DISPERSION OF PARTICLE SYSTEMS IN BROWNIAN FLOWS

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Abstract
We study the dispersion of a collection of particles carried by an isotropic Brownian flow in $\mathbb{R}^d$. Of particular interest are the center of mass and the centered spatial second moments. Their asymptotic behavior depends strongly on the spatial dimension and the largest Lyapunov exponent of the flow. We use estimates for the pair separation process to give a fairly complete picture of this behavior as $t \to \infty$. In particular, for incompressible flows in two dimensions, we show that the variance of the center of mass grows sublinearly, while dispersion relative to the center of mass grows linearly.
STOCHASTIC FLOWS; BROWNIAN FLOWS; MASS TRANSPORT
AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60H10
SECONDARY 60G57, 76F05

1 Introduction

Our object of study is the long-term spatial spreading of a passive substance carried by a Brownian flow on $\mathbb{R}^d$. Specifically, we are interested in the measure-valued random process $\{M_t;\ t \geq 0\}$ where $M_t(A)$ is the (random) mass of the substance contained in the Borel set A at time t. For Brownian flows on compact manifolds, BAXENDALE (1986) and LE JAN (1984), (1985b) have shown the existence of an equilibrium distribution for M_t, that is, weak

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convergence of the probability law of the random measure M_t as $t \to \infty$, under certain conditions on the manifold and the flow. Our interest is in cases where M_t is dissipative and, therefore, has no such limits.

Our motivation comes from applied problems in statistical turbulence (see MONIN and YAGLOM (1971)), oceanography (see DAVIS (1991) for a review), and groundwater hydrology (see DAGAN (1990) for an account and further references). In these cases the random flow F describing particle motion is not Brownian, but rather solves the classical equation

$$\frac{d}{dt} F_t x = v(F_t x, t), \qquad F_0 x = x \tag{1.1}$$

where $F_t x$ stands for the position at time t of a particle that was at x at time 0, and $v(y, t)$ is the random velocity at point y at time t. The 'passive substance' being carried by this flow may be, for example, a pollutant that has fallen into the flow, and whose motion we wish to predict based on the knowledge of the probability law of v. The ultimate goal may be to remove the pollutant from the flow, or at least to prevent it from hitting certain regions in space.

We would like to find the probability law of the process M, but this is far too difficult. Instead, we concentrate on finding the distribution of various descriptors of M_t. For example, the centroid and dispersivity matrix, introduced below, give a rough measure of the location and spatial extent of the substance. Also of interest are the diameter of the support of M_t and the surface area of the boundary between the substance and the surrounding pure fluid. Finally, given a region A in $\mathbb{R}^d$, it is very useful to know the first time the substance hits A, that is, the random time T_A before which $M_t(A) = 0$ and just after which $M_t(A) > 0$.

At the present time, however, these problems are out of reach for random flows satisfying (1.1). The current theory deals mostly with low-order statistics of the density ρ_t of M_t, assuming such a density exists. For example, much effort is expended trying to find partial differential equations for the *mean concentration*, $\mathbb{E}\rho_t(x)$. This is equivalent to finding the probability law of the one-point motion $\{F_t x;\ t \geq 0\}$, and is much harder than one might expect because of the complex spatial and temporal dependence in the velocity field v.

In this paper we consider mass transport by isotropic Brownian flows on $\mathbb{R}^d$. Such flows have complex spatial structure, but simpler temporal dependence than the flows discussed above.

In an isotropic Brownian flow, single particle motions are Brownian motions in $\mathbb{R}^d$, and for each compact set A, Theorem 4.3.10 of KUNITA (1990) shows that $M_t(A) \to 0$ in L^1 as $t \to \infty$. In other words, the flow dissipates the mass. To understand this process better, in this article we concentrate on measures of translation and dispersion for the random measure M_t, namely

$$C_t = \frac{1}{M_t(\mathbb{R}^d)} \int_{\mathbb{R}^d} M_t(dx)\, x \tag{1.2}$$

$$D_t = \frac{1}{M_t(\mathbb{R}^d)} \int_{\mathbb{R}^d} M_t(dx)\, (x - C_t)(x - C_t)^T \tag{1.3}$$

where the superscript T denotes transpose; we call these the *centroid vector* and *dispersivity matrix* of M_t.

The asymptotic behavior of C_t and D_t is intimately connected with the spatial dimension, the top Lyapunov exponent of the underlying flow, and certain characteristics of the pair-distance process. The latter have been studied for isotropic Brownian flows by LE JAN (1985a) and by BAXENDALE and HARRIS (1986). We use their results, supplemented by some estimates for the pair-distance process, to give a reasonably complete picture of the asymptotics of centroid and dispersivity.

In Section 2 we introduce the basic definitions of Brownian flows and review some results of LE JAN (1985a) and BAXENDALE and HARRIS (1986) on isotropic Brownian flows. Section 3 contains the main results (Proposition (3.5) and Theorem (3.9)) and some discussion. In Section 4 we provide some elementary computations for the centroid and dispersivity. Section 5 contains the proof of Theorem (3.9), which relies on an upper bound on the occupation time of a one-dimensional diffusion process. The comparison theorem of IKEDA and WATANABE (1981) gives the upper bound in terms of a simpler diffusion process, which is analyzed in Section 6.

2 Brownian flows

In this section we introduce Brownian flows and provide a brief overview of isotropic Brownian flows. Throughout, $(\Omega, \mathcal{H}, I\!P)$ is the probability space over which all random variables are defined. The Borel σ-algebra will be used on all Euclidean spaces, and $\mathbb{R}_+$ will be used to denote $[0, \infty)$.

A random flow on $\mathbb{R}^d$ is a family F of random transformations $F_{st}, 0 \leq s \leq t < \infty$, from $\mathbb{R}^d$ into $\mathbb{R}^d$ such that, almost surely,

(2.1)
$$\begin{aligned} &(a) \quad F_{ss} \text{ is the identity mapping for each } s \\ &(b) \quad F_{st} \circ F_{rs} = F_{rt}, \qquad 0 \leq r \leq s \leq t \end{aligned}$$

We assume that the map taking (s, t, ω, x) to $F_{st}(\omega, x)$ is jointly measurable. We think of the random variable $F_{st}(x)$ as the position at time t of a particle which was at x at an earlier time s. For simplicity we shall write $F_t x$ in place of $F_{0t}(x)$.

Brownian flows

The flows we consider are solutions of the stochastic equation

(2.2)
$$F_t x = x + \int_0^t U(F_s x, ds)$$

The flow F solves this equation for all x simultaneously. The stochastic integral here generalizes the usual stochastic integral, for U is a random field with both a spatial and a temporal argument. More precisely, U is a continuous Gaussian random field with mean and covariance satisfying

(2.3)
$$\begin{aligned} I\!EU(x,t) &= u(x)t \\ \operatorname{Cov}(U^i(x,s), U^j(y,t)) &= a^{ij}(x,y)(s \wedge t), \end{aligned}$$

where $s \wedge t$ denotes the smaller of s and t. We call u the *drift* and a the *covariance*. Under certain regularity and growth conditions on u and a (see

Section 4), there exists a unique flow F satisfying (2.2) such that the mapping $x \mapsto F_t x$ is a homeomorphism almost surely. See KUNITA (1990) for this result and the general theory of equation (2.2).

The structure of the temporal dependence in U is such that, for $s < t$, the random field $U(\cdot, t) - U(\cdot, s)$ is independent of the history of U to time s. The flow F inherits a similar independence of increments:

$$F_{t_1 t_2}, \ldots, F_{t_{n-1} t_n} \text{ are independent when } t_1 \le t_2 \le \ldots \le t_n$$

For this reason, F is called a *Brownian flow.*

Under minimal assumptions on a we may write U as

$$U(x,t) = u(x)t + \sum_{k=1}^{\infty} u_k(x) W_k(t), \tag{2.4}$$

where $u_1, u_2, \ldots$ are deterministic vector fields and $W_1, W_2, \ldots$ are independent one-dimensional Wiener processes. The covariance a is related to $u_1, u_2, \ldots$ through

$$a(x,y) = \sum_{k=1}^{\infty} u_k(x) u_k(y)^T.$$

Equation (2.2) for F becomes, in differential notation,

$$dF_t x = u(F_t x)dt + \sum_{k=1}^{\infty} u_k(F_t x) W_k(dt) \tag{2.5}$$

This makes clear the relation of (2.2) to standard stochastic differential equations. Note, however, that in this equation there are an infinite number of noise terms, and that we are interested in solving the equation simultaneously for all x.

Isotropic Brownian flows

These are Brownian flows whose probability laws are invariant under all rigid motions of $\mathbb{R}^d$. They are the Brownian versions of random isotropic flows that have been of interest in statistical turbulence ever since the seminal work of KOLMOGOROV (1941). See MONIN and YAGLOM (1971) for an account of the classical case where F is obtained from (1.1) with v a random isotropic vector field; also, see BENNETT (1987), BENNETT and DENMAN (1989), and DAVIS (1991) for applications to oceanography.

Isotropic Brownian flows were characterized and studied by LE JAN (1985a)and BAXENDALE and HARRIS (1986) simultaneously. The following brief review is taken from their work. The reader may also wish to consult the review article by DARLING (1989).

A Brownian flow F obtained from (2.2) is isotropic if and only if

$$u = 0; \qquad a(x, y) = b(x - y), \qquad x, y \in \mathbb{R}^d, \tag{2.6}$$

where b is an *isotropic* covariance tensor, that is, $G^T b(Gz)G = b(z)$ for every orthogonal matrix G and every z in $\mathbb{R}^d$. Under these conditions, for every rigid motion R on $\mathbb{R}^d$, that is, a combination of translation, rotation, and/or reflection, the flow $\tilde{F}$ defined by $\tilde{F}_{st} = R \circ F_{st} \circ R^{-1}$ has the same probability law as F.

When $d = 1$, isotropy requires only that $u = 0$ and $b(z) = b(-z)$. To guarantee the existence of a non-trivial flow of diffeomorphisms, we will assume the following condition.

(2.7) **Condition.** For $d = 1$, both b and b'' are bounded and continuous, and b is not identically constant.

In two or more dimensions, isotropy forces b to have a special form, due to YAGLOM (1957); see also YAGLOM (1987):

$$b^{ij}(z) = b_N(|z|)\delta_{ij} + \frac{z^i z^j}{|z|^2}(b_L(|z|) - b_N(|z|)), \qquad z \in \mathbb{R}^d \tag{2.8}$$

where b_L and b_N are real-valued functions on $\mathbb{R}_+$ defined by two finite measures Φ_P and Φ_S on $\mathbb{R}_+$; see (5.13) and (5.14) for the exact form.

The measure Φ_P gives rise to the potential (irrotational) part of the flow and Φ_S to the solenoidal (incompressible) part. Indeed, when $\Phi_P = 0$, the flow F is incompressible, that is, it preserves Lebesgue measure. See LE JAN (1985a), page 616.

(2.9) **Condition.** For $d \geq 2$, the measures Φ_P and Φ_S have finite fourth moments and put no mass on the set $\{0\}$.

Under this condition, b_L and b_N are C^4-functions decaying to 0 at $+\infty$, there exists an isotropic flow F satisfying (2.2), and for each t, the mapping $x \mapsto F_t x$ is a C^1-diffeomorphism. Moreover, b_L and b_N achieve their maximum value b_0 only at 0, where

$$b_0 = \frac{1}{d}\Phi_P(0,\infty) + \frac{d-1}{d}\Phi_S(0,\infty) \tag{2.10}$$

Near 0, b_L and b_N are quadratic. We set $\beta_L = -b_L''(0)$ and $\beta_N = -b_N''(0)$; these are strictly positive and given explicitly in terms of Φ_P and Φ_S. See BAXENDALE and HARRIS (1986).

Lyapunov exponents

The Lyapunov exponents $\lambda_1 > \lambda_2 > \ldots > \lambda_d$ of the flow are the values taken by the limits

$$\lim_{t\to\infty} \frac{1}{t} \log \mid DF_t(x)\xi \mid$$

as the vector ξ varies over the unit sphere in $\mathbb{R}^d$; here $DF_t(x)$ is the Jacobian matrix of the map F_t evaluated at x. These values are deterministic in an isotropic Brownian flow. For $d = 1$ and b satisfying Condition (2.7), we have $\lambda_1 = -\frac{1}{2}b''(0) \leq 0$. For $d \geq 2$, they are given by:

$$\lambda_i = \frac{d-i}{2}\beta_N - \frac{i}{2}\beta_L\ , \qquad i = 1, \ldots, d. \tag{2.11}$$

When F is incompressible, we have $\Phi_P = 0$ which gives $(d+1)\beta_L = (d-1)\beta_N$, and hence the top Lyapunov exponent is

$$\lambda_1 = \frac{d}{2}\beta_L > 0. \tag{2.12}$$

When F is potential, $\Phi_S = 0$, we have $\beta_L = 3\beta_N$, and

$$\lambda_1 = \frac{d-4}{6}\beta_L, \tag{2.13}$$

which is negative for $d = 2$ and $d = 3$.

Separation process

For fixed x and y in $\mathbb{R}^d$, the distance at time t between two particles started from x and y is

$$Z_t = |F_t x - F_t y|. \tag{2.14}$$

The process $Z = \{Z_t;\ t \geq 0\}$ is called the separation process (or pair-distance process) with initial value $|x-y|$. When F is isotropic, Z is a one-dimensional diffusion on $\mathbb{R}_+$. Under conditions (2.7) and (2.9), Z satisfies the stochastic differential equation

$$dZ_t = \sqrt{2(b_0 - b_L(Z_t))}\,dW_t + (d-1)\frac{b_0 - b_N(Z_t)}{Z_t}dt \tag{2.15}$$

The boundary point 0 is absorbing and inaccessible from $(0,\infty)$. When $d = 1$, the drift part vanishes, we write b in place of b_L, $b(0)$ in place of b_0, and Z becomes a martingale diffusion.

The following proposition lists the asymptotic behavior of the process Z as $t \to \infty$; see LE JAN (1985a) and BAXENDALE and HARRIS (1986) for the proof. Here λ is the top Lyapunov exponent λ_1 of (2.11) and we assume $x \neq y$.

(2.16) **Proposition.** *The process Z is transient on $(0,\infty)$ unless $d = 2$ and $\lambda \geq 0$. More specifically,* *(i) if $d \geq 4$, or if $d = 3$ and $\lambda \geq 0$, then $Z_t \to +\infty$ almost surely;*

(ii) if $d = 3$ and $\lambda < 0$, then Z_t converges to either 0 or $+\infty$, each with strictly positive probability;

(iii) if $d = 2$ and $\lambda > 0$, then Z is null-recurrent and converges to $+\infty$ in probability;

(iv) if $d = 2$ and $\lambda = 0$, then Z is null-recurrent;

(v) if $d = 2$ and $\lambda < 0$, or if $d = 1$, then $Z_t \to 0$ almost surely.

It follows from this proposition and (2.12) that $\mathbb{P}\{Z_t \to 0\} = 0$ for incompressible flows and in general whenever $d \geq 4$. For $d \leq 3$, $\mathbb{P}\{Z_t \to 0\}$ becomes strictly positive if the potential part is large enough.

The case $d = 2$ and $\lambda = 0$ is a bit of a puzzle. Indeed, LE JAN (1985a) shows that the invariant measure π of Z satisfies $\pi(0, \alpha) = \pi(\alpha, \infty) = \infty$ for all $\alpha > 0$, so it is difficult to conclude anything about $\mathbb{P}\{Z_t \leq \alpha\}$ or $\mathbb{P}\{Z_t \geq \alpha\}$ as $t \to \infty$.

3 Main Results

In this section we present our main results concerning the asymptotic behavior of the variance of C_t and the mean of D_t as $t \to \infty$. Proofs will be developed in the following sections.

Let F be an isotropic Brownian flow on $\mathbb{R}^d$ with covariance tensor b. We assume that Conditions (2.7) and (2.9) hold. We write b_0 for the number $b^{11}(0) = \cdots = b^{dd}(0)$ and use λ to denote the top Lyapunov exponent λ_1 of the flow.

Let M_0 be a deterministic measure on $\mathbb{R}^d$ with $M_0(\mathbb{R}^d) = 1$ and such that D_0 is finite. If M_0 has an atomic component, let ρ_0 be the sum of the squares of the masses of the atoms; otherwise, set $\rho_0 = 0$. Then $\rho_0 < 1$ unless M_0 consists of exactly one atom, which is a trivial case.

The measure-valued random process $\{M_t;\ t \geq 0\}$ is defined by setting

$$M_t(A) = M_0(\{x \in \mathbb{R}^d : F_t x \in A\}) = M_0(F_t^{-1}A) \tag{3.1}$$

for each $t > 0$ and each Borel subset A of $\mathbb{R}^d$. Note that $M_t(\mathbb{R}^d) = 1$. The centroid C_t and dispersivity matrix D_t are defined as in (1.2) and (1.3).

We think of M_0 as representing a collection of particles (or a continuous mass distribution), so that $M_0(A)$ is the total mass in the set A at time 0. The particles are carried inertly by the flow, and there is neither creation nor annihilation of mass, so $M_t(A)$ is the total mass in set A at time t.

We begin with formulas for the covariance of C_t and the mean of D_t. Formulas for general Brownian flows are derived in Section 4; this is a special case.

(3.2) **Proposition.** *Suppose F is isotropic and D_0 is finite. Then,*

$$\mathrm{Cov}(C_t^i, C_t^j) = \int M_0(dx) \int M_0(dy) \mathbb{E} \int_0^t ds\, b^{ij}(F_s x - F_s y) \tag{3.3}$$

$$\mathbb{E} D_t^{ij} = D_0^{ij} + b_0 \delta_{ij} t - \mathrm{Cov}(C_t^i, C_t^j) \tag{3.4}$$

Linear growth of Cov(C_t) and $I\!\!E D_t$

The first result is a complete picture of the linear growth rates of $\mathrm{Cov}(C_t^i, C_t^j)$ and $I\!\!E D_t$ in the long run. This proposition covers all dimensions $d \geq 1$ and all cases except the case $d = 2$ and $\lambda \geq 0$. The proof will be given at the end of Section 4; it is a straightforward application of the preceding proposition and Proposition (2.16).

(3.5) **Proposition.** *a) If $d \geq 4$, or if $d = 3$ and $\lambda \geq 0$, then*

$$\lim_{t\to\infty} \frac{d}{dt}\mathrm{Cov}(C_t^i, C_t^j) = b_0\rho_0\delta_{ij}, \qquad \lim_{t\to\infty} \frac{d}{dt} I\!\!E D_t^{ij} = b_0(1-\rho_0)\delta_{ij}$$

b) If $d = 3$ and $\lambda < 0$, then

$$\lim_{t\to\infty} \frac{d}{dt}\mathrm{Cov}(C_t^i, C_t^j) = b_0(1-p(1-\rho_0))\delta_{ij}, \qquad \lim_{t\to\infty} \frac{d}{dt} I\!\!E D_t^{ij} = b_0 p(1-\rho_0)\delta_{ij}$$

where $p \in (0,1)$ is given by

$$p = \int M_0(dx) \int M_0(dy) I\!\!P\{\lim_{t\to\infty} |F_t x - F_t y| = +\infty\}.$$

c) If $d = 1$, or $d = 2$ and $\lambda < 0$, then

$$\lim_{t\to\infty} \frac{d}{dt}\mathrm{Cov}(C_t^i, C_t^j) = b_0\delta_{ij}, \qquad \lim_{t\to\infty} \frac{d}{dt} I\!\!E D_t^{ij} = 0$$

Moreover, all of these results hold with $\frac{d}{dt}$ replaced by $\frac{1}{t}$. □

A benchmark for this result is provided by the following example. Suppose that N_0 is a purely atomic measure with total mass 1. Let ρ_N be the sum of the squares of the masses of the atoms. Each atom moves according to a Brownian motion with zero drift and covariance matrix $b_0 I$, and these Brownian motions are mutually independent. The random measure N_t is formed by the atoms at their locations at time t. This is the classical model of the diffusion of a collection of particles.

A straightforward calculation shows that for this model we have

$$\text{(3.6)} \qquad \mathrm{Cov}(C_t^i, C_t^j) = b_0 \rho_N \delta_{ij} t, \qquad \mathbb{E} D_t^{ij} = D_0^{ij} + b_0(1-\rho_N)\delta_{ij} t$$

Thus, in the classical model, the variance of the centroid increases linearly, as does the expected dispersivity about the centroid.

Similar behavior occurs for isotropic Brownian flows with $\lambda > 0$ (which includes all non-trivial incompressible flows), provided M_0 has an atomic component. Indeed, the long term growth rates are exactly as in the classical model. More interesting effects, however, manifest themselves in the other cases.

First, when $\lambda > 0$ but M_0 is *diffuse*, we have $\rho_0 = 0$ and so $\mathrm{Var}(C_t^i)$ grows sublinearly, while $\mathbb{E} D_t^{ii}$ grows linearly. Hence, at long times, dispersion relative to the centroid is much more pronounced than the translation of the centroid. We will give more detailed estimates of $\mathrm{Var}(C_t^i)$ later in this section.

When $d = 3$ and $\lambda < 0$, both $\mathrm{Var}(C_t^i)$ and $\mathbb{E} D_t^{ii}$ grow linearly, but with rates different from classical diffusion (when the initial measures M_0 and N_0 are atomic and identical). As p decreases, there is more translation and less dispersion. It is worth noting that the value of p is affected by the initial mass distribution: when M_0 is more highly concentrated, p will be smaller. In this way, the initial distribution affects the long-time rate of spreading.

Finally, when $d = 1$, or $d = 2$ and $\lambda < 0$, translation of the centroid is much stronger than dispersion relative to the centroid. In fact, it is shown in ZIRBEL (1993) that for $d = 1$, under certain conditions on b and M_0, the scalar $\mathbb{E} D_t$ grows like $\sqrt{t}$ for large t.

Nearly incompressible flows

We will now sharpen the preceding result in the case $d \geq 2$ and $\lambda > 0$ by analyzing the occupation times of the process $|F_t x - F_t y|$ in more detail.

We require additional conditions on the covariance b and the initial measure M_0. We require that the covariance satisfy:

$$2(d-1)(b_0 - b_N(r)) + r b_L'(r) \geq 0, \qquad \text{all } r > 0 \tag{3.7}$$

This condition is satisfied by incompressible flows and by flows which are nearly incompressible in the sense that their potential component is small enough; see the end of Section 5 for details. We also assume that $|b_L(r)|$ and $|b_N(r)|$ decay faster than $r^{-\delta}$ as $r \to \infty$, for some $\delta > 1$. Finally, we place a condition on the initial measure M_0:

$$\int_{\mathbb{R}^d} M_0(dx) \int_{\mathbb{R}^d \setminus \{x\}} M_0(dy) \log^+ \frac{1}{|x-y|} < \infty \tag{3.8}$$

where $\log^+ z = \max\,(0, \log z)$. Roughly, this requires that the diffuse component of M_0 not have strong concentrations of mass.

Under these conditions the following result holds. The proof will be given in Section 5; see (5.10) for the definition of the function m appearing in the statement.

(3.9) **Theorem.** *There is a continuous function m with $m(0) = 0$ and $\lim_{t\to\infty} m(t) < +\infty$ such that, for $t \geq 0$ and $i, j = 1, \ldots, d$,*

$$|\mathrm{Cov}(C_t^i, C_t^j) - b_0 \rho_0 \delta_{ij} t| \leq m(t)\sqrt{t} \tag{3.10}$$

$$|\mathbb{E} D_t^{ij} - D_0^{ij} - b_0(1-\rho_0)\delta_{ij} t| \leq m(t)\sqrt{t} \tag{3.11}$$

When M_0 has an atomic component, ρ_0 is strictly positive. The new information here is that the deviation of $\mathrm{Cov}(C_t^i, C_t^j)$ and $\mathbb{E} D_t^{ij}$ from their linear asymptotes is no more than $\sqrt{t}$. In particular, the off-diagonal elements are bounded by $\sqrt{t}$.

When M_0 is diffuse, we have a much more interesting situation:

(3.12) **Corollary.** *Suppose M_0 is diffuse. Then, for $t \geq 0$ and $i = 1, \ldots, d$,*

(3.13) $$\mathrm{Var}(C_t^i) \leq m(t)\sqrt{t}$$

(3.14) $$D_0^{ii} + b_0 t - m(t)\sqrt{t} \leq \mathbb{E} D_t^{ii} \leq D_0^{ii} + b_0 t$$

Proof: The upper bound on $\mathbb{E} D_t^{ii}$ follows from Proposition (3.2), the rest is immediate from Theorem (3.9) using $\rho_0 = 0$. □

Thus, a continuous mass distribution in a nearly incompressible isotropic Brownian flow exhibits classical spreading of mass relative to the centroid, but the centroid itself moves much less than in the classical model. It would be interesting to find the *exact* rate of growth of $\mathrm{Var}(C_t^i)$.

4 Centroid and Dispersivity Calculations

Our purpose here is to provide formulas for the mean and variance of the centroid and dispersivity of the random measure M_t as it is carried by a Brownian flow F with drift u and covariance a. We assume that u and a satisfy, for some $K > 0$,

$$|u(x)| \leq K(1+|x|) \tag{4.1}$$

$$|u(x)-u(y)| \leq K|x-y| \tag{4.2}$$

$$|a^{ij}(x,y)| \leq K(1+|x|)(1+|y|) \tag{4.3}$$

$$|a^{ij}(x,y)-a^{ij}(x',y)-a^{ij}(x,y')+a^{ij}(x',y')| \leq K|x-x'||y-y'| \tag{4.4}$$

for all x, x', y, y' in $\mathbb{R}^d$ and $i, j = 1, \ldots, d$. As mentioned in Section 2, under these conditions, there exists a flow F of homeomorphisms satisfying (2.2). See KUNITA (1990), Theorem 4.5.1.

Suppose the measure M_0 is such that C_0 and D_0 are well-defined and finite. Normalize by setting $M_0(\mathbb{R}^d) = 1$. We claim that, for each t, C_t and D_t are well-defined and finite almost surely. We need the following L^p bound from KUNITA (1990), Lemma 4.5.3: for each $p > 0$, there is a constant $C(p)$ such that,

$$\mathbb{E}(1+|F_t x|^2)^p \leq e^{C(p)t}(1+|x|^2)^p, \qquad t \geq 0. \tag{4.5}$$

For C_t, observe that $|x| \leq (1+|x|^2)^{1/2} \leq 1+|x|$, so that

$$\begin{aligned} \mathbb{E}\int_{\mathbb{R}^d} M_t(dx)|x| &\leq \int_{\mathbb{R}^d} M_0(dx)\mathbb{E}(1+|F_t x|^2)^{1/2} \\ &\leq \int_{\mathbb{R}^d} M_0(dx)e^{C(\frac{1}{2})t}(1+|x|^2)^{1/2} \\ &< +\infty \end{aligned}$$

Thus $\int_{\mathbb{R}^d} M_t(dx)|x| < +\infty$ almost surely, so C_t is well-defined and finite almost surely. The proof for D_t is similar.

We will decompose U as

$$U(x,t) = u(x)t + U_0(x,t) \tag{4.6}$$

so that equation (2.2) becomes

$$F_t x = x + \int_0^t ds\ u(F_s x) + \int_0^t U_0(F_s x, ds) \tag{4.7}$$

Since U_0 is a Gaussian random field with mean 0 and covariance kernel $a(x,y)(s \wedge t)$, the last integral in (4.7) defines a square-integrable martingale, and the joint quadratic variation of the semimartingales $F_t^i x$ and $F_t^j y$ is given by

$$\langle F_t^i x, F_t^j y \rangle_t = \int_0^t ds\ a^{ij}(F_s x, F_s y). \tag{4.8}$$

See KUNITA (1990), Theorem 3.2.4. The following summarizes a few results:

(4.9) **Lemma.** *For $t \geq 0$ and $i, j = 1, \ldots, d$,*

$$\mathbb{E} F_t x = x + \mathbb{E} \int_0^t ds\ u(F_s x) \tag{4.10}$$

$$\mathbb{E}(F_t^i x)(F_t^j y) = x^i y^j + \mathbb{E} \int_0^t ds\ A^{ij}(F_s x, F_s y) \tag{4.11}$$

where

$$A^{ij}(x,y) = a^{ij}(x,y) + x^i u^j(y) + u^i(x) y^j. \tag{4.12}$$

Proof: The result for $\mathbb{E} F_t x$ is immediate from (4.7) since the last integral in (4.7) defines a martingale.

For (4.11), we start with the integration by parts formula:

$$(F_t^i x)(F_t^j y) = x^i y^j + \int_0^t F_s^i x\ dF_s^j y + \int_0^t F_s^j y\ dF_s^i x + \langle F^i x, F^j y \rangle_t$$

Using (4.7) and (4.8) and the notation (4.12), we get

$$(F_t^i x)(F_t^j y) = x^i y^j + \int_0^t ds\ A^{ij}(F_s x, F_s y) + \int_0^t F_s^i x\ dY_s^j + \int_0^t F_s^j x\ dX_s^i \tag{4.13}$$

where X_t stands for the martingale part of $F_t x$ and Y_t for that of $F_t y$. The proof of (4.11) follows from (4.13) once we show that the stochastic integrals on the right side are in fact martingales. We will do this for the first; the proof for the second is similar.

Recall that the quadratic variation of Y^j is also that of $F^j y$. By virtue of (4.8), we have

$$\begin{aligned} I\!\!E \int_0^t (F_s^i x)^2 d\langle Y^j, Y^j\rangle_s &= I\!\!E \int_0^t (F_s^i x)^2 a^{jj}(F_s y, F_s y) \\ &\leq K \int_0^t ds\ I\!\!E |F_s x|^2 (1 + |F_s y|)^2, \end{aligned}$$

which is finite by the Cauchy-Schwartz inequality and (4.5). Thus, $\int F_s^i x\, dY_s^j$ is in fact a square-integrable martingale. □

We can now easily derive formulas for C_t.

(4.14) **Proposition.** *The centroid satisfies:*

(4.15) $$I\!\!E C_t = C_0 + \int M_0(dx) I\!\!E \int_0^t ds\ u(F_s x),$$

(4.16) $$I\!\!E C_t^i C_t^j = C_0^i C_0^j + \int M_0(dx) \int M_0(dy) I\!\!E \int_0^t ds\ A^{ij}(F_s x, F_s y)$$

Proof: First we calculate formally, using the result of the previous lemma.

$$\begin{aligned} I\!\!E C_t &= I\!\!E \int M_0(dx) F_t x \\ &= \int M_0(dx) I\!\!E F_t x \\ &= \int M_0(dx)(x + I\!\!E \int_0^t ds\ u(F_s x)) \end{aligned}$$

$$\begin{aligned} I\!\!E C_t^i C_t^j &= I\!\!E \int M_0(dx) F_t^i x \int M_0(dy) F_t^j y \\ &= \int M_0(dx) \int M_0(dy) I\!\!E F_t^i x F_t^j y \\ &= \int M_0(dx) \int M_0(dy) (x^i y^j + I\!\!E \int_0^t ds A^{ij}(F_s x, F_s y)) \end{aligned}$$

These yield the desired results after rearranging. The interchanges of integrals are justified by using (4.5) to show that the integrands are absolutely integrable. □

Finally, we come to D_t.

(4.17) **Proposition.** *For* $t \geq 0$ *and* $i, j = 1, \ldots, d$,

$$\mathbb{E} D_t^{ij} = D_0^{ij} + \int M_0(dx) \int M_0(dy) \mathbb{E} \int_0^t ds [A^{ij}(F_s x, F_s x) - A^{ij}(F_s x, F_s y)]$$

where $A^{ij}(x, y) = a^{ij}(x, y) + x^i u^j(y) + u^i(x) y^j$.

Proof: Write D_t^{ij} as follows

$$\begin{aligned} D_t^{ij} &= \int M_t(dx)(x^i - C_t^i)(x^j - C_t^j) \\ &= \int M_t(dx) x^i x^j - C_t^i C_t^j \\ &= \int M_0(dx) F_t^i x F_t^j x - C_t^i C_t^j \end{aligned} \tag{4.18}$$

Now use the previous two results:

$$\begin{aligned} \mathbb{E} D_t^{ij} &= D_0^{ij} + \int M_0(dx) \mathbb{E} \int_0^t ds A^{ij}(F_s x, F_s x) \\ &\quad - \int M_0(dx) \int M_0(dy) \mathbb{E} \int_0^t ds A^{ij}(F_s x, F_s y) \end{aligned}$$

Interchanging integrals and recalling that $M_0(\mathbb{R}^d) = 1$ yields the desired result. □

Proof of Proposition (3.2)

If F is isotropic, then $u = 0$ and $a(x, y) = b(x - y)$, and it follows that A defined by (4.12) has the form $A(x, y) = a(x, y) = b(x - y)$. Thus, we get $\mathbb{E} C_t = C_0$ from (4.15), the formula (4.16) becomes (3.3), and the formula of (4.17) yields (3.4). □

Proof of Proposition (3.5)

From (3.3) we have

$$\text{(4.19)} \qquad \frac{d}{dt}\mathrm{Cov}(C_t^i, C_t^j) = \int M_0(dx) \int M_0(dy) \mathbb{E} b^{ij}(F_t x - F_t y)$$

Recall that we are excluding the case where $d = 2$ and $\lambda \geq 0$. In view of Proposition (2.16), the Markov process $|F_t x - F_t y|$ converges to 0 or $+\infty$, and $b(z) \to 0$ as $z \to \infty$ by Conditions (2.7) and (2.9). Thus, letting $q(|x-y|) = \mathbb{P}\{|F_t x - F_t y| \to 0\}$, for $x \neq y$ we get

$$\lim_{t\to\infty} \mathbb{E} b^{ij}(F_t x - F_t y) = \begin{cases} 0 & \text{if } d \geq 4, \text{ or if } d = 3 \text{ and } \lambda \geq 0 \\ b_0 q(|x-y|)\delta_{ij} & \text{if } d = 3 \text{ and } \lambda < 0 \\ b_0 \delta_{ij} & \text{if } d = 2 \text{ and } \lambda < 0, \text{ or } d = 1 \end{cases}$$

On the other hand, for $x = y$, we have $b^{ij}(F_t x - F_t y) = b_0 \delta_{ij}$ for all t. Putting these into (4.19) we obtain the results claimed in (3.5) for $\mathrm{Cov}(C_t^i, C_t^j)$. The results for $\mathbb{E} D_t^{ij}$ are now immediate from the formula (3.4). Moreover, $\frac{d}{dt}$ can be replaced by $\frac{1}{t}$, since $\frac{d}{dt} f(t) \to c$ implies $\frac{1}{t} f(t) \to c$. □

5 Nearly incompressible flows

This section is devoted to the proof of Theorem (3.9). The idea of the proof is as follows. Define $\mathcal{I}(t)$ by

$$\mathcal{I}(t) = |\mathrm{Cov}(C_t^i, C_t^j) - b_0\rho_0\delta_{ij}t|, \qquad t \geq 0 \tag{5.1}$$

First, we bound $\mathcal{I}(t)$ above by an occupation time of the separation process Z:

$$\mathcal{I}(t) \leq \int_0^\infty \rho(dz) I\!E^z \int_0^t ds\ f(Z_s)$$

where ρ is a finite measure and f is decreasing. Next, we change variables via $Y_t = \phi(Z_t)$ to obtain a process Y on $\mathbb{R}$ which has a simpler form. Then we use a comparison theorem to find a process X such that $X_t \leq Y_t$ almost surely, which gives an upper bound on the occupation time of Z. The asymptotic analysis of X is done in Section 6.

Proof of Theorem (3.9)

a) Fix i and j such that $1 \leq i, j \leq d$. From (3.3),

$$\begin{aligned} \mathrm{Cov}(C_t^i, C_t^j) &= \int M_0(dx) \int M_0(dy) I\!E \int_0^t ds\ b^{ij}(F_s x - F_s y) \\ &= b_0\rho_0\delta_{ij}t + \int_{\mathbb{R}^d} M_0(dx) \int_{\mathbb{R}^d \setminus \{x\}} M_0(dy) I\!E \int_0^t ds\ b^{ij}(F_s x - F_s y) \end{aligned}$$

Here, ρ_0 is the sum of the squares of the masses of the atoms of M_0. Then,

$$\begin{aligned} \mathcal{I}(t) &= \left| \int_{\mathbb{R}^d} M_0(dx) \int_{\mathbb{R}^d \setminus \{x\}} M_0(dy) I\!E \int_0^t ds\ b^{ij}(F_s x - F_s y) \right| \\ &\leq \int_{\mathbb{R}^d} M_0(dx) \int_{\mathbb{R}^d \setminus \{x\}} M_0(dy) I\!E \int_0^t ds\ |b^{ij}(F_s x - F_s y)| \end{aligned}$$

Now by (2.8), for each z in $\mathbb{R}^d$,

$$b^{ij}(z) = (\delta_{ij} - \frac{z^i z^j}{|z|^2}) b_N(|z|) + \frac{z^i z^j}{|z|^2} b_L(|z|).$$

Note that $\frac{z^i z^j}{|z|^2}$ is in $[-1,1]$, so,

$$|b^{ij}(z)| \;\le\; 2|b_N(|z|)| + |b_L(|z|)| \;\le\; f(|z|) \tag{5.2}$$

for some strictly decreasing function f with $f(0) = 3b_0$ and such that $f(r)$ decays faster than $r^{-\delta}$ as $r \to \infty$, using the assumption made on $|b_L|$ and $|b_N|$ for Theorem (3.9). We have shown that

$$\begin{aligned} \mathcal{I}(t) \;&\le\; \int_{\mathbb{R}^d} M_0(dx) \int_{\mathbb{R}^d\setminus\{x\}} M_0(dy) I\!E^{|x-y|} \int_0^t ds\, f(Z_s) \\ &= \int_0^\infty \rho(dz) I\!E^z \int_0^t ds\, f(Z_s) \end{aligned} \tag{5.3}$$

where Z is the separation process. The notation $I\!E^z$ means that the stochastic process in the expectation (in this case, Z) starts at z. The measure ρ is defined by its integral against Borel $h : \mathbb{R}_+ \to \mathbb{R}_+$:

$$\int_0^\infty \rho(dz) h(z) = \int_{\mathbb{R}^d} M_0(dx) \int_{\mathbb{R}^d\setminus\{x\}} M_0(dy) h(|x-y|) \tag{5.4}$$

Note that ρ has no mass at zero, and that $\rho(\mathbb{R}_+) = 1 - \rho_0$, where ρ_0 is the sum of the squares of the masses of the atoms.

b) Now we set $Y_t = \phi(Z_t)$ to obtain a process Y which is suitable for applying a comparison theorem to bound $I\!E^z f(Z_t)$ above. Recall from (2.15) the stochastic differential equation for Z:

$$dZ_t = \sqrt{2(b_0 - b_L(Z_t))} dW_t + (d-1)\frac{b_0 - b_N(Z_t))}{Z_t} dt$$

Define ϕ by

$$\phi(z) = \int_1^z \frac{dr}{\sqrt{2(b_0 - b_L(r))}}, \qquad 0 < z < \infty \tag{5.5}$$

This integral is well defined, ϕ is strictly increasing, and ϕ maps $\mathbb{R}_+$ to all of $\mathbb{R}$ since $b_0 - b_L$ is quadratic near the origin. Let $Y_t = \phi(Z_t)$. Using Itô's Lemma we see that Y satisfies

$$\begin{aligned} dY_t \;&=\; dW_t + \frac{1}{2Z_t\sigma(Z_t)}[2(d-1)(b_0 - b_N(Z_t)) + Z_t b_L'(Z_t)]dt \\ &=\; dW_t + \mu(Y_t)dt, \end{aligned} \tag{5.6}$$

which defines the drift μ implicitly. Condition (3.7) was imposed to ensure that $\mu(y) \geq 0$ for all y. Moreover, as $y \to -\infty$ we have $z \to 0$, and thus

$$
\begin{aligned}
\lim_{y\to-\infty} \mu(y) &= \lim_{z\to 0} \frac{2(d-1)(b_0 - b_N(z)) + z{b_L}'(z)}{2z\sqrt{2(b_0 - b_L(z))}} \\
&= \lim_{z\to 0} \frac{(d-1)\beta_N z^2 - z\beta_L z}{2z\sqrt{\beta_L z^2}} \\
&= \frac{(d-1)\beta_N - \beta_L}{2\sqrt{\beta_L}} \\
&= \frac{\lambda}{\sqrt{\beta_L}}
\end{aligned}
\tag{5.7}
$$

which we have assumed to be strictly greater than zero. Thus for some a in $\mathbb{R}$ and some $\theta > 0$ we have $\mu(y) > \theta$ for all $y < a$. Thus $\mu(y) \geq \theta 1_{(-\infty,a)}(y)$ for all y in $\mathbb{R}$.

c) Let X solve the same equation as Y but with drift $\theta 1_{(-\infty,a)}$:

$$dX_t = dW_t + \theta 1_{(-\infty,a)}(X_t)dt \tag{5.8}$$

By the comparison theorem of IKEDA and WATANABE (1981) (Theorem 1.1 of Chapter VI), if X and Y start at the same point, then $X_t \leq Y_t$ almost surely since the rightward drift of Y is larger than the drift of X. Thus, if $g : \mathbb{R} \to \mathbb{R}_+$ is decreasing,

$$\mathbb{E}^y g(Y_t) \ \leq \ \mathbb{E}^y g(X_t)$$

Where $\mathbb{E}^y$ means that the stochastic process in the expectation begins at y. Setting $g = f \circ \phi^{-1}$ shows that

$$\mathbb{E}^z f(Z_t) \ \leq \ \mathbb{E}^{\phi(z)} g(X_t)$$

Combining this with (5.3), we obtain

$$\mathcal{I}(t) \ \leq \ \int_0^\infty \rho(dz) \mathbb{E}^z \int_0^t ds \ f(Z_s) \tag{5.9}$$

$$\leq \int_0^\infty \rho(dz) \mathbb{E}^{\phi(z)} \int_0^t ds\, g(X_s)$$
$$= \int_{\mathbb{R}} \eta(dx) \mathbb{E}^x \int_0^t ds\, g(X_s),$$

where we have set $\eta = \rho \circ \phi^{-1}$. Now define m by $m(0) = 0$ and

$$m(t) = \tfrac{1}{\sqrt{t}} \int_{\mathbb{R}} \eta(dx) \mathbb{E}^x \int_0^t ds\, g(X_s), \qquad t > 0 \tag{5.10}$$

Clearly, m is continuous and $\mathcal{I}(t) \leq \sqrt{t} m(t)$. It remains to show that $\lim_{t\to\infty} m(t) < +\infty$.

d) We give a general result on occupation times of the process X in Section 6. In order to apply it, we need only show that

$$\int_0^\infty dx\, g(x) < +\infty, \qquad \int_{-\infty}^0 \eta(dx)|x| < +\infty.$$

Then we will be able to conclude that

$$\lim_{t\to\infty} m(t) = \eta(\mathbb{R}) \sqrt{\tfrac{8}{\pi}} \int_{\mathbb{R}} dx\, g(x) e^{2\theta(x-a)^-} \tag{5.11}$$
$$= (1-\rho_0) \sqrt{\tfrac{8}{\pi}} \int_0^\infty dz \frac{f(z)}{\sqrt{2(b_0 - b_L(z))}} e^{2\theta(\phi(z)-a)^-}$$

where $(\phi(z) - a)^- = \min\,(\phi(z) - a, 0)$. The middle quantity is clearly finite.

First we consider the condition on g. Using $x = \phi(z)$ to change variables,

$$\int_0^\infty dx\, g(x) = \int_1^\infty dz \frac{f(z)}{\sqrt{2(b_0 - b_L(z))}}$$

This is finite since $f(z) \leq Cz^{-\delta}$ for z large enough and $\delta > 1$.

Finally, we show that $\int_{-\infty}^0 \eta(dx)|x| < +\infty$. This requires an estimate on the function ϕ, which in turn depends on the following property of b_L.

(5.12) **Lemma.** *For some $\alpha_1, \alpha_2, \gamma_1, \gamma_2 > 0$, we have*

$$\gamma_2 \wedge \alpha_2 z \leq \sqrt{2(b_0 - b_L(z))} \leq \gamma_1 \wedge \alpha_1 z, \qquad z \in \mathbb{R}_+$$

Proof: The following facts suffice: the function $b_0 - b_L$ is quadratic near the origin, continuous on $\mathbb{R}_+$, and is strictly positive away from the origin. Moreover, $\lim_{z\to\infty} b_0 - b_L(z) = b_0 > 0$. □

Write α, γ for α_1, γ_1. By the definition (5.5) of ϕ and the preceding lemma,

$$\begin{aligned}\phi(z) \geq \int_1^z \frac{dr}{\gamma \wedge \alpha r} &= \int_1^{\frac{\gamma}{\alpha}} \frac{dr}{\gamma \wedge \alpha r} + \int_{\frac{\gamma}{\alpha}}^z \frac{dr}{\gamma \wedge \alpha r} \\ &= C + \begin{cases} \frac{1}{\gamma}(z - \frac{\gamma}{\alpha}) & \text{if } z \geq \frac{\gamma}{\alpha} \\ \frac{1}{\alpha} \log \frac{\alpha z}{\gamma} & \text{if } z < \frac{\gamma}{\alpha} \end{cases}\end{aligned}$$

Finally,

$$\begin{aligned}\int_{-\infty}^{\phi(\gamma/\alpha)} \eta(dx)|x| &= \int_0^{\frac{\gamma}{\alpha}} \rho(dz)(-\phi(z)) \\ &\leq \int_0^{\frac{\gamma}{\alpha}} \rho(dz)(-C - \tfrac{1}{\alpha} \log \tfrac{\alpha z}{\gamma}) \\ &= -(C + \tfrac{1}{\alpha} \log \tfrac{\alpha}{\gamma})\rho(0, \tfrac{\gamma}{\alpha}) + \tfrac{1}{\alpha} \int_0^{\frac{\gamma}{\alpha}} \rho(dz) \log \tfrac{1}{z}\end{aligned}$$

But this is finite by condition (3.8) and the definition (5.4) of ρ. This suffices to show that $\int_{-\infty}^0 \eta(dx)|x| < +\infty$, which ends the proof of Theorem (3.9). □

Condition (3.7)

Finally, we will verify that condition (3.7) always holds in an incompressible isotropic Brownian flow, and we will also see when it holds for a compressible flow.

We will need the following explicit formulas for b_L and b_N:

(5.13)
$$\begin{aligned}b_L(r) &= A_m \int_0^\infty \Phi_P(ds)[L_m(rs) - (rs)^2 L_{m+1}(rs)] \\ &\quad + (d-1)A_m \int_0^\infty \Phi_S(ds) L_m(rs)\end{aligned}$$

(5.14)
$$\begin{aligned}b_N(r) &= A_m \int_0^\infty \Phi_P(ds) L_m(rs) \\ &\quad + A_m \int_0^\infty \Phi_S(ds)[L_{m-1}(rs) - L_m(rs)]\end{aligned}$$

Here $m = d/2$, $A_m = 2^{m-1}\Gamma(m)$, and $L_m(r) = J_m(r)/r^m$, with Γ denoting the gamma function and J_m denoting the Bessel function of the first kind of order m.

We will compute $rb_L'(r)$ directly from (5.13). First, elementary computations and the Bessel identity $xJ_m'(x) = xJ_{m-1}(x) - mJ_m(x)$ can be used to show that for $m > 0$ and $n \geq 0$,

$$r\frac{d}{dr}[(rs)^n L_m(rs)] = (n-2m)(rs)^n L_m(rs) + (rs)^n L_{m-1}(rs)$$

Thus, omitting the argument (rs),

$$\begin{aligned} rb_L'(r) &= A_m \int_0^\infty \Phi_P(ds) r\frac{d}{dr}[L_m - (rs)^2 L_{m+1}] \\ &\quad + (d-1)A_m \int_0^\infty \Phi_S(ds) r\frac{d}{dr} L_m \\ &= A_m \int_0^\infty \Phi_P(ds)[2m(rs)^2 L_{m+1} - (2m + (rs)^2)L_m + L_{m-1}] \\ &\quad + (d-1)A_m \int_0^\infty \Phi_S(ds)[-2mL_m + L_{m-1}] \\ &= (d-1)(b_N(r) - b_L(r)) \\ &\quad + A_m \int_0^\infty \Phi_P(ds)[(rs)^2 L_{m+1} - (2m + (rs)^2)L_m + L_{m-1}] \end{aligned}$$

Another Bessel identity gives $(rs)^2 L_{m+1} - 2mL_m + L_{m-1} = 0$, so we are left with

$$rb_L'(r) = (d-1)(b_N(r) - b_L(r)) - A_m \int_0^\infty \Phi_P(ds)(rs)^2 L_m(rs) \tag{5.15}$$

Thus the quantity in condition (3.7) is

$$\begin{aligned} &2(d-1)(b_0 - b_N(r)) + rb_L'(r) \\ &\quad = (d-1)(2b_0 - b_N(r) - b_L(r)) - A_m \int_0^\infty \Phi_P(ds)(rs)^2 L_m(rs) \end{aligned}$$

Now for all $r > 0$ we have $b_N(r) < b_0$ and $b_L(r) < b_0$, so when $\Phi_P = 0$, this is clearly positive. Moreover, when $\Phi_P \neq 0$, this adds to the right side another continuous function which is quadratic near the origin. If this term is bounded, then for Φ_P small enough, condition (3.7) will still be satisfied. Boundedness is automatic for $d \geq 3$.

6 Diffusion with step drift

In this section we consider the diffusion X which solves the equation

$$dX_t = dW_t + \mu(X_t)dt \tag{6.1}$$

where W is the Wiener process and the drift μ takes a very simple form:

$$\mu(x) = \begin{cases} \theta & \text{if } x < a \\ \tau & \text{if } x \geq a \end{cases} \tag{6.2}$$

The transition function of X is known; see KARATZAS and SHREVE (1988), Chapter 6, Section 5. We are interested in the case $\theta > 0$ and $\tau = 0$. Our purpose is to develop asymptotic results for occupation times of X.

For each x in $\mathbb{R}$, we use $\mathbb{E}^x$ to denote the expectation in which $X_0 = x$ almost surely. Let η be a finite measure on $\mathbb{R}$. The following result concerns the limit as $t \to \infty$ of the occupation time h defined by

$$h(t) \equiv \int_{\mathbb{R}} \eta(dx) \mathbb{E}^x \int_0^t ds\ g(X_s)$$

for a positive bounded function g decaying to 0 as $x \to \infty$. It is not enough to show such a result for each x individually and then integrate with respect to η; much of the proof is concerned with showing enough uniformity of convergence to be able to integrate with respect to η.

(6.3) **Proposition.** *Let $\theta > 0$ and $\tau = 0$. Suppose that*

$$\int_0^\infty dx\ g(x) < +\infty, \qquad \int_{-\infty}^0 \eta(dx)|x| < +\infty \tag{6.4}$$

Then,

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \eta(dx) \mathbb{E}^x \int_0^t ds\ g(X_s) = \eta(\mathbb{R}) \sqrt{\frac{8}{\pi}} \int_{\mathbb{R}} dx\ g(x) e^{2\theta(x-a)^-} \tag{6.5}$$

where $(x-a)^- = \min\ (x-a, 0)$. □

(6.6) **Remark.** In the case $\theta < 0$ and $\tau = a = 0$ we can prove a similar result by similar means. Suppose $g(x)$ decays exponentially (or faster) as $x \to -\infty$, that $\lim_{x\to\infty} g(x) > 0$ and that $\int_{-\infty}^0 \eta(dx)e^{\epsilon|x|} < +\infty$. Then

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \eta(dx) \mathbb{E}^x \int_0^t ds\ g(X_s) = \sqrt{\frac{8}{\pi}} \int_{\mathbb{R}} \eta(dx)(x^+ - \tfrac{1}{2\theta} e^{-2x^-}),$$

which is finite. This was used in ZIRBEL (1993) to prove the result for $d = 1$ mentioned after Proposition (3.5). □

Proof: Proposition (6.3) will follow from consideration of the Laplace transform of the occupation time h. Let $p_t(x, y)$ denote the transition density of X. Then the Laplace transform $\hat{h}$ of h is given by

$$\begin{aligned} \hat{h}(s) &= \int_0^\infty e^{-st} h(dt) \\ &= \int_0^\infty dt\ e^{-st} \int_{\mathbb{R}} \eta(dx) \int_{\mathbb{R}} dy\ p_t(x,y) g(y) \\ &= \int_{\mathbb{R}} \eta(dx) \int_{\mathbb{R}} dy\ g(y) U_s(x,y) \end{aligned} \tag{6.7}$$

where $U_s(x, y) = \int_0^\infty dt\ e^{-st} p_t(x, y)$ is the density of the potential kernel of X.

Let us take $a = 0$ for now. It is straightforward to take the Laplace transform of $p_t(x, y)$ (given in KARATZAS and SHREVE (1988)) to find the following formula for $U_s(x, y)$:

$$U_s(x,y) = \begin{cases} Ae^{-x\tau - x\sqrt{2s+\tau^2}} + 1_{\{y\geq 0\}} B(\tau) & \text{if } x \geq 0 \\ Ae^{-x\theta + x\sqrt{2s+\theta^2}} + 1_{\{y<0\}} B(\theta) & \text{if } x < 0 \end{cases} \tag{6.8}$$

where

$$A = \frac{2}{\tau - \theta + \sqrt{2s+\tau^2} + \sqrt{2s+\theta^2}} [1_{\{y<0\}} e^{y\theta + y\sqrt{2s+\theta^2}} + 1_{\{y\geq 0\}} e^{y\tau - y\sqrt{2s+\tau^2}}]$$

$$B(\alpha) = \frac{e^{(y-x)\alpha}}{\sqrt{2s+\alpha^2}} [e^{-|x-y|\sqrt{2s+\alpha^2}} - e^{-|x+y|\sqrt{2s+\alpha^2}}]$$

Using this expression, we will show that $\sqrt{s}\hat{h}(s) \to C$ as $s \to 0$.

We write $\sqrt{s}\hat{h}(s) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$ where $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$ are defined by

$$\begin{aligned}
\mathcal{I}_1 &= \tfrac{2\sqrt{s}}{-\theta+\sqrt{2s+\theta^2}+\sqrt{2s}}\Big[\int_{-\infty}^{0}\eta(dx)\ e^{-x\theta+x\sqrt{2s+\theta^2}} + \int_0^\infty \eta(dx)\ e^{-x\sqrt{2s}}\Big] \\
&\qquad \times \Big[\int_{-\infty}^{0} dy\ g(y)e^{y\theta+y\sqrt{2s+\theta^2}} + \int_0^\infty dy\ g(y)e^{-y\sqrt{2s}}\Big] \\
\mathcal{I}_2 &= \tfrac{1}{\sqrt{2}}\int_0^\infty \eta(dx)\ \int_0^\infty dy\ g(y)(e^{-|x-y|\sqrt{2s}} - e^{-(x+y)\sqrt{2s}}) \\
\mathcal{I}_3 &= \tfrac{\sqrt{s}}{\sqrt{2s+\theta^2}}\int_{-\infty}^{0}\eta(dx)\ e^{-x\theta}\int_{-\infty}^{0} dy\ g(y)e^{y\theta}(e^{-|x-y|\sqrt{2s+\theta^2}} - e^{(x+y)\sqrt{2s+\theta^2}})
\end{aligned}$$

Now we turn to the treatment of terms $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$.

Consider $\mathcal{I}_1$ first. Multiply the prefactor above and below by $\theta + \sqrt{2s+\theta^2} + \sqrt{2s}$ to obtain

$$\frac{2\sqrt{s}(\theta + \sqrt{2s+\theta^2} + \sqrt{2s})}{4s + 2\sqrt{4s^2 + 2s\theta^2}}$$

The limit of this as $s \to 0$ is $\sqrt{2}$. The sum of the integrals with respect to η converges nicely to $\eta(\mathbb{R})$. The remaining factor converges to its value at $s = 0$. Hence

$$\begin{aligned}
\lim_{s\to 0}\mathcal{I}_1 &= \sqrt{2}\eta(\mathbb{R})\Big[\int_{-\infty}^{0} dy\ g(y)e^{2\theta y} + \int_0^\infty dy\ g(y)\Big] \\
&= \sqrt{2}\eta(\mathbb{R})\int_{\mathbb{R}} dy\ g(y)e^{2\theta y^-}
\end{aligned} \tag{6.9}$$

where $y^- = \min(y, 0)$.

Next, consider $\mathcal{I}_2$. We will show that $\mathcal{I}_2 \to 0$ as $s \to 0$. Both terms in the integrand are monotone as $s \to 0$, so the monotone convergence theorem can be applied to each separately. The first term converges to $\frac{1}{\sqrt{2}}\eta(\mathbb{R}_+)\int_0^\infty dy\ g(y)$, which is finite by condition (6.4). The second term converges to the negative of this; thus $\mathcal{I}_2 \to 0$ as $s \to 0$.

Finally, consider $\mathcal{I}_3$. We will show that $\mathcal{I}_3 \to 0$ as $s \to 0$. The prefactor $\sqrt{s}/\sqrt{2s+\theta^2}$ converges to 0. We will show that the integral converges to a finite, positive value. Note that for $x, y < 0$ we have $x + y < -|x-y|$, so $\mathcal{I}_3$ is positive. Neglect the second term in the integrand. The first term is monotone, so its integral converges to

$$
\begin{aligned}
(6.10)\qquad & \int_{-\infty}^{0} \eta(dx) \int_{-\infty}^{0} dy\ g(y) e^{\theta(-x+y-|x-y|)} \\
&= \int_{-\infty}^{0} \eta(dx) [\int_{-\infty}^{x} dy\ g(y) e^{2\theta(-x+y)} + \int_{x}^{0} dy\ g(y)] \\
&\leq \|g\| \int_{-\infty}^{0} \eta(dx) [e^{-2\theta x} \int_{-\infty}^{x} dy\ e^{2\theta y} + |x|] \\
&= \|g\| \int_{-\infty}^{0} \eta(dx) [\frac{1}{2\theta} + |x|]
\end{aligned}
$$

This is finite by condition (6.4), so $\mathcal{I}_3 \to 0$ as $s \to 0$.

We have shown that

$$
\lim_{s \to 0} \sqrt{s} \hat{h}(s) = \sqrt{2} \eta(\mathbb{R}) \int_{\mathbb{R}} dy\ g(y) e^{2\theta y^-}.
$$

Now we appeal to the Tauberian theorem given in FELLER (1966) vol. 2:

(6.11) **Theorem.** *For* $\gamma > 0$*,* $\lim_{s \to 0} s^{\gamma} \hat{h}(s) = \Gamma(\gamma + 1)$ *is equivalent to* $\lim_{t \to \infty} t^{-\gamma} h(t) = 1$. □

Thus, with $\gamma = \frac{1}{2}$ and $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, we obtain

$$
(6.12)\qquad \lim_{t \to \infty} \tfrac{1}{\sqrt{t}} h(t) = \eta(\mathbb{R}) \sqrt{\tfrac{8}{\pi}} \int_{\mathbb{R}} dx\ g(x) e^{2\theta x^-},
$$

which completes the proof in the case $a = 0$. The general case can be reduced to this one by setting $Y_t = X_t - a$. The drift of Y jumps at $y = 0$. □

Acknowledgement. The first author was supported by a Charlotte Elizabeth Procter Fellowship from Princeton University during the time this work was completed. The second author would like to acknowledge the support of the Office of Naval Research through their Grant No. N00014-92-J-1088 P00001.

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