Markov velocity fields and the generalized Lagrangian velocity

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Abstract. Consider a particle moving in a homogeneous velocity field that is Markov in time. The velocity field may be divergent or non–divergent, and the motion may be with or without molecular diffusion. The generalized Lagrangian velocity is the view of the current velocity field from the point of view of the particle at its current location. This process is shown to be Markov in time. A similar result is shown for Markov motion in a homogeneous Markov random environment under slightly different assumptions. The law of the generalized Lagrangian velocity is found in terms of its generator (for continuous time) or transition kernel (for discrete time).

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Contents

1 Introduction and main result

Consider the motion of a single particle in a random velocity field $U = \{U_t(x),\ x \in \mathbb{R}^d,\ t \geq 0\}$, possibly subject to molecular diffusion. Its trajectory $X_t,\ t \geq 0$ satisfies the stochastic differential equation

$$dX_t = U_t(X_t)dt + \sigma dW_t, \qquad t > 0, \tag{1.1}$$

where W is a Wiener process in $\mathbb{R}^e$ and σ is a constant $d \times e$ matrix. (When $\sigma = 0$, this is an ordinary differential equation.) In integral form,

$$X_t = X_0 + \int_0^t U_s(X_s)ds + \sigma W_t, \qquad t \geq 0. \tag{1.2}$$

These equations make clear the importance of the *Lagrangian velocity process* $U_t(X_t),\ t \geq 0$ for the determination of the law of X.

There is no known general method for deriving the law of the trajectory $X_t,\ t \geq 0$ or even the law of the Lagrangian velocity $U_t(X_t),\ t \geq 0$ from knowledge of the law of the *Eulerian velocity* $U_t(x),\ x \in \mathbb{R}^d,\ t \geq 0$. It has been known at least since Lumley (1957) that it is helpful to assume that U is homogeneous (i.e., the law of U is invariant under spatial translations), for then certain qualitative things may be said about the law of $U_t(X_t),\ t \geq 0$; see Zirbel (2000a).

There has been recent interest in velocity fields U which possess the Markov property in time, by which we mean that the process $U_t,\ t \geq 0$ taking values in the space of velocity fields on $\mathbb{R}^d$ has the Markov property:

$$\mathbb{E}[f(U_t) \mid U_r,\ r \leq s] = \mathbb{E}[f(U_t) \mid U_s], \qquad \text{all } f \geq 0, \text{ all } 0 \leq s \leq t. \tag{1.3}$$

(Velocity fields which evolve according to the Navier–Stokes equation are Markov and deterministic.) In this context, it is natural to consider the view of the field U_t from the location of the particle at time t, which is the *generalized Lagrangian velocity* V_t defined by

$$V_t(x) = U_t(X_t + x), \qquad x \in \mathbb{R}^d,\ t \geq 0. \tag{1.4}$$

It is natural to ask whether V inherits the Markov property from U.

One can imagine situations in which knowledge of V_t alone determines X_t and U_t; for instance, a particle moving in a lake, because beyond the lake shore the velocity is zero. In such a case it is clear that V will be Markov.

The more generic situation, however, is when knowledge of V_t does not allow one to determine U_t. Then homogeneity plays an important role in insuring that V is Markov. Intuitively, one may think of homogeneity as forcing U to evolve in the same way as any spatial translate of U would evolve. This suggests a natural condition (equation (4.1) below) on the transition kernel of U. However, it is not necessary to put any new condition on the transition kernel of U. All that is needed is a slightly stronger statement of the relationship between U and the particle trajectories it generates. In particular, we will assume that U generates a unique flow on $\mathbb{R}^d$, by which we mean that for each x in $\mathbb{R}^d$ and $s \geq 0$, there exists a unique solution $F_{s,t}(x), t \geq s$ of the equation

$$(1.5) \qquad dX_t = U_t(X_t)dt + \sigma dW_t, \qquad t > s; \qquad X_s = x,$$

and that

$$(1.6) \qquad F_{s,t} \circ F_{r,s} = F_{r,t} \qquad \text{almost surely, for each } 0 \leq r \leq s \leq t.$$

Finally, we will assume that U and F are continuous in x to help with some technicalities; see Condition (2.5) below. Note, however, that U may be divergent.

(1.7) **Theorem.** Suppose U is homogeneous, U generates a unique flow F, U and F are continuous in x, and $U_t,\ t \geq 0$ is Markov. Then $V_t,\ t \geq 0$ has the Markov property. □

The proof will be given in Section 2. As a motivating example, imagine a velocity field U which has a homogeneous initial state and evolves over $[0, T)$ according to the Navier–Stokes equation. (The terminal time T may be finite or infinite.) Then $U_t,\ 0 \leq t < T$ is Markov, and would remain so if U were forced in a memoryless way. The instantaneous evolution of U at time t could be determined just as well relative to X_t as relative to 0, so it is plausible that $V_t,\ 0 \leq t < T$ will be Markov.

The fact that V is Markov when U is homogeneous, Markov, and divergence free was recognized by Fannjiang and Komorowski (1999), but no details of the proof were given. Special cases of Theorem (1.7) are Çağlar (1997), Chapter 5, in which U is a velocity field of shot noise type based on a Poisson random measure and V was shown to be Markov assuming only homogeneity of this random measure; Carmona and Xu (1997), in which U is written as a superposition of non–divergent plane waves whose amplitudes are Ornstein–Uhlenbeck processes and the corresponding amplitudes of V are found explicitly to form another diffusion process; and Zirbel (2000b), in which Markov velocity fields in discrete space and time were considered and V was shown directly to be Markov. Finally, Komorowski and Papanicolaou (1997) identified a related embedded Markov chain with a Lagrangian character in connection with particle motion in a velocity field which decorrelates in finite time.

To date, the main use of the fact that V is Markov has been to show that the Lagrangian velocity $U_t(X_t) = V_t(0),\ t \geq 0$ mixes quickly in some sense, so that X converges to Brownian motion when properly rescaled. Fannjiang and Komorowski (1999) and Carmona and Xu (1997) do this in the non–divergent case by computing the generator of V and noting that the part corresponding to U is antisymmetric. In Section 3 we provide formal computations of this generator and its adjoint. In this way we are able to find the law of V. Note that, since $U_t(X_t) = V_t(0)$, we then know the law of the Lagrangian velocity process $U_t(X_t),\ t \geq 0$, and, in principle, the law of the trajectory $X_t,\ t \geq 0$.

There are some cases of Markov motion in a homogeneous Markov random environment which do not naturally generate a flow. We discuss this situation in Section 4 and provide an analogue of Theorem (1.7) under an assumption on the transition kernel of (U, X). This discussion makes clear the role that the flow property plays in simplifying the conditions of Theorem (1.7).

It can be helpful to look at motion in discrete time. Accordingly, the main result is stated and proved in Section 2 in a form that is broad enough to accommodate many such cases. In Section 5 we give a brief description of the discrete time case and compute the transition kernel of V in terms of the transition kernel of U.

Finally, we contrast the generalized Lagrangian velocity V with the canonical Markov process η used by Papanicolaou and Varadhan (1982), Osada (1982), and Landim *et al.* (1998), among others. Motion is determined by (1.1), but U is not required to be Markov. The process η is defined by

$$\eta_t = \{U_{t+r}(X_t + x),\ x \in \mathbb{R}^d,\ r \in \mathbb{R}\}, \qquad t \geq 0. \tag{1.8}$$

Then η_t consists of the past, present, and future of U as seen from the space–time point (X_t, t). Imagine fixing a particular realization of U; this is η_0. When $\sigma = 0$, η evolves deterministically, while when $\sigma \neq 0$, η evolves randomly with W supplying the randomness. In either case it is clear that η is a Markov process. However, the state space of η is much larger than that of V, and the fact that it is Markov bears little relation to the current paper. The canonical Markov process has mostly been used to exploit the white noise in the trajectory equation to bound the mixing rate of $U_t(X_t),\ t \geq 0$, rather than to make use of any special structure possessed by U.

2 General statement and proof of the main result

We may state and prove Theorem (1.7) in a more general setting than in Section 1. Let $\mathbb{D}$ be the spatial domain, which we assume to be an Abelian group with a translation–invariant σ-algebra $\mathcal{D}$ and a translation–invariant measure λ. (In the case $\mathbb{D} = \mathbb{R}^d$, one may use the Borel σ-algebra and the Lebesgue measure.) Let $\mathbb{U}$ be the set of all vector fields on $\mathbb{D}$ taking values in $\mathbb{D}$, and let $\mathcal{U}$ be the smallest σ-algebra on $\mathbb{U}$ for which the mapping $u \mapsto u(x)$ is measurable for all x in $\mathbb{D}$. For each z in $\mathbb{D}$ define a mapping $S_z : \mathbb{U} \to \mathbb{U}$ by

$$(S_z u)(x) = u(x+z), \qquad x \in \mathbb{D}. \tag{2.1}$$

Then S_z is measurable with respect to $\mathcal{U}$. Let $\mathbb{T}$ be the time set, either an interval of the form $[0,T)$ or $[0,\infty)$, or a discrete set such as $\{0,1,2,\ldots,T\}$ or $\{0,1,2,\ldots\}$.

Let $U_t,\ t \in \mathbb{T}$ be a random process taking values in $\mathbb{U}$, and let $W_t,\ t \in \mathbb{T}$ be a process with independent increments that is independent of U. We assume that these are defined on a common probability space $(\Omega, \mathcal{H}, \mathbb{P})$. We now list the conditions that will be needed below.

(2.2) **Condition.** *Unique flow.* The processes U and W generate a unique flow $F_{s,t},\ s,t \in \mathbb{T}, s \le t$ on $\mathbb{D}$ in such a way that for each s and t in $\mathbb{T}$ with $s \le t$, $F_{s,t}$ is determined by the collection $\{U_r, W_r - W_s, s \le r \le t\}$ and for $r \le s \le t$ we have $F_{s,t} \circ F_{r,s} = F_{r,t}$.

Of course, if W is not needed, it may be omitted. We will write F_t in place of $F_{0,t}$.

(2.3) **Condition.** *Homogeneity.* The collection $\{F_t(z) - z, S_z U_t,\ \ t \in \mathbb{T}\}$ has the same distribution for all z in $\mathbb{D}$.

(2.4) **Condition.** *Markov property.* For all measurable $f : \mathbb{U} \to \mathbb{R}_+$ and all s and t in $\mathbb{T}$ with $s \le t$, we have $\mathbb{E}[f(U_t) \mid U_r,\ r \le s] = \mathbb{E}[f(U_t) \mid U_s]$.

(2.5) **Condition.** *Measurability.* The mapping $\omega \mapsto U_t(\omega, x)$ from Ω to $\mathbb{D}$ is measurable for each fixed x in $\mathbb{D}$ and t in $\mathbb{T}$, the mapping $(\omega, z) \mapsto F_t(\omega, z)$ from $\Omega \times \mathbb{D}$ to $\mathbb{D}$ is measurable for each fixed t in $\mathbb{T}$, and the mapping $(u,z) \mapsto S_z u$ from $\mathbb{U} \times \mathbb{D}$ to $\mathbb{U}$ is measurable.

In the particular case of Section 1, Condition (2.2) may be satisfied by making continuity or Lipschitz assumptions on U, even when $\sigma \ne 0$. See Port and Stone (1976) and Fannjiang and Komorowski (1999). Given Condition (2.2) and the assumption that U is homogeneous (i.e., the law of $S_z U_t,\ t \ge 0$ is the same for all z in $\mathbb{R}^d$), Section 6 of Zirbel (2000a) shows that Condition (2.3) is satisfied. In this way Theorem (1.7) will follow from Theorem (2.6) below.

The measurability conditions are more technical than the others, but they are important. For example, they insure that the mapping $\omega \mapsto V_t(\omega)$ is measurable, for this is the composition of $\omega \mapsto (U_t(\omega), F_t(\omega, 0))$ and $(u,z) \mapsto S_z u$, both of which are measurable by Condition (2.5). See Section 3 of Zirbel (2000a) for sufficient conditions on U and F, but note that it is enough for U and F to have realizations that are continuous in x.

(2.6) **Theorem.** Suppose Conditions (2.2) – (2.5) hold. For each t in $\mathbb{T}$, define $V_t = S_{F_t(0)}U_t$. Then $V_t,\ t \in \mathbb{T}$ is Markov.

Proof: Let $(\Omega, \mathcal{H}, \mathbb{P})$ be the probability space for U and W. Fix s and t in $\mathbb{T}$ with $s \le t$. Let $f : \mathbb{U} \to \mathbb{R}_+$ be measurable. We must show that

$$\mathbb{E}[f(V_t) \mid V_r,\ r \le s] = \mathbb{E}[f(V_t) \mid V_s]. \tag{2.7}$$

We will establish the slightly stronger statement

$$\mathbb{E}[f(S_{F_t(0)}U_t) \mid F_r(0), U_r,\ r \le s] = g(S_{F_s(0)}U_s), \tag{2.8}$$

where $g : \mathbb{U} \to \mathbb{R}_+$ is defined in (2.22) below. We do this by showing that

$$\mathbb{E}\Psi f(S_{F_t(0)}U_t) = \mathbb{E}\Psi g(S_{F_s(0)}U_s), \tag{2.9}$$

for all $\Psi : \Omega \to \mathbb{R}_+$ which are $\sigma(F_r(0), U_r,\ r \le s)$–measurable.

We need to consider starting locations other than 0 in order to make use of homogeneity. We would like to write

$$\mathbb{E}[f(S_{F_t(z)}U_t) \mid F_r(z), U_r,\ r \le s] = h(F_s(z), U_s), \tag{2.10}$$

using the same function h for all z. It is difficult to establish the existence of such a function directly, although it is easy to do so for each individual z in $\mathbb{D}$. Our method is to allow z to be random in a certain way, which we introduce next. Eventually we will arrive at (2.20), the analogue of (2.10), and then the remainder of the proof makes use of familiar homogeneity arguments.

Let $\alpha : \mathbb{D} \to \mathbb{R}_+$ be measurable and satisfy $\int_{\mathbb{D}} \lambda(dz)\alpha(z) = 1$. Define a measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{H}) \times (\mathbb{D}, \mathcal{D})$ by

$$\widetilde{\mathbb{P}}(A \times B) = \mathbb{E}1_A \int_{\mathbb{D}} \lambda(dz)\alpha(F_s(z))1_B(z), \qquad A \in \mathcal{H}, B \in \mathcal{D}. \tag{2.11}$$

Here we are using the assumption that $(\omega, z) \mapsto F_s(\omega, z)$ is measurable. Events A in $\mathcal{H}$ with $\mathbb{P}(A) = 1$ also have $\widetilde{\mathbb{P}}(A \times \mathbb{D}) = 1$, as the following computation shows

$$\begin{aligned} \widetilde{\mathbb{P}}(A \times \mathbb{D}) &= \mathbb{E}1_A \int_{\mathbb{D}} \lambda(dz)\alpha(F_s(z)) \\ &= \int_{\mathbb{D}} \lambda(dz)\mathbb{E}\alpha(F_s(z)) \\ &= \int_{\mathbb{D}} \lambda(dz)\mathbb{E}\alpha(z + F_s(0)) \\ &= \mathbb{E}\int_{\mathbb{D}} \lambda(dz)\alpha(z + F_s(0)) \\ &= 1, \end{aligned} \tag{2.12}$$

the third step by Condition (2.3) and the last by translation invariance of λ. Thus, $\widetilde{\mathbb{P}}$ is a probability measure on $\Omega \times \mathbb{D}$. We will write (ω, Z) for the canonical random variables on $\Omega \times \mathbb{D}$ under $\widetilde{\mathbb{P}}$, although ω will be suppressed in the notation as usual.

(2.13) **Remark.** The values of ω and Z are dependent. If $\alpha = 1_C$ for a set C in $\mathcal{D}$ with $\lambda(C) = 1$, then $F_s(Z)$ is uniformly distributed on C, for if D is in $\mathcal{D}$, then

$$\begin{aligned} \widetilde{\mathbb{P}}(F_s(Z) \in D) &= \mathbb{E}\int_{\mathbb{D}} \lambda(dz) 1_C(F_s(z)) 1_{\{F_s(z)\in D\}} \\ &= \mathbb{E}\int_{\mathbb{D}} \lambda(dz) 1_{C\cap D}(F_s(z)) \\ &= \mathbb{E}\int_{\mathbb{D}} \lambda(dz) 1_{C\cap D}(z + F_s(0)), \\ &= \lambda(C \cap D), \end{aligned} \tag{2.14}$$

using Condition (2.3) as above. In the compressible case, it is possible that $F_s(\mathbb{D}) \cap C$ is empty, but this event has probability 0 under $\widetilde{\mathbb{P}}$ since in that case $\int_{\mathbb{D}} \lambda(dz) 1_C(F_s(z)) = 0$, so this possibility does not interfere with (2.14).

When F is incompressible (F_t preserves the measure λ on $\mathbb{D}$ for each t), the marginal of $\widetilde{\mathbb{P}}$ on Ω is equal to $\mathbb{P}$, for if A is in $\mathcal{H}$, then,

$$\begin{aligned} \widetilde{\mathbb{P}}(A \times \mathbb{D}) &= \mathbb{E}1_A \int_{\mathbb{D}} \lambda(dz) \alpha(F_s(z)) \\ &= \mathbb{E}1_A \int_{\mathbb{D}} \lambda(dy) \alpha(y) \\ &= \mathbb{E}1_A, \end{aligned} \tag{2.15}$$

by the change of variable $y = F_s(z)$. Furthermore, if $\alpha = 1_C$ then it is reasonable to say that Z is uniformly distributed on $F_s^{-1}(C)$, since $F_s(Z)$ is uniform on C. □

We claim that, under $\widetilde{\mathbb{P}}$, the σ-algebra $\mathcal{F} = \sigma(Z, U_r, W_r,\ r \le s)$ is conditionally independent of $\mathcal{G} = \sigma(U_r, W_r - W_s,\ r \ge s)$ given U_s. Let $\Gamma : \Omega \times \mathbb{D} \to \mathbb{R}_+$ be $\mathcal{G}$–measurable. By Condition (2.4) and the independence of increments of W, there exists a measurable function $k : \mathbb{U} \to \mathbb{R}_+$ for which

$$\mathbb{E}[\Gamma \mid U_r, W_r,\ r \le s] = k(U_s). \tag{2.16}$$

Note that, by Condition (2.2), for each z in $\mathbb{D}$, $F_s(z)$ is $\sigma(U_r, W_r,\ r \le s)$–measurable. Let $\Phi : \Omega \times \mathbb{D} \to \mathbb{R}_+$ be $\mathcal{F}$–measurable. We may write $\Phi = \varphi(Z, U_r, W_r,\ r \le s)$ for some measurable function φ. Then by (2.16),

$$\mathbb{E}\Gamma\alpha(F_s(z))\varphi(z, U_r, W_r,\ r \le s) = \mathbb{E}k(U_s)\alpha(F_s(z))\varphi(z, U_r, W_r,\ r \le s) \tag{2.17}$$

for each z in $\mathbb{D}$. Integrating over z with respect to λ yields

$$\widetilde{\mathbb{E}}\Gamma\Phi = \widetilde{\mathbb{E}}k(U_s)\Phi, \tag{2.18}$$

where $\widetilde{\mathbb{E}}$ is the expectation associated with $\widetilde{\mathbb{P}}$. This establishes the claim that $\mathcal{F}$ and $\mathcal{G}$ are conditionally independent given U_s.

Recognizing that $F_s(Z)$ is $\mathcal{F}$–measurable, we may write

$$\widetilde{\mathbb{E}}\ell(F_s(Z))\Gamma\Phi = \widetilde{\mathbb{E}}\ell(F_s(Z))k(U_s)\Phi, \tag{2.19}$$

for all measurable $\ell : \mathbb{D} \to \mathbb{R}_+$. We interpret this to mean that, under $\widetilde{\mathbb{E}}$, $\mathcal{F}$ is conditionally independent of $\mathcal{G}_0 \equiv \sigma(F_s(Z), U_r, W_r - W_s,\ r \geq s)$ given $(F_s(Z), U_s)$.

At this point it is clear that

$$\widetilde{\mathbb{E}}[f(S_{F_t(Z)}U_t) \mid \mathcal{F}] = h(F_s(Z), U_s), \tag{2.20}$$

for some jointly measurable function $h : \mathbb{D} \times \mathbb{U} \to \mathbb{R}_+$. This is because $F_t(Z) = F_{s,t}(F_s(Z))$ and $F_{s,t}$ depends only on $\{U_r, W_r - W_s, s \leq r \leq t\}$ by Condition (2.2), and so $S_{F_t(Z)}$ is $\mathcal{G}_0$–measurable. Equation (2.20) is the analogue of (2.10) that we have been driving toward. The remainder of the proof uses h to define g.

We return to the left–hand side of (2.9). Let $\Psi : \Omega \to \mathbb{R}_+$ be $\sigma(F_r(0), U_r,\ r \leq s)$–measurable and write $\Psi = \psi(F_r(0), U_r,\ r \leq s)$ for some function ψ. We begin with a device to exploit homogeneity that was used in the proof of Theorem 4.4 in Zirbel (2000a):

$$\begin{aligned} \mathbb{E}\Psi f(S_{F_t(0)}U_t) &= \mathbb{E}\psi(F_r(0), U_r,\ r \leq s) f(S_{F_t(0)}U_t) \\ &= \mathbb{E}\int_{\mathbb{D}} \lambda(dz)\alpha(z + F_s(0))\psi(F_r(0), U_r,\ r \leq s) f(S_{F_t(0)}U_t) \\ &= \int_{\mathbb{D}} \lambda(dz)\mathbb{E}\alpha(F_s(z))\psi(F_r(z) - z, S_zU_r,\ r \leq s) f(S_{F_t(z)}U_t) \\ &= \widetilde{\mathbb{E}}\psi(F_r(Z) - Z, S_ZU_r,\ r \leq s) f(S_{F_t(Z)}U_t), \end{aligned} \tag{2.21}$$

the third step by Condition (2.3). Condition (2.5) is used throughout to justify joint measurability of the integrands.

Condition inside the expectation on $\mathcal{F}$ and note that $\psi(F_r(Z) - Z, S_ZU_r,\ r \leq s)$ is $\mathcal{F}$–measurable. Using (2.20) we obtain

$$\begin{aligned} \mathbb{E}\Psi f(S_{F_t(0)}U_t) &= \widetilde{\mathbb{E}}\psi(F_r(Z) - Z, S_ZU_r,\ r \leq s) h(F_s(Z), U_s) \\ &= \mathbb{E}\int_{\mathbb{D}} \lambda(dz)\alpha(F_s(z))\psi(F_r(z) - z, S_zU_r,\ r \leq s) h(F_s(z), U_s) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{D}} \lambda(dz)\mathbb{E}\alpha(z+F_s(0))\psi(F_r(0),U_r,\ r\le s)h(z+F_s(0),S_{-z}(U_s)) \\
&= \mathbb{E}\psi(F_r(0),U_r,\ r\le s)\int_{\mathbb{D}} \lambda(dy)\alpha(y)h(y,S_{-y}S_{F_s(0)}U_s) \\
&= \mathbb{E}\Psi g(S_{F_s(0)}U_s),
\end{aligned}
$$

where $g:\mathbb{U}\to\mathbb{R}_+$ is defined by

$$
g(u)=\int_{\mathbb{D}} \lambda(dy)\alpha(y)h(y,S_{-y}u), \tag{2.22}
$$

independent of the choice of Ψ. We have used Condition (2.3) at the third step, and the change of variable $y=z+F_s(0)$ at the fourth. Note that the mapping $(u,y)\mapsto S_{-y}u$ is measurable by Condition (2.5). This establishes (2.9). □

3 Generator of V

Return to the setting of Section 1. Given the law of U, it is possible to find the law of V in the sense that we may write the generator $\mathcal{M}$ of V in terms of the generator $\mathcal{L}$ of U. This has been done by Fannjiang and Komorowski (1999) but with no derivation, and has been done in particular cases by Çağlar (1997) and Carmona and Xu (1997). Here we give a formal computation and indicate where details need to be checked once a model is chosen for U.

Write $\mathbb{P}^u$ for the probability measure on $(\Omega,\mathcal{H})$ under which $U_0=u$ almost surely and $\mathbb{E}^u$ for the corresponding expectation. Let $f:\mathbb{U}\to\mathbb{R}$ be in the domain of $\mathcal{L}$. Define a function $g:\mathbb{U}\times\mathbb{R}^d\to\mathbb{R}$ by $g(u,z)=f(S_zu)$ and assume that g and its first three derivatives in x are continuous for each fixed u. This allows us to avoid making specific continuity assumptions about U.

We compute $\mathcal{M}f$ as follows. Let u be in $\mathbb{U}$. Then,

$$
\begin{aligned}
\mathcal{M}f(u) &= \lim_{t\downarrow 0} t^{-1}(\mathbb{E}^u f(V_t)-f(u)) \\
&= \lim_{t\downarrow 0} t^{-1}(\mathbb{E}^u g(U_t,X_t)-f(u)),
\end{aligned} \tag{3.1}
$$

where we have written X_t for $F_t(0)$. By Taylor's formula, for each x in $\mathbb{R}^d$,

$$
\begin{aligned}
g(u,x)=g(u,0) &+ \sum_{i=1}^d \partial_i g(u,0)x^i+\tfrac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j g(u,0)x^ix^j \\
&+ \tfrac{1}{6}\sum_{i,j,k=1}^d \partial_i\partial_j\partial_k g(u,\alpha x)x^ix^jx^k,
\end{aligned} \tag{3.2}
$$

where $0 \le \alpha \le 1$. Substituting this into (3.1), the first and last terms give

$$\lim_{t \downarrow 0} t^{-1}(\mathbb{E}^u f(U_t) - f(u)) = \mathcal{L}f(u). \tag{3.3}$$

Noting that $X_t = \int_0^t U_s(X_s)ds + \sigma W_t$, the term with first–order derivatives is simplified by observing that $\mathbb{E}^u[W_t \mid U_t] = 0$. What remains is

$$\lim_{t \downarrow 0} \mathbb{E}^u \sum_{i=1}^d \partial_i g(U_t, 0) t^{-1} \int_0^t U_s^i(X_s)ds, \tag{3.4}$$

which should converge to $\sum_{i=1}^d \partial_i g(u,0)u^i(0)$ by dominated convergence and some appropriate right continuity of U. The second–order term is simplified by observing that $\mathbb{E}^u[(\sigma W_t)^i(\sigma W_t)^j \mid U_t] = (\sigma\sigma^T)_{ij}$. What remains should converge to

$$\sum_{i,j=1}^d \partial_i \partial_j g(u,0)(\sigma\sigma^T)_{ij}, \tag{3.5}$$

by dominated convergence and right continuity. The third–order term is similar, with $t^{-1}W_t^i W_t^j W_t^k$ converging to 0 by the law of the iterated logarithm. This term should then converge to 0 by dominated convergence.

In the end, we will arrive at

$$\mathcal{M}f(u) = \mathcal{L}f(u) + u(0) \cdot \nabla_x f(S_x u) \Big|_{x=0} + \tfrac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^T)_{ij} \partial_i \partial_j f(S_x u) \Big|_{x=0}, \tag{3.6}$$

which we abbreviate as $\mathcal{M}f = \mathcal{L}f + \mathcal{M}_1 f + \mathcal{M}_2 f$. The change of perspective brought about by the movement of the particle according to $u(0)$ and diffusion are apparent here.

For the computation of the adjoint of $\mathcal{M}$, suppose that U has an invariant distribution π on $\mathbb{U}$ and that U started with this distribution satisfies Conditions (2.2) and (2.3). We will formally compute the adjoint of $\mathcal{M}$ with respect to π.

Define an inner product on $L^2(\mathbb{U}, \pi)$ by

$$\langle f, g \rangle = \int_{\mathbb{U}} \pi(du) f(u) \overline{g(u)}, \qquad f, g \in L^2(\mathbb{U}, \pi). \tag{3.7}$$

Then for g in the domain of $\mathcal{L}$ and f in the domain of the adjoint $\mathcal{L}^*$ of $\mathcal{L}$,

$$\langle f, \mathcal{L}g \rangle = \langle \mathcal{L}^* f, g \rangle. \tag{3.8}$$

The next term in $\mathcal{M}$ is

$$\langle f, \mathcal{M}_1 g \rangle = \int_{\mathbb{U}} \pi(du) f(u) \sum_{i=1}^d u^i(0) \partial_i g(S_x u) \Big|_{x=0}. \tag{3.9}$$

We may "integrate by parts" by the following lemma:

(3.10) **Lemma.** Let $a, b : \mathbb{U} \to \mathbb{R}$ be bounded and such that $x \mapsto a(S_x u)$ and $x \mapsto b(S_x u)$ are differentiable with respect to x. Then

$$\int_{\mathbb{U}} \pi(du) a(u) \partial_i b(S_x u) \Big|_{x=0} = - \int_{\mathbb{U}} \pi(du) \partial_i a(S_x u) \Big|_{x=0} b(u). \tag{3.11}$$

Proof: Let $\alpha : \mathbb{R}^d \to \mathbb{R}_+$ be infinitely differentiable, have compact support, and satisfy $\int_{\mathbb{R}^d} \alpha(x) dx = 1$. Then,

$$\begin{aligned}
\int_{\mathbb{U}} \pi(du) a(u) \partial_i b(S_x u) \Big|_{x=0} &= \int_{\mathbb{R}^d} dz \alpha(z) \int_{\mathbb{U}} \pi(du) a(u) \partial_i b(S_x u) \Big|_{x=0} \\
&= \int_{\mathbb{R}^d} dz \alpha(z) \int_{\mathbb{U}} \pi(du') a(S_z u') \partial_i b(S_x S_z u') \Big|_{x=0} \\
&= \int_{\mathbb{U}} \pi(du') \int_{\mathbb{R}^d} dz \alpha(z) a(S_z u') \partial_i b(S_x u') \Big|_{x=z} \\
&= - \int_{\mathbb{U}} \pi(du') \int_{\mathbb{R}^d} dz \partial_i (\alpha(y) a(S_y u')) \Big|_{y=z} b(S_z u') \\
&= - \int_{\mathbb{R}^d} dz \int_{\mathbb{U}} \pi(du') \Big(\partial_i \alpha(z) a(S_z u') b(S_z u') \\
&\qquad + \alpha(z) \partial_i a(S_y u') \Big|_{y=z} b(S_z u') \Big) \\
&= - \int_{\mathbb{R}^d} dz \int_{\mathbb{U}} \pi(du) \Big(\partial_i \alpha(z) a(u) b(u) + \alpha(z) \partial_i a(S_y u) \Big|_{y=0} b(u) \Big) \\
&= - \int_{\mathbb{U}} \pi(du) \partial_i a(S_y u) \Big|_{y=0} b(u),
\end{aligned}$$

where we have used the change of variable $u = S_z u'$ and invariance of π under S_z. □

Using this result,

$$\begin{aligned}
\langle f, \mathcal{M}_1 g \rangle &= - \int_{\mathbb{U}} \pi(du) \sum_{i=1}^{d} \partial_i (f(S_x u) u^i(x)) \Big|_{x=0} g(u) \\
&= - \langle \mathcal{M}_1 f, g \rangle - \langle f (\text{div } u)(0), g \rangle.
\end{aligned} \tag{3.12}$$

Similarly,

$$\begin{aligned}
\langle f, \mathcal{M}_2 g \rangle &= \tfrac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T)_{ij} \int_{\mathbb{U}} \pi(du) \sum_{i=1}^{d} f(u) \partial_i \partial_j g(S_x u) \Big|_{x=0} \\
&= \tfrac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T)_{ij} \int_{\mathbb{U}} \pi(du) \sum_{i=1}^{d} \partial_i \partial_j f(S_x u) \Big|_{x=0} g(u) \\
&= \langle \mathcal{M}_2 f, g \rangle.
\end{aligned} \tag{3.13}$$

Thus, the adjoint of $\mathcal{M}$ is

$$\mathcal{M}^* f(u) = \mathcal{L}^* f(u) - \mathcal{M}_1 f(u) - (\text{div } u)(0) f(u) + \mathcal{M}_2 f(u). \tag{3.14}$$

Supposing that U is non–divergent,

$$\mathcal{M}^* = \mathcal{L}^* - \mathcal{M}_1 + \mathcal{M}_2, \tag{3.15}$$

so that the symmetrization $\mathcal{M}^s = \frac{1}{2}(\mathcal{M} + \mathcal{M}^*)$ is

$$\mathcal{M}^s = \mathcal{L}^s + \mathcal{M}_2, \tag{3.16}$$

which generalizes a similar result in Carmona and Xu (1997).

4 Transition kernel of U

Return to the general setting of Section 2. Given that U is homogeneous and Markov in time, it is clear that the initial distribution ν of U on $\mathbb{U}$ is invariant under S_z for all z in $\mathbb{D}$. It is natural to imagine that U has a transition kernel P satisfying

$$P_t(u, A) = P_t(S_z u, S_z A), \tag{4.1}$$

for all u in $\mathbb{U}$, A in $\mathcal{U}$, and z in $\mathbb{D}$. However, we have been unable to show the existence of such a transition kernel under general assumptions on U. Under mild conditions, U *has* a transition kernel (cf. Kuznetsov (1980)), but it is indeed difficult to get the detailed structure (4.1) from this general existence result. The main difficulty is that the transition kernel $P_t(u, \cdot)$ can be modified on a set of ν measure 0 without affecting the law of U. Following the idea of the proof of Theorem (2.6), one would naturally define

$$\hat{P}_t(u, A) = \int_{\mathbb{D}} \lambda(dz)\alpha(z) P_t(S_z u, S_z A), \tag{4.2}$$

in order to smooth out such extraneous fluctuations, and then show that $\hat{P}$ satisfies (4.1) and serves as the transition kernel of U. But it is not clear how to carry this through.

However, when $\mathbb{U}$ is discrete, we may write, for all z,

$$\begin{aligned} \mathbb{E} f(U_0) g(U_t) &= \mathbb{E} f(S_{-z} U_0) g(S_{-z} U_t) \\ &= \sum_{u \in \mathbb{U}} \nu(u) f(S_{-z} u) \sum_{v \in \mathbb{U}} P_t(u, v) g(S_{-z} v) \\ &= \sum_{u' \in \mathbb{U}} \nu(S_z u') f(u') \sum_{v' \in \mathbb{U}} P_t(S_z u', S_z v') g(v') \\ &= \sum_{u' \in \mathbb{U}} \nu(u') f(u') \sum_{v' \in \mathbb{U}} P_t(S_z u', S_z v') g(v'), \end{aligned} \tag{4.3}$$

by the invariance of ν under S_z. For this to be equal for all z in $\mathbb{D}$ and all f and g requires that $P_t(S_z u, S_z v) = P_t(u, v)$ for all u with $\nu(u) > 0$.

It is the assumption that U generates a unique flow F that allows us to prove Theorem (2.6) without a detailed homogeneity property such as (4.1). However, in certain instances it is unnatural to require the flow property. For example, imagine this mechanism for Markov motion in a homogeneous random environment. Let $U_t(x),\ x \in \mathbb{Z}^d,\ t \geq 0$ be a positive–valued random field which is homogeneous and Markov in time. Interpret $U_t(x)$ as the rate at which a particle at x jumps away from x. Given U, let $X_0 = 0$ and let $X_t,\ t \geq 0$ be a Markov process in $\mathbb{Z}^d$ which jumps according to the rates U and moves to each of its nearest neighbors with equal probability.

It is not natural to be forced to construct a flow on $\mathbb{Z}^d$ which would describe the simultaneous motion of particles at all sites in $\mathbb{Z}^d$. However, it is not enough to observe that U is homogeneous and Markov and X is Markov given U, for we could modify the model above so that X_t jumps with rate $U_t(X_t)$, but always *toward a specified point* y. Then $S_{X_t}U_t$ would not be Markov, for the past values of $S_{X_t}U_t$ could indicate how close X_t is to y, and so give information about the future evolution of $S_{X_t}U_t$. The trouble is that the transition mechanism of X given U introduces inhomogeneity.

One notes that the paired process $(U_t, X_t),\ t \geq 0$ is Markov in this case. By placing a condition on the transition kernel of (U, X), we may prove an analogue of Theorem (2.6).

(4.4) **Theorem.** Suppose that $(U_t, X_t),\ t \in \mathbb{T}$ is Markov, that $X_0 = 0$, that the distribution of $S_z U_0$ is the same for all z in $\mathbb{D}$, and that (U, X) has a transition kernel K satisfying

$$K_{s,t}(u, x; A, B) = K_{s,t}(S_z u, x - z; S_z A, -z + B) \tag{4.5}$$

for all x and z in $\mathbb{D}$, all u in $\mathbb{U}$, A in $\mathcal{U}$, and B in $\mathcal{D}$. Define $V_t = S_{X_t}U_t$. Then V is Markov.

Proof: Fix s and t in $\mathbb{T}$ with $s \leq t$. Let $f : \mathbb{U} \to \mathbb{R}_+$ be measurable. As in the proof of Theorem (2.6), we wish to establish the slightly stronger statement

$$\mathbb{E}[f(S_{X_t}U_t) \mid U_r, X_r,\ r \leq s] = g(S_{X_s}U_s). \tag{4.6}$$

Now

$$\mathbb{E}[f(S_{X_t}U_t) \mid U_r, X_r,\ r \leq s] = h(U_s, X_s), \tag{4.7}$$

by the Markov property of (U, X), where h satisfies

$$h(u, x) = \int_{\mathbb{U}\times\mathbb{D}} K_{s,t}(u, x; dv, dy) f(S_y v). \tag{4.8}$$

But by (4.5),

$$h(u, x) = \int_{\mathbb{U}\times\mathbb{D}} K_{s,t}(S_x u, 0; dv', dy') f(S_{y'} v'), \tag{4.9}$$

so that $h(u, x) = g(S_x u)$ for some measurable $g : \mathbb{U} \to \mathbb{R}_+$. This completes the proof. □

5 Discrete time

It is interesting to consider the discrete time analogue of the motion discussed in Section 1, partly because it can shed new insight on the theory, and partly because numerical simulations of fluids are done in discrete time.

Let $\mathbb{T} = \{0, 1, 2, \ldots\}$. Let $U_t(x),\ x \in \mathbb{D},\ t \in \mathbb{T}$, be a random field taking values in $\mathbb{D}$ and let $\Delta W_t,\ t \in \mathbb{T}$ be independent, identically distributed random variables taking values in $\mathbb{D}$ and independent of U. Define $F_{t,t}(x) = x$ and

$$F_{t,t+1}(x) = x + U_t(x) + \Delta W_t, \qquad x \in \mathbb{D},\ t \in \mathbb{T}, \tag{5.1}$$

and define $F_{s,t}$ for arbitrary s and t in $\mathbb{T}$ with $s \le t$ by composition. Then Condition (2.2) is met. Moreover, if U is homogeneous, then Condition (2.3) is met, cf. Remark (6.14) of Zirbel (2000a).

Although (5.1) resembles the Euler method for the motion of particles in a continuous–time velocity field U with molecular diffusion, it is not that simple, for if U_t is an arbitrary vector field on $\mathbb{R}^d$, the mapping $F_{t,t+1}$ will tend to be non–invertible. That is a defect of the Euler method, one might say. Instead, it is better to choose U_t to be the displacement resulting from integrating through some other velocity field between times t and $t+1$.

Assuming that U is homogeneous and Markov, the generalized Lagrangian velocity V defined by $V_t = S_{F_t(0)} U_t,\ t \ge 0$ will be Markov by Theorem (2.6). In order to compute its law, let P be the transition kernel of U and write Q for the transition kernel of V. Supposing that $U_0 = u$, we have $F_t(0) = u(0) + \Delta W_0$ and $V_1 = S_{\Delta W_0} S_{u(0)} U_1$. Fix A in $\mathcal{U}$. Then,

$$\begin{aligned} Q(u, A) &= \mathbb{P}^u(S_{\Delta W_0} S_{u(0)} U_1 \in A) \\ &= \mathbb{P}^u(U_1 \in S_{-\Delta W_0} S_{u(0)} A) \\ &= \mathbb{E} P(u, S_{-\Delta W_0} S_{u(0)} A) \\ &= \int_{\mathbb{D}} \mu(dw) P(u, S_{-w} S_{u(0)} A), \end{aligned} \tag{5.2}$$

where μ is the distribution of ΔW_0.

However, there is a better way to proceed if P satisfies (4.1), for then we continue with

$$Q(u, A) = \int_{\mathbb{D}} \mu(dw) P(S_w S_{u(0)} u, A). \tag{5.3}$$

We may write

$$Q = \Sigma D P, \tag{5.4}$$

where Σ is the transition kernel under which u makes a deterministic transition to $S_{u(0)}u$ and D is the transition kernel

$$D(u, A) = \int_{\mathbb{D}} \mu(dw) 1_A(S_w u), \tag{5.5}$$

which corresponds to the transition $u \mapsto S_{\Delta W_0} u$. See Zirbel (2000b) for concrete examples and interpretations of Σ and D.

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