)(x^i - C_t^i)(x^j - C_t^j), \tag{1.7}$$

(in the case of continuous mass, $M_t(dx)$ may be replaced by $\rho_t(x)dx$).

Our aim in this paper is to present our results regarding M_t in the case where the random velocity field v is given by

$$v(x,t) = u(x,t) + \dot{U}_0(x,t) \tag{1.8}$$

where u is a deterministic velocity field and $\dot{U}_0$ is the formal time derivative of a Brownian motion with a spatial parameter. Formally, $\dot{U}_0$ is a Gaussian random vector field with mean zero and covariance given by

$$\operatorname{Cov}(\dot{U}_0^i(x,s), \dot{U}_0^j(y,t)) = a^{ij}(x,y)\delta(t-s), \tag{1.9}$$

where δ is Dirac's delta function at 0, and a is a spatial covariance tensor. The corresponding flow is then called a *Brownian flow*.

In the next section, we put the equation (1.1) of motion in rigorous form in the case of Brownian velocity fields (1.8). Also in that section, we give a brief introduction to isotropic Brownian flows, which figure prominently in the rest of the paper.

In Section 3 we discuss numerical simulations of Brownian flows, which are complicated by the presence of stochastic integrals. We prove an error estimate justifying the use of the Euler method for simulating trajectories. We use this method to generate pictures of the evolution of mass distributions in compressible and incompressible isotropic Brownian flows.

In Section 4 we list some results concerning the evolution of M_t, including the stochastic differential equation it satisfies, the partial differential equations for the mean and covariance of the density ρ, and some formulas for the mean and covariance of the centroid and dispersion matrix.

Finally, in Section 5 we specialize to the case of isotropic Brownian flows. We report on results we have obtained elsewhere concerning the behavior of C_t and D_t as $t \to \infty$.

2. Brownian flows. A *random flow* on $\mathbb{R}^d$ is a collection $F = \{F_{st}; 0 \le s \le t < \infty\}$ of random transformations from $\mathbb{R}^d$ into $\mathbb{R}^d$ satisfying, almost surely, the flow equations (1.2) and (1.3). These F_{st} are defined on a probability space $(\Omega, \mathcal{H}, \mathbf{P})$ in such a way that the mapping $(\omega, s, t, x) \mapsto F_{st}(\omega, x)$ is jointly measurable with respect to the product σ-algebra on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$. We call F a *flow of homeomorphisms* if, almost surely, all the mappings $F_{st} : x \mapsto F_{st}(x)$ are homeomorphisms.

The phrase 'almost surely' means 'for all ω in a set $\Omega_0 \in \mathcal{H}$ such that $\mathbf{P}(\Omega_0) = 1$.' One may also substitute 'with probability one.' The realization ω is usually suppressed as an argument of random variables; for example, $F_{st}(\omega, x)$ will always be written $F_{st}(x)$.

In this section we will be concerned with flows which solve stochastic differential equations. First, however, we must introduce the notion of a Brownian motion with a spatial parameter and define the stochastic integral with respect to such a Brownian motion. For a comprehensive introduction to this theory, we refer the reader to KUNITA (1990).

2.1. Brownian motion with a spatial parameter. Let $u = \{u^i(x,t); i = 1, \dots, d,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+\}$ be a deterministic vector field and let $a = \{a^{ij}(x,y);\ i,j = 1, \dots, d,\ x,y \in \mathbb{R}^d\}$ be the covariance tensor of a vector-valued random field on $\mathbb{R}^d$. Let U be a Gaussian random vector field on $\mathbb{R}^d$ with mean and covariance given by

$$\mathbf{E}U^i(x,t) = \int_0^t dr\ u^i(x,r) \tag{2.1}$$

$$\mathrm{Cov}(U^i(x,s), U^j(y,t)) = a^{ij}(x,y)(s \wedge t) \tag{2.2}$$

where $s \wedge t$ is the smaller of s and t. If u and a satisfy conditions (2.8) through (2.11) below, then there exists a version of U which is continuous in (x,t). The field U is called a *Brownian motion with a spatial parameter* because it is continuous and has independent increments: if $s \leq t$, the random field

$$U(\cdot,t) - U(\cdot,s) \equiv \{U(x,t) - U(x,s);\ x \in \mathbb{R}^d\}$$

is independent of the history of U before s. The field u is called the *drift* of U and a is called its *covariance*. We could have allowed the covariance to depend on t also, but this generality is not particularly important to us here. We will often decompose U as

$$U(x,t) = \int_0^t dr\ u(x,r) + U_0(x,t), \tag{2.3}$$

where U_0 is a Brownian motion with zero drift and covariance a.

Let $X_t,\ t \geq 0$ be a process in $\mathbb{R}^d$ which is predictable with respect to the natural filtration of U, and such that $\int_0^T dr\ a^{ii}(X_r, X_r) < +\infty$ almost surely. The integral of U along X, denoted $\int_s^t U(X_r, dr)$, is defined to be the limit in probability of

$$\sum_{k=0}^{N-1} U(X_{t_k \wedge t}, t_{k+1} \wedge t) - U(X_{t_k \wedge t}, t_k \wedge t)$$

as the width of the partition $s = t_0 \leq t_1 \leq \cdots \leq t_N = T$ tends to zero. Here $T > t$ is some fixed terminal time. This is similar to the usual stochastic integral, except that all the action takes place in the spatial argument of U.

For practical purposes there are only a few things we need to know concerning the integral of U along X. First, using the decomposition (2.3), the integral can be written as

$$\int_s^t U(X_r, dr) = \int_s^t dr\ u(X_r, r) + \int_s^t U_0(X_r, dr) \tag{2.4}$$

The first integral on the right side is the usual Lebesgue-Stieltjes integral. Second, the process $\int_s^t U_0(X_r, dr),\ t \geq s$ is a continuous mean-zero martingale, and if Y is another predictable process, then the joint quadratic variation of the integrals of U along X and along Y satisfies

$$\left\langle \int_0^t U_0^i(X_r, dr), \int_0^t U_0^j(Y_r, dr) \right\rangle = \int_0^t dr\ a^{ij}(X_r, Y_r) \tag{2.5}$$

These facts will be used repeatedly in the rest of the paper.

2.2. Stochastic differential equation for F. Now we are able to write a stochastic differential equation for a random flow F based on the Brownian motion U:

$$F_{st}(x) = x + \int_s^t U(F_{sr}, dr), \qquad t \geq s \tag{2.6}$$

Using (2.4) this can also be written as

$$F_{st}(x) = x + \int_s^t dr\ u(F_{sr}(x), r) + \int_s^t U_0(F_{sr}, dr), \qquad t \geq s \tag{2.7}$$

Suppose that u and a satisfy the conditions

$$|u(x,t)| \ \leq \ K_1(1 + |x|) \tag{2.8}$$

$$|u(x,t) - u(y,t)| \ \leq \ K_2|x - y| \tag{2.9}$$

$$|a^{ij}(x,y)| \ \leq \ K_3(1 + |x|)(1 + |y|) \tag{2.10}$$

$$|a^{ij}(x,y) - a^{ij}(x',y) - a^{ij}(x,y') + a^{ij}(x',y')| \leq K_4|x - x'||y - y'| \tag{2.11}$$

for all x, x', y, y' in $\mathbb{R}^d$ and t in $\mathbb{R}_+$, where K_1, K_2, K_3, K_4 are finite constants. Then by Theorem 4.2.5 of KUNITA (1990), there exists a flow $F = \{F_{st};\ 0 \leq s \leq t < \infty\}$ of homeomorphisms satisfying (2.6) and such that

$$\begin{array}{ll} (a) & \text{the mapping } (x,t) \mapsto F_t x \text{ is continuous almost surely} \\ (b) & F_{t_1 t_2}, \ldots, F_{t_{n-1} t_n} \text{ are independent when } t_1 \leq t_2 \leq \ldots \leq t_n \end{array} \tag{2.12}$$

Because of (b), the independence of multiplicative increments, F is called a *Brownian flow.*

As a consequence of (2.12), for fixed $x_1, \ldots, x_n$ in $\mathbb{R}^d$, the n-point motion

$$\{F_{0t}(x_1), \ldots, F_{0t}(x_n);\ t \geq 0\}$$

is a continuous, strong Markov process, or diffusion, in $\mathbb{R}^{d \times n}$. The drift of each component is u, and the joint quadratic variation satisfies

$$\langle F_{0t}^i(x), F_{0t}^j(y) \rangle = \int_0^t dr\ a^{ij}(F_{0r}(x), F_{0r}(y)) \tag{2.13}$$

which can be seen from (2.4), (2.5), and (2.6). In particular, the generator A of the one-point motion $F_{0t}(x),\ t \geq 0$ satisfies

$$(2.14)\quad A_t f(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_d^{ij}(x) \partial_i \partial_j f(x) + \sum_{i=1}^{d} u^i(x,t) \partial_i f(x), \qquad t \geq 0$$

where $a_d(x) = a(x,x)$ is the 'diagonal' of a.

We can interpret (2.13) in this way: if two particles are near each other, they will move together in concert, since the joint quadratic variation increases rapidly when $F_{0t}(x)$ is near $F_{0t}(y)$. If they are far apart, the joint quadratic variation increases slowly, indicating that their motions are nearly independent.

2.3. Alternative formulations. We now describe three alternative ways of thinking about a Brownian flow F satisfying (2.6). First, under conditions (2.10) and (2.11), we may write U_0 as

$$(2.15)\qquad U_0(x,t) = \sum_{k=1}^{\infty} u_k(x) W_k(t),$$

where $u_1, u_2, \ldots$ are deterministic Lipschitz vector fields on $\mathbb{R}^d$ and W_1, $W_2, \ldots$ are independent one-dimensional Wiener processes. The covariance a of U_0 is related to the u_k through

$$(2.16)\qquad a^{ij}(x,y) = \sum_{k=1}^{\infty} u_k^i(x) u_k^j(y)$$

Equation (2.7) for F becomes, in differential notation,

$$(2.17)\ F_{ss}(x) = x;\quad dF_{st}(x) = u(F_{st}(x),t)dt + \sum_{k=1}^{\infty} u_k(F_{st}(x)) W_k(dt),\ t > s$$

This makes clear the relationship of (2.6) and (2.7) to standard stochastic differential equations. The difference is that in this equation there are an infinite number of noise terms (otherwise we would limit the class of covariances obtainable via (2.16)) and we are interested in solving the equation for all x in $\mathbb{R}^d$ simultaneously.

Second, if we write (2.7) in differential notation, formally divide through by dt, and write $\dot{U}_0(x,t)$ in place of $U_0(x,dt)/dt$, we obtain

$$(2.18)\quad F_{ss}(x) = x;\quad \frac{d}{dt} F_{st}(x) = u(F_{st}(x),t) + \dot{U}_0(F_{st}(x),t), \quad t > s$$

This is of the form (1.1) for velocity fields of the form (1.8). Now $\dot{U}_0$ is what is sometimes called a 'generalized process,' so its covariance must be inferred from integrals of $\dot{U}_0$ against test functions. For convenience we choose these to be of the form $1_{[s,t]}$:

$$\begin{aligned}\mathbf{E}\int_s^t dr\ \dot{U}_0(x,r)\int_{s'}^{t'} dr'\ \dot{U}_0(y,r')^T \\ &= \mathrm{E}(U_0(x,t)-U_0(x,s))(U_0(y,t')-U_0(y,s')) \\ &= a(x,y)\mathrm{Leb}([s,t]\cap[s',t']) \\ &= \int_s^t dr\int_{s'}^{t'} dr'\ a(x,y)\delta(r-r')\end{aligned}$$

This justifies the following heuristic for equation (2.18): the flow F is driven by a deterministic velocity field u plus noise that is δ-correlated in time but spatially correlated with covariance tensor a.

Finally, suppose that v is a mean-zero random velocity field on $\mathbb{R}^d$ whose law is stationary in time. For each $\varepsilon > 0$, define a flow G^ε via the classical equation

$$G^\varepsilon_{ss}(x) = x; \quad \frac{d}{dt}G^\varepsilon_{st}(x) = \varepsilon v(G^\varepsilon_{st}(x),t), \quad t > s \tag{2.19}$$

Define another flow F^ε by $F^\varepsilon_{st} = G^\varepsilon_{s/\varepsilon^2,t/\varepsilon^2}$ and consider the limit of F^ε as $\varepsilon \to 0$. Under reasonable mixing conditions on v, plus some draconian restrictions on the moments of its derivatives, the flows F^ε will converge in law to a Brownian flow F having drift u given by

$$u^i(x) = \sum_{j=1}^d \int_0^\infty dr\ \mathrm{E}v^j(x,0)\frac{\partial v^i}{\partial x^j}(x,r) \tag{2.20}$$

and covariance a given by $a(x,y) = \bar{a}(x,y) + \bar{a}(y,x)^T$, where

$$\bar{a}^{ij}(x,y) = \int_0^\infty dr\ \mathrm{E}v^i(x,0)v^j(y,r) \tag{2.21}$$

This result is shown in Section 5.6 of KUNITA (1990), along with precise conditions on v.

2.4. Isotropic Brownian flows. Isotropic Brownian flows are the Brownian versions of random isotropic flows that have been of interest in statistical turbulence ever since the seminal work of KOLMOGOROV (1941). See MONIN and YAGLOM (1971) for an account of the classical case where F is obtained from (1.1) with v an isotropic random vector field.

Isotropic Brownian flows were characterized and studied by LE JAN (1985) and BAXENDALE and HARRIS (1986) simultaneously. The following brief review is taken from their work. The reader may also wish to consult the review article by DARLING (1989).

A Brownian flow F based on a Brownian motion U with drift u and covariance a is isotropic if and only if

$$u \equiv 0; \qquad a(x,y) = b(x-y), \qquad x, y \in \mathbb{R}^d, \tag{2.22}$$

where b is an *isotropic* covariance tensor, that is, $O^T b(Oz) O = b(z)$ for every orthogonal matrix O and every z in $\mathbb{R}^d$. Under these conditions, for every rigid motion R on $\mathbb{R}^d$, that is, a combination of translation, rotation, and/or reflection, the flow G defined by $G_{st} = R \circ F_{st} \circ R^{-1}$ has the same probability law as F. In other words, the law of F is invariant under all rigid motions of $\mathbb{R}^d$.

When $d = 1$, isotropy requires only that $u = 0$ and $b(z) = b(-z)$. The following condition guarantees the existence of a non-trivial isotropic flow F satisfying (2.6) and such that, almost surely, all the maps $F_{st} : x \mapsto F_{st}(x)$ are C^1-diffeomorphisms. See KUNITA (1990), Theorem 4.6.5.

CONDITION 2.1. *For $d = 1$, both b and b'' are bounded and continuous, and b is not identically constant.*

In two or more dimensions, isotropy forces b to have a special form, due to YAGLOM (1957); see also YAGLOM (1987):

$$b^{ij}(z) = b_N(|z|)\delta_{ij} + \frac{z^i z^j}{|z|^2}(b_L(|z|) - b_N(|z|)) \tag{2.23}$$

where b_L and b_N are real-valued functions on $\mathbb{R}_+$ defined in terms of two finite measures, Φ_P and Φ_S, on $\mathbb{R}_+$:

$$b_N(r) = A \int_0^\infty \Phi_P(ds) L_m(rs) + A \int_0^\infty \Phi_S(ds)[L_{m-1}(rs) - L_m(rs)] \tag{2.24}$$

$$b_L(r) = A \int_0^\infty \Phi_P(ds)[L_m(rs) - (rs)^2 L_{m+1}(rs)] + A(d-1) \int_0^\infty \Phi_S(ds) L_m(rs) \tag{2.25}$$

Here $m = d/2$, $A = 2^{m-1}\Gamma(m)$, and $L_m(r) = J_m(r)/r^m$, with Γ denoting the gamma function and J_m denoting the Bessel function of the first kind of order m. The measure Φ_P gives rise to the potential (irrotational) part of the flow and Φ_S to the solenoidal (incompressible) part. Indeed, when $\Phi_P = 0$, the flow F is incompressible, that is, it preserves Lebesgue measure.

CONDITION 2.2. *For $d \geq 2$, the measures Φ_P and Φ_S have finite fourth moments and put no mass on the set $\{0\}$.*

Under this condition, b_L and b_N are C^4-functions decaying to 0 at $+\infty$, there exists an isotropic flow F satisfying (2.6), and, almost surely, all the maps $F_{st} : x \mapsto F_{st}(x)$ are a C^1-diffeomorphisms. See BAXENDALE and HARRIS (1986).

The behavior of b_L and b_N near 0 is quadratic:

$$b_L(r) = b_0 - \tfrac{1}{2}\beta_L r^2 + O(r^4), \qquad r \to 0, \tag{2.26}$$

$$b_N(r) = b_0 - \tfrac{1}{2}\beta_N r^2 + O(r^4), \qquad r \to 0, \tag{2.27}$$

where b_0, β_L, β_N are strictly positive constants given by

$$b_0 = \frac{1}{d}\Phi_P(0,\infty) + \frac{d-1}{d}\Phi_S(0,\infty) \tag{2.28}$$

$$\beta_L = \frac{3}{d(d+2)}\int_0^\infty \Phi_P(ds)s^2 + \frac{d-1}{d(d+2)}\int_0^\infty \Phi_S(ds)s^2 \tag{2.29}$$

$$\beta_N = \frac{1}{d(d+2)}\int_0^\infty \Phi_P(ds)s^2 + \frac{d+1}{d(d+2)}\int_0^\infty \Phi_S(ds)s^2 \tag{2.30}$$

Moreover, the maxima of b_L and b_N occur at 0.

Let us set $b_0 = b(0)$ for $d = 1$. Then for all dimensions, the generator A of the one-point motion as defined in (2.14) becomes

$$Af(x) = \tfrac{1}{2}\sum_{i,j=1}^{d} b^{ij}(0)\partial_i\partial_j f(x) = \tfrac{1}{2}b_0\Delta f(x), \tag{2.31}$$

where Δ is the Laplacian operator. We recognize this as the generator of a Brownian motion in $\mathbb{R}^d$ with zero drift and covariance matrix $b_0 I$.

2.4.1. Lyapunov exponents. The Lyapunov exponents $\lambda_1 > \lambda_2 > \ldots > \lambda_d$ of the flow are the values taken by the limits

$$\lim_{t\to\infty}\frac{1}{t}\log \mid DF_{0t}(x)\xi \mid$$

as the vector ξ varies over the unit sphere in $\mathbb{R}^d$; here $DF_{0t}(x)$ is the Jacobian matrix of the map F_{0t} evaluated at x. The Lyapunov exponents are deterministic in an isotropic Brownian flow. For $d = 1$ and b satisfying 2.1, we have $\lambda_1 = \frac{1}{2}b''(0) \leq 0$. For $d \geq 2$, they are given by:

$$\lambda_i = \frac{d-i}{2}\beta_N - \frac{i}{2}\beta_L\ , \qquad i = 1,\ldots,d. \tag{2.32}$$

When F is incompressible, we have $\Phi_P = 0$ which gives $(d+1)\beta_L = (d-1)\beta_N$, and hence the top Lyapunov exponent is

$$\lambda_1 = \frac{d}{2}\beta_L\ > 0. \tag{2.33}$$

When F is potential, $\Phi_S = 0$, we have $\beta_L = 3\beta_N$, and

$$\lambda_1 = \frac{d-4}{6}\beta_L, \tag{2.34}$$

which is negative for $d = 2$ and $d = 3$.

2.4.2. Separation process. For fixed x and y in $\mathbb{R}^d$, the distance at time t between two particles started from x and y is

$$Z_t = |F_t x - F_t y|. \tag{2.35}$$

The process $Z = \{Z_t;\ t \geq 0\}$ is called the *separation process* (or pair distance process) with initial value $|x - y|$. When F is isotropic, Z is a one-dimensional diffusion on $\mathbb{R}_+$. Under conditions 2.1 and 2.2, Z satisfies the stochastic differential equation

$$dZ_t = \sqrt{2(b_0 - b_L(Z_t))}dW_t + (d-1)\frac{b_0 - b_N(Z_t)}{Z_t}dt \tag{2.36}$$

The boundary point 0 is absorbing and inaccessible from $(0, \infty)$. When $d = 1$, the drift part vanishes, we write b in place of b_L, and Z becomes a martingale diffusion.

The following proposition lists the asymptotic behavior of the process Z as $t \to \infty$; see LE JAN (1985) and BAXENDALE and HARRIS (1986) for the proof. Conditions 2.1 and 2.2 are in force.

PROPOSITION 2.3. *The process Z is transient on $(0, \infty)$ unless $d = 2$ and $\lambda_1 \geq 0$. More specifically, if $Z_0 > 0$,*

(i) if $d \geq 4$ or if $d = 3$ and $\lambda_1 \geq 0$, then $Z_t \to +\infty$ almost surely;
(ii) if $d = 3$ and $\lambda_1 < 0$, then Z_t converges to either 0 or $+\infty$, each with strictly positive probability;
(iii) if $d = 2$ and $\lambda_1 > 0$, then Z is null-recurrent and converges to $+\infty$ in probability;
(iv) if $d = 2$ and $\lambda_1 = 0$, then Z is null-recurrent;
(v) if $d = 2$ and $\lambda_1 < 0$, or if $d = 1$, then $Z_t \to 0$ almost surely

It follows from this proposition and (2.33) that $\mathrm{P}\{Z_t \to 0\} = 0$ for incompressible flows and in general whenever $d \geq 4$. For $d \leq 3$, $\mathrm{P}\{Z_t \to 0\}$ becomes strictly positive if the potential part is large enough.

3. Numerical simulations of Brownian flows. Our purpose in this section is to describe and implement a procedure for the numerical simulation of Brownian flows. Our emphasis is on gaining insight into the nature of Brownian flows through an adequate simulation method, rather than on developing high-order schemes or making extremely accurate numerical estimates.

Let F be a Brownian flow satisfying the equation

$$F_{st}(x) = x + \int_s^t U(F_{sr}(x), dr), \qquad t \geq s \tag{3.1}$$

where U is a Brownian motion with drift u and covariance a satisfying conditions (2.8) to (2.11). In this section we will always take $s = 0$; for brevity we will write F_t for the map F_{0t} and $F_t x$ in place of $F_{0t}(x)$.

3.1. Euler approximation method. We will use the Euler approximation to define a collection $\{G_t;\ 0 \le t \le T\}$ of maps approximating F on the interval $[0,T]$. First, fix a time-step $\delta > 0$ for the approximation. Define a partition $\{t_k\}$ of $[0,T]$ by $t_k = k\delta,\ k = 0,1,\dots$. The definition of G is recursive: for each x in $\mathbb{R}^d$,

$$G_0 x = x \tag{3.2}$$

$$G_t x = G_{t_k} x + U(G_{t_k} x, t) - U(G_{t_k} x, t_k), \qquad t \in (t_k, t_{k+1}] \tag{3.3}$$

Note that the map $(x,t) \mapsto G_t x$ is continuous.

For the purpose of visualizing the action of the flow, an important issue is how closely the n-point trajectories under G approximate those under F when both are based on the same Brownian motion U. The following estimate shows that the approximate solution G is accurate to order $\sqrt{\delta}$.

THEOREM 3.1. *Let $T > 0$. Then there exists a constant C depending on T such that, for all n in $\mathbb{N}$, $x_1,\dots,x_n$ in $\mathbb{R}^d$, and $\delta < T$, we have*

$$\mathbf{E} \sup_{t \le T} \sum_{m=1}^{n} |F_t x_m - G_t x_m|^2 \ \le\ C\delta(n + \sum_{m=1}^{n} |x_m|^2) \tag{3.4}$$

Proof. First observe that

$$\mathbf{E} \sup_{t \le T} \sum_{m=1}^{n} |F_t x_m - G_t x_m|^2 \ \le\ \sum_{m=1}^{n} \mathbf{E} \sup_{t \le T} |F_t x_m - G_t x_m|^2,$$

so we need only consider $\mathbf{E}\sup_{t \le T} |F_t x - G_t x|^2$ for fixed x in $\mathbb{R}^d$. We will show that this is less than $C\delta(1 + |x|^2)$, which will complete the proof.

To simplify notation, let $X_t = F_t x$ and $Y_t = G_t x$. We need to consider piecewise constant approximations to X and Y, defined by

$$\bar{X}_t = X_{t_k}, \qquad \bar{Y}_t = Y_{t_k}, \qquad t \in (t_k, t_{k+1}]$$

We can represent Y compactly in terms of $\bar{Y}$, for note that by (3.3),

$$Y_t = Y_{t_k} + \int_{t_k}^{t} U(\bar{Y}_r, dr)$$

Writing the same expression for $Y_{t_k}, Y_{t_{k-1}}, \dots$ yields

$$Y_t = x + \int_0^t U(\bar{Y}_r, dr)$$

¿From (3.1) we have $X_t = x + \int_0^t U(X_r, dr)$, so we can apply the Lemma following this proof to obtain:

$$\mathbf{E} \sup_{r \le t} |X_r - Y_r|^2 \le C_1(T)\mathbf{E} \int_0^t dr\ |X_r - \bar{Y}_r|^2, \tag{3.5}$$

for all $t \leq T$. In order to estimate $\mathbf{E}|X_t - \bar{Y}_t|^2$, we add and subtract $\bar{X}_t$:

$$\mathbf{E}|X_t - \bar{Y}_t|^2 \leq 2\mathbf{E}|X_t - \bar{X}_t|^2 + 2\mathbf{E}|\bar{X}_t - \bar{Y}_t|^2 \tag{3.6}$$

For the second term, it will suffice to use the inequality $\mathbf{E}|\bar{X}_t - \bar{Y}_t|^2 \leq \mathbf{E}\sup_{r\leq t}|X_r - Y_r|^2$.

Next, we obtain an upper bound for $\mathbf{E}|X_t - \bar{X}_t|^2$. First, note that by the definitions of X and $\bar{X}$, for t in $(t_k, t_{k+1}]$ we have

$$X_t - \bar{X}_t = \int_{t_k}^{t} U(X_r, dr)$$

Thus, using the decomposition (2.4),

$$\mathbf{E}|X_t - \bar{X}_t|^2 \leq 2\mathbf{E}\left|\int_{t_k}^{t} dr\ u(X_r, r)\right|^2 + 2\mathbf{E}\left|\int_{t_k}^{t} U_0(X_r, dr)\right|^2 \tag{3.7}$$

We treat these two terms separately, writing each as a sum over components $1, \ldots, d$.

First, by Jensen's inequality,

$$\begin{aligned}\mathbf{E}\left|\int_{t_k}^{t} dr\ u^i(X_r, r)\right|^2 &\leq (t - t_k)\mathbf{E}\int_{t_k}^{t} dr\ (u^i(X_r, r))^2 \\ &\leq \delta K_1 \int_{t_k}^{t} dr\ \mathbf{E}(1 + |X_r|)^2,\end{aligned}$$

the last inequality by condition (2.8) on u. Second, since $\int_0^t U_0^i(X_r, dr)$ is a mean-zero martingale,

$$\begin{aligned}\mathbf{E}\left|\int_{t_k}^{t} U_0^i(X_r, dr)\right|^2 &= \mathbf{E}\left\langle \int_{t_k}^{t} U_0^i(X_r, dr)\right\rangle \\ &= \mathbf{E}\int_{t_k}^{t} dr\ a^{ii}(X_r, X_r) \\ &\leq K_3 \int_{t_k}^{t} dr\ \mathbf{E}(1 + |X_r|)^2,\end{aligned}$$

by condition (2.10) on a. Finally, by Lemma 4.5.3 of KUNITA (1990), there exists a constant $C_2 > 0$ such that

$$\mathbf{E}(1 + |X_r|)^2 \leq 2e^{C_2 T}(1 + |x|^2)$$

for all $r \leq T$. We have shown that

$$\begin{aligned}\mathbf{E}|X_t - \bar{X}_t|^2 &\leq 2d(\delta K_1 + K_3)\int_{t_k}^{t} dr\ 2e^{C_2 T}(1 + |x|^2) \\ &\leq C_3(T)\delta(1 + |x|^2)\end{aligned} \tag{3.8}$$

for some $C_3(T) > 0$.

Now let $f(t) = \mathrm{E}\sup_{r\le t}|X_r - Y_r|^2$. Equations (3.5) through (3.8) show that for $t \le T$,

$$\begin{aligned} f(t) &\le 2C_1(T)\left[\int_0^t dr\ C_3(T)\delta(1+|x|^2) + \int_0^t dr\ f(r)\right] \\ &\le C_4(T)\delta(1+|x|^2) + 2C_1(T)\int_0^t dr\ f(r) \end{aligned}$$

By Gronwall's Lemma,

$$\begin{aligned} f(t) &\le C_4(T)\delta(1+|x|^2)\exp(2C_1(T)t) \\ &\le C_5(T)\delta(1+|x|^2), \end{aligned}$$

which completes the proof. □

LEMMA 3.2. *Let U be a Brownian motion with drift u and covariance a satisfying (2.9) and (2.11), respectively. Let X and Y be predictable processes such that $\int_0^T dr\ a^{ii}(X_r, X_r) < \infty$ almost surely; similarly for Y. Then for each $T \ge 0$, there is a positive constant $C(T)$, not depending on X and Y, such that for all $t \le T$,*

$$(3.9)\ \mathrm{E}\sup_{s\le t}\left|\int_0^s U(X_r, dr) - \int_0^s U(Y_r, dr)\right|^2 \le C(T)\mathrm{E}\int_0^t dr\ |X_r - Y_r|^2$$

The proof is found in the proof of Lemma 3.4.2 of KUNITA (1990). Our proof of Theorem 3.1 was suggested by the proof of Theorem 10.2.2 of KLOEDEN and PLATEN (1992).

We conclude from Theorem 3.1 that the second moment of the error between the n-point trajectories of F and G converges to 0 as $\delta \to 0$. This is good enough for the purpose of illustration, but if we were interested in more complicated functionals of the trajectories we would want to choose a higher order method. See TALAY (1990).

For numerical simulations, we want to generate the values of the trajectories at discrete times $t_1, t_2, \ldots$ using formula (3.3), which reads

$$(3.10)\qquad G_{t_{k+1}}x = G_{t_k}x + U(G_{t_k}x, t_{k+1}) - U(G_{t_k}x, t_k)$$

Consider the random field $U(x, t_{k+1}) - U(x, t_k)$ appearing in this equation. This is a Gaussian random field whose mean and covariance are $\int_{t_k}^{t_{k+1}} dr\ u(x,r)$ and δa. Thus, the random field V_k defined implicitly by

$$(3.11)\qquad U(x, t_{k+1}) - U(x, t_k) = \int_{t_k}^{t_{k+1}} dr\ u(x,r) + \sqrt{\delta}V_k(x)$$

is Gaussian with mean 0 and covariance a. Moreover, the collection $\{V_k, k = 0, 1, \ldots\}$ is independent and identically distributed.

Our outline of the numerical simulation procedure will be complete once we describe how to generate the fields V_k. For the simulation of n-point motion, the field V_k only needs to be generated at the points $G_{t_k}x_1, \ldots, G_{t_k}x_n$, and this can be done in general if need be. A particularly simple case occurs when a is isotropic, so that $a(x,y) = b(x-y)$ for some isotropic covariance b. We now discuss this case in detail.

3.2. Spectral method for isotropic vector fields. The *spectrum* of a covariance matrix $(b^{j\ell})$ is a matrix-valued measure f on $\mathbb{R}^d$ such that

$$b^{j\ell}(z) = \int_{\mathbb{R}^d} e^{iz\cdot k}\, \mathrm{f}^{j\ell}(dk) \tag{3.12}$$

(YAGLOM (1987) 22.4.68). The spectrum of an isotropic covariance can be expressed in terms of the measures Φ_P and Φ_S which define b via (2.23), (2.24), and (2.25). We have

$$\mathrm{f}^{j\ell}(dk) = \frac{\sigma_k(ds)}{\Sigma_d |k|^{d-1}} \left[\tfrac{k^j k^\ell}{|k|^2} \Phi_P(d|k|) + (\delta_{jl} - \tfrac{k^j k^\ell}{|k|^2}) \Phi_S(d|k|) \right] \tag{3.13}$$

where $\sigma_k(ds)d|k|$ is the volume element in spherical coordinates, $\sigma_k(ds)$ is the area element at k on the sphere of radius $|k|$, and Σ_d is the area of the unit sphere in $\mathbb{R}^d$, $\Sigma_d = 2\pi^{d/2}\Gamma(d/2)^{-1}$ (YAGLOM (1987) 22.4.174).

To generate a realization of V, first fix the number N of modes desired and let $\xi_n^m, \zeta_n^m,\ n = 1, \ldots, N,\ m = 1, \ldots, d$ be independent standard Gaussian variables. Let $k_n,\ n = 1, \ldots, N$ be independent random vectors in $\mathbb{R}^d$ with the following common distribution: $\frac{k}{|k|}$ is uniform on the unit sphere and $|k|$ has distribution μ, where μ is some probability measure on $(0, \infty)$ such that Φ_S and Φ_P are absolutely continuous with respect to μ. We may choose μ. Define functions ψ_S and ψ_P on $\mathbb{R}_+$ in terms of the Radon-Nikodym derivatives of Φ_S and Φ_P with respect to μ:

$$\psi_S^2 = \frac{d\Phi_S}{d\mu}, \qquad \psi_P^2 = \frac{d\Phi_P}{d\mu} \tag{3.14}$$

Now define the random field V by setting $V^j(x)$ equal to

$$\frac{1}{\sqrt{N}} \sum_{m,n} (\xi_n^m \cos k_n \cdot x + \zeta_n^m \sin k_n \cdot x)(\tfrac{k_n^j k_n^m}{|k_n|^2} \psi_P(|k_n|) + (\delta_{jm} - \tfrac{k_n^j k_n^m}{|k_n|^2}) \psi_S(|k_n|))$$

The following proposition is the justification of this method.

PROPOSITION 3.3. *For each N in $\mathbb{N}$, V has mean 0 and isotropic covariance b based on Φ_P and Φ_S as in Section 2. Moreover, as $N \to \infty$, the finite dimensional distributions of V converge to those of a Gaussian random field.*

Proof. Consider $\mathrm{E}V(x)$. Condition on k_n. The variables ξ and ζ have mean 0, so $\mathrm{E}V(x) = 0$. Next consider $\mathrm{E}V^j(x)V^\ell(y)$, writing $V^j(x)$ as a sum over n, m, and $V^\ell(y)$ as a sum over n', m'. Condition on k_n and $k_{n'}$ to bring out the factor:

$$\mathrm{E}(\xi_n^m \cos k_n \cdot x + \zeta_n^m \sin k_n \cdot x)(\xi_{n'}^{m'} \cos k_{n'} \cdot x + \zeta_{n'}^{m'} \sin k_{n'} \cdot x)$$

$$= \delta_{mm'} \delta_{nn'} \cos k_n(x - y)$$

The equality follows by independence of the ξ and ζ and by the angle addition formula. Each of the remaining N terms are identical, so we drop $\frac{1}{N}\sum_{n=1}^N$ and the index n. A sum over m remains.

Multiplying the factors containing ψ_P and ψ_S and summing over m yields, after some algebra,

$$\begin{aligned}
&\mathrm{E}V^j(x)V^\ell(y) \\
&= \mathrm{E}(\tfrac{k^j k^\ell}{|k|^2}\psi_P(|k|)^2 + (\delta_{j\ell} - \tfrac{k^j k^\ell}{|k|^2})\psi_S(|k|)^2) \cos k \cdot (x - y) \\
&= \int \mu(d|k|) \frac{\sigma_k(ds)}{\Sigma_d |k|^{d-1}} [\tfrac{k^j k^\ell}{|k|^2} \frac{d\Phi_P}{d\mu}(|k|) + (\delta_{j\ell} - \tfrac{k^j k^\ell}{|k|^2}) \frac{d\Phi_S}{d\mu}(|k|)] \cos k \cdot (x - y) \\
&= \int_{\mathbb{R}^d} e^{ik\cdot(x-y)} \mathfrak{f}^{j\ell}(dk) \\
&= b^{j\ell}(x - y)
\end{aligned}$$

The integral against $i \sin k \cdot (x - y)$ is zero because the factor inside the brackets is even under $k \to -k$. By the central limit theorem the finite dimensional distributions of V converge to Gaussians as $N \to \infty$. □

3.3. Simulations of an isotropic flow. We now describe the particulars of our simulations of a two-dimensional isotropic Brownian flow. We begin by choosing the measures Φ_P and Φ_S which define the isotropic covariance b via (2.24) and (2.25):

$$\Phi_P(d\alpha) = 4(1 - \eta)p^2 \alpha^3 e^{-p\alpha^2} d\alpha \tag{3.15}$$

$$\Phi_S(d\alpha) = 4\eta s^2 \alpha^3 e^{-s\alpha^2} d\alpha, \tag{3.16}$$

where η, p, and s are positive constants. We will use η to vary the relative strengths of the potential and solenoidal components. In particular, $\eta = 1$ corresponds to a solenoidal (incompressible) flow and $\eta = 0$ to a potential flow.

In two dimensions, formulas (2.24) and (2.25) become

$$b_N(r) = \int_0^\infty \frac{J_1(r\alpha)}{r\alpha} \Phi_P(d\alpha) + \int_0^\infty J_1'(r\alpha)\Phi_S(d\alpha) \tag{3.17}$$

$$b_L(r) = \int_0^\infty J_1'(r\alpha)\Phi_P(d\alpha) + \int_0^\infty \frac{J_1(r\alpha)}{r\alpha}\Phi_S(d\alpha) \tag{3.18}$$

Making use of a table of integrals allows us to show that

$$b_N(r) = \eta(1-\frac{r^2}{2s})\exp(-\frac{r^2}{4s}) + (1-\eta)\exp(-\frac{r^2}{4p}) \tag{3.19}$$

$$b_L(r) = \eta\exp(-\frac{r^2}{4s}) + (1-\eta)(1-\frac{r^2}{2p})\exp(-\frac{r^2}{4p}) \tag{3.20}$$

$$b_0 = 1 \tag{3.21}$$

$$\lambda_1 = \eta(\frac{1}{2s}+\frac{1}{2p}) - \frac{1}{2p} \tag{3.22}$$

In particular, $\lambda_1 = 0$ when $\eta = \frac{s}{s+p}$.

We turn to generating isotropic random fields using the spectral method described above. We give, for general d, a procedure which is very easy to implement. First, let $k^1, \ldots, k^d$ be independent Gaussian random variables with mean zero and variance γ. Then the vector $\frac{k}{|k|}$ is uniform on the unit sphere, and $|k|^2 = (k^1)^2 + \cdots + (k^d)^2$ has the gamma distribution with shape index $\frac{d}{2}$ and scale parameter $\frac{1}{2\gamma}$. Writing the gamma density explicitly and changing variables shows that the distribution μ of $|k|$ is given by

$$\mu(dr) = \frac{2}{(2\gamma)^{\frac{d}{2}}\Gamma(\frac{d}{2})} r^{d-1}\exp(-\frac{r^2}{2\gamma})dr \tag{3.23}$$

Now one calculates ψ_S and ψ_P from (3.14).

Here we are interested in $d = 2$. The functions ψ_S and ψ_P are given by

$$\psi_S^2(r) = \frac{d\Phi_S}{d\mu}(r) = 4\eta\gamma s^2 r^2 \exp(r^2(\frac{1}{2\gamma} - s)) \tag{3.24}$$

$$\psi_P^2(r) = \frac{d\Phi_P}{d\mu}(r) = 4(1-\eta)\gamma p^2 r^2 \exp(r^2(\frac{1}{2\gamma} - p)) \tag{3.25}$$

We should choose γ to make the exponential factor close to 1.

In what follows, we chose $p = 1$, $s = 2$, $\gamma = \frac{3}{4}$, $\delta = 0.005$, and $N = 64$.

It is important to keep in mind the temporal and spatial scales of the flow. As noted above, for this flow $b_0 = 1$, so that the one-point motion $F_t x,\ t \geq 0$ is Brownian with zero drift and variance given by

$$\mathrm{Var}(F_t^i x) = t, \qquad i = 1,2,\ t \geq 0$$

The coordinate processes $F_t^1 x,\ t \geq 0$ and $F_t^2 x,\ t \geq 0$ are independent.

The covariance b defines another spatial scale. The functions $b_L(r)$ and $b_N(r)$ are positive for r less than about 1.5, and they are essentially zero for r larger than about 7. Recalling the discussion following equation (2.13), we see that if two points are within about one unit distance of each other, their motions will be somewhat coherent and they will be likely to stay together for some time. Points further than about 7 units apart diffuse almost independently.

3.3.1. The density of a mass distribution. The purpose of the next two figures is to illustrate the effect compressibility has on the density of a mass distribution carried by a flow. We follow the motion of 2500 points. Initially, they are evenly spaced on a rectangular lattice with 50 rows and 50 columns. The x and y coordinates both range from -10 to 10. This approximates a uniform continuous mass distribution.

Figure 1 shows the motion of these points in an incompressible flow ($\eta = 1$) while Figure 2 shows them in a pure potential flow ($\eta = 0$). In the incompressible case, the number of atoms per unit area stays roughly constant, as it should. The potential case is strikingly different. By time 1.0, there are clearly demarcated cells containing little or no mass, the mass being concentrated on the cell boundaries.

Two comments on the presentation: The axes are adjusted in each frame to keep all the points in view. This makes the mass distribution look less spread out than it really is. Second, due to limited resolution, closely-spaced points are plotted as though they overlap.

3.3.2. The boundary of a mass distribution. We have seen the effect of a flow on the density *inside* a mass distribution. Now we consider how the *boundary* of a mass distribution is deformed as the mass spreads into the surrounding space.

Recall that for fixed t, the map F_t is a homeomorphism from $\mathbb{R}^d$ to $\mathbb{R}^d$, so the boundary will be a connected, non-self-intersecting curve for all time, provided that it was so at time 0. We model the boundary with a large number of points initially evenly spaced around a circle. We draw it at each time by connecting the points with lines. The parameters of the simulation are the same as above.

We begin by presenting simulations of the deformation of a circle of radius 1, made up of 2500 evenly-spaced points. The incompressible case is shown first (Figure 3). Recall that the Lebesgue measure of the set enclosed by the curve remains constant in time. Note how elongated the distribution becomes by time 7.

The potential case (Figure 4) shows much more elongation combined with an apparent decrease in the area enclosed by the image of the circle. Note that between time 4 and time 7, the points are compressed from a scale of about 3.5 by 4 to a scale of 0.2 by 0.2.

In Figure 5, we increase the radius of the initial circle to 10 and the number of points to 10,000. This results in an apparent increase in spatial

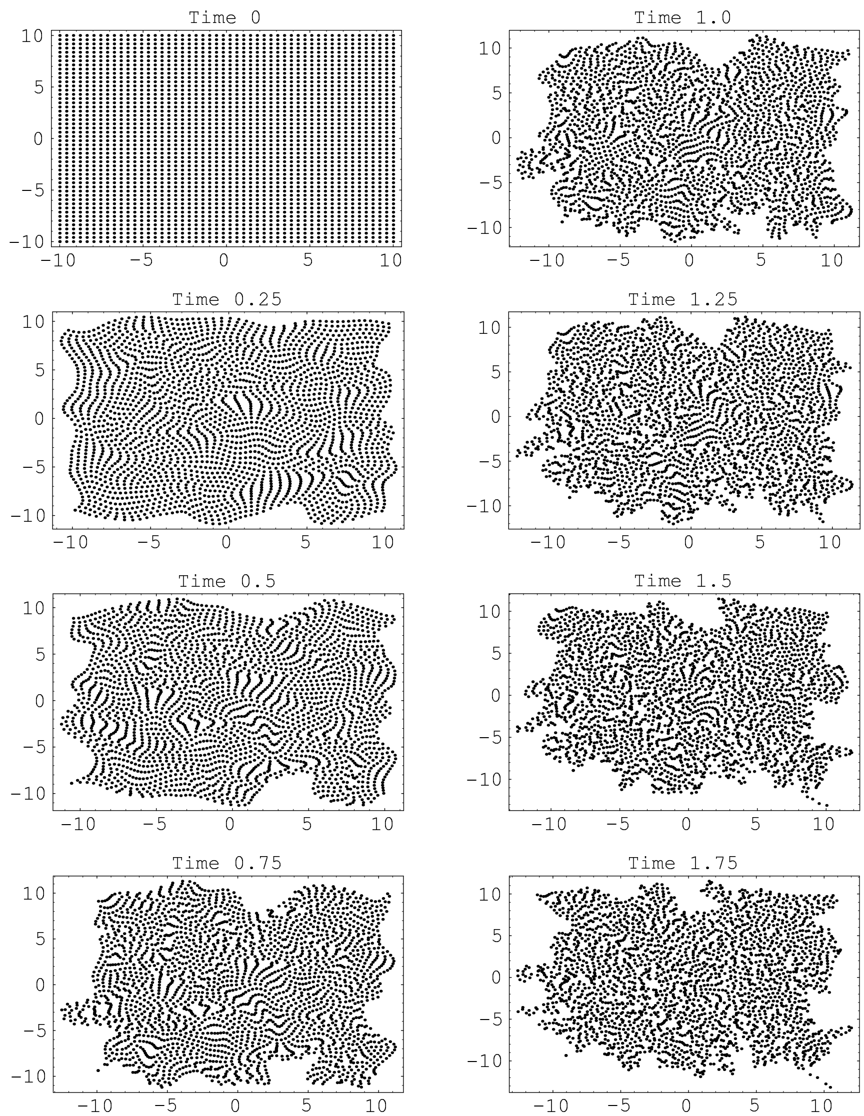

FIG. 1. *Incompressible case, with* $\eta = 1$ *and* $\lambda_1 > 0$. *Note that the mass density remains fairly constant in time.*

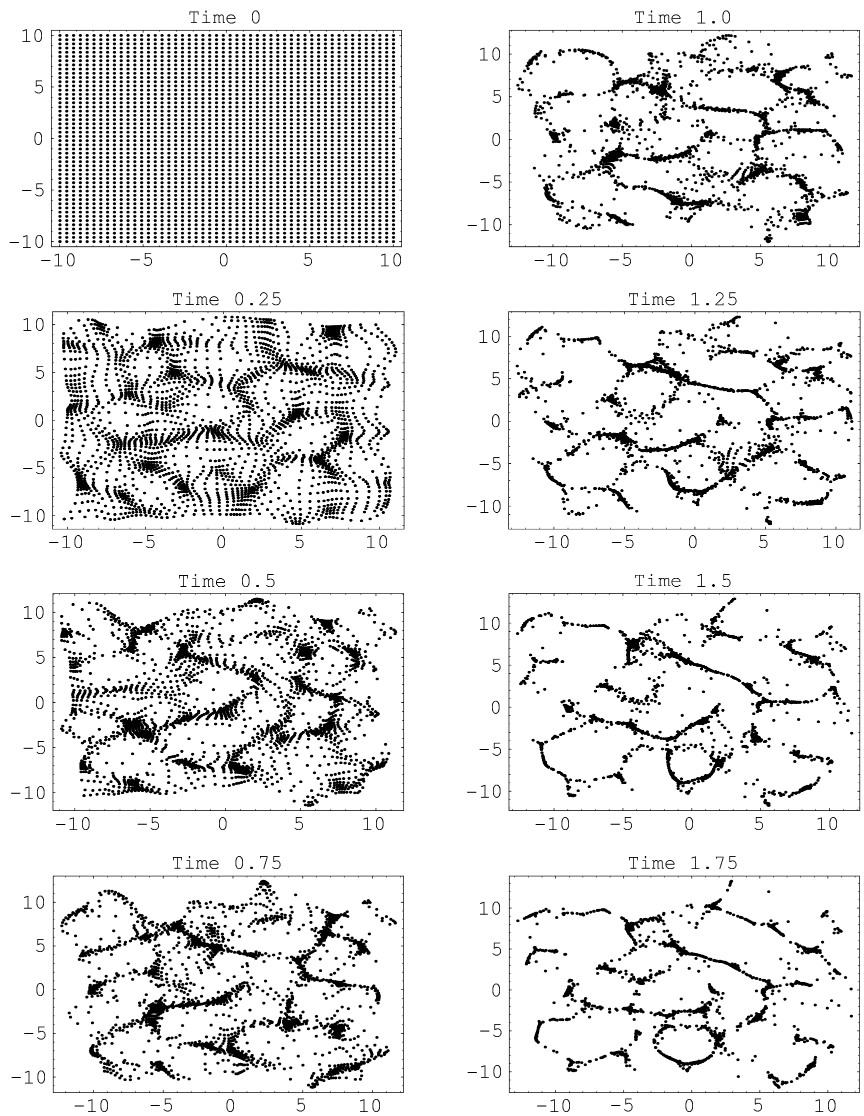

FIG. 2. *Potential case, with* $\eta = 0$ *and* $\lambda_1 < 0$. *Note how well-defined the cell boundaries are at times 1.5 and 1.75.*

structure. This structure is resolved reasonably well in the incompressible case – the points we followed did not separate too far, except in a few cases, by time 7. In the potential case, however, some neighboring points are very rapidly separated from one another, so that we quickly see a few small clumps of points separated by long distances. We have omitted the picture since it is not very informative.

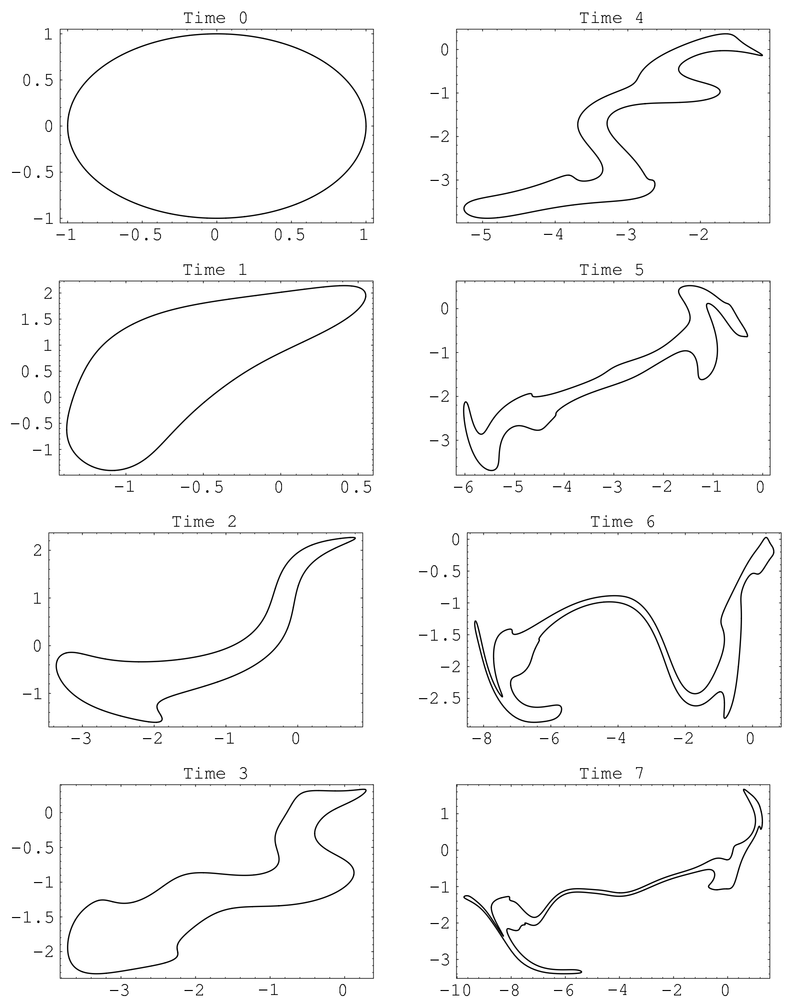

FIG. 3. *Incompressible case, with* $\eta = 1$ *and* $\lambda_1 > 0$. *Initial radius = 1. Note how elongated the curve is by time 7.*

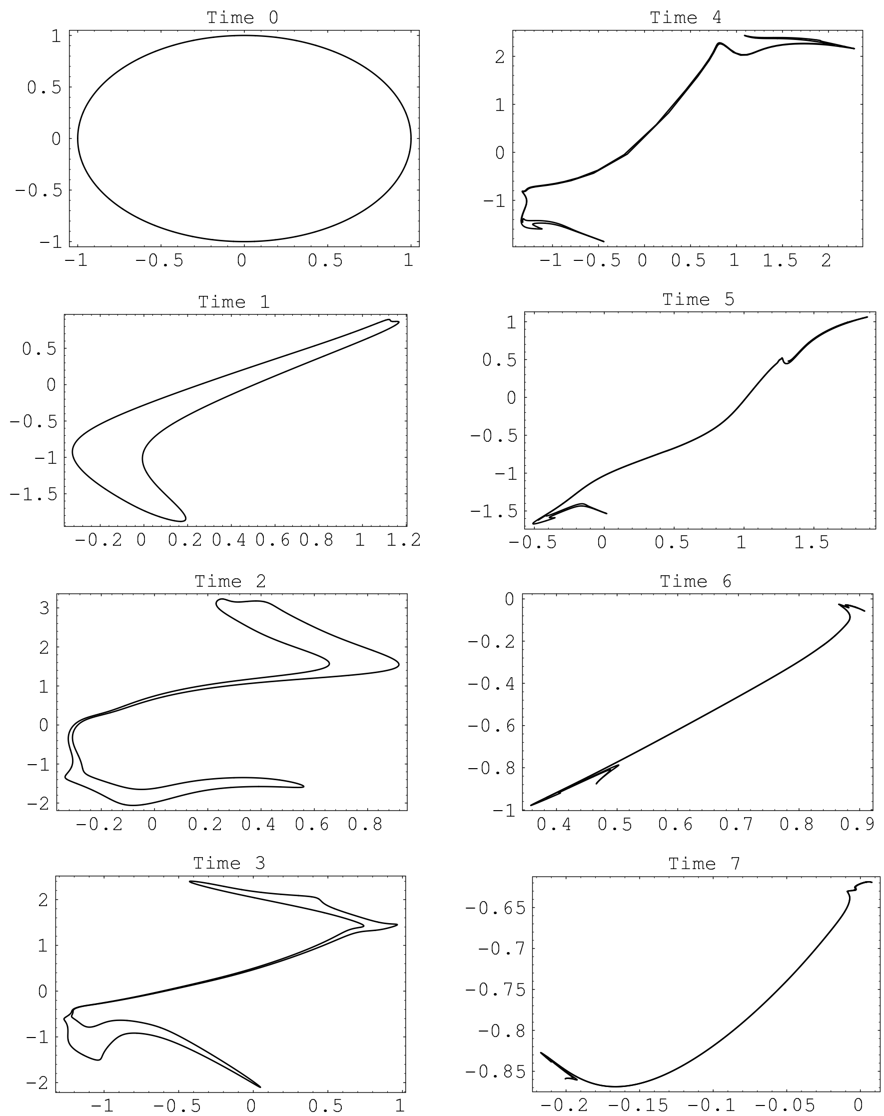

FIG. 4. *Potential case, with $\eta = 0$ and $\lambda_1 < 0$. Initial radius = 1. Note the rapid compression between times 4 and 7.*

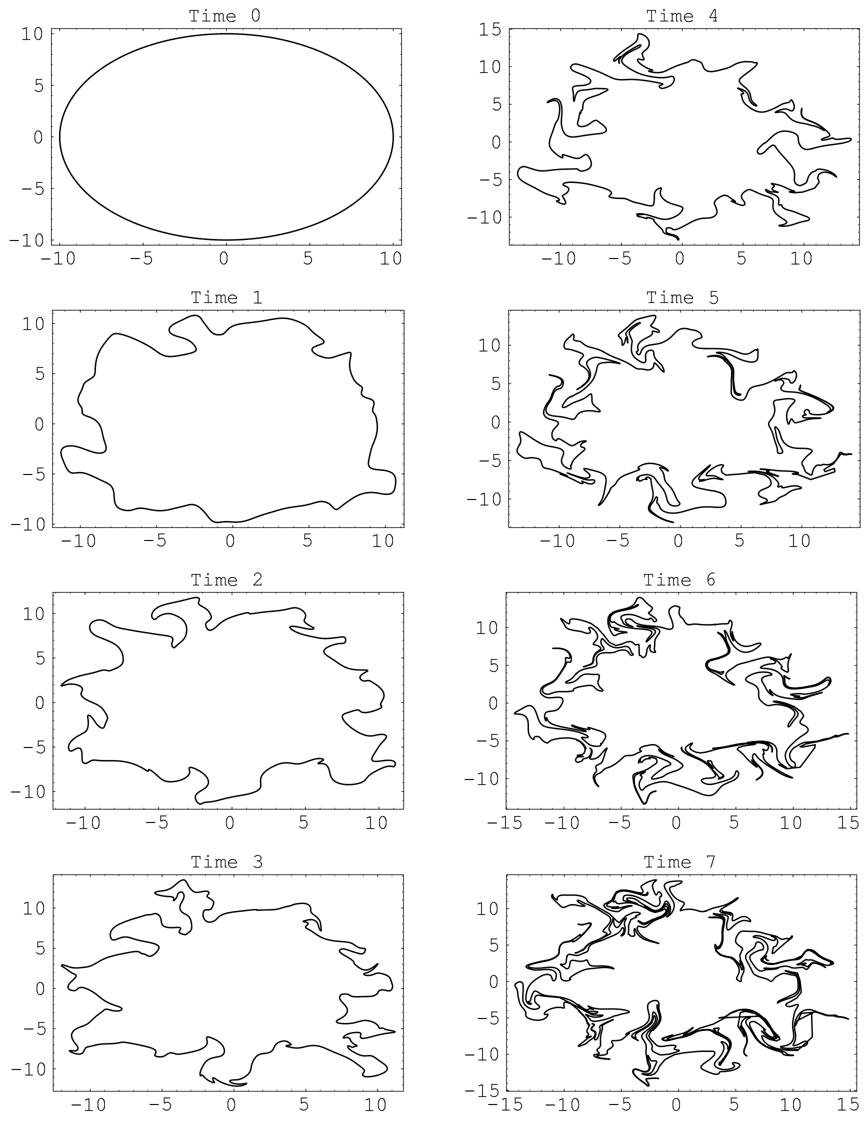

FIG. 5. *Incompressible case, with $\eta = 1$ and $\lambda_1 > 0$. Initial radius = 10.*

4. Formulas for the evolution of M_t. In this section we present a number of rigorous results and formulas describing the evolution of a mass distribution in a Brownian flow. Recall from Section 1 that M_0 is a finite, deterministic measure on $\mathbb{R}^d$ describing a mass distribution at time 0, and at time t the mass is described by the random measure M_t given by

$$M_t(R) = M_0(\{x \in \mathbb{R}^d \ : \ F_{0t}(x) \in R\}) \tag{4.1}$$

Throughout this section we will assume, without further mention, that F is a Brownian flow on $\mathbb{R}^d$ satisfying equation (2.6) where U is a Brownian motion with drift u and covariance a which satisfy (2.8)-(2.11). We will also write $F_t x$ as a shorthand for $F_{0t}(x)$.

4.1. Limiting behavior. To get our bearings, we first list a result on the limiting behavior of M_t as $t \to \infty$. See KUNITA (1990), Theorems 4.3.9 and 4.3.10 for the proof.

THEOREM 4.1. *a) Suppose that for all x in $\mathbb{R}^d$, the matrix $a(x,x)$ is strictly positive definite:*

$$\sum_{i,j=1}^{d} a^{ij}(x,x)\xi^i\xi^j > 0, \qquad \xi \in \mathbb{R}^d,\ \xi \neq 0 \tag{4.2}$$

Moreover, suppose that the one-point motion under F has an invariant distribution π with $\pi(\mathbb{R}^d) = 1$. Then for each Borel subset R of $\mathbb{R}^d$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t ds\ M_s(R) = \pi(R) \quad \textit{almost surely} \tag{4.3}$$

b) Suppose the one-point motion does not have a finite invariant measure. Then for each bounded Borel subset R of $\mathbb{R}^d$,

$$\lim_{t\to\infty} \mathrm{E}M_t(R) = 0 \tag{4.4}$$

□

4.2. Stochastic calculus of M_t. To understand the evolution of the random measure M_t, it is useful to consider the process $\int M_t(dx)f(x), t \geq 0$, for various choices of f. For convenience we will write $M_t f$ in place of $\int M_t(dx)f(x)$. Note that by the definition of M_t, we have

$$M_t f = \int M_t(dx)f(x) = \int M_0(dx)f(F_t x) \tag{4.5}$$

For bounded f, the integral $M_t f$ will always exist and be finite. To deal with unbounded f, we introduce the following condition:

CONDITION 4.2. *There exist constants $p > 2$ and $C < +\infty$ such that $\int M_0(dx)|x|^p < +\infty$ and both $|f|$ and $|Af|$ are bounded by the function*

$C(1+|x|)^p$. Moreover, $f \in C^2(\mathbb{R}^d, \mathbb{R})$ and for each $i = 1, \dots, d$, the function $\partial_i f$ is of polynomial growth.

If this condition is satisfied, then $M_t f,\ t \geq 0$ is well defined almost surely, and the expectation of $M_t f$ is finite. The same is true of $\int_0^t ds\ M_s f$. Moreover, if f is continuous, then $M_t f,\ t \geq 0$ is also continuous. These facts and the following result on the stochastic calculus of $M_t f$ appear in ZIRBEL (1995a).

THEOREM 4.3. *Suppose that M_0 and f satisfy Condition 4.2.*

a) The process $M_t f$ satisfies the following equation:

$$M_t f = M_0 f + \int_0^t ds\ M_s(Af) + L_t, \tag{4.6}$$

where L is a continuous martingale with mean 0, given by

$$L_t = \sum_{i=1}^{d} \int M_0(dx) \int_0^t \partial_i f(F_s x) U_0^i(F_s x, ds). \tag{4.7}$$

The quadratic variation of L can be computed using part (b).

b) Let N_0 be a finite deterministic measure on $\mathbb{R}^d$ and define N_t by $N_t = N_0 \circ F_t^{-1}$. Suppose that N_0 and a function g satisfy Condition 4.2 with $\bar{p}$ in place of p, and in addition that for $x, y \in \mathbb{R}^d, i, j = 1, \dots, d$,

$$|\partial_i f(x) \partial_j g(y) a^{ij}(x,y)| \leq C(1+|x|)^p (1+|y|)^{\bar{p}} \tag{4.8}$$

Then the joint quadratic variation of $M_t f$ and $N_t g$ satisfies

$$\langle M_t f, N_t g \rangle = \int_0^t ds \int M_s(dx) \int N_s(dy) \sum_{i,j=1}^{d} \partial_i f(x) \partial_j g(y) a^{ij}(x,y). \tag{4.9}$$

4.3. Mean measure, spatial covariance, Laplace functional. It is quite natural to inquire into the behavior of such simple quantities as the mean, covariance, and Laplace transform of random variables. Here we discuss the analogous quantities for the random measure M_t.

The *mean measure* μ_t of M_t is a deterministic measure defined for each $t \geq 0$ by

$$\mu_t(R) = \mathbf{E} M_t(R), \qquad \text{Borel } R \subseteq \mathbb{R}^d \tag{4.10}$$

Often, μ_t will have a density m_t with respect to Lebesgue measure. Such a density is called the *mean concentration.*

The *spatial covariance measure* η_t is a deterministic measure on $\mathbb{R}^d \times \mathbb{R}^d$ defined for each $t \geq 0$ by

$$\eta_t(R_1 \times R_2) = \mathbf{E} M_t(R_1) M_t(R_2), \qquad \text{Borel } R_1, R_2 \subseteq \mathbb{R}^d \tag{4.11}$$

It is clear that this definition can be extended to any number of factors, yielding higher moments of M_t. If M_t has the (random) density $\rho_t(x)$, then η_t will have a density $c_t(x,y)$ which satisfies

$$c_t(x,y) = \mathbf{E}\rho_t(x)\rho_t(y) \tag{4.12}$$

The covariance of $\rho_t(x)$ and $\rho_t(y)$ is then equal to $c_t(x,y) - \mathbf{E}\rho_t(x)\mathbf{E}\rho_t(y)$.

The *Laplace functional* $\mathcal{L}_t$ of M_t is defined for each Borel map f from $\mathbb{R}^d$ to $\mathbb{R}_+$ by

$$\mathcal{L}_t(f) = \mathbf{E}e^{-M_t f} \tag{4.13}$$

This is the analogue of the Laplace transform. The Laplace functional uniquely characterizes the law of M_t.

The following result concerns the evolution of μ_t, η_t, and $\mathcal{L}_t$ as t increases. Recall that A denotes the generator of the one-point motion $F_t x,\ t \geq 0$; see (2.14). Recall the notation $a_d(x) = a(x,x)$. The adjoint A^* of A is the operator defined implicitly in equation (4.14) below.

We will need the following condition in connection with the density of the mean measure.

CONDITION 4.4. *The function a_d is uniformly elliptic: For some $\delta > 0$,*

$$\sum_{i,j=1}^{d} a_d^{ij}(x)\xi^i\xi^j \geq \delta|\xi|^2, \qquad x, \xi \in \mathbb{R}^d,\ t \in [0,T]$$

Also, the functions a_d^{ij}, $\frac{\partial a_d^{ij}}{\partial x^i}$, $\frac{\partial^2 a_d^{ij}}{\partial x^i \partial x^j}$, u^i, $\frac{\partial u^i}{\partial x^i}$ exist, are bounded, and are uniformly Holder continuous in $\mathbb{R}^d \times [0,T]$. □

For the spatial covariance measure, we will need:

CONDITION 4.5. *For some $\delta > 0$,*

$$\sum_{p,q=1}^{2} \sum_{i,j=1}^{d} a^{ij}(x_p, x_q)\xi_p^i\xi_q^j \geq \delta(|\xi|_1^2 + |\xi|_2^2), \quad x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}^d,\ t \in [0,T]$$

Also, the functions a^{ij}, $\frac{\partial a^{ij}}{\partial x_p^i}$, $\frac{\partial^2 a^{ij}}{\partial x_p^i \partial x_q^j}$, u^i, $\frac{\partial u^i}{\partial x^i}$ exist, are bounded, and are uniformly Holder continuous in $\mathbb{R}^d \times \mathbb{R}^d \times [0,T]$ and $\mathbb{R}^d \times [0,T]$. □

PROPOSITION 4.6. *a) Suppose that M_0 has compact support and Condition 4.4 holds. Then μ_t has a density m_t in $C^{2,1}(\mathbb{R}^d \times (0,\infty))$ which satisfies the* advection-diffusion *equation:*

$$\frac{\partial m_t}{\partial t} = A^* m_t = \tfrac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x^i \partial x^j}(a_d^{ij} m_t) - \sum_{i=1}^{d} \frac{\partial}{\partial x^i}(u^i m_t) \tag{4.14}$$

b) Suppose that M_0 has compact support and Condition 4.5 holds. Then η_t has a density c_t in $C^{2,2,1}(\mathbb{R}^d \times \mathbb{R}^d \times (0,\infty))$ which satisfies

$$\frac{\partial c_t}{\partial t} = A_x^* c_t + A_y^* c_t + \tfrac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x^i \partial y^j}(c_t a^{ij}) \tag{4.15}$$

where A_x^* is A^* acting on the first argument of c_t, etc.

c) Suppose that M_0 and f satisfy Condition 4.2 and that f satisfies (4.8) with $g = f$ and $\bar{p} = p$. Then the derivative of $\mathbf{E}\exp(-M_t f)$ with respect to t equals

$$\mathbf{E}\exp(-M_t f)[-M_t(Af) + \tfrac{1}{2} \int M_t(dx) \int M_t(dy) \sum_{i,j=1}^{d} \partial_i f(x) \partial_j f(y) a^{ij}(x,y)] \tag{4.16}$$

Proof. a) Denote by P_t the transition kernel of the one-point motion under F, that is, $P_t(x,A) = \mathbf{P}(F_t x \in A)$. By Condition 4.4, the measure $P_t(x,\cdot)$ has a density $p_t(x,y)$ which is once continuously differentiable in t and twice in y (KARATZAS and SHREVE (1988), Section 5.7). Moreover, p satisfies,

$$\frac{\partial p}{\partial t} = A^* p, \qquad t > 0$$

where A^* acts on the second argument of p_t and the first argument is held fixed.

By the definition of M_t and μ_t, for Borel $R \subseteq \mathbb{R}^d$,

$$\begin{aligned} \mu_t(R) = \mathbf{E}M_t(R) &= \mathbf{E}\int M_0(dx) 1_R(F_t x) \\ &= \int M_0(dx) \mathbf{E} 1_R(F_t x) \\ &= \int M_0(dx) \int p_t(x,y) 1_R(y) dy, \end{aligned}$$

which shows that μ_t has a density $m_t(y) = \int M_0(dx) p_t(x,y)$. Now M_0 has compact support and the time and space derivatives of p_t are continuous. This justifies the interchanges of integrals with derivatives in the following computation, and also shows that m_t is in $C^{2,1}(\mathbb{R}^d \times (0,\infty))$.

$$\begin{aligned} \frac{\partial m}{\partial t}(y) &= \int M_0(dx) \frac{\partial p}{\partial t}(x,y) \\ &= \int M_0(dx) A^* p_t(x,y) \\ &= A^* m_t(y) \end{aligned}$$

This completes the proof of part (a).

b) This is quite similar to the proof of part (a), except that here we use the two-particle transition kernel $P_t^{(2)}(x_1, x_2; A_1, A_2)$. Condition 4.5 guarantees the existence of a density for this transition kernel. The density satisfies an evolution equation of the form $\frac{\partial p^{(2)}}{\partial t} = A_2^* p^{(2)}$, where A_2^* is the adjoint of the generator of the two-point motion under F. This operator is defined implicitly in (4.15).

c) Let g be a twice differentiable function from $\mathbb{R}$ to $\mathbb{R}$. By Itô's Lemma,

$$\begin{aligned} g(M_t f) &= g(M_0 f) + \int_0^t g'(M_s f) dM_s f + \tfrac{1}{2} \int_0^t g''(M_s f) d\langle M_s f, M_s f\rangle \\ &= g(M_0 f) + \int_0^t g'(M_s f) M_s(Af) ds + \int_0^t g'(M_s f) dL_s \\ + & \tfrac{1}{2} \int_0^t ds\, g''(M_s f) \int M_s(dx) \int M_s(dy) \sum_{i,j=1}^d \partial_i f(x) \partial_j f(y) a^{ij}(x,y) \end{aligned}$$

The second equality follows from parts (a) and (b) of Theorem 4.3. Now let $g(x) = \exp(-x)$. Take expectations and the time derivative to give

$$\frac{d}{dt}\mathbf{E} \exp(-M_t f) = -\mathbf{E} \exp(-M_t f) M_t(Af)$$

$$+ \tfrac{1}{2}\mathbf{E} \exp(-M_t f) \int M_t(dx) \int M_t(dy) \sum_{i,j=1}^d \partial_i f(x) \partial_j f(y) a^{ij}(x,y)$$

This is the desired result. □

4.4. Centroid and Dispersion Calculations. The centroid C_t and the dispersion matrix D_t were defined in Section 1. We are now able to elucidate their semimartingale structure and provide some formulas for their means and covariances. All of these results are consequences of Theorem 4.3. See ZIRBEL (1995a) for more details.

For simplicity we take $M_0(\mathbb{R}^d) = 1$. Suppose that for some $p > 2$, $\int M_0(dx)|x|^p < +\infty$. Define $f^i(x) = x^i$. Then $C_t^i = M_t f^i$, and by Theorem 4.3,

$$C_t = C_0 + \int_0^t ds\, M_s u + \int M_0(dx) \int_0^t U_0(F_s x, ds), \tag{4.17}$$

the last term of which is a martingale. We also obtain

$$\langle C^i, C^j\rangle_t = \int_0^t ds \int M_s(dx) \int M_s(dy) a^{ij}(x,y) \tag{4.18}$$

Applying Itô's formula to $C_t^iC_t^j$, we get

$$C_t^iC_t^j = C_0^iC_0^j + \int_0^t C_s^i dC_s^j + \int_0^t C_s^j dC_s^i + \langle C^i, C^j\rangle_t$$

Assume that $\int M_0(dx)|x|^4 < +\infty$. Taking expectations drops out the martingale terms and yields

$$\begin{aligned}\mathbf{E}C_t^iC_t^j &= C_0^iC_0^j + \mathbf{E}\int_0^t ds\ C_s^i M_s u^j + \mathbf{E}\int_0^t ds\ C_s^j M_s u^i \\ &\quad + \mathbf{E}\int_0^t ds \int M_s(dx) \int M_s(dy) a^{ij}(x,y)\end{aligned}$$

In other words,

$$\mathbf{E}C_t^iC_t^j = C_0^iC_0^j + \mathbf{E}\int_0^t ds \int M_s(dx)\int M_s(dy) E^{ij}(x,y), \tag{4.19}$$

where $E^{ij}(x,y) = a^{ij}(x,y) + x^i u^j(y) + u^i(x) y^j$. The covariance of C_t^i and C_t^j can now be computed easily.

By the definition of D_t, we have

$$D_t^{ij} = \int M_t(dx)(x^i - C_t^i)(x^j - C_t^j) = \int M_t(dx) x^i x^j - C_t^iC_t^j$$

Thus, if we set $g^{ij}(x) = x^i x^j$, we have $D_t^{ij} = M_t g^{ij} - C_t^iC_t^j$. By Theorem 4.3 and the fact that $Ag^{ij}(x) = E^{ij}(x,x)$, we have

$$M_t g^{ij} = M_0 g^{ij} + \int_0^t ds \int M_s(dx) E^{ij}(x,x) + L_t^{ij},$$

the last term of which is a martingale. Taking the expectation of D_t^{ij} yields

$$\mathbf{E}D_t^{ij} = D_0^{ij} + \mathbf{E}\int_0^t ds \int M_s(dx) \int M_s(dy)(E^{ij}(x,x) - E^{ij}(x,y)). \tag{4.20}$$

The calculation of $\langle D^{ij}, D^{k\ell}\rangle_t$ begins by substituting the expressions above for $M_t g^{ij}$ and $C_t^iC_t^j$ into $\langle D^{ij}, D^{k\ell}\rangle_t$. Standard properties of quadratic variations and part (b) of Theorem 4.3 then yield

$$\langle D^{ij}, D^{k\ell}\rangle_t = \int_0^t ds \int M_s(dx) \int M_s(dy)(x - C_s)^T H(x,y)(y - C_s), \tag{4.21}$$

where the matrix H is defined by

$$H^{mn} = \delta_{im}\delta_{kn}a^{j\ell} + \delta_{jm}\delta_{kn}a^{i\ell} + \delta_{im}\delta_{\ell n}a^{jk} + \delta_{jm}\delta_{\ell n}a^{ik} \tag{4.22}$$

with the argument (x,y) suppressed. Note that H has at most four non-zero entries. One can now calculate $\mathbf{E}D_t^{ij}D_t^{k\ell}$ and then the covariance of D_t^{ij} and $D_t^{k\ell}$, as above with C_t.

5. Centroid and dispersion in isotropic flow. In this section we specialize to the case of isotropic Brownian flows. We will quickly run through the results of Section 4 for this case, then report our results concerning the limiting behavior of C_t and D_t as $t \to \infty$.

Throughout this section, F is an isotropic Brownian flow with covariance b which satisfies conditions 2.1 and 2.2. The initial measure M_0 is deterministic and satisfies $M_0(\mathbb{R}^d) = 1$ and $\int M_0(dx)|x|^p < +\infty$ for some $p > 2$.

5.1. Formulas for M_t. ¿From Theorem 4.1, we see that $\mathrm{E}M_t(R) \to 0$ for all bounded Borel $R \subseteq \mathbb{R}^d$, since the one-point motion is d-dimensional Brownian motion, which has the Lebesgue measure as its invariant measure. Thus, the mass 'moves off to ∞' in some sense.

More precisely, since the transition density of Brownian motion is smooth, we may rework the proof of Proposition 4.6 without further conditions on b and M_0 to yield the evolution equation for the mean tracer concentration m_t:

$$\frac{\partial m_t}{\partial t} = \tfrac{1}{2} b_0 \Delta m_t, \tag{5.1}$$

which is the heat equation. Similarly, the evolution equation for the spatial covariance density c_t becomes

$$\frac{\partial c_t}{\partial t} = \tfrac{1}{2} b_0 \Delta_x c_t + \tfrac{1}{2} b_0 \Delta_y c_t + \tfrac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x^i \partial y^j} (c_t a^{ij}) \tag{5.2}$$

The formulas for the centroid and dispersion matrix simplify considerably for isotropic flows. The centroid is a martingale satisfying

$$C_t = C_0 + \int M_0(dx) \int_0^t U(F_s x, ds) \tag{5.3}$$

Its covariance is given by

$$\mathrm{Cov}(C_t^i, C_t^j) = \mathbf{E} \int_0^t ds \int M_s(dx) \int M_s(dy) b^{ij}(x-y) \tag{5.4}$$

The expectation of the dispersion matrix can be written compactly as

$$\mathrm{E}D_t^{ij} = D_0^{ij} + b_0 t \delta_{ij} - \mathrm{Cov}(C_t^i, C_t^j). \tag{5.5}$$

5.2. Behavior of C_t and D_t as $t \to \infty$. The centroid and dispersion matrix are meant to be rough descriptors of where the mass distribution is and how spread out it is, respectively. These are very important matters in any real-world mass transport problem.

Consider, for example, a common pollution remediation problem. A certain quantity of an industrial chemical has been spilled and is spreading underground. One would like to know whether the plume it forms is tightly concentrated or widely dispersed and, in either case, how far it has moved from the site of the spill. In the absence of detailed information about the soil layers and groundwater flow patterns, one needs to take a statistical approach to design a soil sampling and remediation strategy. The relevant concerns then are how dispersed the chemical is *likely* to be, and how far it *may* have moved from the spill site.

This is a notoriously difficult problem.

The results we present here are rigorous and exact, but of course are limited to isotropic Brownian flows. Nevertheless, we believe that this case captures many of the essential difficulties of the general problem, and may serve as a useful point of comparison for other models.

It happens that the continuous or discrete nature of M_0 plays an important role in the asymptotics of C_t and D_t. Generally speaking, the presence of a discrete (atomic) component masks more subtle effects which occur with a non-atomic distribution. If M_0 has an atomic component, we let σ denote the sum of the squares of the masses of the atoms of M_0. Otherwise, M_0 is diffuse and we set $\sigma = 0$. In either case, we have $\sigma < 1$, unless M_0 consists of exactly one atom, which is a trivial case we henceforth exclude.

One further point deserves mention. If we fix the dimension d and the variance b_0 of the one-point motion, we completely specify the one-point motion and the evolution equation (5.1) for the mean concentration m_t. Even so, there remains a wide range of possible behaviors for the centroid and dispersion matrix. Thus, it is not enough to know the mean concentration of M_t.

5.2.1. Linear growth rates. The first result is a fairly complete picture of the linear growth rates of $\mathrm{Cov}(C_t^i, C_t^j)$ and $\mathrm{E}D_t$ in the long run. This proposition covers all dimensions $d \geq 1$ and all cases except the case $d = 2$ and $\lambda_1 \geq 0$. The proof appears in ZIRBEL and CINLAR (1996). It is an easy consequence of (5.4) and Proposition 2.3.

PROPOSITION 5.1. *a) If $d \geq 3$ and $\lambda_1 \geq 0$, then*

$$\lim_{t\to\infty} \frac{d}{dt}\mathrm{Cov}(C_t^i, C_t^j) = b_0\sigma\delta_{ij}, \qquad \lim_{t\to\infty} \frac{d}{dt}\mathrm{E}D_t^{ij} = b_0(1-\sigma)\delta_{ij}$$

b) If $d = 3$ and $\lambda_1 < 0$, then

$$\lim_{t\to\infty} \frac{d}{dt}\mathrm{Cov}(C_t^i, C_t^j) = b_0(1-p(1-\sigma))\delta_{ij}, \quad \lim_{t\to\infty} \frac{d}{dt}\mathrm{E}D_t^{ij} = b_0 p(1-\sigma)\delta_{ij}$$

where $p \in (0,1)$ equals $\int M_0(dx) \int M_0(dy) \mathrm{P}\{\lim_{t\to\infty} |F_t x - F_t y| = +\infty\}$.

c) If $d = 1$, or $d = 2$ and $\lambda_1 < 0$, then

$$\lim_{t\to\infty} \frac{d}{dt}\mathrm{Cov}(C_t^i, C_t^j) = b_0\delta_{ij}, \qquad \lim_{t\to\infty} \frac{d}{dt}\mathrm{E}D_t^{ij} = 0$$

Moreover, all these results hold with $\frac{d}{dt}$ replaced by $\frac{1}{t}$. □

We are interested in how quickly $\mathrm{Var}(C_t^i)$ and $\mathrm{E}D_t^{ii}$ increase, for these tell how far the centroid may be from its initial location and how spread out the mass is, respectively. The gist of this result is that for large d and $\lambda_1 \geq 0$, dispersion increases at the maximum rate possible, and for small d and $\lambda_1 < 0$, dispersion is not so pronounced, but the centroid moves a great deal.

We are able to give more detailed information in two cases, $d \geq 3$ and $\lambda_1 > 0$, and $d = 2$ and $\lambda_1 > 0$.

5.2.2. Three or more dimensions. Next, we consider isotropic Brownian flows in three or more dimensions and with strictly positive Lyapunov exponent. We will give very exact bounds on the deviation of $\mathrm{Cov}(C_t^i, C_t^j)$ and $\mathrm{E}D_t^{ij}$ from their asymptotic linear growth rates. The proof is given in ZIRBEL (1995b).

We need to assume that the covariance b satisfies:

$$\int_0^\infty dz\ z(|b_N(z)| + |b_L(z)|) < +\infty \tag{5.6}$$

Essentially, we need b_N and b_L to decay faster than $\frac{1}{z^2}$ as $z \to \infty$. Also, we require

$$\int_{\mathbb{R}^d} M_0(dx) \int_{\mathbb{R}^d\setminus\{x\}} M_0(dy) \log^+ \frac{1}{|x-y|} < +\infty \tag{5.7}$$

where $\log^+ z = \max\,(0, \log z)$. This says that the diffuse component of M_0 must not have strong concentrations of mass.

Let s denote the scale function of the separation process Z_t, and let m be the density of the speed measure of Z. Note that $s(\infty) < \infty$ when $\lambda_1 > 0$, since in that case $Z_t \to \infty$ almost surely.

THEOREM 5.2. *Suppose $d \geq 3$, $\lambda_1 > 0$, and the two conditions above hold. Then for all $t \geq 0$, and $i, j = 1 \ldots d$, we have*

$$|\mathrm{Cov}(C_t^i, C_t^j) - b_0\sigma t\delta_{ij}| \leq K \tag{5.8}$$

$$|\mathrm{E}D_t^{ij} - D_0^{ij} - b_0(1-\sigma)t\delta_{ij}| \leq K, \tag{5.9}$$

where K is a finite constant equal to

$$\int M_0(dx) \int_{y\neq x} M_0(dy) \int_0^\infty dz\ m(z)(2|b_N(z)|+|b_L(z)|)(s(\infty)-s(|x-y|\wedge z))$$

Thus, $\mathrm{Cov}(C_t^i, C_t^j)$ and $\mathrm{E}D_t^{ij}$ stay extremely close to their linear asymptotes as $t \to \infty$. More remarkable is what occurs when M_0 is diffuse, so the 'linear asymptote' of $\mathrm{Cov}(C_t^i, C_t^j)$ is 0.

COROLLARY 5.3. *Suppose M_0 is diffuse. Then, for all $t \geq 0$ and $i, j = 1 \ldots d$,*

$$|\mathrm{Cov}(C_t^i, C_t^j)| \leq K \tag{5.10}$$

$$|\mathrm{E}D_t^{ij} - D_0^{ij} - b_0 t \delta_{ij}| \leq K \tag{5.11}$$

Moreover, the martingale C_t converges almost surely as $t \to \infty$ to a random vector C_∞ with mean C_0 and variance bounded by K.

Proof. The inequalities are immediate from the Theorem. The variance of C_t^i is bounded, so C^i is a uniformly integrable martingale, which means it converges almost surely. □

This is a remarkable result. Apparently, in this case, dispersion relative to the centroid is strong enough that the centroid, a spatial average over the mass distribution, converges. This is a sort of strong law of large numbers for the spatial average, which indicates that the particles making up the mass distribution have quite weakly dependent motions at long times.

On the other hand, as soon as we have $\lambda_1 < 0$, the variance of C_t grows linearly in time, by Proposition 5.1.

5.2.3. Two dimensions with positive Lyapunov exponent. Finally, we give a limit theorem for the case $d = 2$ and $\lambda_1 > 0$. The proof appears in ZIRBEL (1995b).

We will need two conditions on M_0. For some $\epsilon > 0$,

$$\int M_0(dx) e^{\epsilon |x|} < +\infty \tag{5.12}$$

$$\int_{\mathbb{R}^d} M_0(dx) \int_{\mathbb{R}^d \setminus \{x\}} M_0(dy) \frac{1}{|x - y|^\epsilon} < +\infty \tag{5.13}$$

The first condition requires the mass to be localized in space. The second condition will be satisfied, for example, if the diffuse component of M_0 has bounded density with respect to Lebesgue measure. We also need a condition on the covariance b:

$$\int_0^\infty dz \; z \, |b_L(z) + b_N(z)| < +\infty \tag{5.14}$$

We introduce the notation $\mathrm{Var}(C_t)$ for $\mathrm{Var}(C_t^1) + \mathrm{Var}(C_t^2)$, and B_t for $D_t^{11} + D_t^{22}$.

THEOREM 5.4. *Suppose $d = 2$, $\lambda_1 > 0$, and the conditions above hold. Then,*

$$\lim_{t \to \infty} \frac{\mathrm{Var}(C_t) - 2 b_0 \sigma t}{\log t} = \frac{1 - \sigma}{2 b_0} K \int_0^\infty dz \; m(z)(b_L(z) + b_N(z)) \tag{5.15}$$

$$(5.16) \lim_{t\to\infty} \frac{2b_0(1-\sigma)t - \mathrm{E}B_t}{\log t} = \frac{1-\sigma}{2b_0} K \int_0^\infty dz\, m(z)(b_L(z) + b_N(z))$$

where $K = \lim_{z\to\infty} \frac{z}{m(z)}$. *Moreover, if the flow is incompressible and* $\lim_{z\to\infty} z^2 b_L(z) = 0$, *then the right side equals 0.* □

When M_0 is diffuse, we have $\sigma = 0$ and so (5.15) becomes,

$$(5.17) \qquad \lim_{t\to\infty} \frac{\mathrm{Var}(C_t)}{\log t} = \frac{K}{2b_0} \int_0^\infty dz\, m(z)(b_L(z) + b_N(z))$$

In the incompressible case, if $\lim_{z\to\infty} z^2 b_L(z) = 0$, we even get

$$(5.18) \qquad \mathrm{Var}(C_t) \ll \log t$$

$$(5.19) \qquad 0 \le 2b_0 t - \mathrm{E}B_t \ll \log t$$

as $t \to \infty$.

Consider what we know now about two-dimensional isotropic Brownian flows. Beginning with the incompressible case, $\mathrm{Var}(C_t)$ increases much slower than $\log t$. As we add a potential component to the flow, λ_1 decreases, and $\mathrm{Var}(C_t)$ goes as $\log t$. We cross the point $\lambda_1 = 0$ (for which we have no result as yet), and suddenly $\mathrm{Var}(C_t)$ is asymptotically linear. Similar (but opposite) comments apply to relative dispersion by formula (5.5). Thus it is that the degree of compressibility in the flow exerts a huge effect on the behavior of a mass distribution in the flow.

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