Lagrangian observations of homogeneous random environments

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Abstract. This article deals with the distribution of the view of a random environment as seen by an observer whose location at each moment is determined by the environment. The main application is in statistical fluid mechanics, where the environment consists of a random velocity field and the observer is a particle moving in the velocity field, possibly subject to molecular diffusion. Several results on such Lagrangian observations of the environment have appeared in the literature, beginning with the 1957 dissertation of J. L. Lumley. This article unites these results into a simple unified framework and rounds out the theory with new results in several directions.

When the environment is homogeneous, one may re–cast the problem in terms of certain random mappings on the physical space that are based on the random location of the observer. If these mappings preserve the invariant measure on the physical space, then the view from the random location has the same distribution as the view from the origin. If these mappings satisfy the flow property and the environment is stationary, then the succession of Lagrangian observations over time forms a strictly stationary process. In particular, for motion in a homogeneous, stationary, and non–divergent velocity field, the Lagrangian velocity (the velocity of the particle) is strictly stationary, which was first observed by Lumley. In the compressible case, the distribution of a Lagrangian observation has a density with respect to the distribution of the view from the origin, and in some cases one can show convergence in distribution of the Lagrangian observations as time tends to infinity.

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1 Introduction

Consider the view of a random environment as seen by an observer whose location is determined by the environment. Given the probability law of the environment and the mechanism by which the observer's location is determined, it is very difficult in general to determine the law of the environment as seen by the observer. Geman and Horowitz (1975) observed that if the environment is homogeneous (i.e., its law is invariant under spatial translations), the problem may be re–cast in terms of a random mapping on the space and some qualitative results may be obtained. Among these, they showed that if the mapping preserves the invariant measure on the space, then the view from the random location has the same distribution as the view from, say, the origin.

One goal of this article is to present a simpler proof of this result and to extend it in a natural way to cover a succession of observations of the environment as the observer moves in a way determined by the environment. Under the right circumstances, these observations form a strictly stationary process in time. This is, of course, at a high level of generality.

The other goal of the article is to make clear how these results relate to statistical fluid mechanics, which has its own history of results along these lines. Let $U(x,t),\ x \in \mathbb{R}^d,\ t \geq 0$ be a random vector field and consider the position of a particle moving in U over time according to:

$$\frac{dX_t}{dt} = U(X_t, t), \qquad t > 0. \tag{1.1}$$

A long–standing problem is to use the law of U to derive the law of X, or at least certain aspects of that law such as the asymptotic growth rate of the covariance matrix of X_t as $t \to \infty$. A natural place to start is to try to determine the law of the *Lagrangian velocity process* $U(X_t,t),\ t \geq 0$, which consists of observations of U at random locations which are determined by U itself. If this can be done, then the law of X is essentially known, since $X_t = X_0 + \int_0^t U(X_s,s)ds$.

Lumley (1957, 1962) first argued that if U is homogeneous and non–divergent (div $U \equiv 0$), then for each $t \geq 0$, $U(X_t,t)$ has the same distribution as $U(0,t)$. Port and Stone (1976) supplied a rigorous proof of this fact and extended it in two ways. First, the view of all of U relative to the space–time point (X_t,t) has the same distribution as the view of U from $(0,t)$. Second, the motion may be determined by U plus molecular diffusion with diffusion coefficient β, according to the integral equation

$$X_t = X_0 + \int_0^t U(X_s, s)ds + \beta W_t, \qquad t \geq 0. \tag{1.2}$$

Here W is a Wiener process independent of U.

Lumley (1962) argued that when U is also stationary (meaning that its law is invariant under temporal translation), the process $U(X_t, t),\ t \geq 0$ is stationary in the wide sense. Osada (1982) proved that when $\beta \neq 0$, the view of all of U from (X_t, t) forms a strictly stationary process in time, by considering the view of U from $(X_t, t),\ t \geq 0$ to be the state at time t of a canonical Markov process. Zirbel (1993) and Komorowski and Papanicolaou (1997) showed that even when $\beta = 0$, the view of U from $(X_t, t),\ t \geq 0$ forms a strictly stationary process as a consequence of Port and Stone's result.

Qualitative results of this type are an important first step toward determining the law of $U(X_t, t),\ t \geq 0$, since they considerably narrow the range of possibilities. But they do not solve the problem: stationarity and thus the one–point marginals of $U(X_t, t),\ t \geq 0$ are known, but we have no further general results on the law of this process, which may differ substantially from that of, say, $U(0, t),\ t \geq 0$, depending on the space–time structure of U. Even less is known when U is divergent, for then the view of U from $(X_t, t),\ t \geq 0$ is not stationary, but becomes progressively more biased toward the values of U prevailing in regions of convergence. Davis (1982) first gave a quantitative analysis of this phenomenon and conjectured that Lagrangian observations would converge in distribution as $t \to \infty$.

In the mean time, the purpose of this article is to collect together and synthesize the various works quoted above into a single comprehensive framework that can form the basis for further work into the specifics of the distribution of Lagrangian observations. The emphasis throughout is on completeness, simplicity to the extent possible, and applicability of the results. This is a synthesis, not a review. There are several new results and new proofs of known results.

We emphasize that this article concerns the view from particles undergoing motion that is *determined* by the environment, rather than, for example, Markov motion in a random environment.

Motion subject to molecular diffusion can be considered to be motion determined by the environment by including the Wiener process in the environment as is explained in Section 7 below. For Markov motion in a random environment that cannot be reduced to the present case, one may refer to Lyons and Schramm (1999) and Zirbel (1997).

We conclude with a sketch of the structure of the paper. Section 2 reviews shift maps on the probability space and Section 3 resolves some technical issues involving the measurability of these maps. Section 4 derives the distribution of the view of the environment from a random location X in terms of a random mapping G on $\mathbb{D}$. Section 5 gives the result on the stationarity of Lagrangian observations over time. Section 6 specializes these results to the case of motion in a velocity field and Section 7 does the same for motion generated by a stochastic differential equation. Section 8 provides density computations for compressible flows while Section 9 addresses the evolution in time and possible convergence of the distribution of Lagrangian observations in compressible flows.

2 Homogeneity

The main results of this paper (in Sections 4 and 5) make substantial use of the formulation of homogeneity in terms of shift maps on a canonical probability space of environments. This section reviews homogeneity, canonical probability spaces, and shift maps associated with a random field and with a random measure. It concludes by listing the generic setup for homogeneity that will be used in the remainder of the paper. These topics are standard, but we develop them here for completeness and lack of a simple reference.

We denote by $\mathbb{D}$ the spatial domain. In Section 1 this was $\mathbb{R}^d$. While $\mathbb{R}^d$ is the natural state space for physical flows, periodic boundary conditions and discrete lattices are often used for numerical simulations and theoretical work, and the theory encompasses all these cases with few modifications. We only need $\mathbb{D}$ to be an Abelian group with a translation–invariant σ-algebra $\mathcal{D}$ (i.e., if $x \in \mathbb{D}$ and $A \in \mathcal{D}$, then $x + A \in \mathcal{D}$) and a translation–invariant measure λ (i.e., $\lambda(x + A) = \lambda(A)$). Standard continuous cases are $\mathbb{R}^d$ and the d–dimensional torus ($[0,1]^d$ with addition modulo 1 in each component), each with the Borel or Lebesgue σ-algebras and the Lebesgue measure. Discrete cases are $\mathbb{Z}^d$, the discrete torus ($\{0,1,\ldots,\ell-1\}^d$ with addition modulo ℓ), and the hexagonal lattice, each with the discrete σ-algebra and counting measure.

In this section, we will include a time parameter in the random fields and random measures because they are usually needed but cause no difficulty. The nature of the time set $\mathbb{T}$ is unimportant.

2.1 Homogeneity of random fields

Let $A(x,t), x \in \mathbb{D},\ t \in \mathbb{T}$ be a random field taking values in a measurable space $(\mathbb{F}, \mathcal{F})$. The field A is *homogeneous* if the collection $\{A(x+z,t), x \in \mathbb{D},\ t \in \mathbb{T}\}$ has the same law for all z in $\mathbb{D}$. When there are two such random fields A and B, we will say for brevity that A and B are *jointly homogeneous* when we mean that (A, B) is homogeneous.

The canonical probability space $(\Omega, \mathcal{H}, \mathbb{P})$ for A is as follows. The set Ω is nominally taken to consist of all mappings from $\mathbb{D} \times \mathbb{T}$ to $\mathbb{F}$, but can be restricted to those mappings having the same regularity properties that the sample paths of A enjoy almost surely, cf. Revuz and Yor (1991), Section I.3. We assume without further mention that Ω is closed under translation, meaning that if ω is in Ω, then the mapping $(x,t) \mapsto \omega(x+z,t)$ is also in Ω for all z in $\mathbb{D}$. The σ-algebra $\mathcal{H}$ is the smallest one for which all mappings $\omega \mapsto \omega(x,t)$ are measurable. If the random field A is originally defined on a probability space $(\Omega_0, \mathcal{H}_0, \mathbb{P}_0)$, we recognize now that A is a measurable mapping from $(\Omega_0, \mathcal{H}_0)$ to $(\Omega, \mathcal{H})$. We denote the distribution of A by $\mathbb{P}$. Homogeneity of A is a matter only of the distribution of A in Ω, and so may be checked under $\mathbb{P}_0$ or $\mathbb{P}$. From now on we assume that A is defined on $(\Omega, \mathcal{H}, \mathbb{P})$ and write $A(\omega, x, t) = \omega(x,t)$. We will often refer to ω as the *environment*, as it may represent all physical variables in space and time.

The manipulation of homogeneity is facilitated by the introduction of *shift maps* on Ω. For each z in $\mathbb{D}$, define $\sigma_z : \Omega \to \Omega$ by $(\sigma_z\omega)(x,t) = \omega(x+z,t)$ for x in $\mathbb{D}$ and t in $\mathbb{T}$. We think of $\sigma_z\omega$ as the view of the environment ω relative to the spatial location z. Note that

$$A(\omega, x+z, t) = \omega(x+z,t) = (\sigma_z\omega)(x,t) = A(\sigma_z\omega, x, t) \tag{2.1}$$

for all ω in Ω, x and z in $\mathbb{D}$, and t in $\mathbb{T}$. As a special case, $A(\omega, z, t) = A(\sigma_z\omega, 0, t)$, so that A depends on ω and z only through $\sigma_z\omega$. Note that σ_z is measurable and $\sigma_{z+y} = \sigma_z\sigma_y$.

Because of (2.1), we may write $\omega = \{A(\omega, x, t) : x \in \mathbb{D},\ t \in \mathbb{T}\}$ and $\sigma_z\omega = \{A(\omega, x+z, t) : x \in \mathbb{D},\ t \in \mathbb{T}\}$. If A is homogeneous, then the collections indicated here have the same distribution, so the random variables ω and $\sigma_z\omega$ both have distribution $\mathbb{P}$. Thus, σ_z preserves $\mathbb{P}$, which we write $\mathbb{P} \circ \sigma_z^{-1} = \mathbb{P}$. Intuitively, the distribution of the view of the environment ω relative to z is the same as the distribution of the view relative to the origin, which is a succinct statement of homogeneity.

(2.2) **Remark.** Suppose $B(x), x \in \mathbb{D}$ is a random field which is of the form $B(\omega, x) = f(\sigma_x\omega)$, where f is a measurable function from Ω to an arbitrary measurable space. Then for each z in $\mathbb{D}$, B satisfies $B(\omega, x+z) = B(\sigma_z\omega, x)$ so the collections $B = \{B(x), x \in \mathbb{D}\}$ and $B \circ \sigma_z = \{B(x+z), x \in \mathbb{D}\}$ have the same distributions. Thus, B is homogeneous, and, in fact, A and B are jointly homogeneous. □

2.2 Homogeneity of random measures

Some flow models and velocity field models are based on random measures, such as Lee (1974), Matsumoto and Shigekawa (1985), Harris (1981), and Çağlar (1997). Here we briefly review the formulation of homogeneity and shift maps for arbitrary random measures.

Let M be a random measure on $(\mathbb{D} \times \mathbb{T}, \mathcal{D} \times \mathcal{T})$, where $\mathcal{T}$ is a σ-algebra on $\mathbb{T}$ and $\mathcal{D} \times \mathcal{T}$ denotes the product σ-algebra. For each z in $\mathbb{D}$, let $S_z : \mathbb{D} \times \mathbb{T} \to \mathbb{D} \times \mathbb{T}$ be defined by $S_z(x, t) = (x - z, t)$. Then M is said to be *homogeneous* if, for all z in $\mathbb{D}$, $M \circ S_z^{-1}$ has the same distribution as M.

The canonical probability space for M is nominally the space Ω of all measures on $(\mathbb{D}\times\mathbb{T}, \mathcal{D}\times\mathcal{T})$, but may be restricted according to the almost–sure properties of M, so long as the resulting space is closed under S_z for each z in $\mathbb{D}$. We write $M(\omega, A) = \omega(A)$. The σ-algebra $\mathcal{H}$ is the smallest one for which all mappings $\omega \to M(\omega, A)$ are measurable, as A ranges over $\mathcal{D} \times \mathcal{T}$.

The shift σ_z on Ω is defined by $\sigma_z\omega = \omega \circ S_z^{-1}$. Then σ_z is measurable and homogeneity of M is equivalent to σ_z preserving $\mathbb{P}$ for all z in $\mathbb{D}$. The definition of σ_z is compatible with the definition in the previous section in the sense that if a random field A is defined by $A(\omega, x) = \int_{\mathbb{D}} M(\omega, dy, dt)\varphi(x - y, t)$, where $\varphi : \mathbb{D} \times \mathbb{T} \to \mathbb{R}$ is measurable and bounded, then $A(\sigma_z\omega, x) =$

$A(\omega, x+z)$. Alternatively, if M is defined in terms of a random density m, then there is no incompatibility between the shifts in this section for M and the shifts in the previous section for m.

2.3 Usual setup for homogeneity

The basic results in Sections 4 and 5 are formulated in the general setting of observations of a random environment ω from a random location X determined by ω, which encompasses both the random field and random measure situations above. The *usual setup for homogeneity* includes a probability space $(\Omega, \mathcal{H}, \mathbb{P})$, a measurable space $(\mathbb{D}, \mathcal{D}, \lambda)$ in which $\mathbb{D}$ is an Abelian group and the σ-algebra $\mathcal{D}$ and the measure λ are invariant under translations, and measurable shift maps $\sigma_z, z \in \mathbb{D}$ on $(\Omega, \mathcal{H})$ which each preserve $\mathbb{P}$ and which satisfy $\sigma_0 =$ identity and $\sigma_z\sigma_y = \sigma_{z+y}$ for all y and z in $\mathbb{D}$. For the purposes of Sections 4 and 5, the nature of the environment is unimportant.

3 Measurability

Begin with the usual setup for homogeneity of Section 2.3. We wish to observe the random environment ω from a location $X(\omega)$ determined by ω. The following conditions must be met to insure the existence of various integrals and to allow interchanges of integrals in Section 4.

$$\text{(3.1)} \qquad \begin{array}{lcccl} (i) & \omega & \mapsto & X(\omega) & \text{from } (\Omega, \mathcal{H}_0) \text{ to } (\mathbb{D}, \mathcal{D}) \text{ is measurable} \\ (ii) & \omega & \mapsto & \sigma_{X(\omega)}\omega & \text{from } (\Omega, \mathcal{H}_0) \text{ to } (\Omega, \mathcal{H}) \text{ is measurable} \\ (iii) & (\omega, z) & \mapsto & X(\sigma_z\omega) & \text{from } (\Omega \times \mathbb{D}, (\mathcal{H} \times \mathcal{D})_0) \text{ to } (\mathbb{D}, \mathcal{D}) \text{ is measurable.} \end{array}$$

Here $\mathcal{H}_0$ is the completion of $\mathcal{H}$ with respect to $\mathbb{P}$, $\mathcal{H} \times \mathcal{D}$ is the product σ-algebra on $\Omega \times \mathbb{D}$, and $(\mathcal{H} \times \mathcal{D})_0$ is the completion of $\mathcal{H} \times \mathcal{D}$ with respect to $\mathbb{P} \times \lambda$.

Let $\ell : \Omega \times \mathbb{D} \to \Omega$ be defined by $\ell(\omega, z) = \sigma_z\omega$. It is not difficult to see that the following conditions are sufficient to imply (3.1):

$$\text{(3.2)} \qquad \begin{array}{ll} (i) & X \text{ is } \mathcal{H} \text{ to } \mathcal{D} \text{ measurable} \\ (ii) & \ell \text{ is } \mathcal{H} \times \mathcal{D} \text{ to } \mathcal{H} \text{ measurable.} \end{array}$$

Verifying the measurability of X must be done on a case–by–case basis. For motion based on velocity fields, X is usually defined by the method of successive approximations and so is measurable. (This is the case in Sections 6 and 7 below.) On the other hand, in previous articles, in particular Geman and Horowitz (1975), Zirbel (1993), and Komorowski and Papanicolaou (1997), the measurability of ℓ was simply assumed. In this section we find more natural sufficient conditions for (3.2) and (3.1).

3.1 Continuous random fields

Let A be a random field taking values in $(F, \mathcal{F})$ and let $(\Omega, \mathcal{H}, \mathbb{P})$ be the canonical probability space. If ℓ is $\mathcal{H} \times \mathcal{D}$ to $\mathcal{H}$ measurable, then for each t in $\mathbb{T}$, the mapping $(\omega, z) \mapsto A(\omega, z, t)$ is measurable. We say that A is *jointly measurable*. The next result gives the converse. The argument is not difficult, but we have been unable to find it in the literature.

(3.3) **Proposition.** (i) Suppose that for each t in $\mathbb{T}$, the mapping $(\omega, z) \mapsto A(\omega, z, t)$ is $\mathcal{H} \times \mathcal{D}$ to $\mathcal{F}$ measurable. Then ℓ is $\mathcal{H} \times \mathcal{D}$ to $\mathcal{H}$ measurable. (ii) Suppose that for each t in $\mathbb{T}$, the mapping $(\omega, z) \mapsto A(\omega, z, t)$ is $(\mathcal{H} \times \mathcal{D})_0$ to $\mathcal{F}$ measurable. Then ℓ is $(\mathcal{H} \times \mathcal{D})_0$ to $\mathcal{H}$ measurable.

Proof: Fix y in $\mathbb{D}$, t in $\mathbb{T}$, and K in $\mathcal{F}$, and let $C = \{\omega \in \Omega : \omega(y, t) \in K\}$. Such sets C generate $\mathcal{H}$. Define $b : \Omega \times \mathbb{D} \to \Omega \times \mathbb{D}$ by $b(\omega, x) = (\omega, x + y)$. Then b is $\mathcal{H} \times \mathcal{D}$ to $\mathcal{H} \times \mathcal{D}$ measurable by the invariance of $\mathcal{D}$ under translations. Then,

$$\ell^{-1}(C) = \{(\omega, x) : (\sigma_x \omega)(y, t) \in K\} = \{(\omega, x) : A(b(\omega, x), t) \in K\}.$$

This set is $\mathcal{H} \times \mathcal{D}$–measurable since $(\omega, x) \mapsto A(b(\omega, x), t)$ is the composition of measurable mappings.

For the proof of (ii), note that $(\mathcal{H} \times \mathcal{D})_0$ consists of sets of the form $C \cup N$, where $C \in \mathcal{H} \times \mathcal{D}$ and N is negligible. Now $b^{-1}(C \cup N) = b^{-1}(C) \cup b^{-1}(N)$ and because b preserves $\mathbb{P} \times \lambda$, $b^{-1}(N)$ is negligible as well. Thus b is $(\mathcal{H} \times \mathcal{D})_0$ to $(\mathcal{H} \times \mathcal{D})_0$ measurable, and the result follows as above. $\square$

Next, we prove the familiar result that the continuity of realizations of A insures the joint measurability of A. This in turn implies Condition (3.2)(ii). The second statement appears to be less well known. It will be needed in Section 3.2.

(3.4) **Proposition.** Suppose that $\mathbb{D}$ and $\mathbb{F}$ are metric spaces whose σ-algebras contain the Borel σ-algebras and that $\mathbb{D}$ has a countable dense set $\mathbb{D}_0$. Fix t in $\mathbb{T}$. (i) Suppose that for each realization of A, the mapping $x \mapsto A(x,t)$ is continuous. Then the mapping $(\omega, z) \mapsto A(\omega, z, t)$ from $\Omega \times \mathbb{D}$ to $\mathbb{F}$ is $\mathcal{H} \times \mathcal{D}$ to $\mathcal{F}$ measurable. (ii) Suppose that, with probability one, the mapping $x \mapsto A(x,t)$ is continuous λ almost everywhere. Then the mapping $(\omega, z) \mapsto A(\omega, z, t)$ from $\Omega \times \mathbb{D}$ to $\mathbb{F}$ is $(\mathcal{H} \times \mathcal{D})_0$ to $\mathcal{F}$ measurable.

Proof: Let $x_1, x_2, \ldots$ be an enumeration of $\mathbb{D}_0$. For each x in $\mathbb{D}$ and n in $\mathbb{N}$, let $\pi_n(x)$ be the element of $\{x_1, \ldots, x_n\}$ whose distance to x is the smallest, using the element x_i having the smallest index i in the case of ties. Then π_n is a measurable mapping from $\mathbb{D}$ to $\mathbb{D}$.

Set $A_n(\omega, x) = A(\omega, \pi_n(x))$. We claim that A_n is $\mathcal{H} \times \mathcal{D}$ to $\mathcal{F}$ measurable. Fix K in $\mathcal{F}$. Then

$$\{(\omega, x) : A_n(\omega, x) \in K\} = \bigcup_{i=1}^{n} \{\omega \in \Omega : A(\omega, x_i) \in K\} \times \{x \in \mathbb{D} : \pi_n(x) = x_i\},$$

which is clearly $\mathcal{H} \times \mathcal{D}$ measurable. As $n \to \infty$, $\pi_n(x) \to x$, so $A_n(\omega, x) = A(\omega, \pi_n(x)) \to A(\omega, x)$. Thus $A_n \to A$ pointwise, so A is measurable with respect to $\mathcal{H} \times \mathcal{D}$, which proves (i).

To prove (ii), fix ω for which $x \mapsto A(\omega, x)$ is continuous for λ almost every x in $\mathbb{D}$. As $n \to \infty$, $A_n(\omega, x) \to A(\omega, x)$ for λ almost every x in $\mathbb{D}$. Thus $A_n \to A$ pointwise almost everywhere with respect to $\mathbb{P} \times \lambda$, so A is $(\mathcal{H} \times \mathcal{D})_0$ to $\mathcal{F}$ measurable. □

3.2 Discontinuous random fields

Given a random field A on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$, there are weaker conditions than continuity (e.g., stochastic continuity) under which one can assert the existence of a jointly measurable process A^*, called a version of A, for which $\mathbb{P}(A(x,t) = A^*(x,t)) = 1$ for each x in $\mathbb{D}$ and t in $\mathbb{T}$. In the present context, it is inadvisable to change the version because X typically takes on uncountably many values, so there is no assurance that $A^*(X,t)$ will have the same distribution as $A(X,t)$.

Instead, we take the perspective that A has sufficient regularity to allow the definition of the random location X, and this regularity alone should be used to establish Conditions (3.1).

(3.5) **Proposition.** Let $\mathbb{D}$ and $\mathbb{F}$ satisfy the conditions of Proposition (3.4) and suppose that Condition (3.1)(i) holds. (i) Suppose that, for each t in $\mathbb{T}$, with probability one, the mapping $x \mapsto A(x,t)$ is continuous at $(z+X,t)$. Then Condition (3.1)(ii) holds. (ii) Suppose that A satisfies Condition (ii) of Proposition (3.4). Then Condition (3.1)(iii) holds. (iii) Suppose that, with probability one, the mapping $z \mapsto X \circ \sigma_z$ is continuous λ–almost everywhere. Then Condition (3.1)(iii) holds.

Proof: For (i), fix z in $\mathbb{D}$ and let $Z_n = \pi_n(z+X)$, where π_n is as in the proof of Proposition (3.4). Then $Z_n : \Omega \to \mathbb{D}$ is $\mathcal{H}_0$ to $\mathcal{D}$ measurable. Also $\omega \mapsto A(\omega, Z_n(\omega))$ is $\mathcal{H}_0$ to $\mathcal{F}$ measurable, for

$$\{\omega \in \Omega : A(\omega, Z_n(\omega)) \in K\} = \bigcup_{i=1}^{n} \left(\{\omega \in \Omega : A(\omega, x_i) \in K\} \cap \{\omega \in \Omega : Z_n(\omega) = x_i\}\right),$$

and all of these sets are $\mathcal{H}_0$ measurable. As $n \to \infty$, $Z_n \to z + X$, so that $A(Z_n)$ converges to $A(z+X)$ almost surely. Thus the mapping $\omega \mapsto A(\omega, z + X(\omega))$ is $\mathcal{H}_0$ to $\mathcal{F}$ measurable. Now fix t in $\mathbb{T}$ and K in $\mathcal{F}$. The set $\{\omega \in \Omega : (\sigma_{X(\omega)}\omega)(z,t) \in K\} = \{\omega \in \Omega : A(\omega, z + X(\omega), t) \in K\}$ is $\mathcal{H}_0$ measurable, which establishes Condition (3.1)(ii).

For (ii), by Propositions (3.3) and (3.4), ℓ is $(\mathcal{H}\times\mathcal{D})_0$ to $\mathcal{H}$ measurable. We claim that it is $(\mathcal{H}\times\mathcal{D})_0$ to $\mathcal{H}_0$ measurable. Let $N \subset \Omega$ be negligible. We must show that $\ell^{-1}(N)$ is a negligible subset of $\Omega \times \mathbb{D}$. Let $\tilde{N} \in \mathcal{H}$ contain N and satisfy $\mathbb{P}(\tilde{N}) = 0$. Then $\ell^{-1}(\tilde{N}) = \{(\omega, z) : \omega \in \sigma_z^{-1}(\tilde{N})\}$. For each fixed z in $\mathbb{D}$, the section $\sigma_z^{-1}(\tilde{N})$ of $\ell^{-1}(\tilde{N})$ has $\mathbb{P}(\sigma_z^{-1}(\tilde{N})) = \mathbb{P}(\tilde{N}) = 0$ since A is homogeneous and thus σ_z preserves $\mathbb{P}$. Because every section has $\mathbb{P}$ measure zero, $\ell^{-1}(\tilde{N})$ has $\mathbb{P} \times \lambda$ measure 0, and so $\ell^{-1}(N)$ is negligible. Thus $X \circ \ell$ is $(\mathcal{H} \times \mathcal{D})_0$ to $\mathcal{D}$ measurable, establishing Condition (3.1)(iii).

For (iii), apply part (ii) of Proposition (3.4) to the random field $A(\omega, z) = X(\sigma_z \omega)$. □

3.3 Random measures

Suppose that the environment is based on a random measure M on $\mathbb{D} \times \mathbb{T}$. By considering the density of M with respect to λ to be a homogeneous random field which can be handled by the methods above, we can reasonably restrict attention to purely atomic random measures such as the Poisson random measure. Without loss of generality, suppose that M is defined on the canonical probability space $(\Omega, \mathcal{H}, \mathbb{P})$ described in Section 2.2, so that $M(\omega, A) = \omega(A)$.

The proof of the following result is straightforward. Note that if M is a Poisson random measure whose mean measure on $\mathbb{D} \times \mathbb{T}$ is σ–finite, then its conditions are satisfied.

(3.6) **Proposition.** Suppose that M can be written in the form

$$M(A) = \sum_{i=1}^{N} 1_A(L_i, T_i), \qquad A \in \mathcal{D} \times \mathcal{T}, \tag{3.7}$$

where the random variables N, L_i, and T_i take values in the sets $\{0, 1, 2, \ldots, \infty\}$, $\mathbb{D}$, and $\mathbb{T}$, respectively. Then Condition (3.2)(ii) is satisfied.

4 The view from a random location

This section concerns the distribution of the view of a homogeneous random environment from the point of view of an observer at a random location X determined by the environment. We will see how homogeneity allows certain simplifications whose nature is determined by a random mapping G from $\mathbb{D}$ to $\mathbb{D}$ which is based on X. We give a short, simple proof that when G preserves λ, the distribution of the view of the environment from X is the same as from 0. We begin with the usual setup for homogeneity described in Section 2.3 and a random location X satisfying Conditions (3.1). This justifies the existence of the integrals below and the interchanges of integrals we need.

We write σ_X for the mapping $\omega \mapsto \sigma_{X(\omega)}\omega$ and regard σ_X as the view of the environment ω from location $X(\omega)$. Let $f : \Omega \to \mathbb{R}_+$ be $\mathcal{H}$ to $\mathcal{B}_\mathbb{R}$ measurable and let $\alpha : \mathbb{D} \to \mathbb{R}_+$ be $\mathcal{D}$ to $\mathcal{B}_\mathbb{R}$ measurable and satisfy $\int_\mathbb{D} \lambda(dz)\alpha(z) = 1$. Then,

$$\int_\Omega \mathbb{P}(d\omega) f(\sigma_X(\omega)) = \int_\Omega \mathbb{P}(d\omega) \int_\mathbb{D} \lambda(dz)\alpha(z + X(\omega)) f(\sigma_{X(\omega)}\omega) \tag{4.1}$$

$$
\begin{aligned}
&= \int_{\mathbb{D}} \lambda(dz) \int_{\Omega} \mathbb{P}(d\omega')\alpha(z + X(\sigma_z\omega'))f(\sigma_{X(\sigma_z\omega')}\sigma_z\omega') \\
&= \int_{\mathbb{D}} \lambda(dz) \int_{\Omega} \mathbb{P}(d\omega')\alpha(G(\omega', z))f(\sigma_{G(\omega',z)}\omega'),
\end{aligned}
$$

where we have used translation invariance of λ, an interchange of integrals, the change of variable $\omega' = \sigma_{-z}\omega$, invariance of $\mathbb{P}$ under σ_{-z}, and (3.1), and we have set

$$
G(\omega, z) = z + X(\sigma_z\omega), \qquad \omega \in \Omega,\ z \in \mathbb{D}. \tag{4.2}
$$

The mapping $(\omega, z) \mapsto G(\omega, z)$ is $(\mathcal{H} \times \mathcal{D})_0$ to $\mathcal{D}$ measurable by (3.1).

We think of G as a random mapping from $\mathbb{D}$ to $\mathbb{D}$. It arises by thinking of $X(\omega)$ as a displacement from the origin, so that $X(\sigma_z\omega)$ represents the displacement starting from location z in the same environment ω. The idea of (4.1) is that the displacement X from the initial location 0 is replaced by an integral over possible starting locations z conditioned (by α) on their image under G. (In Section 6 we show that when X is determined by motion in a random velocity field over time t, then G coincides with the random mapping F_t which describes the motion of all particles over time t.)

Supposing that G preserves the measure λ on $\mathbb{D}$ almost surely, meaning that $\lambda \circ G^{-1} = \lambda$ with probability one, we say that G is *incompressible* and continue the calculation with an interchange of integrals and the change of variable $y = G(\omega', z)$:

$$
\begin{aligned}
\int_{\Omega} \mathbb{P}(d\omega)f(\sigma_X(\omega)) &= \int_{\Omega} \mathbb{P}(d\omega') \int_{\mathbb{D}} \lambda(dy)\alpha(y)f(\sigma_y\omega') \\
&= \int_{\mathbb{D}} \lambda(dy)\alpha(y) \int_{\Omega} \mathbb{P}(d\omega')f(\sigma_y\omega'),
\end{aligned}
$$

which equals $\int_{\Omega} \mathbb{P}(d\omega)f(\omega)$ by the change of variable $\omega = \sigma_y\omega'$, invariance of $\mathbb{P}$ under σ_y, and the definition of α. Thus σ_X has distribution $\mathbb{P}$, and so *the view of the environment from the random location X has the same distribution as the view from 0.* We have proven the first basic result:

(4.3) **Theorem.** Assume the usual setup for homogeneity and conditions (3.1). If the random mapping G defined by (4.2) preserves λ almost surely, then σ_X has distribution $\mathbb{P}$. □

(4.4) **Remark.** This result can be recovered from Mecke (1975), Satz 4.3 by setting $\vartheta = \delta_0$ and from Port and Stone (1976) by setting $\Phi(\omega, z, \cdot) = \delta_z$. In both cases, the environment determines a transition kernel for X rather than a single value for X. Under slightly stronger conditions, Geman and

Horowitz (1975) showed Theorem (4.3) and its converse, that σ_X having distribution $\mathbb{P}$ implies that G preserves λ almost surely. $\square$

When incompressibility of G fails, it may do so in a mild way, so that $\lambda \circ G^{-1}$ is absolutely continuous with respect to λ, almost surely. We say that G is *compressible*. From Theorem 10 and Lemmas 3, 4, and 8 of Geman and Horowitz (1975), one may deduce the existence of a homogeneous density ρ of $\lambda \circ G^{-1}$ with respect to λ (it satisfies $\rho(\omega, y) = \rho(\sigma_y \omega, 0), y \in \mathbb{D}$). Then the computation continues from the last line of (4.1) by making the change of variables $y = G(\omega', z)$ and using ρ:

(4.5)
$$\begin{aligned}\int_\Omega \mathbb{P}(d\omega) f(\sigma_X(\omega)) &= \int_\Omega \mathbb{P}(d\omega') \int_{\mathbb{D}} \lambda(dy)\alpha(y) f(\sigma_y \omega')\rho(\omega', y) \\ &= \int_{\mathbb{D}} \lambda(dy)\alpha(y) \int_\Omega \mathbb{P}(d\omega') f(\sigma_y \omega')\rho(\sigma_y \omega', 0) \\ &= \int_\Omega \mathbb{P}(d\omega) f(\omega)\rho(\omega, 0),\end{aligned}$$

and so the distribution of σ_X has density $\rho(0)$ with respect to $\mathbb{P}$, where $\rho(0)$ is the density at the origin of the measure $\lambda \circ G^{-1}$ with respect to λ. Thus, the distribution of σ_X may be understood in terms of the joint distribution of the environment and the density at the origin of particles which begin uniformly distributed on $\mathbb{D}$ and then are moved by G, cf. Section 9. We have proven:

(4.6) **Theorem.** Assume the usual setup for homogeneity and conditions (3.1). Suppose that the measure $\lambda \circ G^{-1}$ is absolutely continuous with respect to λ, almost surely, with density ρ depending on ω and y only through $\sigma_y \omega$. Then σ_X has density $\rho(0)$ with respect to $\mathbb{P}$. $\square$

Finally, when $\lambda \circ G^{-1}$ is not almost surely absolutely continuous with respect to λ, we say that G is *singular*. Theorem 10 of Geman and Horowitz (1975) shows that the distribution of σ_X has a component that is singular with respect to $\mathbb{P}$. The view from such random locations is entirely different than the view from the origin, and we can give no further general result.

(4.7) **Remark.** *Conditions for non–singularity.* Let $\mathbb{D} = \mathbb{R}^d$. If G is invertible and G^{-1} is Lipschitz, then by Geman and Horowitz (1975), Example 5, the Jacobian of G^{-1} exists almost everywhere and by their Equation (12) with $g \equiv 1$, $\int_{\mathbb{R}^d} h(x)J_0(x)\lambda(dx) = \int_{\mathbb{R}^d} h(G(y))\lambda(dy)$, for all Borel $h : \mathbb{R}^d \to \mathbb{R}_+$, where J_0 is the absolute value of the Jacobian. Thus, $\lambda \circ G^{-1}$ has density $\rho = J_0$ with respect to λ, so G is not singular. The argument may be extended to the case in which G^{-1} is locally Lipschitz, cf. Federer (1969), p. 241. □

(4.8) **Example.** *Observations of the maximum value.* Let $\mathbb{D} = \{0, 1, 2, \ldots, 9\}$ with addition modulo 10. Let $A(x),\ x \in \mathbb{D}$ be a homogeneous random field taking values in $\{1, 2, \ldots, 10\}$ in such a way that $A(0), \ldots, A(9)$ are always distinct. Define X so that $A(\omega, X(\omega)) = 10$ for all ω. Then $A(0)$ and $A(X)$ have very different distributions. Because $A(\omega, X(\omega)) = A(\sigma_z\omega, X(\sigma_z\omega)) = A(\omega, z + X(\sigma_z\omega))$, we see that $G(\omega, z) = X(\omega)$ for all ω and z. Thus, G compresses all points onto X. □

(4.9) **Example.** *Views of a Poisson random measure.* Let N be a Poisson random measure on $\mathbb{R}^d$ with Lebesgue measure λ as its mean measure. Let Y denote the location of the atom nearest 0.

(i) Set $X = Y$. Then $\lambda \circ G^{-1}$ is concentrated at the atoms of N, so G is singular. The distribution of σ_X is singular with respect to $\mathbb{P}$, because relative to X, there is always an atom at the origin.

(ii) Set $X = \frac{1}{2}Y$. Then $G(\omega, z) = z + \frac{1}{2}Y(\sigma_z\omega)$, so that $G(\omega, z)$ is equal to the midpoint between z and the atom closest to z. The random mapping G is compressible. The density of $\lambda \circ G^{-1}$ at the origin may be zero, if the atoms near 0 are roughly equidistant from 0, but it will be 2^d if there is a single atom near 0. Thus, the view from X will tend to have one atom near the origin.

(iii) Let $d = 2$ and define G so that around each atom of N, a disk halfway to the nearest atom is drawn and all points within the disk are rotated around its center by angle θ. This mapping is one to one, onto, incompressible, and discontinuous. The distribution of the view from $X = G(0)$ is the same as from the origin. □

(4.10) **Example.** *Inhomogeneous deterministic component.* Consider motion in a velocity field $U = u + U_0$, where u is deterministic and U_0 is continuous and homogeneous, cf. (1.1). This is the situation in Komorowski and Papanicolaou (1997), where u decays rapidly to zero in time. The question is how the presence of u affects the distribution of $U_0(X_t, t)$. Suppose that the solution of (1.1) exists for each ω. For each $t \geq 0$ set $G_t(\omega, z) = z + X_t(\sigma_z\omega)$. Then for $t > 0$, G_t satisfies

$\frac{d}{dt}G_t(\omega, z) = u(-z + G_t(\omega, z), t) + U(\omega, G_t(\omega, z), t)$. The collection $G_t,\ t \geq 0$ does not constitute a flow because of the dependence of u on the initial location z. Generally, the mapping $z \mapsto G_t(\omega, z)$ does not preserve Lebesgue measure and is not invertible, but in certain circumstances the measure $\lambda \circ G_t^{-1}$ is absolutely continuous with respect to λ. Then $\lambda \circ G_t^{-1}$ has a density ρ_t with respect to λ, and one may understand $U(X_t, t)$ by studying $\rho_t(0)$, cf. Komorowski and Papanicolaou (1997). □

5 Stationarity

In statistical fluid mechanics, observations of the velocity field are made over time from the location of a moving particle. In this section we will see conditions under which such views of a homogeneous random environment form a stationary process.

We begin with a brief discussion of stationarity. The time set $\mathbb{T}$ may be $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}$, or $\mathbb{Z}_+$. A random field $A(x,t), x \in \mathbb{D},\ t \in \mathbb{T}$ is *stationary* if the collection $\{A(x, t+s), x \in \mathbb{D},\ t \in \mathbb{T}\}$ has the same distribution for all s in $\mathbb{T}$. On the canonical probability space $(\Omega, \mathcal{H}, \mathbb{P})$ for A, define a time shift τ_s by $(\tau_s\omega)(x,t) = \omega(x, t+s)$. We think of $\tau_s\omega$ as the view of the environment ω relative to time s. Each τ_s is $\mathcal{H}$ measurable. Clearly $\tau_s\tau_r = \tau_{s+r}$ for all s and r in $\mathbb{T}$, and $\tau_s\sigma_z = \sigma_z\tau_s$, so these shifts commute. If U is stationary, then τ_s preserves $\mathbb{P}$, and conversely. The construction of time shifts on the canonical probability space for a random measure is similar to the construction of spatial shifts in Section 2.2.

The *usual setup for stationarity* includes a probability space $(\Omega, \mathcal{H}, \mathbb{P})$, a time set $\mathbb{T}$ from among the four choices above, and measurable shifts which preserve $\mathbb{P}$ and satisfy $\tau_0 =$ identity and $\tau_s\tau_r = \tau_{s+r}$ for all s and r in $\mathbb{T}$. We assume that the shifts σ_z and τ_s commute, which is always true when using the canonical probability space.

Given the usual setups for homogeneity and stationarity, we wish to observe the random environment ω from locations X_t determined by ω as t varies over $\mathbb{T}$. We assume that $X_0 = 0$ and that for each t in $\mathbb{T}$, the measurability conditions (3.1) are met for $X = X_t$.

The view of ω from location $X_t(\omega)$ and time t is given by $\theta_t(\omega)$, where θ_t is defined by

$$\theta_t(\omega) = \tau_t \sigma_{X_t(\omega)} \omega, \qquad \omega \in \Omega. \tag{5.1}$$

It is clear that θ_t is $\mathcal{H}_0$ to $\mathcal{H}$ measurable. For each t in $\mathbb{T}$ define a random mapping G_t by

$$G_t(\omega, z) = z + X_t(\sigma_z \omega), \qquad \omega \in \Omega, z \in \mathbb{D}, \tag{5.2}$$

as in Section 4. If $\lambda \circ G_t^{-1} = \lambda$ almost surely, then $\mathbb{P} \circ \theta_t^{-1} = \mathbb{P} \circ \sigma_{X_t^{-1}} \circ \tau_t^{-1} = \mathbb{P}$, by Theorem (4.3) and stationarity. Thus, the process $\theta_t, t \in \mathbb{T}$ has one–point distributions all equal to $\mathbb{P}$, but this is not enough to guarantee that the process is stationary. Stationarity requires that the displacements X_t be determined from the environment in a certain way, which is explained by the result below. First, we extend the definition of G by setting

$$G_{s,s+t}(\omega, z) = z + X_t(\tau_s \sigma_z \omega), \qquad s, t \in \mathbb{T}, \omega \in \Omega, z \in \mathbb{D}. \tag{5.3}$$

We will continue to write G_t for $G_{0,t}$.

(5.4) **Theorem.** Assume the usual setup for homogeneity and stationarity and that (3.1) is met by $X = X_t$ for each t in $\mathbb{T}$. Suppose that for each t in $\mathbb{T}$, $\lambda \circ G_t^{-1} = \lambda$ almost surely. Suppose that X satisfies $X_{s+t} = X_t + X_s \circ \theta_t$ for all s and t in $\mathbb{T}$, or, equivalently, that G satisfies the flow property

$$G_{s+t} = G_{s,s+t} \circ G_s, \qquad s, t \in \mathbb{T}. \tag{5.5}$$

Then $\theta_t, t \in \mathbb{T}$ is stationary.

Proof: It is straightforward to show that the shifts $\theta_t, t \in \mathbb{T}$ satisfy the semigroup property $\theta_{s+t} = \theta_s \circ \theta_t$. Let C denote the collection $\{\theta_t, t \in \mathbb{T}\}$ and let $s \in \mathbb{T}$. Then $C \circ \theta_s$ is the collection $\{\theta_{t+s}, t \in \mathbb{T}\}$. But C and $C \circ \theta_s$ have the same distribution by the fact that $\lambda \circ G_s^{-1} = \lambda$ almost surely, so $\theta_t, t \in \mathbb{T}$ is stationary. The equivalence of the conditions on X and G is easy to check. □

(5.6) **Remarks.** (i) This result first appeared in Zirbel (1993), Chapter 5 and another proof appears in Komorowski and Papanicolaou (1997), Lemma 3.1. The proof of Lumley (1957, 1962) was not as rigorous or general. No proof of stationarity appeared in Port and Stone (1976). The natural extension of their "random transition probability distributions" is motion that is Markov given the environment. It is possible to formulate the analogues of homogeneity and stationarity for such motion and show a result analogous to Theorem (5.4), cf. Zirbel (1997).

(ii) If G is not incompressible, stationarity of $\theta_t, t \in \mathbb{T}$ will fail. See Remark (4.4) and Section 9.

(iii) If A is of the form $A(\omega, x, t) = f(\tau_t \sigma_x \omega)$, then A is both homogeneous and stationary under the conditions of Theorem (5.4). The process $A(X_t, t), t \in \mathbb{T}$ is stationary, for $A(X_t, t) = f \circ \theta_t$.

(iv) The motion process of Example (4.10) satisfies (5.5) only if u is constant in space and time.

(v) It is natural to attempt to establish ergodicity of the process $A(X_t, t), t \in \mathbb{T}$ by showing ergodicity of the shift θ_t on Ω. This appears to be difficult to accomplish in general, even if one assumes that σ_z and τ_s are ergodic for all nonzero z in $\mathbb{D}$ and all s in $\mathbb{T}$. There is a related result concerning the ergodicity of the canonical Markov process for motion in a velocity field with noise as in (1.2), cf. Remark (8.9). However, in that case, the observed environment consists of only the velocity field, with the noise W acting independently. As such, the setup is different than in the current paper, in which particle motion is determined by the environment alone. If one includes the noise in the environment as in Section 7 below, then it is not known whether θ_t is ergodic or not. □

6 Classical flows on $\mathbb{R}^d$

Consider the motion of a single particle in $\mathbb{D} = \mathbb{R}^d$ with time set $\mathbb{T} = [0, \infty)$ according to

(6.1) $$\frac{dX_t}{dt} = U(X_t, t), \qquad t > 0; \qquad X_0 = 0,$$

where U is a random velocity field. Following Section 5, we may use X to define random mappings $G_{s,s+t}$ by (5.3). But it is more familiar to consider the flow F generated by the trajectory equation

(6.2) $$\frac{d}{dt} F_{s,s+t}(z) = U(F_{s,s+t}(z), s+t), \qquad t > 0; \qquad F_{s,s}(z) = z,$$

or the more robust integral form (which allows U to be discontinuous in t),

(6.3) $$F_{s,s+t}(z) = z + \int_s^{s+t} U(F_{s,r}(z), r) dr, \qquad t \geq 0.$$

We will show now that F and G coincide. Thus, the incompressibility condition of Theorems (4.3) and (5.4) applies to F, and G satisfies the flow condition (5.5) because F is a flow.

Without loss of generality, we work on the canonical space for U. It is not necessary to assume homogeneity or stationarity at this point. That F equals G is merely a consequence of the nature of equations (6.2) and (6.3) and the shifts on Ω.

(6.4) **Proposition.** Suppose that for each $s \geq 0$ and z in $\mathbb{R}^d$, (6.3) has a unique solution. Then,

$$F_{s,s+t}(\omega, z) = z + X_t(\tau_s \sigma_z \omega) = z + F_{s,s+t}(\sigma_z \omega, 0) = z + F_t(\tau_s \sigma_z \omega, 0), \tag{6.5}$$

where $F_t = F_{0,t}$. In particular, F coincides with G.

Proof: We give the proof in differential form because it is somewhat cleaner. Note that $z + X_0(\tau_s \sigma_z \omega) = z$ and that by (6.1), (2.1), and the definition of τ_s,

$$\begin{aligned} \frac{d}{dt}(z + X_t(\tau_s \sigma_z \omega)) &= U(\tau_s \sigma_z \omega, X_t(\tau_s \sigma_z \omega), t), \qquad t > 0 \\ &= U(\omega, z + X_t(\tau_s \sigma_z \omega), s + t), \qquad t > 0, \end{aligned}$$

and so $z + X_t(\tau_s \sigma_z \omega),\ t \geq 0$ satisfies (6.2). By uniqueness of solutions, the first equality of (6.5) follows. The second equality follows from the first by setting $\omega = \sigma_{z'} \omega'$ and $z = 0$, and the third by setting $\omega = \tau_{s'} \sigma_{z'} \omega'$ and $s = 0$ and $z = 0$. □

(6.6) **Remark.** *Some homogeneous fields.* Assume that U is homogeneous, so that the shifts σ_z on the canonical probability space for U preserve $\mathbb{P}$. The random field $L(z,t) = U(F_t(z), t),\ z \in \mathbb{R}^d,\ t \geq 0$ is commonly called the *Lagrangian velocity field* as it is made up of the velocities at time t of particles being moved by U, as indexed by their initial locations z, cf. Monin and Yaglom (1971), Volume I, Section 9.5. This field is also homogeneous, for one may use (6.5) and (2.1) to check that $L(\omega, z, t) = L(\sigma_z \omega, 0, t)$ and then appeal to Remark (2.2). However, L is not stationary, for the joint distribution of $U(F_t(y), t)$ and $U(F_t(z), t)$ is almost never the same as that of $U(y, 0)$ and $U(z, 0)$. Similarly, the displacement field $D_{s,s+t}(z) = F_{s,s+t}(z) - z$ is homogeneous. Such observations were made by Lumley (1962), but with stronger conditions on U. □

The original stationarity results of Lumley (1957, 1962) are superseded by the following result.

(6.7) **Corollary.** Suppose that U is a homogeneous random vector field on $\mathbb{R}^d$ whose realizations are continuous in x. Suppose that for each $s \geq 0$ and z in $\mathbb{R}^d$, (6.3) has a unique solution and that the mapping $\omega \mapsto F_t(\omega, x)$ is measurable. Finally, suppose that for each $t \geq 0$, F_t preserves the Lebesgue measure λ with probability one. Then for each fixed $t \geq 0$, the view of U from (X_t, t) has the same distribution as the view of U from $(0, t)$. In particular, $U(X_t, t)$ has the same distribution as $U(0, t)$.

If, in addition, U is stationary, then the view of U from (X_t, t) is a stationary process as t ranges over $[0, \infty)$. In particular, $U(X_t, t),\ t \geq 0$ is stationary.

Proof: Without loss of generality, we may work on the canonical space for U described in Section 2.1. The flow F coincides with the mappings G of (4.2), so G satisfies the flow condition (5.5) and the incompressibility condition. The results now follow from (3.3), (3.4), (4.3), and (5.4). □

(6.8) **Remark.** *The law of* $U(X_t, t),\ t \geq 0$. In the homogeneous, stationary, and non–divergent case, $U(X_t, t)$ has the same distribution as $U(0, 0)$ for all $t \geq 0$. Thus, $U(X_t, t),\ t \geq 0$ and $U(0, t),\ t \geq 0$ are both stationary processes with the same one–dimensional marginal distributions. However, the laws of these two processes are different in general. In fact, stationarity and equality of one–dimensional marginals is as far as our knowledge goes in general, for there is no known way of deriving the law of $U(X_t, t),\ t \geq 0$ from the law of the original Eulerian velocity field U, except in some special Markov cases, cf. Carmona and Xu (1997), Fannjiang and Komorowski (1999), and Bennett and Zirbel (2000). □

(6.9) **Remark.** *Generalized Lagrangian velocity.* A useful intermediate between $U(X_t, t)$ and the view of all of U from (X_t, t) is the *generalized Lagrangian velocity* V defined by $V(x, t) = U(x + X_t, t)$ for x in $\mathbb{R}^d$ and $t \geq 0$. Then $V(\cdot, t)$ is the velocity field at time t relative to location X_t. One may check that $V(\cdot, t) = V(\cdot, 0) \circ \theta_t$. Thus, when U is homogeneous, stationary, and non–divergent, we see that V is a stationary random field. □

(6.10) **Remark.** *Conditions for existence, uniqueness, and measurability of F.* Suppose that U is continuous in x for all t, almost surely and that $U(x,t)$ is left continuous in t for all x. (If U is right–continuous in t with left–hand limits, one may redefine U to be left–continuous without changing the truth of (6.3).) Finally, assume that, for each $n = 0, 1, 2, \dots$ and compact $K \subset \mathbb{R}^d$,

$$\int_n^{n+1} \sup_{x \in \mathbb{R}^d} \frac{|U(x,t)|}{1+|x|} dt < \infty \tag{6.11}$$

$$\int_n^{n+1} \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|U(x,t) - U(y,t)|}{|x-y|} dt < \infty, \tag{6.12}$$

almost surely. Then by Theorems 3.4.5, 3.4.6, and 4.7.1 of Kunita (1990), equation (6.3) has a unique solution for all time and each $F_{s,s+t}$ is a (random) homeomorphism of $\mathbb{R}^d$. Moreover, the mapping $\omega \mapsto F_{s,s+t}(\omega, x)$, being the limit of successive approximations, is measurable. □

(6.13) **Remark.** *Nonsingular flows.* Assume the conditions of the preceding remark and fix $T > 0$. Let $K_0 \subset \mathbb{R}^d$ be compact and let $K \subset \mathbb{R}^d$ be a compact set containing $F_t(x),\ x \in K_0, 0 \leq t \leq T$. Let L_t be the integrand in (6.12). Then $\int_0^T L_t dt < \infty$ almost surely. Fix ω for which this integral is finite. Fix $x, y \in K_0$. Then $|F_t(x) - F_t(y)| \leq |x-y| + \int_0^t (1+L_s)|F_s(x) - F_s(y)| ds$ for all $0 \leq t \leq T$. Making the change of variable $u(s) = \int_0^s (1+L_r) dr$ and applying Gronwall's inequality, we find that $|F_T(x) - F_T(y)| \leq |x-y| e^{u(T)}$. Thus, F_T is locally Lipschitz. Similarly, noting that $F_{s,t}^{-1}(x) = x - \int_s^t U(F_{r,t}^{-1}(x), r) dr$, we find that F_T^{-1} is locally Lipschitz. By Remark (4.7), $\lambda \circ F_T^{-1}$ is absolutely continuous with respect to λ. The density ρ may be computed as in Section 8 below. □

(6.14) **Remark.** *Uniform launching in periodic velocity fields.* Let U be a deterministic or random velocity field on the unit cell $\mathbb{D} = [0,1)^d$. Impose periodic boundary conditions, so that addition of elements of $\mathbb{D}$ is done modulo 1 in each component. Let Y_0 be uniformly distributed on the cell, independent of U. Then the field $U(x+Y_0, t),\ x \in \mathbb{D},\ t \geq 0$ is homogeneous, for $z + Y_0$ is uniformly distributed on $\mathbb{D}$ and independent of U for each fixed z in $\mathbb{D}$. Suppose now that U is stationary (steady in the deterministic case) and satisfies the conditions of Remark (6.10) when repeated periodically throughout $\mathbb{R}^d$. Setting $X_t = F_t(Y_0) - Y_0$, we see that X satisfies $X_0 = 0$ and $\frac{dX_t}{dt} = U(X_t + Y_0, t),\ t > 0$. By repeating the argument of Corollary (6.7) on the cell, we see that the view of $U(\cdot + Y_0, \cdot)$ from $(X_t, t),\ t \geq 0$ is stationary.

More to the point, because $F_t(Y_0) = X_t + Y_0$, the view of U from $(F_t(Y_0), t),\ t \geq 0$ is also stationary. Thus, homogeneity of U is exchanged for making Y_0 uniformly distributed (uniformly launched) on the cell. In this way, the stationarity result applies to Lagrangian observations of non–homogeneous periodic velocity fields. See Example (9.6) for a similar case with a divergent velocity field. □

(6.15) **Remark.** *Motion with diffusion.* Motion with diffusion may be handled using similar techniques. Port and Stone (1976) show that if U is non–divergent and U and its first x derivatives are jointly continuous in (x, t), then for each ω in Ω, there exist unique solutions of the flow equation

(6.16) $$F_{s,s+t}(z) = z + \int_0^t U(F_{s,s+r}(z), s + r)dr + \beta(W_{s+t} - W_s), \qquad t \geq 0,$$

corresponding to (1.2), and the flow F preserves Lebesgue measure on $\mathbb{R}^d$. Set $X_t = F_t(0)$.

As a special case of Section 7 below, we may define shifts σ_z and τ_s which preserve $\mathbb{P}$ and for which $W_t(\sigma_z\omega) = W_t(\omega)$ and $W_t(\tau_s\omega) = W_{s+t}(\omega) - W_s(\omega)$ for all ω in Ω. Noting that

$$z + X_t(\tau_s\sigma_z\omega) = z + \int_0^t U(\omega, z + X_r(\tau_s\sigma_z\omega), s + r)dr + \beta(W_{s+t}(\omega) - W_s(\omega)),$$

which is the same equation that $F_{s,s+t}(\omega, z),\ t \geq 0$ satisfies, we establish the analogue of Proposition (6.4). Thus, the view of U from $(X_t, t),\ t \geq 0$ will be stationary. See also Remark (8.9) below.
□

(6.17) **Remark.** *Flows in discrete time.* Let F be a flow on a general state space $\mathbb{D}$ with time set $\mathbb{T} = \{0, 1, 2, \ldots\}$. The results of this section may be understood in terms of the "velocity field" U associated with F, which we define as $U(x, t) = F_{t,t+1}(x) - x$, for all x in $\mathbb{D}$ and t in $\mathbb{T}$. Suppose that U is homogeneous and stationary and use the canonical probability space. If we agree that $\frac{dh}{dt}$ means $h(t + 1) - h(t)$ in discrete time, then X is obtained from U by (6.1) and F is obtained from U via (6.2). Proposition (6.4) and its proof are unchanged except that the condition $t > 0$ is replaced by $t \geq 0$. The incompressibility requirement for F is unchanged. (When $\mathbb{D}$ is discrete, incompressibility means that F is a flow of permutations on $\mathbb{D}$.) Under these conditions, the last conclusion of Corollary (6.7) will hold in this form: the view of U from (X_t, t) is a stationary process as t ranges over $\mathbb{T}$. In particular, $U(X_t, t),\ t \in \mathbb{T}$ is stationary. See Bennett and Zirbel (2000). □

7 Stochastic flows on $\mathbb{R}^d$

The purpose of this section is to extend the results of Section 6 to stochastic flows solving stochastic differential equations. We begin with motivation for studying stochastic flows.

In statistical fluid mechanics, it is often of interest to model the effect of molecular diffusion or other effects having a relatively short time scale, such as wind forcing on the upper ocean. The most common method of doing so is to add a white noise term as in (1.2), which we repeat here in differential form:

$$dX_t = U(X_t, t)dt + \beta dW_t, \qquad t > 0; \qquad X_0 = 0. \tag{7.1}$$

This models the motion of one particle in $\mathbb{R}^d$. How can we extend this to model the simultaneous motion of all particles? For the motion of n particles, one might use $dX_t^{(i)} = U(X_t^{(i)}, t)dt + \beta dW_t^{(i)}$, for $t > 0$, where the $W^{(i)}$ are independent Wiener processes. But one cannot use the same approach to model the simultaneous motion of *all* particles in $\mathbb{R}^d$ because that would require uncountably many independent Wiener processes, and all spatial regularity would be lost.

One can use a single Wiener process to drive the motion of all particles, as in Remark (6.15). Although this maintains (7.1) as the equation of motion for each individual particle, it is not a physically satisfying model for the simultaneous motion of all particles.

A middle ground between these two extremes uses *spatially correlated noise* in such a way that the simultaneous motion of all particles is described by a continuous random flow in $\mathbb{R}^d$. The flow F is the solution of the stochastic integral equation

$$F_{s,s+t}(x) = x + \int_s^{s+t} U(F_{s,r}(x), r)dr + \int_s^{s+t} M(F_{s,r}(x), dr), \qquad t \geq 0, \tag{7.2}$$

where $M(x, \cdot),\ x \in \mathbb{R}^d$ is a collection of $\mathbb{R}^d$–valued martingales. (The case of Remark (6.15) has $M(x,t) = \beta W_t$.) The stochastic integral in (7.2) is the limit in probability of

$$\sum_{k=0}^{n-1} [M(F_{s,r_k}(x), r_{k+1}) - M(F_{s,r_k}(x), r_k)] \tag{7.3}$$

as the width of the partition $s = r_0 < r_1 < \cdots < r_n = s + t$ goes to zero (3.2.19 of Kunita (1990)).

The one–particle motion will satisfy (7.1) provided that $M(x,\cdot),\ \ x \in \mathbb{R}^d$ is a collection of Brownian motions such that M is a jointly Gaussian random field with mean 0 and covariance $\mathrm{Cov}(M^i(x,s), M^j(y,t)) = a^{ij}(x,y)\min(s,t)$, where the spatial covariance a satisfies $a(x,x) = \beta^2 I$. Then F is called a *Brownian* flow. The covariance $a(x,y)$ can be made to go quickly to zero as $|x-y|$ increases, even while the flow F maintains spatial continuity and differentiability. For more background on stochastic and Brownian flows, see Kunita (1990) and Zirbel and Çinlar (1997).

The setup for (7.2) is as follows. Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a complete probability space and let $\mathcal{H}_t,\ t \geq 0$ be a right–continuous filtration on it. Let U be a random velocity field adapted to $\mathcal{H}$ and satisfying the conditions of Remark (6.10). Let $M(x,t),\ x \in \mathbb{R}^d,\ t \geq 0$ take values in $\mathbb{R}^d$, be jointly continuous in (x,t) almost surely, and be such that for each fixed x in $\mathbb{R}^d$, $M(x,t),\ t \geq 0$ is an $\mathcal{H}$ martingale. We assume that the joint quadratic variation of M satisfies $\langle M^i(x,\cdot), M^j(y,\cdot)\rangle_t = \int_0^t a^{ij}(x,y,s)ds$, for a predictable process a, called the *local characteristic* of M. Finally, we require that for each compact $K \subset \mathbb{R}^d$ and $n = 0,1,2,\ldots$,

$$\int_n^{n+1} \sup_{x,y\in\mathbb{R}^d} \frac{\|a(x,y,t)\|}{(1+|x|)(1+|y|)} dt < \infty \tag{7.4}$$

$$\int_n^{n+1} \sup_{\substack{x,x',y,y'\in\mathbb{R}^d \\ x\neq x', y\neq y'}} \frac{\|a(x,y,t) - a(x',y,t) - a(x,y',t) + a(x',y',t)\|}{|x-x'||y-y'|} dt < \infty, \tag{7.5}$$

almost surely, where $\|\cdot\|$ denotes the matrix norm. These conditions guarantee that the local characteristic (U,a) satisfies the conditions of Theorem 3.4.6 and belongs to the class $B^{0,1}$ of Kunita (1990). By Theorems 3.4.6 and 4.7.1 of Kunita (1990), there exists a unique flow of homeomorphisms $F_{s,s+t},\ \ s,t \geq 0$ of $\mathbb{R}^d$ satisfying (7.2). Uniqueness means that for each fixed x in $\mathbb{R}^d$ and $s,t \geq 0$, the value of $F_{s,s+t}(x)$ is unique up to sets of probability zero. Also, to be precise, Kunita gives the existence of flows on time intervals $[n, n+1]$ which may be stitched together at integer times.

7.1 Homogeneity

As in Section 6, we wish to show that the mappings $F_t,\ t \geq 0$ coincide with the random mappings $G_t,\ t \geq 0$ defined in (5.2) on the basis of the one–point motion $X_t \equiv F_t(0)$. This is more difficult here because the uniqueness of solutions of (7.2) is weaker than the uniqueness for (6.2). We will use homogeneity and the existence of a canonical probability space to get around this difficulty.

(7.6) **Proposition.** Suppose that U and M are jointly homogeneous and satisfy the conditions above. Let D be the displacement field $D_{s,s+t}(x) = F_{s,s+t}(x) - x$. Then (U, M, D) is homogeneous.

Proof: Without loss of generality, assume that the pair (U, M) is defined on a canonical probability space with measure–preserving shifts σ_z as in Section 2.1. Fix x and z in $\mathbb{R}^d$ and s and T in $[0, \infty)$. Consider the continuous process Y defined by $Y_t(\omega) = z + F_{s,s+t}(\sigma_z\omega, x)$, for $0 \le t \le T$, which we may also write $Y_t = z + F_{s,s+t}(x) \circ \sigma_z$. We claim that Y satisfies the same equation as $F_{s,s+t}(x + z), 0 \le t \le T$. Set $N(x, t) = \int_0^t U(x, r)dr + M(x, t)$ for $0 \le t < \infty$. Then N is a family of semimartingales. It is homogeneous because $N(\omega, x, t) = N(\sigma_x\omega, 0, t)$. Now by (7.2),

$$F_{s,s+t}(x) \circ \sigma_z = x + \left(\int_s^{s+t} N(F_{s,r}(x), dr) \right) \circ \sigma_z,$$

where the integral on the right may be approximated as in (7.3). Making use of this approximation, it is clear that $\int_s^{s+t} (N \circ \sigma_z)(F_{s,r}(x) \circ \sigma_z, dr)$ is a version of $\left(\int_s^{s+t} N(F_{s,r}(x), dr) \right) \circ \sigma_z$. Substituting $N(\sigma_z\omega, x, t) = N(\omega, x+z, t)$ and using the definitions of Y and N, we see that $Y_t, 0 \le t \le T$ satisfies the same equation as $F_{s,s+t}(x + z), 0 \le t \le T$.

By Theorem 3.4.1 of Kunita (1990) and the fact that T was arbitrary, for each x and z in $\mathbb{R}^d$ and all $s, t \ge 0$, we have $z + F_{s,s+t}(x) \circ \sigma_z = F_{s,s+t}(x+z)$ almost surely. Thus $D_{s,s+t}(x) \circ \sigma_z = D_{s,s+t}(x+z)$ almost surely. Now let $x_i \in \mathbb{R}^d$ and $s_i, t_i \ge 0$ for $i = 1, \ldots, n$. Then for each z in $\mathbb{R}^d$,

$$(U(x_i, s_i), M(x_i, s_i), D_{s_i,s_i+t_i}(x_i)) \circ \sigma_z = (U(x_i + z, s_i), M(x_i + z, s_i), D_{s_i,s_i+t_i}(x_i + z))$$

almost surely. Thus, by homogeneity, the distribution of the collection $\{U(x_i + z, s_i), M(x_i + z, s_i), D_{s_i,s_i+t_i}(x_i + z), i = 1, \ldots, n\}$ does not depend on z. Thus, the triple (U, M, D) is homogeneous. $\square$

(7.7) **Corollary.** The mappings F_t of (7.2) coincide with the random mappings G_t defined by (5.2). Moreover, if F_t preserves the Lebesgue measure on $\mathbb{R}^d$ almost surely, then σ_{X_t} has distribution $\mathbb{P}$.

Proof: By changing the probability space if necessary, we may assume that the triple (U, M, D) is defined on a canonical probability space with measure–preserving shifts such that $D_t(\sigma_z\omega, 0) =$

$D_t(\omega, z)$. On this space, set $F_t(\omega, z) = z + D_t(\omega, z)$. Then F satisfies (7.2), since the equation relies on limits in probability, which are not affected by changing the probability space. Finally, $F_t(\omega, z) = z + D_t(\sigma_z\omega, 0) = G_t(\omega, z)$ by the invariance of D. Now use (4.3). □

7.2 Stationarity

In order to show that the succession of observations of the environment from space–time points (X_t, t) forms a stationary process, we must first formulate stationarity for the fields U and M. This differs slightly from Section 5 because M will have stationary increments instead of being stationary. Then we must show that the mappings $F_{s,s+t}$ solving (7.2) coincide with the random mappings $G_{s,s+t}$ of (5.3). This is similar to the results of the previous subsection, but is not the same.

We say that (U, M) is stationary if the collection $\{U(x, s+t), M(x, s+t)-M(x, s),\ x \in \mathbb{R}^d,\ t \geq 0\}$ has the same distribution for all $s \geq 0$. Let $(\Omega, \mathcal{H}, \mathbb{P})$ be the canonical probability space for (U, M), with outcomes (ω, μ). The time shift $\tau_s : \Omega \to \Omega$ is defined by

(7.8) $$(\tau_s(\omega, \mu))(x, t) = (\omega(x, t + s), \mu(x, t + s) - \mu(x, s)), \qquad x \in \mathbb{R}^d,\ t \geq 0.$$

It is clear that τ_s preserves $\mathbb{P}$ when (U, M) is stationary. We say that (U, M, F) is stationary if the collection $\{U(x, s + t), M(x, s + t) - M(x, s), F_{s,s+t}(x),\ x \in \mathbb{R}^d,\ t \geq 0\}$ has the same distribution for all $s \geq 0$.

(7.9) **Proposition.** Suppose that U and M satisfy the regularity conditions in and above (6.11) to (7.5) and that (U, M) is stationary. Then (U, M, F) is stationary.

Proof: Without loss of generality we work on the canonical space for (U, M). Fix x in $\mathbb{R}^d$ and $s \geq 0$, and set $Y_t = F_{0,t}(x) \circ \tau_s$ for $t \geq 0$. We claim that Y solves the same equation as $F_{s,s+t}(x),\ t \geq 0$.

Define N as in the proof of Proposition (7.6). Then $F_{0,t}(x) = x + \int_0^t N(F_{0,r}(x), dr)$. Now $\int_0^t N(F_{0,r}(x), dr)$ is the limit in probability of $\sum_{k=0}^{n-1} [N(F_{0,r_k}(x), r_{k+1}) - N(F_{0,r_k}(x), r_k)]$, where

$0 = r_0 < \cdots < r_n = t$. Composing with τ_s yields

$$\sum_{k=0}^{n-1} \left[(N \circ \tau_s)(F_{0,r_k}(x) \circ \tau_s, r_{k+1}) - (N \circ \tau_s)(F_{0,r_k}(x) \circ \tau_s, r_k) \right]. \tag{7.10}$$

Now, by (7.8) and a change of variables, $(N \circ \tau_s)(y, r_{k+1}) - (N \circ \tau_s)(y, r_k)$ is equal to $N(y, r_{k+1} + s) - N(y, r_k + s)$. Thus, (7.10) equals

$$\sum_{k=0}^{n-1} \left[N(F_{0,r_k}(x) \circ \tau_s, r_{k+1} + s) - N(F_{0,r_k}(x) \circ \tau_s, r_k + s) \right].$$

Thus, considering the limit in probability as $n \to \infty$, we see that $\int_s^{s+t} N(F_{0,r}(x) \circ \tau_s, dr)$ is a version of $\left(\int_0^t N(F_{0,r}(x), dr) \right) \circ \tau_s$, and so $Y_t = F_{0,t} \circ \tau_s$ satisfies $Y_t = x + \int_s^{s+t} N(Y_r, dr)$ for $t \geq 0$. This is the same equation as for $F_{s,s+t}(x),\ t \geq 0$. Thus, $F_{s,s+t}(x)$ equals $F_{0,t}(x) \circ \tau_s$ almost surely for each fixed x in $\mathbb{R}^d$ and $s, t \geq 0$. The argument that (U, M, F) is stationary is entirely similar to the conclusion of the proof of Proposition (7.6). □

Now we are in a position to conclude the stationarity of Lagrangian observations. Suppose that (U, M) is homogeneous and stationary. Then (U, M, D) is homogeneous and (U, M, F) is stationary by Propositions (7.6) and (7.9). On the canonical space for (U, M, F), the definition (7.8) of τ_s is extended so that $F_{0,t}(x) \circ \tau_s = F_{s,s+t}(x)$. Then,

$$F_{s,s+t}(\omega, z) = F_{0,t}(\tau_s \omega, z) = z + D_{0,t}(\tau_s \omega, z) = z + D_{0,t}(\tau_s \sigma_z \omega, 0) = z + X_t(\tau_s \sigma_z \omega),$$

for all z in $\mathbb{R}^d$, $s, t \geq 0$, and $\omega \in \Omega$. This shows that the flow F coincides with the random mappings G of (5.3). Define the shift θ_t as in (5.1).

(7.11) **Corollary.** Suppose the pair (U, M) is homogeneous and stationary and that the flow F solving (7.2) preserves the Lebesgue measure on $\mathbb{R}^d$ almost surely. Then on the canonical space for (U, M, F), we have that $\theta_t,\ t \geq 0$ is stationary. □

8 Density computations on $\mathbb{R}^d$

Return to the setting of Section 4. A random location X in $\mathbb{D}$ induces a random mapping G from $\mathbb{D}$ to $\mathbb{D}$ by $G(\omega, z) = z + X(\sigma_z \omega)$. If G preserves the measure λ on $\mathbb{D}$ almost surely, then the view

σ_X from X has distribution $\mathbb{P}$, the same as the view from 0. In this section G need not preserve λ, but we will usually assume that the random measure $\mu = \lambda \circ G^{-1}$ is absolutely continuous with respect to λ almost surely.

We mentioned in passing in Section 4 that μ has a random density ρ which depends on ω and x only through $\sigma_x \omega$, but we gave no suggestion how to compute ρ. But we must do so, for it is more delicate than it may at first appear. Supposing we found a function $\rho(\omega, \cdot)$ which, for every ω, served as the density of $\mu(\omega, \cdot)$ with respect to λ, we could arbitrarily set $\rho(\omega, 0) = 5$ for every ω without changing this fact. But $\rho(0)$ cannot be set arbitrarily for its role in Theorem (4.6). Indeed, ρ must be homogeneous; it must depend on ω and x only through $\sigma_x \omega$.

The purpose of this section is to give concrete computations of ρ which make clear that ρ depends only on $\sigma_x \omega$. Throughout, the state space is $\mathbb{R}^d$ and we assume that Conditions (3.1) are satisfied. Criteria for incompressibility of classical and stochastic flows are given toward the end.

8.1 Set differentiation

Set differentiation may be used when G is not differentiable in x. Let $B_\varepsilon(x)$ denote $\{y \in \mathbb{R}^d : |x - y| < \varepsilon\}$. For each ω in Ω and x in $\mathbb{R}^d$, set $\rho(\omega, x) = \lim_{\varepsilon \downarrow 0} \dfrac{\mu(\omega, B_\varepsilon(x))}{\lambda(B_\varepsilon(x))}$ if the limit exists and is finite and 0 otherwise.

(8.1) **Proposition.** The function ρ is $(\mathcal{H} \times \mathcal{D})_0$ to $\mathcal{B}_\mathbb{R}$ measurable, satisfies $\rho(\omega, x) = \rho(\sigma_x \omega, 0)$, and for almost every ω, it serves as the density with respect to λ of the absolutely continuous component of $\mu(\omega, \cdot)$. If $\mu \ll \lambda$ almost surely, then $\rho(0)$ serves as the density of the distribution of σ_X with respect to $\mathbb{P}$.

Proof: For each k in $\mathbb{Z}^d$ set $D_k = k + [0, 1]^d$. Then $\mathbb{R}^d = \bigcup_k D_k$. For each k in $\mathbb{Z}^d$, we have

$$\mathbb{E}\mu(D_k) = \int_\Omega \mathbb{P}(d\omega) \int_{\mathbb{R}^d} \lambda(dz) 1_{D_k}(G(\omega, z)) = 1,$$

by reversing (4.1) with $f \equiv 1$. Thus $\mu(D_k) < \infty$ almost surely. Let Ω_0 be the set of ω for which $\mu(D_k) < \infty$ for all k in $\mathbb{Z}^d$. Then $\mathbb{P}(\Omega_0) = 1$. For each ω in Ω_0, $\rho(\omega, \cdot)$ serves as the density with respect to λ of the absolutely continuous component of $\mu(\omega, \cdot)$ and is finite–valued λ almost everywhere because $\mu(\omega, \cdot)$ is σ–finite (cf. Theorems 6.10 and 7.14 of Rudin (1987)).

The mapping $\omega \mapsto \rho(\omega, 0)$ is $\mathcal{H}$ to $\mathcal{B}_{\mathbb{R}}$ measurable, for it is the limit of a sequence of measurable functions. By the definition (4.2) of G used twice, $\mu(\omega, B_\varepsilon(x)) = \mu(\sigma_x\omega, B_\varepsilon(0))$. Clearly $\lambda(B_\varepsilon(0)) = \lambda(B_\varepsilon(x))$, so by the definition of ρ, $\rho(\omega, x) = \rho(\sigma_x\omega, 0)$. Finally, $(\omega, x) \mapsto \rho(\omega, x)$ is $(\mathcal{H} \times \mathcal{D})_0$ to $\mathcal{B}_{\mathbb{R}}$ measurable because $(\omega, x) \mapsto \sigma_x\omega$ is $(\mathcal{H} \times \mathcal{D})_0$ to $\mathcal{H}_0$ measurable by (3.1)(iii). □

8.2 Jacobian

When G is differentiable in x, one can use the Jacobian J of G, defined by $J(x) = \left|\det\left[\frac{\partial G^i}{\partial x^j}(x)\right]\right|$. The result below is largely a restatement of Example 5 of Geman and Horowitz (1975). We write ν for counting measure on $\mathbb{R}^d$ and set $\rho(x) = \int_{G^{-1}(\{x\})} \frac{1}{J(z)}\nu(dz)$.

(8.2) **Proposition.** Suppose that $x \mapsto G(\omega, x)$ is a Lipschitz mapping for almost all ω. Then J exists λ almost everywhere and the random field ρ serves as the density of the component of μ which is absolutely continuous with respect to λ. Moreover, ρ depends on ω and x only through $\sigma_x\omega$.

Proof: Only the last statement is not given by Geman and Horowitz (1975), but may be readily checked. The Jacobian J satisfies $J(\omega, x) = J(\sigma_x\omega, 0)$ since $G(\omega, x + z) = x + G(\sigma_x\omega, z)$ from (4.2), which we differentiate with respect to z and then evaluate at $z = 0$. Then,

$$\rho(\omega, x) = \int_{\mathbb{R}^d} \frac{1}{J(\sigma_x\omega, z - x)} 1_{\{y: G(\sigma_x\omega, y-x)=0\}}(z)\nu(dz) = \rho(\sigma_x\omega, 0),$$

using the change of variable $z' = z - x$. □

(8.3) **Example.** *A non–invertible mapping.* Let U be uniformly distributed on $[0, 1)^d$. Let G take each cell of the form $U + k + [0, 1)^d$ and dilate it by a factor of two, keeping the point $U + k$ fixed. Then, under G, each point has 2^d preimages and the Jacobian is identically 2^d. Conditions (3.1) are met and G preserves λ for all ω. Thus, the views from 0 and $G(0)$ have the same distribution. □

8.3 Formulas for the Jacobian

Return to the setting of Section 7, where the flow F is generated by a stochastic differential equation. The setting of Section 6 is included as the special case $M \equiv 0$. The next result says that if U and M are smooth enough, the Jacobian J_t of F_t can be computed by an integral along the trajectory of a particle.

(8.4) **Theorem.** Suppose that U and M satisfy the conditions in and above (7.5) and that for some $\delta > 0$, the local characteristic a of M is in the class $B^{1,\delta}$ and U is in the class $B_b^{1,\delta}$ (cf. Kunita (1990), pp. 79, 85 and 335). Then, F is a flow of C^1 diffeomorphisms and

(8.5) $$J_t(x) = \exp\left(\int_0^t (\operatorname{div} U)(F_s(x), s)ds + \int_0^t (\operatorname{div} M)(F_s(x), ds) - \int_0^t \alpha(F_s(x), s)ds\right),$$

where div M has the usual meaning and $\alpha(x,t) = \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 a^{ij}}{\partial x^j \partial y^i}(x,x,t)$.

Proof: In this proof, all external references and terminology are from Kunita (1990). The proof is similar to that of Lemma 4.3.1 of Kunita, which uses Stratonovich integrals. Unfortunately, a straightforward conversion of that result to Itô integrals imposes additional requirements on U and a, so we provide a direct proof.

Define N as in the proof of Proposition (7.6). By Theorem 3.1.2 of Kunita, M is a $C^{1,\beta}$ process for each $\beta < \delta$, and by our assumption on U, the same is true of N. By Theorems 3.4.6 and 4.7.2 of Kunita, F is a flow of C^1 diffeomorphisms and $F_t,\ t \geq 0$ is a $C^{1,\beta}$ semimartingale for each $\beta < \delta$.

Fix x in $\mathbb{R}^d$. By Theorem 3.3.3 of Kunita applied to (7.2) with $s = 0$, we have

(8.6) $$\frac{\partial F_t^i}{\partial x^j}(x) = \delta_{ij} + \sum_{m=1}^d \int_0^t \frac{\partial F_s^m}{\partial x^j}(x)\frac{\partial N^i}{\partial x^m}(F_s(x), ds).$$

Apply Itô's formula to $\det\left(\frac{\partial F_t}{\partial x}(x)\right)$ to see that $\det\left(\frac{\partial F_t}{\partial x}(x)\right) - 1$ equals

(8.7) $$\sum_{i,j=1}^d \sum_{\substack{\sigma \\ \sigma(i)=j}} \varepsilon(\sigma)\int_0^t \frac{\partial F^1}{\partial x^{\sigma(1)}} \cdots \overset{\vee}{\frac{\partial F^i}{\partial x^j}} \cdots \frac{\partial F^d}{\partial x^{\sigma(d)}} d\left(\frac{\partial F^i}{\partial x^j}\right)$$
$$+\frac{1}{2}\sum_{i,j=1}^d \sum_{\substack{k,\ell=1 \\ k\neq i}}^d \sum_{\substack{\sigma \\ \sigma(i)=j \\ \sigma(k)=\ell}} \int_0^t \frac{\partial F^1}{\partial x^{\sigma(1)}} \cdots \overset{\vee}{\frac{\partial F^i}{\partial x^j}} \cdots \overset{\vee}{\frac{\partial F^k}{\partial x^\ell}} \cdots \frac{\partial F^d}{\partial x^{\sigma(d)}} d\left\langle \frac{\partial F^i}{\partial x^j}, \frac{\partial F^k}{\partial x^\ell}\right\rangle,$$

where the sum is over all permutations σ of $\{1,2,\ldots,d\}$, $\varepsilon(\sigma)$ denotes the sign of σ, the symbol $\vee$ over a factor indicates that it is left out of the product, and we have suppressed the arguments x and s of F. The sums in the first term reduce to $\sum_{i=1}^d \sum_\sigma$, while those in the second term reduce to $\sum_{i=1}^d \sum_{\substack{k=1\\k\neq i}}^d \sum_\sigma$. Plugging (8.6) into the first term of (8.7) gives

$$\sum_{i=1}^d \sum_{m=1}^d \int_0^t \left(\sum_\sigma \varepsilon(\sigma) \frac{\partial F_s^1}{\partial x^{\sigma(1)}} \cdots \frac{\partial F_s^m}{\partial x^{\sigma(i)}} \cdots \frac{\partial F_s^d}{\partial x^{\sigma(d)}} \right) \frac{\partial N^i}{\partial x^m}(F_s(x), ds).$$

The quantity inside parentheses is a determinant in which the row $\frac{\partial F^m}{\partial x}$ appears twice unless $i = m$. Thus, this term equals $\int_0^t \det(\frac{\partial F_s}{\partial x}(x))(\operatorname{div} N)(F_s(x), ds)$.

The second term in (8.7) is treated similarly. First, by (8.6) and Theorems 3.1.2, 3.2.4, and 3.3.3,

$$\left\langle \frac{\partial F_\cdot^i}{\partial x^j}(x), \frac{\partial F_\cdot^k}{\partial x^\ell}(x) \right\rangle_t = \sum_{m,n=1}^d \int_0^t \frac{\partial F_s^m}{\partial x^j}(x) \frac{\partial F_s^n}{\partial x^\ell}(x) \frac{\partial^2 a^{ik}}{\partial x^m \partial y^n}(F_s(x), F_s(x), s) ds.$$

With this, the second term of (8.6) becomes

$$\frac{1}{2} \sum_{i=1}^d \sum_{\substack{k=1\\k\neq i}}^d \sum_{m,n=1}^d \int_0^t \left(\sum_\sigma \varepsilon(\sigma) \frac{\partial F^1}{\partial x^{\sigma(1)}} \cdots \frac{\partial F^m}{\partial x^{\sigma(i)}} \cdots \frac{\partial F^n}{\partial x^{\sigma(k)}} \cdots \frac{\partial F^d}{\partial x^{\sigma(d)}} \right) \frac{\partial^2 a^{ik}}{\partial x^m \partial y^n}(F_s(x), F_s(x), s) ds.$$

The factor in parentheses is a determinant. It is non–zero when $m = i$ and $n = k$, and also when $m = k$ and $n = i$. The expression equals

$$\frac{1}{2} \sum_{i=1}^d \sum_{\substack{k=1\\k\neq i}}^d \int_0^t \det\left(\frac{\partial F_s}{\partial x}(x)\right) \left(\frac{\partial^2 a^{ik}}{\partial x^i \partial y^k}(F_s(x), F_s(x), s) - \frac{\partial^2 a^{ik}}{\partial x^k \partial y^i}(F_s(x), F_s(x), s) \right) ds,$$

the sign reversal being due to an interchange of rows when $m = k$ and $n = i$. When $i = k$, the terms in parentheses cancel, and so we may lift the restriction $k \neq i$ in the sum.

Making these substitutions in (8.7), we may check that (8.5) is a solution of the resulting equation by Itô's formula and Theorems 3.2.4 and 3.1.2 of Kunita (1990). □

(8.8) **Remark.** *Conditions for incompressibility.* Under the conditions of Theorem (8.4), F_t is a diffeomorphism, so F_t^{-1} is locally Lipschitz. By Remark (4.7), $\lambda \circ F_t^{-1}$ is absolutely continuous with respect to λ. By Proposition (8.2), F is incompressible if and only if $J \equiv 1$. If div M does not vanish identically, then the flow F will not preserve Lebesgue measure, for the bounded variation terms in (8.5) cannot cancel the integral of div M. Second, if div M vanishes but U and M are dependent in such a way that $\operatorname{div} U = \alpha$ everywhere, then F is incompressible even though U may be divergent. Finally, suppose that U and M are homogeneous and independent. To have $\operatorname{div} U = \alpha$, we must have $\operatorname{div} U$ and α identically equal to a constant, by homogeneity and independence. But by homogeneity of U, $0 = \frac{\partial}{\partial x^j}\mathbb{E}U^i(x,t) = \mathbb{E}\frac{\partial U^i}{\partial x^j}(x,t)$, so this constant must be zero. Thus, in this case, incompressibility of F is equivalent to $\operatorname{div} M \equiv 0, \alpha \equiv 0$, and $\operatorname{div} U \equiv 0$. □

(8.9) **Remark.** *Canonical Markov process.* Suppose that U is homogeneous, stationary, and non–divergent. Recall Remark (6.15) and the discussion at the beginning of Section 7. In the case of (7.1), Papanicolaou and Varadhan (1982), Osada (1982), and others have observed that the process $U \circ \theta_t,\ t \geq 0$ is a stationary Markov process, with W supplying the randomness. It is commonly called the *canonical Markov process*. Moreover, if for some $t > 0$ the shifts $\tau_t\sigma_z, z \in \mathbb{R}^d$ are all ergodic, then $U \circ \theta_t,\ t \geq 0$ is ergodic as well. See also Remark (5.6)(v).

Now use (7.2) to model the simultaneous motion of all particles in $\mathbb{R}^d$, such that M has independent increments, is independent of U, and has local characteristic a satisfying $a(x,x,t) = \beta^2 I$. Then the one–point motion $F_t(0),\ t \geq 0$ satisfies (7.1) with $W_t = \frac{1}{\beta}\int_0^t M(F_s(0), ds)$, and so $U \circ \theta_t,\ t \geq 0$ is stationary, as above. Yet we may choose M in such as way that $\operatorname{div} M \neq 0$, and so F will be compressible and θ_t will not preserve $\mathbb{P}$.

There is no contradiction here. The compressibility due to M does not bias the view of U, but it does bias the view of M. However, in the canonical Markov process, M is not observed. □

(8.10) **Conjecture.** *Compressibility and independence.* The previous remark suggests the following conjecture. Let U and $\hat{U}$ be independent homogeneous velocity fields and suppose $\operatorname{div} U \equiv 0$. Let X move in the velocity field $U + \hat{U}$. The conjecture is that the view $U \circ \theta_t$ of U from (X_t, t) has the same distribution as the view $U \circ \tau_t$ of U from $(0, t)$, even if $\operatorname{div} \hat{U} \neq 0$. □

9 Evolution of statistics in compressible flows

Return to the setting of Theorem (5.4). If G is compressible (not singular, cf. Section 4), the Lagrangian observations $\theta_t,\ t \in \mathbb{T}$ will not be stationary, but θ_t will have a density $\rho_t(0)$ with respect to $\mathbb{P}$. Davis (1982) appears to have been the first to recognize that *compressibility* biases Lagrangian observations of a homogeneous velocity field toward features seen in regions where particle density is higher due to convergence. He cites the Stokes drift and freely–drifting floating instruments such as SOFAR floats which are prescribed to remain at the surface or at a certain depth, and so the two–dimensional velocity field in which they move may well be divergent even though the three–dimensional field is not. Indeed, Middleton and Garrett (1986) confirm this effect on Lagrangian trajectories of icebergs, which are found to rotate preferentially cyclonically (anticlockwise in the Northern Hemisphere) even though the Eulerian field U has predominately cyclonic rotation, because cyclonic motion occurs in regions of downwelling, which are regions of convergence.

Davis (1982) derived a version of Theorem (4.6) and asserted that "Except in pathologically simple flows, it can be expected that long after deployment in homogeneous and stationary Eulerian fields the Lagrangian statistics will become stationary." This still appears to be reasonable, but has resisted proof except in a few cases which we discuss below. Such results would have wider ramifications, for stationarity of Lagrangian observations plays a crucial role in most homogenization results for $X_t, \quad t \geq 0$. Showing the existence of a limiting distribution for the particle velocity $U(X_t, t)$ as $t \to \infty$ is a necessary first step toward homogenization results for compressible flows.

One can make the existence of a limiting distribution plausible by considering a homogeneous, stationary, divergent velocity field U defined with time set $\mathbb{T} = \mathbb{R}$. We assume that U satisfies the conditions of Remark (6.10) and let F be the solution of (6.3). Let $\rho_{r,s}$ denote the density of $\lambda \circ F_{r,s}^{-1}$ with respect to Lebesgue measure λ. Let U^+ be defined by $U^+(\omega) = \{U(\omega, x, t),\ x \in \mathbb{R}^d, t \geq 0\}$, so that U^+ pulls out from ω the present and future values of U. Then by (5.1) and Theorem (4.6), $\mathbb{E}f(U^+ \circ \theta_t) = \mathbb{E}f(U^+ \circ \tau_t)\rho_{0,t}(0)$. By stationarity, this equals $\mathbb{E}f(U^+)\rho_{-t,0}(0)$. If we condition inside the expectation on U^+, we come naturally to consider the function h_t which satisfies

(9.1) $$h_t(U^+) = \mathbb{E}[\rho_{-t,0}(0)|U^+].$$

Thus, $\mathbb{E}f(U^+ \circ \theta_t) = \mathbb{E}f(U^+)h_t(U^+)$. To interpret, when observing the present and future value of U from the location of a moving particle, the distribution of U^+ so observed has a density h_t with respect to the Eulerian distribution of U^+. To calculate h_t, one fixes possible values of U^+ and averages $\rho_{-t,0}(0)$ over the part of U before time 0. If U is smooth enough, we may write $\rho_{-t,0}(0) = \exp\left(-\int_{-t}^{0}(\operatorname{div} U)(F_{s,0}^{-1}(0), s)ds\right)$ using Proposition (8.2) and (8.5). Substituting into (9.1),

$$h_t(U^+) = \mathbb{E}\left[\exp\left(-\int_{-t}^{0}(\operatorname{div} U)(F_{s,0}^{-1}(0), s)ds\right)\middle| U^+\right]. \tag{9.2}$$

Here one fixes U^+, then integrates the divergence back in time along the trajectory of the particle which is at 0 at time 0. This will emphasize outcomes for which the divergence is negative along this trajectory, that is, outcomes for which the particle arriving at 0 at time 0 has experienced convergence ($\operatorname{div} U < 0$).

To the extent that $U(\cdot, -t)$ depends more weakly on U^+ as t increases, it is plausible that h_t will converge as $t \to \infty$ to a function h, which would serve as the long–time density of $U^+ \circ \theta_t$ with respect to the Eulerian distribution of U^+. But this is more difficult to prove than one might hope.

9.1 Examples from homogenization theory

A few articles (see below) have given results on the convergence to Brownian motion of the rescaled particle location process $\varepsilon X_{t/\varepsilon^2},\ t \geq 0$ as $\varepsilon \to 0$ in the presence of compressibility. The first step in each case is to establish the convergence in distribution of Lagrangian observations. Here we simply place these results in the present context of flows in order to show the role of compressibility and the specialized nature of the examples.

(9.3) **Example.** *Spatially varying diffusion coefficient.* Let $\beta^{(1)}, \ldots, \beta^{(n)}$ be homogeneous, stationary, jointly continuous random vector fields on $\mathbb{R}^d \times \mathbb{R}_+$. Write β for the $d \times n$ matrix $[\beta^{(1)} \cdots \beta^{(n)}]$. Let W be an n–dimensional Wiener process independent of β. Define a collection of martingales by $M(x,t) = \int_0^t \beta(x,s)dW_s$ for $t \geq 0$ and x in $\mathbb{R}^d$. Then M has local characteristic a given by $a(x,y,t) = \beta(x,t)\beta^T(y,t)$. One readily checks that a satisfies (7.4) and (7.5) provided that

$$\int_n^{n+1}\left(\sup_{x\in\mathbb{R}^d}\frac{|\beta^{(i)}(x,t)|}{1+|x|}\right)^2 dt < \infty, \tag{9.4}$$

$$\int_n^{n+1}\left(\sup_{x,y\in K, x\neq y}\frac{|\beta^{(i)}(x,t)-\beta^{(i)}(y,t)|}{|x-y|}\right)^2 dt < \infty, \tag{9.5}$$

almost surely. By Theorem 3.1.1 of Kunita (1990), M has a modification which is jointly continuous. Let F be the flow based on M via (7.2) with $U \equiv 0$. Then F will be compressible unless the $\beta^{(i)}$ satisfy further conditions, cf. Remark (8.8).

The one–point motion $X_t,\ t \geq 0$ satisfies $dX_t = \beta(X_t, t)dW_t$ for $t > 0$. It can be considered to be a diffusion on $\mathbb{R}^d$ with random diffusion coefficient β. Its generator at time t is $(L_t f)(x) = \frac{1}{2}\sum_{i,j=1}^d a^{ij}(x, x, t)\frac{\partial^2 f}{\partial x^i \partial x^j}(x)$. The question is whether $\beta \circ \theta_t$ converges in some sense as $t \to \infty$.

Suppose now that the $\beta^{(i)}$ do not depend on t, are ergodic with respect to all non–trivial spatial translations, and are such that $C_1 \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d a^{ij}(\omega, x, x)\xi_i\xi_j \leq C_2 \sum_{i=1}^d \xi_i^2$, for all $x, \xi \in \mathbb{R}^d$ and $\omega \in \Omega$, where $0 < C_1 \leq C_2 < \infty$ are constant. Then Papanicolaou and Varadhan (1982) have shown that $\beta \circ \theta_t,\ t \geq 0$ is an ergodic Markov process which converges in distribution as $t \to \infty$. □

(9.6) **Example.** *Periodic divergent velocity field.* Let $\mathbb{D} = [0, 1)^d$ with addition modulo 1 in each component. Let $u, \beta^{(1)}, \ldots, \beta^{(n)}$ be deterministic vector fields on $\mathbb{D}$, where u is continuous and Lipschitz and $\beta = [\beta^{(1)} \cdots \beta^{(n)}]$ satisfies the same conditions as in the previous example. Let Z be uniformly distributed on $\mathbb{D}$. Set $U(x) = u(x + Z)$ and $B(x) = \beta(x + Z)$ for x in $\mathbb{D}$. Then U and B are jointly homogeneous random fields on $\mathbb{D}$ as in Remark (6.14). Let W be a Wiener process independent of Z and consider the flow F generated by the equation $F_{s,s+t}(x) = x + \int_s^{s+t} U(F_{s,r}(x))dr + \int_s^{s+t} B(F_{s,r}(x))dW_r$, for $t \geq 0$. Then F is compressible unless u and β satisfy further conditions, and so $(U, B) \circ \theta_t,\ t \geq 0$ may be non–stationary. Let $Y_t = F_t(0) + Z$ modulo $\mathbb{D}$. Then Y_0 is uniformly distributed on $\mathbb{D}$ and Y satisfies $dY_t = u(Y_t)dt + \beta(Y_t)dW_t$ for $t > 0$. Bhattacharya (1985) shows that Y_t converges in distribution as $t \to \infty$. But $U \circ \theta_t = U(\cdot + F_t(0)) = u(\cdot + Y_t)$, and in the same way, $B \circ \theta_t = B(\cdot + Y_t)$, so that $(U, B) \circ \theta_t$ converges in distribution as $t \to \infty$ as well. The key here is that the divergence of U is overcome by the diffusion. □

9.2 Regenerative velocity field

Here we give a concrete instance in which the program outlined at the beginning of this section can be carried out directly. It makes clear how the loss of memory in U over time limits the effect of the integral in (9.2) to times near 0. Unfortunately, we must make a drastic independence assumption.

This example first appeared in Zirbel (1993), but the formulation is improved here.

Let $U(\cdot, t),\ t \geq 0$ be a (possibly delayed) regenerative process. That is, let there be a sequence $T_0, T_1, T_2, \ldots$ of random times which form a (possibly delayed) renewal process and suppose that the collection $\{U(\cdot, T_i + s), s \geq 0\}$ has the same distribution for $i = 0, 1, 2, \ldots$ and is independent of $\sigma(U_t, t < T_i)$ for $i = 0, 1, 2, \ldots$. Suppose also that U is homogeneous, in the sense that for all n and k in $\mathbb{N}$, $x_1, \ldots, x_n$ in $\mathbb{R}^d$, $t_1, \ldots, t_n \geq 0$, and $i_1, \ldots, i_k$ in $\mathbb{N}$, the collection $\{U(x_1 + z, t_1), \ldots, U(x_n + z, t_n), T_{i_1}, \ldots, T_{i_k}\}$ has the same distribution for all z in $\mathbb{R}^d$. On the canonical space for U and T we may assume that $T_i \circ \sigma_z = T_i$. Assume that U satisfies the conditions of Remark (6.10) (note that U as given is right continuous) and let $\rho_{s,s+t}$ be the density of $\lambda \circ F_{s,s+t}^{-1}$ guaranteed by Remark (6.13) and Proposition (8.1).

The next result shows that, at time t, the effect of compressibility extends back in time only to the time of last regeneration.

(9.7) **Theorem.** Fix $t \geq 0$ and let $L = 0$ if $T_0 > t$, otherwise $L = \max(T_i : T_i \leq t)$. Then $\mathbb{E}f(U^+ \circ \theta_t) = \mathbb{E}f(U^+ \circ \tau_t)\rho_{L,t}(0)$ for all positive measurable f.

Proof: We may work on the canonical space for U and T. As in the proof of Theorem (5.4), we see that $\sigma_{F_{L,t}(0)}\left(\sigma_{F_{0,L}(0)}\omega\right) = \sigma_{F_{0,t}(0)}\omega$, since L is unaffected by spatial shifts. Thus,

$$\mathbb{E}f(U^+ \circ \theta_t) = \mathbb{E}f(U^+ \circ \tau_t \circ \sigma_{F_{0,t}(0)}) = \mathbb{E}f(U^+ \circ \tau_t \circ \sigma_{F_{L,t}(0)})\rho_{0,L}(0), \tag{9.8}$$

by Theorem (4.6) with $X = F_{0,L}(0)$ and thus $G = F_{0,L}$. Conditioning inside the expectation on $U^+ \circ \tau_t, F_{L,t}$, and L, we are left to consider $\mathbb{E}[\rho_{0,L}(0)|U^+ \circ \tau_t, F_{L,t}, L]$. We show below that $F_{0,L}$ is conditionally independent of $(U^+ \circ \tau_t, F_{L,t})$ given L, and so this conditional expectation equals $\mathbb{E}[\rho_{0,L}(0)|L]$. But this equals 1 almost surely, for let $h : \mathbb{R} \to \mathbb{R}_+$ be measurable and consider that $\mathbb{E}h(L) = \mathbb{E}h(L \circ \sigma_{F_{0,L}(0)}) = \mathbb{E}h(L)\rho_{0,L}(0)$ by Theorem (4.6). Returning to (9.8), we may now write $\mathbb{E}f(U^+ \circ \theta_t) = \mathbb{E}f(U^+ \circ \tau_t)\rho_{L,t}(0)$, by Theorem (4.6) applied now with $X = F_{L,t}(0)$.

It remains to establish that $F_{0,L}$ is conditionally independent of $(U^+ \circ \tau_t, F_{L,t})$ given L. We must show that for all positive measurable g, $\mathbb{E}[g(U^+ \circ \tau_t, F_{L,t})|L, F_{0,L}]$ is a function of L alone. This is trivially true when $L = 0$, for then $F_{0,L}$ is the identity mapping. Thus, we may restrict attention to the event $\{L > 0\} = \{T_0 > t\}$.

Let $I = \max(i : T_i < t)$. Then $L = T_I$ and $I \geq 0$. Let h be a function for which $h(I, L, F_{0,L}) = \mathbb{E}[g(U^+ \circ \tau_t, F_{L,t})|I, L, F_{0,L}]$. We will show that $h(I, L, F_{0,L})$ depends only on L, which is stronger than what we need to show. First, we claim that for $i = 0, 1, 2, \ldots,$

$$\mathbb{E}[1_i(I)g(U^+ \circ \tau_t, F_{L,t})|T_i, F_{0,T_i}] = h(i, T_i, F_{0,T_i})\mathbb{E}[1_i(I)|T_i, F_{0,T_i}]. \tag{9.9}$$

Indeed, for all positive, measurable φ,

$$\begin{aligned} \mathbb{E}\varphi(T_i, F_{0,T_i})1_i(I)g(U^+ \circ \tau_t, F_{L,t}) &= \mathbb{E}\varphi(L, F_{0,L})1_i(I)\mathbb{E}[g(U^+ \circ \tau_t, F_{L,t})|I, L, F_{0,L}] \\ &= \mathbb{E}\varphi(T_i, F_{0,T_i})h(i, T_i, F_{0,T_i})\mathbb{E}[1_i(I)|T_i, F_{0,T_i}], \end{aligned}$$

which proves (9.9). Now we work with the left–hand side of (9.9):

$$\begin{aligned} \mathbb{E}[1_i(I)g(U^+ \circ \tau_t, F_{L,t})|T_i, F_{0,T_i}] &= 1_{\{T_i \leq t\}}\mathbb{E}[1_{\{T_{i+1} > t\}}g(U^+ \circ \tau_t, F_{T_i,t})|T_i, F_{0,T_i}] \\ &= 1_{\{T_i \leq t\}}\mathbb{E}[\tilde{g}(U^+ \circ \tau_{T_i}, T_i, T_{i+1} - T_i)|T_i, F_{0,T_i}], \end{aligned}$$

for some function $\tilde{g}$ (which may depend on t, which is fixed). By the regenerative property of U, this equals $1_{\{T_i \leq t\}}\ell(T_i)$ for some function ℓ. Similarly, on the right–hand side of (9.9), $\mathbb{E}[1_i(I)|T_i, F_{0,T_i}]$ equals $1_{\{T_i \leq t\}}k(T_i)$ for some k. Neither ℓ nor k depend on i. In light of (9.9), for all i we have $\ell(T_i) = h(i, T_i, F_{0,T_i})k(T_i)$ on the set $\{T_i \leq t\}$. Thus, on the set $\{L > 0\}$, we have $\ell(L) = h(I, L, F_{0,L})k(L)$, which shows that $h(I, L, F_{0,L})$ depends only on L, which was to be shown. □

In addition to what preceded Theorem (9.7), assume now that U is stationary with time set $\mathbb{T} = \mathbb{R}$. Let K denote the time of the last renewal before time 0.

(9.10) **Corollary.** Suppose that $\mathbb{E} \sup_{K \leq r \leq 0} \rho_{r,0}(0) < \infty$ and that K is finite almost surely. Then $U^+ \circ \theta_t$ converges in distribution as $t \to \infty$, and the limiting distribution has density h with respect to the distribution of U^+, where h satisfies $h(U^+) = \mathbb{E}[\rho_{K,0}(0)|U^+]$.

Proof: Let f be a positive measurable function on the space in which U^+ takes its values and suppose that f is bounded by C. Then, $\mathbb{E}f(U^+ \circ \theta_t) = \mathbb{E}f(U^+)\rho_{\max(K,-t),0}(0)$ by (9.7) and the stationarity of U. As $t \to \infty$, $\max(K, -t) \to K$ almost surely. Also, $f(U^+)\rho_{\max(K,-t),0}(0)$ is dominated by $C \sup_{K \le r \le 0} \rho_{r,0}(0)$, which is assumed to be integrable. Thus, by the dominated convergence theorem, $\lim_{t\to\infty} \mathbb{E} f(U^+ \circ \theta_t) = \mathbb{E} f(U^+)\rho_{K,0}(0) = \mathbb{E} f(U^+)h(U^+)$ by the definition of h. $\square$

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