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TRANSLATION AND DISPERSION OF MASS BY ISOTROPIC BROWNIAN FLOWS

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Abstract. We study the long-term translation and dispersion of a mass distribution carried by an isotropic Brownian flow on $\mathbb{R}^d$. We use the variance of the center of mass as a measure of translation and the mean of the centered spatial second moments as a measure of dispersion. We find the exact growth rates of these quantities in essentially all cases. When the mass distribution is diffuse, the rates of growth depend strongly on the degree of compressibility in the flow. For example, in two dimensions we show that the variance of the center of mass grows more slowly than logarithmically in the incompressible case, while it grows linearly in highly compressible flows. Our method involves a detailed analysis of the two-point separation process and uses methods developed in a companion paper for the analysis of mean occupation times of one-dimensional Markov processes.

Key words. Stochastic flows; Brownian flows; mass transport

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1 Introduction and overview of results

We are interested in the long-term translation and spreading of a passive tracer carried by an isotropic Brownian flow on $\mathbb{R}^d$. The initial tracer distribution is given by a measure M_0 normalized so that $M_0(\mathbb{R}^d) = 1$. The tracer is moved according to the flow F, so that a particle starting at x at time 0 arrives at $F_t x$ at time t. The image of M_0 under F_t is a random measure M_t which tells the tracer's location at time t.

It is known (see KUNITA (1990), Theorem 4.3.10) that for each compact set A, the expected value of $M_t(A)$ goes to 0 as $t \to \infty$, so the tracer mass is dissipated by the flow. We want to know *how* the mass is dissipated, that is, how M_t develops as a spatial object. In this paper we study two descriptors of M_t:

$$C_t \equiv \int M_t(dx)x = \int M_0(dx)F_t x \tag{1.1}$$

$$D_t \equiv \int M_t(dx)(x - C_t)(x - C_t)^T = \int M_0(dx)(F_t x - C_t)(F_t x - C_t)^T \tag{1.2}$$

We call C_t the *centroid* or *center of mass* and D_t the *dispersion matrix*. We will assume, without loss of generality, that $C_0 = 0$. The goal of the paper is to say what we can about the random variables C_t and D_t as $t \to \infty$.

Our results concern the rate at which $\mathrm{E}|C_t|^2$ and the trace of the matrix $\mathrm{E}D_t$ grow as $t \to \infty$. The quantity $|C_t|^2$ measures the distance of the centroid from the origin, and thus measures the amount of translation of the tracer by the flow. The trace of D_t is a single number reflecting how spread out the tracer is relative to its center of mass, and so measures dispersion.

ZIRBEL and ÇINLAR (1996a) consider the second moments of C_t and the mean of D_t in general Brownian flows, obtaining the formulas we begin with here (see Section 2) and some results on growth rates as $t \to \infty$ in the isotropic case. ZIRBEL (1995) considers the processes C and D themselves, including the stochastic equations they satisfy and computations of their quadratic variations.

We are motivated by applied problems of the following sort: a tracer (such as a quantity of crude oil) falls into a complex fluid flow (such as the ocean during stormy weather). We want to understand the motion of the tracer and predict where it will go and when. Such problems are studied in statistical turbulence (see MONIN and YAGLOM (1971), Section 24.3), oceanography (see DAVIS (1991)), and groundwater hydrology (see DAGAN (1990)). A common element in these works is that the motion is determined by a random velocity field v with known statistical characteristics through the trajectory equation $\frac{d}{dt}F_t x = v(F_t x, t)$.

Our model departs from these in that our flow F is based on a stochastic differential equation which gives the flow a very limited temporal dependence. Heuristically, we may say that the flow is determined by a random velocity field which is "δ-correlated" in time (white noise in time). Such flows appear in the statistical turbulence literature; see, for example, KIMURA and KRAICHNAN (1993). The rigorous formulation of this idea in terms of Brownian flows and stochastic equations has been developed since roughly 1980. We provide a brief description and further references in Section 2.

A perennial topic of interest in the statistical turbulence literature is to find an evolution equation for the mean concentration m_t (i.e., the density of the mean measure of M_t). See, for example, LIPSCOMBE et al. (1991). The equation found is often some variant of the heat equation. For Brownian flows this is always the case because of the simple temporal dependence. In the isotropic case, we get the heat equation exactly:

$$\frac{\partial m_t}{\partial t} = \tfrac{1}{2} b_0 \Delta m_t, \qquad t \geq 0 \tag{1.3}$$

with initial condition $m_0 = M_0$. (The one-point motion, $F_t x,\ t \geq 0$, is Brownian motion on $\mathbb{R}^d$ with $\mathrm{Var}(F_t^i x) = b_0 t$ for some constant $b_0 > 0$.) Equation (1.3) applies for every flow in this article.

The virtue of isotropic Brownian flows is that there are many different flows which are identical at the level of the mean concentration but which actually treat the mass distribution in strikingly different ways. Understanding these differences gives us a frame of reference for the problem of mass transport problem due to a classical random velocity field.

Isotropic Brownian flows differ from one another in their two-point motions, $(F_t x, F_t y)$, $t \geq 0$, for x and y in $\mathbb{R}^d$. These are Markov processes. The quantities we consider here, namely the second moments of C_t and the mean of D_t, can be reduced to the study of the two-point motion. This is similar to the classical case (see MONIN and YAGLOM (1971), Section 24.3), except that here the law of the two-point motion is known and is amenable to exact analysis.

In fact, we find in Section 2 that we can focus attention on the expected amount of time the pair-separation process $Z_t = |F_t x - F_t y|$ spends in certain regions. This process also has the Markov property. The companion article ZIRBEL (1997) gives a general method for finding the asymptotic behavior of such "occupation times" of one-dimensional Markov processes. It includes a number of new estimates which are crucial for the cases we encounter here. See the end of Section 2 for further discussion.

Benchmark examples

It will be useful to keep in mind two extreme examples.

(1.4) **Example. Lock-step motion**. Let $B_t,\ t \geq 0$, be a standard Brownian motion on $\mathbb{R}^d$ starting at 0. The flow F defined by

$$F_t x = x + B_t, \qquad t \geq 0,\ x \in \mathbb{R}^d$$

is an incompressible isotropic Brownian flow on $\mathbb{R}^d$. It is a trivial case because all points move in lock step. In particular, for any initial measure M_0 with $C_0 = 0$, we have

$$C_t = B_t, \qquad D_t = D_0, \qquad t \geq 0.$$

Thus,

$$\mathrm{E}|C_t|^2 = dt, \qquad \mathbf{E}D_t = D_0, \qquad t \geq 0.$$

The center of mass moves like a Brownian particle and there is no dispersion. The mean concentration m_t still solves equation (1.3) with $b_0 = 1$. This example is excluded from the results below because the spatial correlations in F do not drop off quickly enough (in fact, they do not drop off at all). Nevertheless, it sets the standard for how fast the centroid can move: $\mathbf{E}|C_t|^2$ can grow no faster than $b_0 dt$.

(1.5) **Example. Independent motion**. Suppose the initial measure M_0 is purely atomic, with n atoms of equal mass. Allow each atom to move according to a standard Brownian motion, and let these be mutually independent. Let M_t be the measure formed by the atoms at time t. Then we have

$$\mathrm{E}|C_t|^2 = \tfrac{1}{n} dt, \qquad \mathrm{E}D_t = D_0 + (1 - \tfrac{1}{n}) tI, \qquad t \geq 0,$$

where I is the identity matrix. There is no flow F, but we can think of this case as the limit of flows with ever shorter spatial correlation length. Dispersion now grows linearly in time, but the center of mass moves slowly when the number of atoms is large. The mean concentration m_t solves the heat equation (1.3) with $b_0 = 1$. This example sets the standard for how fast dispersion can grow: $\mathrm{trE}D_t$ can grow no faster than $b_0 dt$.

Overview of results

In order to simplify the discussion of the results, we will assume for now that M_0 is diffuse (i.e., $M_0(\{x\}) = 0$ for all x in $\mathbb{R}^d$). The precise statements of the results (Theorems (4.6), (4.3), (6.4), and (7.5)) allow M_0 to have an atomic component, which has the effect of adding linear terms to the behavior we will see here. We also assume that the one-point motion is a *standard* Brownian motion, so the constant b_0 appearing in (1.3) equals 1.

The precise conditions for these results are listed in Sections 4, 6, and 7. Roughly speaking, the spatial correlations in F must decay quickly enough (like $1/r^2$), the initial measure M_0 must be somewhat localized in space, and in some cases, it must not have strong concentrations of mass (apart from the atoms).

The results are broken into several cases depending on the spatial dimension d and the sign of the top Lyapunov exponent λ of the flow, defined in (3.6) below. The sign of λ is closely connected to the two-point motion: For an isotropic Brownian flow on $\mathbb{R}^d$,

$$\lambda \geq 0 \quad \text{if and only if} \quad \mathrm{P}(\lim_{t\to\infty} |F_t x - F_t y| = 0) = 0$$
$$\lambda < 0 \quad \text{if and only if} \quad \mathrm{P}(\lim_{t\to\infty} |F_t x - F_t y| = 0) > 0,$$

where x and y are distinct points in $\mathbb{R}^d$. In this manner, our results can be understood in terms of the possibility that the distance between two points in the flow converges to 0. (One caveat: we give no results for the boundary case $\lambda = 0$, so we must take care in applying the first condition.)

A flow F is said to be *incompressible* if it preserves Lebesgue measure on $\mathbb{R}^d$; that is, with probability one,

$$\mathrm{Leb}(F_t A) = \mathrm{Leb}(A) \text{ for all } t \geq 0, \text{ all Borel } A \subseteq \mathbb{R}^d.$$

Non-trivial incompressible flows have $\lambda > 0$, and so fall into the first category above. The trivial flow of Example (1.4) has $\lambda = 0$.

First, consider flows with $\lambda > 0$. This includes all non-trivial incompressible flows. For $d \geq 3$, we have

$$\begin{aligned} \mathbf{E}|C_t|^2 &\leq A_1, & t \geq 0 \\ \mathrm{tr}\mathbf{E}D_t &\sim dt, & t \to \infty, \end{aligned}$$

where A_1 is a finite constant. The precise result and the definition of A_1 are given in Theorem (4.3). As a consequence of the first statement, the centroid C_t converges almost surely to a random vector C_∞. Heuristically, the mass distribution spreads as quickly as possible (since $\mathrm{tr}\mathbf{E}D_t$ grows linearly), but it translates very little – so little in fact that the center of mass essentially ceases to move for large t.

This remarkable result depends crucially on the assumption that M_0 is diffuse. If M_0 has even a small atomic component, the center of mass will not converge; rather, its second moment will grow linearly in time.

Suppose now that $d = 2$ and the flow F is incompressible (so that $\lambda > 0$). We find in Section 6 that

$$\begin{aligned} \mathbf{E}|C_t|^2 &= o(\log t), & t \to \infty \\ \mathrm{tr}\mathbf{E}D_t &\sim 2t, & t \to \infty \end{aligned}$$

Translation of the center of mass is still very weak in this case while dispersion is strong. The exact rate of growth of $\mathbf{E}|C_t|^2$ is not known. When $d = 2$ and $\lambda > 0$ but the flow is compressible (does not preserve Lebesgue measure),

$$\begin{aligned} \mathbf{E}|C_t|^2 &\sim A_2 \log t, & t \to \infty \\ \mathrm{tr}\mathbf{E}D_t &\sim 2t, & t \to \infty \end{aligned}$$

Thus, introducing compressibility into the flow increases the amount of translation of the mass distribution, but does not reduce dispersion (to first order).

Next, consider flows with $\lambda < 0$. In Section 7 we show that, for $d = 2$,

$$\begin{aligned} \mathbf{E}|C_t|^2 &\sim 2t, & t \to \infty \\ \mathrm{tr}\mathbf{E}D_t &\sim A_3 \tfrac{t}{\log t}, & t \to \infty \end{aligned}$$

where A_3 is a finite constant. Thus, making λ cross 0 to negative values results in the center of mass having a second moment which increases linearly, as it does for the pace-setting lock-step motion of Example (1.4). Dispersion relative to the center of mass is of a lower order than it is for flows with positive top Lyapunov exponent.

The two remaining cases are treated elsewhere. For $d = 3$ and $\lambda < 0$, ZIRBEL and ÇINLAR (1996a), Proposition 3.5, shows that

$$\begin{aligned} \mathrm{E}|C_t|^2 &\sim (3 - A_4)t, && t \to \infty \\ \mathrm{tr}\mathbf{E}D_t &\sim A_4 t, && t \to \infty \end{aligned}$$

where A_4 is a constant with $0 < A_4 < 3$. For the case $d = 1$ and $\lambda < 0$, ZIRBEL (1993), Theorem 2.20, shows that

$$\begin{aligned} \mathbf{E}|C_t|^2 &\sim t, && t \to \infty \\ \mathbf{E}D_t &= A_5(t)\sqrt{t}, && t \to \infty \end{aligned}$$

where $0 < a \leq A_5(t) \leq b$ for all t large enough, a and b being finite constants.

There are no isotropic Brownian flows with $d = 1$ and $\lambda > 0$ or with $d > 3$ and $\lambda < 0$, so we have exhausted essentially all the cases. The case $\lambda = 0$ has been systematically excluded, although it could be treated using the methods in this paper if so desired. The example in Section 6 of ZIRBEL (1997) suggests that in the two-dimensional case we might have $\mathrm{E}|C_t|^2 \sim A_6\sqrt{t}\log t$ as $t \to \infty$, which would be intermediate between the rates for $\lambda > 0$ and $\lambda < 0$.

Overview of the paper

Section 2 contains background material on Brownian flows and formulas for the second moments of C_t and the mean of D_t. These are written in terms of an expected occupation time of the pair-distance process $Z_t = |F_t x - F_t y|$, which is a one-dimensional diffusion.

Sections 3 and 5 compile technical results concerning the separation process Z. Sections 4, 6, and 7 use these results and the methods of ZIRBEL (1997) to prove the main results of the paper (Theorems (4.3), (6.4), and (7.5)).

2 Random measures on isotropic Brownian flows

A Brownian flow on $\mathbb{R}^d$ is a collection $\{F_{st},\ 0 \le s \le t < \infty\}$ of random homeomorphisms from $\mathbb{R}^d$ to $\mathbb{R}^d$ with the following properties:

(*i*) $F_{st} \circ F_{rs} = F_{rt}, \qquad 0 \le r \le s \le t$

(*ii*) F_{ss} is the identity mapping for each $s \ge 0$

(*iii*) the mapping $(s,t,x) \mapsto F_{st}x$ is continuous almost surely

(*iv*) $F_{t_1 t_2}, \ldots, F_{t_{n-1} t_n}$ are independent when $t_1 \le t_2 \le \ldots \le t_n$

In the remainder of this article we will consider only the maps $F_{0t},\ t \ge 0$. For brevity, we will write F_t in place of F_{0t}.

The Brownian flows we consider are solutions of the stochastic differential equation

$$dF_t x = u(F_t x)dt + U(F_t x, dt), \tag{2.1}$$

where u is a deterministic vector field on $\mathbb{R}^d$ and U is a continuous Gaussian random vector field with mean 0 and covariance

$$\mathrm{Cov}(U^i(x,s), U^j(y,t)) = a^{ij}(x,y)\min(s,t).$$

The functions u and a characterize F; they are called the *drift* and *covariance* of the flow.

The precise meaning of (2.1) is given in KUNITA (1990) in terms of stochastic integral equations. Various interpretations of the equation are discussed in ZIRBEL and ÇINLAR (1996b). It is appropriate to think of the motion as being driven by a classical velocity field u plus a spatially correlated "white noise" term $U(F_t x, dt)$.

Isotropic Brownian flows

Isotropic Brownian flows were introduced and studied by BAXENDALE and HARRIS (1986) and LE JAN (1985). They are Brownian flows with

$$u(x) = 0, \qquad a(x,y) = b(x-y), \qquad x, y \in \mathbb{R}^d$$

where b is an *isotropic* covariance tensor, that is, $G^T b(Gz) G = b(z)$ for every orthogonal matrix G and every z in $\mathbb{R}^d$. These conditions insure that the law of F is invariant under translations, rotations, and reflections of $\mathbb{R}^d$.

In two or more dimensions, isotropy requires b to have the form

$$b^{ij}(z) = b_N(|z|)\delta_{ij} + \frac{z^i z^j}{|z|^2}(b_L(|z|) - b_N(|z|)), \qquad z \in \mathbb{R}^d \tag{2.2}$$

(See YAGLOM (1957) or YAGLOM (1987), Section 22.) The scalar functions b_L and b_N are given by explicit formulae in terms of finite measures Φ_P and Φ_S on $(0,\infty)$ (see BAXENDALE and HARRIS (1986), equation (2.16) or ZIRBEL and ÇINLAR (1996b), equation (2.24)). By varying our choices of Φ_P and Φ_S, we may change the degree of compressibility in the flow.

In this paper we will assume that $d \geq 2$ and that b satisfies the following condition, which guarantees that F_t is a C^1-diffeomorphism almost surely.

(2.3) **Condition.** The functions b_L and b_N are C^4-functions decaying to 0 at ∞.

The restriction that the covariance b decay to 0 at ∞ can be weakened. Suppose that F is an isotropic flow satisfying Condition (2.3), and let B be a standard Brownian motion on $\mathbb{R}^d$, and let $\mu_0 > 0$ be a constant. Define a flow $\tilde{F}$ by $\tilde{F}_t x = F_t x + \sqrt{\mu_0} B_t$. Then $\tilde{F}$ is isotropic with covariance $\tilde{b}^{ij}(z) = b^{ij}(z) + \mu_0 \delta_{ij}$. Moreover, the centroid and dispersion matrix under $\tilde{F}$ are related to those under F by

$$\tilde{C}_t = C_t + \sqrt{\mu_0} B_t, \qquad \tilde{D}_t = D_t, \tag{2.4}$$

with B independent of C.

Of central concern in this paper is the *pair-distance* or *separation* process Z, defined for fixed x and y in $\mathbb{R}^d$ by

$$Z_t = |F_t x - F_t y|, \qquad t \geq 0. \tag{2.5}$$

Because of isotropy, Z is a diffusion on $[0,\infty)$, with 0 inaccessible from $(0,\infty)$. See Section 3 for more information.

Centroid and dispersion

Suppose that $\int M_0(dx)|x|^2 < \infty$. The results of ZIRBEL (1995) show that the processes C and D defined in (1.1) and (1.2) are well defined and continuous. Moreover, C is a square-integrable martingale with mean 0.

We will need the following formulas for the covariance of C_t and the mean of D_t (see ZIRBEL and ÇINLAR (1996a) or ZIRBEL (1995)):

$$\mathrm{Cov}(C_t^i, C_t^j) = \int M_0(dx) \int M_0(dy) \mathbf{E} \int_0^t ds\, b^{ij}(F_s x - F_s y) \tag{2.6}$$

$$\mathbf{E} D_t^{ij} = D_0^{ij} + b_0 \delta_{ij} t - \mathrm{Cov}(C_t^i, C_t^j) \tag{2.7}$$

A particularly convenient expression arises if we consider the expected value of $|C_t|^2$, where $|\cdot|$ denotes the vector norm:

$$
\begin{aligned}
(2.8) \qquad \mathbf{E}|C_t|^2 &= \sum_{i=1}^{d} \mathbf{E}(C_t^i)^2 = \sum_{i=1}^{d} \mathrm{Cov}(C_t^i, C_t^i) \\
&= \int M_0(dx) \int M_0(dy) \mathbf{E} \int_0^t ds \; ((d-1)b_N + b_L)(|F_s x - F_s y|)
\end{aligned}
$$

Thus, this measure of the second moment of C_t depends only on the separation process $Z_t = |F_t x - F_t y|$. The trace of $\mathbf{E}D_t$ may be treated the same way, since

$$
(2.9) \qquad \mathrm{tr}\mathbf{E}D_t = \sum_{i=1}^{d} \mathbf{E}D_t^{ii} = \mathrm{tr}D_0 + b_0 dt - \mathbf{E}|C_t|^2,
$$

by (2.7).

A more compact expression will also be useful. Define a measure ρ on $[0, \infty)$ by specifying the integral ρh of each Borel function $h : [0, +\infty) \to [0, +\infty)$ with respect to ρ:

$$
(2.10) \qquad \rho h \equiv \int_{[0,\infty)} \rho(dz) h(z) = \int M_0(dx) \int M_0(dy) h(|x-y|)
$$

Note that if M_0 has an atomic component, ρ will have an atom at 0. Set $\rho_0 = \rho(\{0\})$. Then ρ_0 is the sum of the squares of the masses of the atoms of M_0. If M_0 is diffuse, then $\rho_0 = 0$.

We may write $\mathbf{E}|C_t|^2$ in terms of ρ and the separation process Z as follows:

$$
(2.11) \qquad \mathbf{E}|C_t|^2 = \int_{[0,\infty)} \rho(dz) \mathbf{E}^z \int_0^t ds \; f(Z_s),
$$

where $f = (d-1)b_N + b_L$ and the superscript z of $\mathbf{E}^z$ means $Z_0 = z$ almost surely. This is an expected occupation time of Z, where Z has initial distribution ρ and "occupation time" is measured according to the function f. It will be useful to separate out the atoms by writing this as

$$
(2.12) \qquad \mathbf{E}|C_t|^2 = b_0 d \rho_0 t + \int_{(0,\infty)} \rho(dz) \mathbf{E}^z \int_0^t ds \; f(Z_s),
$$

which is true since $Z_0 = 0$ implies $Z_t = 0$ for all $t > 0$.

Techniques for studying such occupation times are developed in the companion article ZIRBEL (1997). Note that the support of f is all of $[0, \infty)$ and the support of ρ includes 0 and extends as far as the diameter of the support of M_0.

Here is a brief explanation of how we will use the results of ZIRBEL (1997). References are to sections in that article. Let us denote the expected occupation time in (2.12) by

$L(t)$. The asymptotic behavior of L near $+\infty$ is connected to the behavior of its Laplace-Stieltjes transform $\hat{L}$ near 0 via the Tauberian theorem (Section 1). The transform $\hat{L}$ has the decomposition

$$\hat{L}(\alpha) = h(\alpha) \int_{(0,\infty)} \rho(dz) \int_{(0,\infty)} v_1(z \wedge y, \alpha) v_2(z \vee y, \alpha) f(y) M(dy) \tag{2.13}$$

where M is the speed measure of Z and the functions h, v_1, and v_2 are defined solely in terms of Z (Section 2). The asymptotic behavior of $h(\alpha)$ as $\alpha \to 0$ can be found by examining the speed and scale of Z near 0 and $+\infty$ (Section 3). Because f is supported on $[0, \infty)$, we must get bounds on $v_1(z, \alpha)$ and $v_2(z, \alpha)$ as $z \to 0$ and $z \to \infty$ in order to analyze the double integral in (2.13) as $\alpha \to 0$. This can be done (Sections 4 and 5), but may require more information about Z than just the asymptotic behavior of the speed and scale.

3 Separation process – speed and scale

This is a technical section concerning the separation process $Z_t = |F_t x - F_t y|$. The main reference is Section 3 of BAXENDALE and HARRIS (1986).

In equation (2.12), we wrote $\mathrm{E}|C_t|^2$ and $\mathrm{tr}\mathbf{E}D_t$ in terms of an expected occupation time of Z. This can be studied using the method of ZIRBEL (1997). For the case of a three-dimensional flow with $\lambda > 0$ (Section 4), it suffices to know the asymptotics of the speed and scale of Z at 0 and ∞. We quote these in Lemmas (3.9) and (3.10). We add some exact computations for incompressible flows (Proposition (3.11)) to be used in Sections 4 and 6.

From the form of its generator, we can consider the separation process to be a strong solution of the stochastic differential equation

$$dZ_t = \sigma(Z_t) dW_t + \mu(Z_t) dt, \tag{3.1}$$

where W is a one-dimensional Wiener process, and

$$\sigma(z) = \sqrt{2(b_0 - b_L(z))}, \qquad \mu(z) = \frac{1}{z}(d-1)(b_0 - b_N(z)) \tag{3.2}$$

where b_L and b_N are related to the covariance of the flow (see (2.2)). The functions b_L and b_N are quadratic near 0:

$$b_0 - b_L(z) \sim \tfrac{1}{2}\beta_L z^2, \qquad b_0 - b_N(z) \sim \tfrac{1}{2}\beta_N z^2, \qquad z \to 0 \tag{3.3}$$

where both β_L and β_N are strictly positive constants. Thus both σ and μ are approximately linear near the origin:

$$\sigma(z) \sim \sqrt{\beta_L} z, \qquad \mu(z) \sim \tfrac{1}{2}(d-1)\beta_N z, \qquad z \to 0 \tag{3.4}$$

Also, since b_L and b_N decay to 0 at $+\infty$ by Condition (2.3),

(3.5)
$$\sigma(z) \to \sqrt{2b_0}, \qquad \mu(z) \sim \frac{(d-1)b_0}{z}, \qquad z \to \infty$$

Finally, the functions σ and μ are strictly positive on $(0,\infty)$ since b_L and b_N are less than b_0 on this set.

The top Lyapunov exponent λ of the flow is given by

(3.6)
$$\lambda = \tfrac{1}{2}((d-1)\beta_N - \beta_L)$$

From equation (2.13) of BAXENDALE and HARRIS (1986), we have

(3.7)
$$\frac{d-4}{6}\beta_L \le \lambda \le \frac{d}{2}\beta_L,$$

with $\lambda = \frac{d}{2}\beta_L$ for an incompressible flow.

The point 0 is inaccessible from $(0,\infty)$, so Z is a continuous, regular, strong Markov process on the interval $(0,\infty)$. As such, it satisfies the conditions of Section 2 of ZIRBEL (1997). Note also that we can modify the speed measure M of Z so that it is inextensible; see Section 3 of ZIRBEL (1997). This modification does not affect the behavior of Z.

The scale function s and the density m of the speed measure of Z satisfy

(3.8)
$$s'(z) = \exp\left(-\int_{z_0}^{z} dy\, \frac{2\mu(y)}{\sigma^2(y)}\right), \qquad m(z) = \frac{2}{\sigma^2(z)s'(z)} \qquad 0 < z < \infty$$

The number z_0 is arbitrary but must be held fixed. This specifies s up to additive and multiplicative constants and determines m up to a multiplicative constant. The values of these constants do not affect any of the final results of the paper (as one would expect).

Asymptotics of speed and scale

We quote the asymptotic behavior of s', s, and m as $z \to 0$ and as $z \to \infty$ from pages 1162-1163 of BAXENDALE and HARRIS (1986). For notational convenience, define $\nu = \frac{2\lambda}{\beta_L}$. The constant z_0 is chosen to make the constants in the first result work out nicely.

(3.9) **Lemma.** *Suppose $d \ge 2$. As $z \to 0$, $s'(z) \sim 1/z^{\nu+1}, m(z) \sim \frac{2}{\beta_L} z^{\nu-1}$, and*

$$s(z) \sim \begin{cases} C, & \text{if } \lambda < 0 \\ \log z, & \text{if } \lambda = 0 \\ -\frac{1}{\nu}(\frac{1}{z})^{\nu}, & \text{if } \lambda > 0 \end{cases}$$

where C is a finite constant. □

(3.10) **Lemma.** *Suppose $d \geq 2$. As $z \to \infty$, $s'(z) \sim Kz^{1-d}, m(z) \sim z^{d-1}/Kb_0$, and*

$$s(z) \sim \begin{cases} K \log z, & \text{if } d = 2 \\ C', & \text{if } d > 2 \end{cases}$$

where C' and K are finite constants. □

Speed and scale in incompressible flows

We can get an exact expression for s' and m in the incompressible case. This will be used in Sections 4 and 6. Note that $\lambda = \frac{d}{2}\beta_L$ and so $\nu = d$ in this case.

(3.11) **Proposition.** *Let $d \geq 2$ and suppose F is incompressible. Then z_0 can be chosen so that*

$$s'(z) = \frac{1}{z^{d-1}(b_0 - b_L(z))}, \qquad m(z) = z^{d-1}, \qquad z > 0.$$

Moreover, if $\lim_{z\to\infty} z^d b_L(z) = 0$, then,

$$\int_0^\infty dz\; m(z)((d-1)b_N(z) + b_L(z)) = 0.$$

Proof: First let us note that, in the incompressible case,

$$zb_L'(z) = (d-1)(b_N(z) - b_L(z)) \tag{3.12}$$

See YAGLOM (1987), Equation 4.176 of Section 22 or ZIRBEL and ÇINLAR (1996a).

From (3.8), we have

$$-\log s'(z) = \int_{z_0}^z dy\; \frac{(d-1)(b_0 - b_N(y))}{y(b_0 - b_L(y))}$$

It can be verified by differentiation that this equals

$$\log z^{d-1}(b_0 - b_L(z))/C$$

where $C = z_0^{d-1}(b_0 - b_L(z_0))$. We may choose z_0 so that $C = 1$, which yields the claims made concerning s' and m.

For the last claim, note that, by (3.12),

$$\frac{d}{dz}(z^d b_L(z)) = z^{d-1}((d-1)b_N(z) + b_L(z)).$$

Thus, since $m(z) = z^{d-1}$,

$$\begin{aligned}\int_0^\infty dz\ m(z)((d-1)b_N(z) + b_L(z)) &= \int_0^\infty dz\ \frac{d}{dz}(z^d b_L(z)) \\ &= \lim_{z\to\infty} z^d b_L(z) - 0,\end{aligned}$$

which equals 0 by our assumption on the decay of b_L. □

4 Centroid and dispersion for $d \geq 3$ and $\lambda > 0$

In this section, we consider isotropic Brownian flows in three or more dimensions and with strictly positive top Lyapunov exponent. We find the exact limit of the deviation of $\mathrm{E}|C_t|^2$ and $\mathrm{trE}D_t$ from their linear asymptotes. Going a little further, we obtain finite bounds on the deviations of $\mathrm{Cov}(C_t^i, C_t^j)$ and $\mathrm{E}D_t^{ij}$ from their linear asymptotes. These results improve greatly on the bound on the deviation from linearity given in ZIRBEL and ÇINLAR (1996a), which was of the form $A\sqrt{t}$.

For Theorem (4.3), we assume that the covariance b satisfies the decay condition:

(4.1)
$$\int_0^\infty dz\ z|(d-1)b_N(z) + b_L(z)| < +\infty.$$

Essentially, we need b_L and b_N to decay faster than $\frac{1}{z^2}$ as $z \to \infty$. Also, we require M_0 to satisfy

(4.2)
$$\int M_0(dx) \int_{y\neq x} M_0(dy) \log^+ \frac{1}{|x-y|} < +\infty$$

where $\log^+ z = \max\ (0, \log z)$. This says that the diffuse component of M_0 must not have strong concentrations of mass, but does not prevent M_0 from having an atomic component.

We recall that s denotes the scale function of the separation process $Z_t = |F_t x - F_t y|$, and m is the density of the speed measure of Z. Lemma (3.10) shows that $s(\infty) < \infty$. Let ρ_0 be the sum of the squares of the masses of the atoms of M_0. If M_0 is diffuse, then $\rho_0 = 0$.

(4.3) **Theorem.** *Suppose $d \geq 3$, $\lambda > 0$, and conditions (4.1) and (4.2) hold. Then,*

$$\begin{aligned}\lim_{t\to\infty} \mathbf{E}|C_t|^2 - b_0 d\rho_0 t &= A_1 \\ \lim_{t\to\infty} \mathrm{trE}D_t - \mathrm{tr}D_0 - b_0 d(1-\rho_0)t &= -A_1\end{aligned}$$

where A_1 is a finite constant equal to

$$\int M_0(dx) \int_{y\neq x} M_0(dy) \int_0^\infty dz\ m(z)((d-1)b_N(z) + b_L(z))(s(\infty) - s(|x-y| \vee z)).$$

(4.4) **Remark.** If the flow is incompressible, we may substitute $m(z) = z^{d-1}$, subject to the normalization of s in Proposition (3.11). If, in addition, $\lim_{z\to\infty} z^d b_L(z) = 0$, we may replace $s(\infty)$ with 0 in the definition of A_1. See Proposition (3.11). □

When M_0 is diffuse, the linear term subtracted from $\mathbf{E}|C_t|^2$ vanishes, with the following consequence:

(4.5) **Corollary.** *Suppose $d \geq 3$, $\lambda > 0$, M_0 is diffuse and satisfies (4.2), and b satisfies (4.1). Then $\mathbf{E}|C_t|^2$ is increasing and*

$$\lim_{t\to\infty} \mathbf{E}|C_t|^2 = A_1$$

$$\lim_{t\to\infty} \mathrm{tr}\mathbf{E}D_t - \mathrm{tr}D_0 - b_0 dt = -A_1$$

Moreover, the martingale C_t converges almost surely as $t \to \infty$ to a random vector C_∞ with mean 0 and $\mathbf{E}|C_\infty|^2 = A_1$.

Proof: The process C_t^i, $t \geq 0$ is a square-integrable martingale, so $\mathbf{E}|C_t^i|^2$ is increasing and bounded by A_1 (by Theorem (4.3)). Thus C_t^i converges almost surely as $t \to \infty$ to some C_∞^i. Moreover, the submartingale $(C_t^i)^2$, $t \geq 0$ is uniformly bounded in L^1 and so converges in L^1, yielding $\mathbf{E}C_\infty = 0$ and $\mathbf{E}|C_\infty|^2 = A_1$. □

(4.6) **Theorem.** *Suppose $d \geq 3$, $\lambda > 0$, and M_0 satisfies (4.2). Suppose the covariance b satisfies*

$$\int_0^\infty dz\, z(|b_L(z)| + |b_N(z)|) < +\infty. \tag{4.7}$$

Then for all $t \geq 0$ and $i, j = 1, \ldots, d$, we have

$$|\mathrm{Cov}(C_t^i, C_t^j) - b_0 \rho_0 t \delta_{ij}| \leq A_1' \tag{4.8}$$

$$|\mathbf{E}D_t^{ij} - D_0^{ij} - b_0(1-\rho_0)t\delta_{ij}| \leq A_1', \tag{4.9}$$

where A_1' is a finite constant equal to

$$\int M_0(dx) \int_{y\neq x} M_0(dy) \int_0^\infty dz\, m(z)(2|b_N(z)| + |b_L(z)|)(s(\infty) - s(|x-y| \wedge z)).$$

The proofs of Theorems (4.3) and (4.6) begin by writing the deviation from linearity in terms of an expected occupation time of the separation process Z. Lemma (4.11) below gives a formula for the limit of this expected occupation time as $t \to \infty$. Lemma (4.12) goes on to give conditions under which the limit is finite. These are satisfied by virtue of our assumptions on b and M_0.

Proof of Theorem (4.3)

Recall from (2.12) that

$$\mathrm{E}|C_t|^2 = b_0 d\rho_0 t + \int_{(0,\infty)} \rho(dz)\mathrm{E}^z \int_0^t ds\ f(Z_s)$$

where $f = (d-1)b_N + b_L$. Also, by (2.9),

$$\mathrm{tr}\mathrm{E}D_t - \mathrm{tr}D_0 - b_0 d(1-\rho_0)t = -\mathrm{E}|C_t|^2 + b_0 d\rho_0 t$$

so both claims of the theorem are about the limit of the expected occupation time above.

Write f in terms of its positive and negative parts: $f = f^+ - f^-$. By Lemma (4.11) below,

$$\begin{aligned} &\lim_{t\to\infty} \int_{(0,\infty)} \rho(dz)\mathrm{E}^z \int_0^t ds\ f^\pm(Z_s) \\ &= \int_{(0,\infty)} \rho(dz) \int_0^\infty dy\ f^\pm(y)m(y)(s(\infty) - s(z\vee y)), \end{aligned}$$

and by Lemma (4.12) and our assumptions (4.1) and (4.2), this expression is finite for f^+ and f^-. Subtracting the two gives the result claimed. □

Proof of Theorem (4.6)

Fix $t \geq 0$ and $i, j = 1, \ldots, d$. Recall from (2.6) that

$$\mathrm{Cov}(C_t^i, C_t^j) = b_0\rho_0 t\delta_{ij} + \int M_0(dx) \int_{y\neq x} M_0(dy)\mathrm{E}\int_0^t ds\ b^{ij}(F_s x - F_s y)$$

We have separated out the linear term caused by the atomic component of M_0 (if any). Let us define $\mathcal{I}(t)$ by

$$\begin{aligned} \mathcal{I}(t) &= |\mathrm{Cov}(C_t^i, C_t^j) - b_0\rho_0 t\delta_{ij}| \\ &= |\int M_0(dx)\int_{y\neq x} M_0(dy)\mathrm{E}\int_0^t ds\ b^{ij}(F_s x - F_s y)| \end{aligned} \tag{4.10}$$

Note that, by (2.7), we have $\mathcal{I}(t) = |\mathrm{E}D_t^{ij} - D_0^{ij} - b_0(1-\rho_0)t\delta_{ij}|$. Define A_1' as in the statement of Theorem (4.6). The claim of the theorem is that $\mathcal{I}(t) \leq A_1'$.

From the special form (2.2) of b, we have

$$|b^{ij}(v)| \leq 2|b_N(|v|)| + |b_L(|v|)|, \qquad v \in \mathbb{R}^d$$

Set $f = 2|b_N| + |b_L|$. Then,

$$\begin{aligned}\mathcal{I}(t) &\leq \int M_0(dx) \int_{y\neq x} M_0(dy) \mathrm{E} \int_0^t ds\, f(|F_s x - F_s y|) \\ &= \int_{(0,\infty)} \rho(dz) \mathrm{E}^z \int_0^t ds\, f(Z_s),\end{aligned}$$

where ρ is the measure defined in (2.10), E^z is the expectation under which $Z_0 = z$ almost surely, and Z is the separation process of Section 3.

This occupation time of Z is increasing with t. A formula for its limit is given in Lemma (4.11); this coincides with the definition of A_1'. Lemma (4.12) shows that A_1' is finite, since we assumed that $\int_0^\infty dz\, z f(z) < \infty$. ∎

The following lemma is stated in the general setting of ZIRBEL (1997). The process X is assumed to be a continuous, regular, strong Markov process on an interval $I \subseteq \mathbb{R}$, having scale s and speed measure M. If the left endpoint ℓ is a regular boundary point, it must be absorbing, and similarly for the right endpoint r. The process Z satisfies these conditions, and it satisfies the additional conditions of the lemma by virtue of Lemmas (3.9) and (3.10).

(4.11) **Lemma.** *Let X be as above. Suppose that $s(\ell) = -\infty$, $s(r) < \infty$, and $M[c, r) = +\infty$. Let $f : I \to \mathbb{R}$ be a non-negative Borel function. Then for all $x \in (\ell, r)$,*

$$\mathrm{E}^x \int_0^\infty dt\, f(X_t) = \int_I (s(r) - s(x \vee y)) f(y) M(dy).$$

Proof: By the decomposition (2.2) of ZIRBEL (1997),

$$\mathbf{E}^c \int_0^\infty dt\, e^{-\alpha t} f(X_t) = h(\alpha) \int_I v_1(c \wedge y, \alpha) v_2(c \vee y, \alpha) f(y) M(dy)$$

where $c \in (\ell, r)$ is a fixed reference point. Equation (3.4) of ZIRBEL (1997) shows that $h(\alpha) \to s(r) - s(c)$ as $\alpha \to 0$. Also, $v_1(c \wedge y, \alpha)$ increases to 1 and $v_2(c \vee y, \alpha)$ increases to $(s(r) - s(c \vee y))/(s(r) - s(c))$ by (2.10) and (2.11) of ZIRBEL (1997). Thus, by the monotone convergence theorem,

$$\begin{aligned}\mathbf{E}^c \int_0^\infty dt\, f(X_t) &= (s(r) - s(c)) \int_I \frac{s(r) - s(c \vee y)}{s(r) - s(c)} f(y) M(dy) \\ &= \int_I (s(r) - s(c \vee y)) f(y) M(dy).\end{aligned}$$

The functions in this expression are defined without reference to c. Since the choice of $c \in (\ell, r)$ is arbitrary, we may now substitute x for c. ∎

We return to consideration of the separation process Z.

(4.12) **Lemma.** *Suppose $d \geq 3, \lambda > 0$, and M_0 satisfies (4.2). Let f be a non-negative, bounded, Borel function satisfying $\int_0^\infty dz\, zf(z) < +\infty$. Then,*

$$\int_{(0,\infty)} \rho(dz) \int_0^\infty dy\ f(y)m(y)(s(\infty) - s(z \vee y)) \tag{4.13}$$

is finite, where ρ is the measure defined in (2.10).

Proof: Note that the inside integral is decreasing as a function of z. We will bound its growth as $z \to 0$. Suppose for definiteness that $z \leq 1$. Then,

$$\begin{aligned} &\int_0^\infty dy\ f(y)m(y)(s(\infty) - s(z \vee y)) \\ &\quad = s(\infty) \int_0^1 dy\ f(y)m(y) - s(z) \int_0^z dy\ f(y)m(y) \\ &\qquad - \int_z^1 dy\ f(y)m(y)s(y) + \int_1^\infty dy\ f(y)m(y)(s(\infty) - s(y)) \end{aligned} \tag{4.14}$$

Now for $y \leq 1$, the boundedness of f and Lemma (3.9) give $f(y)m(y) \leq C_1 y^{\nu-1}$ and $-s(y) \leq C_2 y^{-\nu}$ for positive C_1 and C_2. Thus, the first three terms are bounded above by

$$\begin{aligned} &s(\infty)C_1 \int_0^1 dy\ y^{\nu-1} + C_1 C_2 z^{-\nu} \int_0^z dy\ y^{\nu-1} + C_1 C_2 \int_z^1 dy\ y^{\nu-1} y^{-\nu} \\ &= s(\infty) C_1 \frac{1}{\nu} + C_1 C_2 \frac{1}{\nu} + C_1 C_2 \log \frac{1}{z} \end{aligned}$$

The first two are finite since $\nu = \frac{2\lambda}{\beta_L} > 0$ by assumption. For the fourth term in (4.14), note that by Lemma (3.10), $m(y) \leq C_3 y^{d-1}$ for $y \geq 1$ and

$$s(\infty) - s(y) = \int_y^\infty dx\ s'(x) \leq C_4 \int_y^\infty dx\ x^{1-d} = \frac{C_4}{d-2} y^{2-d}, \qquad y \geq 1.$$

Thus, the fourth term in (4.14) is less than

$$\frac{C_3 C_4}{d-2} \int_1^\infty dy\ f(y) y^{d-1} y^{2-d} = \frac{C_3 C_4}{d-2} \int_1^\infty dy\ f(y) y,$$

which we assumed to be finite.

Using these bounds, the quantity in (4.13) is less than

$$\int_{(0,\infty)} \rho(dz)(C_5 + C_1 C_2 \log \frac{1}{z \wedge 1}),$$

which is finite by assumption (4.2). This completes the proof. □

5 The separation process in two dimensions

We continue the project begun in Section 3, namely, collecting technical results concerning the separation process $Z_t = |F_t x - F_t y|$. We confine the discussion to two-dimensional flows. The estimates we obtain here will be used in Sections 6 and 7 to establish the asymptotic behavior of $\mathbf{E}|C_t|^2$ and $\mathrm{tr}\mathbf{E}D_t$ in two-dimensional flows.

Throughout this section, c is a fixed reference location in $(0, \infty)$. Here are a few facts concerning v_1 and v_2, taken from Section 2 of ZIRBEL (1997). For each $\alpha > 0$, $v_1(z, a)$ is increasing in z and $v_2(z, \alpha)$ is decreasing in z. They are normalized by $v_1(c, \alpha) = v_2(c, \alpha) = 1$. Suppose that $z \geq c$. Then as α decreases, $v_1(z, \alpha)$ decreases and $v_2(z, \alpha)$ increases. On the other hand, if $z \leq c$, then $v_1(z, \alpha)$ increases while $v_2(z, \alpha)$ decreases as α decreases.

Behavior of $h(\alpha)$ as $\alpha \to 0$

In Section 3 of ZIRBEL (1997), the function h is written as

$$\frac{1}{h(\alpha)} = \frac{1}{h_+(\alpha)} + \frac{1}{h_-(\alpha)}$$

The asymptotic behavior of $h_+(\alpha)$ as $\alpha \to 0$ can be found from knowledge of the asymptotics of the speed and scale of Z near ∞. Similarly, h_- can be found from s and m near 0.

From Lemma (3.10), we have $s(z) \sim K \log z$ and $m(z) \sim z/Kb_0$ as $z \to \infty$. Thus, the speed measure M satisfies $M[c, z] \sim z^2/2Kb_0$. Let $g(x) = \frac{K}{2} \log x$. Then

$$\lim_{z\to\infty} \frac{g(s(z)M[c, z])}{s(z)} = \lim_{z\to\infty} \frac{K}{2} \frac{\log s(z) + \log M[c, z]/z^2 + \log z^2}{s(z)} = 1,$$

so by Theorem (3.9) of ZIRBEL (1997), $h_+(\alpha) \sim g(\frac{1}{\alpha}) = \frac{K}{2} \log \frac{1}{\alpha}$ as $\alpha \to 0$.

The behavior of $h_-(\alpha)$ as $\alpha \to 0$ depends on the sign of λ. Equation (3.5) of ZIRBEL (1997) is

$$\lim_{\alpha \downarrow 0} h_-(\alpha) = s(c) - s(0).$$

When $\lambda < 0$, $s(0)$ is finite by Lemma (3.9), so $h_-(\alpha)$ has a finite limit. When $\lambda > 0$, Lemma (3.9) implies that $M(0, c)$ is finite and $s(z) \to -\infty$ as $z \to 0$. Letting $g(x) = x/M(0, c)$,

$$\lim_{z\to 0} \frac{g(-s(z)M(z, c))}{-s(z)} = \lim_{z\to 0} \frac{M(z, c)}{M(0, c)} = 1,$$

so by Corollary (3.13) of ZIRBEL (1997), $h_-(\alpha) \sim g(\frac{1}{\alpha}) = M(0, c)/\alpha$ as $\alpha \to 0$.

The implications for h are clear. We summarize them in a lemma.

(5.1) **Lemma.** *Suppose $d = 2$. If $\lambda < 0$, then $h(\alpha) \to s(c) - s(0)$ as $\alpha \to 0$. If $\lambda > 0$, then $h(\alpha) \sim \frac{K}{2} \log \frac{1}{\alpha}$ as $\alpha \to 0$, where $K = \lim_{z\to\infty} \frac{s(z)}{\log z}$.* □

Growth of $v_1(z, \alpha)$ as $z \to \infty$

To obtain bounds on $v_1(z, \alpha)$ for $z \geq c$, first observe that the speed and scale of Z have the same asymptotic behavior as the speed and scale of the BES^2 process as $z \to \infty$. Indeed, the speed and scale of the BES^2 process satisfy

$$\tilde{s}'(z) = \frac{1}{z}, \qquad \tilde{m}(z) = 2z.$$

In addition, the analogous function $\tilde{v}_1$ equals $I_0(\sqrt{2\alpha x})/I_0(\sqrt{2\alpha c})$. See Example (2.7) of ZIRBEL (1997) for details. Lemma (3.10) shows that $s'/\tilde{s}'$ and $m/\tilde{m}$ have finite limits at $+\infty$. Corollary (4.7) of ZIRBEL (1997) allows us to conclude that there exist constants R_1 and R_2 and functions $B_1(\alpha)$ and $B_2(\alpha)$ such that

$$B_1(\alpha)\frac{I_0(R_1\sqrt{\alpha}z)}{I_0(R_1\sqrt{\alpha}c)} \leq v_1(z, \alpha) \leq B_2(\alpha)\frac{I_0(R_2\sqrt{\alpha}z)}{I_0(R_2\sqrt{\alpha}c)} \tag{5.2}$$

for all $\alpha > 0$ and $z \geq c$.

Growth of $v_2(z, \alpha)$ as $z \to 0$

We will use the method employed above to find upper and lower bounds on $v_2(z, \alpha)$ for z near 0. To find a process $\tilde{Z}$ suitable for comparison to Z, recall the stochastic differential equation (3.1) for Z. The coefficients σ and μ are approximately linear near 0, so we take $\tilde{Z}$ to be generated by the linear approximation:

$$d\tilde{Z}_t = \sqrt{\beta_L}\tilde{Z}_t dW_t + \tfrac{1}{2}(d-1)\beta_N \tilde{Z}_t \, dt$$

It is easy to see that the scale function $\tilde{s}$ of $\tilde{Z}$ satisfies

$$\tilde{s}'(z) = \exp(-\int_1^z dy \, \frac{(d-1)\beta_N}{\beta_L y}) = \frac{1}{z^{\nu+1}}, \qquad 0 < z < \infty,$$

where $\nu = \frac{(d-1)\beta_N}{\beta_L} - 1 = \frac{2\lambda}{\beta_L}$ as in Lemma (3.9). The density $\tilde{m}$ of the speed measure of $\tilde{Z}$ is $\tilde{m}(z) = \frac{2}{\beta_L} z^{\nu-1}$. Clearly, from Lemma (3.9), $s'/\tilde{s}'$ and $m/\tilde{m}$ have finite limits at 0.

Let $\tilde{\mathcal{A}}$ denote the generator of $\tilde{Z}$. The function $\tilde{v}_2$ is the unique decreasing solution of the differential equation $\tilde{\mathcal{A}}v = \alpha v$ satisfying $\tilde{v}_2(c, \alpha) = 1$ (see Section 2 of ZIRBEL (1997)). This equation is

$$\tfrac{1}{2}\beta_L z^2 v'' + \tfrac{1}{2}(d-1)\beta_N z v' - \alpha v = 0$$

The solutions are of the form $v(z,\alpha) = z^{\gamma_\pm(\alpha)}$, where

$$\gamma_\pm(\alpha) = \tfrac{1}{2}\left(-\nu \pm \sqrt{\nu^2 + \frac{8\alpha}{\beta_L}}\right)$$

Now $\tilde{v}_2$ is decreasing and $\tilde{v}_2(c,\alpha) = 1$, so we must have

$$\tilde{v}_2(z,\alpha) = \left(\frac{z}{c}\right)^{\gamma_-(\alpha)}, \qquad 0 < z < \infty.$$

By Corollary (4.7) of ZIRBEL (1997), for some constants R_3 and R_4 and functions $B_3(\alpha)$ and $B_4(\alpha)$, we have

$$B_3(\alpha) z^{\gamma_-(R_3\alpha)} \le v_2(z,\alpha) \le B_4(\alpha) z^{\gamma_-(R_4\alpha)} \tag{5.3}$$

for all $\alpha > 0$ and z in $(0, c]$.

Decay of $v_2(z,\alpha)$ as $z \to \infty$

It will be necessary to have upper and lower bounds on $v_2(z,\alpha)$ as $z \to \infty$ for the analysis of two-dimensional flows with negative top Lyapunov exponent (Section 7). We do this in two steps. First, we define an auxiliary process Y by $Y_t = \phi(Z_t)$ for a function ϕ chosen to make the diffusion coefficient of Y simpler than the diffusion coefficient σ of Z. Second, we bound the drift of Y above and below by a simpler drift and let $\tilde{Y}$ be a process with the same diffusion coefficient as Y, but with the simpler drift. Both Y and $\tilde{Y}$ have a decomposition of the form (2.13), with u_2 and $\tilde{u}_2$ in place of v_2. We can bound u_2 above and below by $\tilde{u}_2$ using Proposition (5.4) of ZIRBEL (1997). Unraveling the change of variables gives the bounds on v_2.

Recall from (3.1) that Z satisfies the stochastic differential equation $dZ_t = \sigma(Z_t)dW_t + \mu(Z_t)dt$. Define a function ϕ by

$$\phi(z) = \int_1^z \frac{dx}{\sigma(x)}, \qquad 0 < z < \infty. \tag{5.4}$$

Note that ϕ is strictly increasing and maps $(0,\infty)$ to $\mathbb{R}$. Suppose $Z_0 > 0$. Define a new process Y by $Y_t = \phi(Z_t)$. Using Itô's Lemma we find that Y satisfies

$$dY_t = dW_t + \eta(Y_t)dt,$$

where η is defined implicitly by the relation

$$\eta(\phi(z)) = \frac{2\mu(z) + b_L'(z)}{2\sigma(z)}, \qquad 0 < z < \infty. \tag{5.5}$$

The following lemma shows that the drift η of Y is bounded above and below by the drift of a shifted BES^2 process.

(5.6) **Lemma.** *Let* $d = 2$. *Suppose that* $\lim_{z\to\infty} zb'_L(z) = 0$ *and*

$$\begin{aligned}\int_0^\infty dz\ |b_N(z) - b_L(z) - zb'_L(z)| &< \infty \\ \int_0^\infty dz\ z|2b'_N(z) - b'_L(z) - zb''_L(z)| &< \infty\end{aligned}$$

Then there exist finite constants y^* *and* δ *such that* $y^* > \delta > 0$ *and*

$$\frac{1}{2(y+\delta)} \leq \eta(y) \leq \frac{1}{2(y-\delta)}, \qquad y \geq y^*.$$

Proof: The assertion is equivalent to

$$\frac{1}{2(\phi(z)+\delta)} \leq \eta(\phi(z)) \leq \frac{1}{2(\phi(z)-\delta)}, \qquad \phi(z) \geq y^*,$$

which is the same as

$$\left|\frac{1}{2\eta(\phi(z))} - \phi(z)\right| \leq \delta, \qquad \phi(z) \geq y^*.$$

Let h be the function inside the absolute value bars. We claim that $|h(z)|$ is bounded for z greater than some $\hat{z}$.

Write h as $h(z) = \frac{z\sigma}{\pi} - \phi$, where $\pi(z) = 2b_0 - 2b_N(z) + zb'_L(z)$. As $z \to \infty$, $b_N(z)$ converges to 0 by Condition (2.3) and $zb'_L(z)$ converges to 0 by the first assumption of the Lemma. Thus, $\pi(z)$ converges to $2b_0$ as $z \to \infty$, so we can pick a point $\hat{z}$ such that $\pi(z) > 0$ for all $z \geq \hat{z}$. Clearly, for $z \geq \hat{z}$ we have

$$|h(z)| \leq |h(\hat{z})| + \int_{\hat{z}}^\infty dx\ |h'(x)|$$

Set δ equal to the right side of this expression. We will show that $\delta < \infty$.

First compute h', noting that $\sigma\sigma' = -b'_L$:

$$\begin{aligned} h' &= \frac{(\sigma + z\sigma')\pi - z\sigma\pi'}{\pi^2} - \frac{1}{\sigma} \\ &= \frac{1}{\pi^2\sigma}[\pi(\sigma^2 + z\sigma\sigma' - \pi) - z\sigma^2\pi'] \\ &= \frac{1}{\pi^2\sigma}[2\pi(b_N - b_L - zb'_L) + z\sigma^2(2b'_N - b'_L - zb''_L)] \end{aligned}$$

Take the absolute value, use the triangle inequality, recall that σ is bounded above and below and π is bounded below for $z \geq \hat{z}$, and integrate from $\hat{z}$ to ∞. The result is finite by our assumptions on b_L and b_N, so δ is finite.

We have shown that $|h(z)| \leq \delta < \infty$ for all $z \geq \hat{z}$. Now set $y^* = (\delta \vee \phi(\hat{z})) + 1$. Then $|h(z)| \leq \delta$ for z such that $\phi(z) \geq y^*$, and also $y^* > \delta$. □

(5.7) **Proposition.** *Under the conditions of the preceding lemma, there exist finite constants z^* and δ such that $\phi(z^*) > \delta \geq 0$ and for all $\alpha > 0$ and $z \geq z^*$,*

$$\frac{K_0(\sqrt{2\alpha}(\phi(z)-\delta))}{K_0(\sqrt{2\alpha}(\phi(z^*)-\delta))} \leq \frac{v_2(z,\alpha)}{v_2(z^*,\alpha)} \leq \frac{K_0(\sqrt{2\alpha}(\phi(z)+\delta))}{K_0(\sqrt{2\alpha}(\phi(z^*)+\delta))}$$

Proof: We show the lower bound; the upper bound is similar. By the preceding lemma, the drift η of Y is bounded above by $\frac{1}{2(y-\delta)}$ for $y \geq y^*$. Let $\tilde{Y}$ solve

$$d\tilde{Y}_t = dW_t + \frac{1}{2(\tilde{Y}_t - \delta)}dt$$

By Proposition (5.4) of ZIRBEL (1997),

$$\frac{\tilde{u}_2(y,\alpha)}{\tilde{u}_2(y^*,\alpha)} \leq \frac{u_2(y,\alpha)}{u_2(y^*,\alpha)}, \qquad y \geq y^*.$$

Now $\tilde{Y}$ is a BES^2 process shifted by δ, and by Example (2.7) of ZIRBEL (1997), the function $\tilde{u}_2(y,\alpha)$ is equal to $K_0(\sqrt{2\alpha}(y-\delta))/K_0(\sqrt{2\alpha}(\tilde{c}-\delta))$, where $\tilde{c} > \delta$ is fixed. Thus, we have shown that

$$\frac{K_0(\sqrt{2\alpha}(y-\delta))}{K_0(\sqrt{2\alpha}(y^*-\delta))} \leq \frac{u_2(y,\alpha)}{u_2(y^*,\alpha)}, \qquad y \geq y^*.$$

Finally, Equation (2.13) of ZIRBEL (1997) shows that $v_2(z,\alpha) = u_2(\phi(z),\alpha)$ since $Y_t = \phi(Z_t)$. Setting $z^* = \phi^{-1}(y^*)$ finishes the proof. □

6 Centroid and dispersion for $d = 2$ and $\lambda > 0$

In this section we consider the asymptotic growth rate of $\mathrm{E}|C_t|^2$ and $\mathrm{trE}D_t$ in a two-dimensional isotropic Brownian flow with strictly positive top Lyapunov exponent λ. We impose two conditions on the initial measure M_0: for some $\epsilon > 0$,

(6.1) $$\int M_0(dx)e^{\epsilon|x|} < +\infty$$

(6.2) $$\int M_0(dx)\int_{y\neq x} M_0(dy)\frac{1}{|x-y|^\epsilon} < +\infty$$

The first condition requires the mass to be localized in space. The second condition will be satisfied, for example, if the diffuse component of M_0 has a bounded density with respect to Lebesgue measure. We also need a condition on the covariance b of the flow (recall (2.2)):

(6.3) $$\int_0^\infty dz\ z\ |b_L(z) + b_N(z)| < +\infty$$

This requires a moderate rate of decay of the spatial correlations in the flow.

Recall that ρ_0 is the sum of the squares of the masses of the atoms of M_0 and m is the density of the speed measure of the separation process $Z_t = |F_t x - F_t y|$.

(6.4) **Theorem.** *Suppose $d = 2$, $\lambda > 0$, and conditions (6.1), (6.2), and (6.3) hold. Then, for some finite C and all $t \geq 2$,*

$$|\mathbf{E}|C_t|^2 - 2b_0\rho_0 t| \leq C \log t$$

$$|\mathrm{tr}\mathbf{E}D_t - 2b_0(1-\rho_0)t| \leq C \log t$$

Moreover, if M_0 is diffuse, then, for K given in Lemmas (3.10) and (5.1),

$$\mathbf{E}|C_t|^2 \sim \frac{K}{2}\int_0^\infty dy\ (b_L(y) + b_N(y))m(y)\log t, \qquad t \to \infty$$

$$2b_0 t - \mathrm{tr}\mathbf{E}D_t \sim \frac{K}{2}\int_0^\infty dy\ (b_L(y) + b_N(y))m(y)\log t, \qquad t \to \infty$$

When the flow is incompressible and $\lim_{z\to\infty} z^2 b_L(z) = 0$, the constant on the right side equals 0 and these read

$$\mathbf{E}|C_t|^2 = o(\log t), \qquad t \to \infty$$

$$0 \leq 2b_0 t - \mathrm{tr}\mathbf{E}D_t = o(\log t), \qquad t \to \infty$$

This result is almost as striking as the result for $d \geq 3$ and $\lambda > 0$, except that here the deviations of $\mathbf{E}|C_t|^2$ and $\mathrm{tr}\mathbf{E}D_t$ from their linear asymptotes are of the order $\log t$. Still, this is very small! The exact rate of growth of $\mathbf{E}|C_t|^2$ when the flow is incompressible and M_0 is diffuse is not known.

Proof of Theorem (6.4)

Equation (2.9) shows that the statements concerning $\mathrm{tr}\mathbf{E}D_t$ are identical to the corresponding statements about $\mathbf{E}|C_t|^2$.

We treat the diffuse case first. By equation (2.12),

$$\mathbf{E}|C_t|^2 = \int_{(0,\infty)} \rho(dz)\mathbf{E}^z \int_0^t ds\ f(Z_s)$$

where ρ is defined in (2.10), $f = b_L + b_N$, and Z is the separation process $Z_t = |F_t x - F_t y|$ of Section 3. Let us call the right hand side $L(t)$. It is increasing since C is a martingale. By (2.13), the Laplace-Stieltjes transform $\hat{L}$ of L has the decomposition

$$\hat{L}(\alpha) = h(\alpha)\int_{(0,\infty)} \rho(dz)\int_{(0,\infty)} dy\ v_1(z\wedge y, \alpha)v_2(z\vee y, \alpha)f(y)m(y) \tag{6.5}$$

We will make use of the results on h, v_1, v_2, and m established in Sections 3 and 5 to find the asymptotic behavior of $\hat{L}(\alpha)$ as $\alpha \to 0$. We set the reference value c equal to 1 in this section.

We will use the dominated convergence theorem to show that the iterated integral in (6.5) converges to $\rho(0,\infty)\int_0^\infty dy\ f(y)m(y)$ as $\alpha \to 0$. We break the integral into three parts, over

$$\begin{array}{ll} (i) & 0 < z < 1, 1 \le y < \infty \text{ or } 1 \le z < \infty, 0 \le y < 1 \\ (ii) & 0 < z < 1,\ \ 0 \le y < 1 \\ (iii) & 1 \le z < \infty,\ \ 1 \le y < \infty \end{array}$$

We will go region by region giving a function dominating the integrand, then showing that this function is integrable.

(i) In this region, we have $z \wedge y < 1$ and $z \vee y \ge 1$, so

$$v_1(z \wedge y, \alpha) \le 1 \text{ and } v_2(z \vee y, \alpha) \le 1$$

Thus, the integrand is dominated by $|f(y)|m(y)$. This is integrable at ∞ by Lemma (3.10) and assumption (6.3). It is integrable near 0 by Lemma (3.9) and our assumption that $\lambda > 0$. The integral over z is no problem since ρ is a finite measure.

(ii) In this region $v_1(z \wedge y, \alpha) \le 1$ as above and $v_2(z \vee y, \alpha)$ is decreasing in α, so for all $\alpha < \bar{\alpha}$ the integrand is dominated by

$$v_2(z \vee y, \bar{\alpha})|f(y)|m(y).$$

We need to choose $\bar{\alpha}$ so that this function is integrable. From (5.3) we have

$$v_2(z \vee y, \bar{\alpha}) \le B_4(\bar{\alpha})(z \vee y)^{\gamma_-(R_4\bar{\alpha})}, \qquad y, z \in (0,1).$$

The exponent $\gamma_-(R_4\bar{\alpha})$ increases to $-\nu$ as $\bar{\alpha} \downarrow 0$. Choose $\bar{\alpha}$ small enough that $\gamma_-(R_4\bar{\alpha}) = -\nu - \epsilon$ for the ϵ of condition (6.2). In addition, f is bounded and $m(y) \sim \frac{2}{\beta_L} y^{\nu-1}$ as $y \to 0$ by Lemma (3.9), so

$$|f(y)|m(y) \le Cy^{\nu-1}, \qquad y < 1.$$

Integrability of $v_2(z \vee y, \bar{\alpha})|f(y)|m(y)$ is shown by the computation

$$\int_{(0,1)} \rho(dz) \int_0^1 dy\ (z \vee y)^{-\nu-\epsilon} y^{\nu-1} = \int_{(0,1)} \rho(dz)((\tfrac{1}{\nu} + \tfrac{1}{\epsilon})z^{-\epsilon} - \tfrac{1}{\epsilon}),$$

which is finite by condition (6.2) on M_0.

(iii) In this region, $v_2(z \vee y, \alpha) \le 1$, so the integrand is dominated by

$$v_1(z \wedge y, \bar{\alpha})|f(y)|m(y)$$

for all α less than some $\bar{\alpha}$ to be chosen shortly. By (5.2) and the asymptotics of Bessel functions, v_1 satisfies

$$v_1(z \wedge y, \bar{\alpha}) \leq B_2'(\bar{\alpha}) e^{K_2 \sqrt{\bar{\alpha}}(z \wedge y)}, \qquad y, z \geq 1$$

Now choose $\bar{\alpha}$ strictly less than $(\epsilon/K_2)^2$, using ϵ from condition (6.1). Also, Lemma (3.10) shows that for some constant C, $m(y) \leq Cy$ for $y \geq 1$. Integrability is shown by the calculation

$$\int_1^\infty \rho(dz) \int_1^\infty dy\ e^{K_2 \sqrt{\bar{\alpha}}(z \wedge y)} |f(y)| y$$
$$\leq \int_1^\infty \rho(dz) \left[2b_0 \int_1^z dy\ y e^{K_2 \sqrt{\bar{\alpha}} y} + e^{K_2 \sqrt{\bar{\alpha}} z} \int_z^\infty dy\ |f(y)| y \right]$$

The first integral inside the brackets may be evaluated explicitly. The second integral is decreasing in z, and is finite for $z = 1$ by condition (6.3). Condition (6.1) then shows that the integral with respect to z is finite.

We now apply the dominated convergence theorem to the iterated integral in (6.5). Note that $v_1(z, \alpha)$ and $v_2(z, \alpha)$ go to 1 as $\alpha \to 0$ since $s(0)$ and $s(\infty)$ are infinite (see Lemmas (3.9) and (3.10) and Equations (2.10) and (2.11) of ZIRBEL (1997)). Thus,

$$\lim_{\alpha \downarrow 0} \frac{\hat{L}(\alpha)}{h(\alpha)} = \rho(0, \infty) \int_0^\infty dy\ f(y) m(y)$$

The right side is finite by our assumption (6.3) and the fact that m is linear near ∞ (Lemma (3.10)). Lemma (5.1) gives $h(\alpha) \sim \frac{K}{2} \log \frac{1}{\alpha}$ as $\alpha \to 0$ since $\lambda > 0$. We have assumed that $\rho_0 = 0$, so that $\rho(0, \infty) = 1$. Thus,

$$\lim_{\alpha \downarrow 0} \frac{\hat{L}(\alpha)}{\log \frac{1}{\alpha}} = \frac{K}{2} \int_0^\infty dy\ (b_L(y) + b_N(y)) m(y)$$

Recall from Lemma (3.11) that the integral on the right side equals 0 for incompressible flows, provided that $\lim_{z \to \infty} z^2 b_L(z) = 0$.

The Tauberian theorem (see (1.3) of ZIRBEL (1997)) yields

$$\lim_{t \to \infty} \frac{L(t)}{\log t} = \frac{K}{2} \int_0^\infty dy\ (b_L(y) + b_N(y)) m(y),$$

even when the right side is 0. This settles the diffuse case.

Finally, we establish the first two claims of the theorem. First, by (2.12),

$$|\mathrm{E}|C_t|^2 - 2b_0 \rho_0 t| \leq \int_{(0,\infty)} \rho(dz) \mathrm{E}^z \int_0^t ds\ |f(Z_s)|.$$

Call the right hand side L^a_t. It is increasing. Its Laplace-Stieltjes transform $\hat{L}^a$ has a decomposition of the form (6.5), but with $|f|$ replacing f. As above, the integrand is dominated by an integrable function, so we find

$$\lim_{\alpha\downarrow 0}\frac{\hat{L}^a(\alpha)}{h(\alpha)} = \rho(0,\infty)\int_0^\infty dy\ |f(y)|m(y)$$

The right side is finite by (6.3). As above, the Tauberian theorem yields

$$L^a(t) \sim \frac{K}{2}\rho(0,\infty)\int_0^\infty dy\ |b_L(y)+b_N(y)|m(y)\log t, \qquad t\to\infty$$

whence $L^a(t)\le C\log t$ for $t\ge 2$ and some finite C. □

7 Centroid and dispersion for $d=2$ and $\lambda<0$

In this section we consider the asymptotic growth rate of $\mathrm{E}|C_t|^2$ and $\mathrm{trE}D_t$ in two-dimensional isotropic Brownian flows with strictly negative top Lyapunov exponent λ. We impose a localization condition on the initial measure M_0:

(7.1) $$\int M_0(dx)e^{\epsilon|x|} < +\infty$$

for some $\epsilon>0$. We need three conditions on the covariance b of the flow:

(7.2) $$\lim_{z\to\infty} zb'_L(z) = 0$$

(7.3) $$\int_0^\infty dz\ |b_N(z)-b_L(z)-zb'_L(z)| \ < \ \infty$$

(7.4) $$\int_0^\infty dz\ z|2b'_N(z)-b'_L(z)-zb''_L(z)| \ < \ \infty$$

These are exactly the conditions of Lemma (5.6). Essentially we need b_L and b_N to decay like $z^{-1-\delta}$.

(7.5) **Theorem.** *Suppose $d=2$, $\lambda<0$, and conditions (7.1) – (7.4) hold. Then, as $t\to\infty$,*

(7.6) $$2b_0t-\mathrm{E}|C_t|^2 \sim \frac{A_3t}{\log t},$$

(7.7) $$\mathrm{trE}D_t \sim \frac{A_3t}{\log t},$$

where A_3 is a finite constant given by

(7.8) $$A_3 = \frac{4b_0}{K}\int M_0(dx)\int M_0(dy)(s(|x-y|)-s(0))$$

Here s is the scale function of Z, and $K = \lim_{z\to\infty} \frac{s(z)}{\log z}$. □

Whether M_0 is diffuse or not, $\mathbf{E}|C_t|^2$ grows linearly. The $\frac{t}{\log t}$ term is a big correction, but still of lower order. Thus, translation of the mass distribution is strong. On the other hand, dispersion relative to the center of mass is sublinear, though barely so. Dispersion is thus weaker than in the case of a positive top Lyapunov exponent. The leading constant A_3 is larger if the flow itself is more energetic (b_0 larger), or if the initial mass distribution is more spread out.

Proof of Theorem (7.5)

The two claims of the theorem are identical by equation (2.9). By (2.11),

$$\begin{aligned} 2b_0 t - \mathbf{E}|C_t|^2 &= 2b_0 t - \int_{[0,\infty)} \rho(dz)\mathbf{E}^z \int_0^t ds\ f(Z_s) \\ &= \int_{(0,\infty)} \rho(dz)\mathbf{E}^z \int_0^t ds\ g(Z_s), \end{aligned}$$

where ρ is defined in (2.10) and we have set $g = 2b_0 - f = 2b_0 - b_L - b_N$. Note that $g(0) = 0$, g is quadratic near 0, g is positive, and $g(z) \to 2b_0$ as $z \to \infty$, in contrast to the function f in the proof of Theorem (6.4).

Let $L(t) = \int_{(0,\infty)} \rho(dz)\mathbf{E}^z \int_0^t ds\ g(Z_s)$. Then by (2.13), the Laplace-Stieltjes transform of L satisfies

(7.9) $$\hat{L}(\alpha) = h(\alpha) \int_{(0,\infty)} \rho(dz) \int_{(0,\infty)} dy\ v_1(z \wedge y, \alpha) v_2(z \vee y, \alpha) g(y) m(y)$$

We claim that $\lim_{\alpha\downarrow 0} \alpha \log \frac{1}{\alpha} \hat{L}(\alpha) = A_3$, the constant defined in (7.8). Once we have shown this, the Tauberian theorem (see (1.3) of ZIRBEL (1997)) yields $\lim_{t\to\infty} \frac{1}{t} \log t L(t) = A_3$, which is the claim of the theorem.

Our task is more difficult than it was in the proof of Theorem (6.4) because the double integral in (7.9) blows up as $\alpha \to 0$. Finding its asymptotic behavior is a delicate matter. On the other hand, by Lemma (5.1), $h(\alpha) \to s(c) - s(0)$ as $\alpha \to 0$, so this factor is no problem.

Recall Lemma (5.7), from which we obtain bounds on $v_2(z, \alpha)$ for $z \ge z^*$. We will set the reference location c equal to z^*. We will break the region of integration for the iterated integral in (7.9) into four parts:

$$\begin{array}{lll} (i) & 0 \le z < z^*, & z \le y < z^* \\ (ii) & 0 \le z < \infty, & 0 < y < z \wedge z^* \\ (iii) & z^* \le z < \infty, & z^* \le y < z \\ (iv) & 0 \le z < \infty, & z^* \vee z \le y < \infty \end{array}$$

We will show that the integrals over regions $(i), (ii)$, and (iii) are finite, so they do not contribute to the limit of $\alpha \log \frac{1}{\alpha}\hat{L}(\alpha)$. Then we deal with region (iv).

We need the following bounds for regions (i) and (ii): As $y \to 0$, g is quadratic (see Section 3) and $m(y)$ is asymptotic to $\frac{2}{\beta_L}y^{\nu-1}$ from Lemma (3.9), so $g(y)m(y)$ is asymptotic to $C_1 y^{\nu+1}$ as $y \to 0$. But $\nu + 1 = \frac{2\lambda}{\beta_L} + 1 = \frac{\beta_N}{\beta_L}$, and $\frac{\beta_N}{\beta_L} \geq \frac{1}{3}$ from BAXENDALE and HARRIS (1986). Thus, for some constant C_2, we have

$$g(y)m(y) \leq C_2 y^{1/3}, \qquad 0 < y \leq z^*.$$

Also, from equation (5.3), there exist functions B_4 and γ_- such that

$$v_2(y,\alpha) \leq B_4(\alpha) y^{\gamma_-(K_4\alpha)}, \qquad y \leq z^*$$

Here $\gamma_-(K_4\alpha)$ is negative but increases to 0 as $\alpha \downarrow 0$.

Finally, from the discussion at the beginning of Section 5,

$$\begin{aligned} v_1(z,\alpha) &\leq 1, \qquad && z \leq z^* \\ v_2(z,\alpha) &\leq 1, \qquad && z \geq z^*, \end{aligned}$$

with equality at z^*.

(i) In this region $v_1(z \wedge y, \alpha) \leq 1$ and $v_2(z \vee y, \alpha)$ is decreasing, so the portion of the double integral in (7.9) corresponding to region (i) is less than

$$\int_0^{z^*} \rho(dz) \int_z^{z^*} dy\ v_2(y,\alpha) g(y) m(y)$$

This decreases as α decreases. We only need to show that it becomes finite at some point. The integrand is bounded above by $B_4(\alpha)C_2 y^{\gamma_-(K_4\alpha)+1/3}$, and for α small enough, this is less than $B_4(\alpha)C_2(z^* \vee 1)^{1/3}$. For such α, the integral over this region is finite, so the contribution from region (i) is finite.

(ii) In this region, $v_1(z \wedge y, \alpha) \leq 1$. In addition, $v_2(z \vee y, \alpha)$ is decreasing in z, so the integral is less than

$$\int_0^{\infty} \rho(dz) v_2(z \wedge z^*, \alpha) \int_0^{z \wedge z^*} dy\ g(y) m(y)$$

This is decreasing as α decreases. Using the bounds given above on $g(y)m(y)$ and $v_2(y,\alpha)$, we see that this integral is less than

$$\int_0^{\infty} \rho(dz) B_4(\alpha)(z \wedge z^*)^{\gamma_-(K_4\alpha)} \tfrac{3}{4} C_2 (z \wedge z^*)^{4/3},$$

which is finite for α small enough. Thus, the contribution from region (ii) is finite.

(iii) In this region $v_2(z \vee y, \alpha) \le 1$ and $g(y)$ is bounded, so we consider

$$\int_{z^*}^{\infty} \rho(dz) \int_{z^*}^{z} dy \; v_1(y,\alpha) m(y)$$

This decreases as α decreases. By Lemma (3.10), m is approximately linear as $y \to \infty$, so $m(y)$ is bounded above by $C_3 y$ for all $y \ge z^*$. Also, by equation (5.2) and the asymptotics of Bessel functions,

$$v_1(y,\alpha) \le B_2'(\alpha) C_4 e^{K_2 \sqrt{\alpha} y}, \qquad y \ge z^*.$$

Thus, the integral above is bounded by

$$\int_{z^*}^{\infty} \rho(dz) \int_{z^*}^{z} dy \; B_2'(\alpha) C_4 e^{K_2 \sqrt{\alpha} y} C_3 y$$

which is finite for sufficiently small α by condition (7.1) on M_0.

(iv) In this region $z \wedge y = z$ and $z \vee y = y$. We claim that

$$\lim_{\alpha \downarrow 0} \alpha \log \frac{1}{\alpha} \int_0^{\infty} \rho(dz) v_1(z,\alpha) \int_{z \vee z^*}^{\infty} dy \; v_2(y,\alpha) g(y) m(y) = \frac{A_3}{s(z^*) - s(0)}. \tag{7.10}$$

Here is our approach: By Lemma (5.7), for $y \ge z^*$, $v_2(y,\alpha)$ is bounded above and below by

$$\frac{K_0(\sqrt{2\alpha}(\phi(y) \pm \delta))}{K_0(\sqrt{2\alpha}(\phi(z^*) \pm \delta))}$$

where $\phi(z^*) > \delta \ge 0$. The positive sign is used for the upper bound, the negative sign for the lower bound. This gives upper and lower bounds on the quantity in the limit in (7.10):

$$\alpha \log \frac{1}{\alpha} \int_0^{\infty} \rho(dz) v_1(z,\alpha) \int_{z \vee z^*}^{\infty} dy \; \frac{K_0(\sqrt{2\alpha}(\phi(y) \pm \delta))}{K_0(\sqrt{2\alpha}(\phi(z^*) \pm \delta))} g(y) m(y)$$

We will show that, regardless of the sign of δ, this converges to the limit claimed above.

First, an easy simplification. By the asymptotics of Bessel functions,

$$\lim_{\alpha \downarrow 0} \frac{\log \frac{1}{\alpha}}{K_0(\sqrt{2\alpha}(\phi(z^*) \pm \delta))} = \lim_{\alpha \downarrow 0} \frac{\log \frac{1}{\alpha}}{-\log(\frac{1}{2}\sqrt{2\alpha}(\phi(z^*) \pm \delta))} = 2$$

We are left with evaluating the limit

$$\lim_{\alpha \downarrow 0} \int_0^{\infty} \rho(dz) v_1(z,\alpha) \int_{z \vee z^*}^{\infty} dy \; 2\alpha K_0(\sqrt{2\alpha}(\phi(y) \pm \delta)) g(y) m(y)$$

Change variables by letting $x = \sqrt{2\alpha}(\phi(y) \pm \delta)$. Recall that $\phi(y)$ equals $\int_1^y ds/\sigma(s)$, where $\sigma(s) = \sqrt{2(b_0 - b_L(s))}$. We obtain

$$\lim_{\alpha \downarrow 0} \int_0^{\infty} \rho(dz) v_1(z,\alpha) \int_0^{\infty} dx \; I_z(x) K_0(x) \sqrt{2\alpha} (g m \sigma)(\phi^{-1}(\frac{x}{\sqrt{2\alpha}} \mp \delta)). \tag{7.11}$$

The function I_z is the indicator of the interval $[\sqrt{2\alpha}(\phi(z \vee z^*) \pm \delta), \infty)$. We will use the dominated convergence theorem. First we find a function dominating the integrand and show that it is integrable. Then we evaluate the limit of the integrand.

The function $z \mapsto v_1(z, \alpha)$ is increasing, so $v_1(z, \alpha) \leq v_1(z \vee z^*, \alpha)$ for all $z > 0$. The function ϕ is asymptotically linear near ∞, so its inverse ϕ^{-1} is also. Thus, for some positive constants C_1 and C_2,

$$\phi^{-1}(u) \leq C_1 u + C_2, \qquad u \geq \phi(z^*)$$

Also, g and σ are bounded and $m(y)$ is linear as $y \to \infty$, so for positive constants C_3 and C_4,

$$(gm\sigma)(y) \leq C_3 y + C_4, \qquad y \geq z \vee z^*$$

Combining these two inequalities, we see that for all x in the support of I_z (i.e., $x \geq \sqrt{2\alpha}(\phi(z \vee z^*) \pm \delta)$),

$$\begin{aligned} (gm\sigma)(\phi^{-1}(\frac{x}{\sqrt{2\alpha}} \mp \delta)) &\leq C_3 \phi^{-1}(\frac{x}{\sqrt{2\alpha}} \mp \delta) + C_4 \\ &\leq C_3(C_1(\frac{x}{\sqrt{2\alpha}} \mp \delta) + C_2) + C_4 \end{aligned}$$

Moreover, we may choose $C_2 \geq C_1 \delta$, so that the right hand side is positive for *all* $x \geq 0$. Thus, for all $x, z \geq 0$, the integrand in (7.11) is bounded above by

$$v_1(z \vee z^*, \alpha) K_0(x)(C_5 x + C_6 \sqrt{2\alpha}) \tag{7.12}$$

where $C_5 = C_1 C_3$ and $C_6 = C_3(C_2 \mp C_1 \delta) + C_4$. This bound is decreasing as a function of α, so we need only show that it is integrable for α small enough.

The x and z integrals separate. The z integral, $\int_0^\infty \rho(dz) v_1(z \vee z^*, \alpha)$, is finite for α small enough by the bound (5.2) on v_1 and the exponential condition (7.1) on M_0. The x integral is

$$\int_0^\infty dx\; K_0(x)(C_5 x + C_6 \sqrt{2\alpha})$$

This is finite since $\int_0^\infty dx\; K_0(x) x = -\int_0^\infty (x K_1(x))' dx = 1$ and $\int_0^\infty dx\; K_0(x) < \infty$.

Now we evaluate the limit as $\alpha \to 0$ of the integrand in (7.11). First, the limit of $v_1(z, \alpha)$ is given in (2.10):

$$\lim_{\alpha \downarrow 0} v_1(z, \alpha) = \frac{s(z) - s(0)}{s(z^*) - s(0)}$$

The function I_z converges to 1 as $\alpha \to 0$. Finally, letting $\beta = x/\sqrt{2\alpha}$ and then $\gamma = \phi^{-1}(\beta)$,

$$\begin{aligned} \lim_{\alpha \downarrow 0} \sqrt{2\alpha}(gm\sigma)(\phi^{-1}(\frac{x}{\sqrt{2\alpha}} \mp \delta)) &= \lim_{\beta \to \infty} \frac{x}{\beta}(gm\sigma)(\phi^{-1}(\beta)) \\ &= \lim_{\gamma \to \infty} \frac{x}{\phi(\gamma)}(gm\sigma)(\gamma) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\gamma\to\infty} x\frac{\gamma}{\phi(\gamma)} g(\gamma)\frac{m(\gamma)}{\gamma}\sigma(\gamma) \\
&= x \cdot \sqrt{2b_0} \cdot 2b_0 \cdot \frac{1}{Kb_0} \cdot \sqrt{2b_0} \\
&= \frac{4b_0}{K} x,
\end{aligned}
$$

using the asymptotic behavior of ϕ from (5.4), of m from (3.10), and of σ (defined above (7.11)).

Applying the dominated convergence theorem to (7.11), the limit is

$$
\frac{4b_0}{K}\int_0^\infty \rho(dz)\frac{s(z)-s(0)}{s(z^*)-s(0)} = \frac{A_3}{s(z^*)-s(0)}
$$

since $\int_0^\infty dx\, xK_0(x) = 1$. Thus, the limits of our upper and lower bounds are the same, and we have proven claim (7.10) about the integral over region (iv).

We refer again to equation (7.9). We know from Lemma (5.1) that $h(\alpha) \to s(z^*) - s(0)$ as $\alpha \to 0$. As for the double integral in (7.9), the integrals over regions (i), (ii), and (iii) are finite, so they do not contribute. The limit of the integral over region (iv) is given by (7.10). Thus, $\lim_{\alpha\downarrow 0} \alpha \log \frac{1}{\alpha}\hat{L}(\alpha) = A_3$, as claimed below (7.9). Now, the Tauberian theorem yields $\lim_{t\to\infty} \frac{1}{t} \log tL(t) = A_3$, which is the claim of the theorem.

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