Random velocity fields with known Lagrangian law

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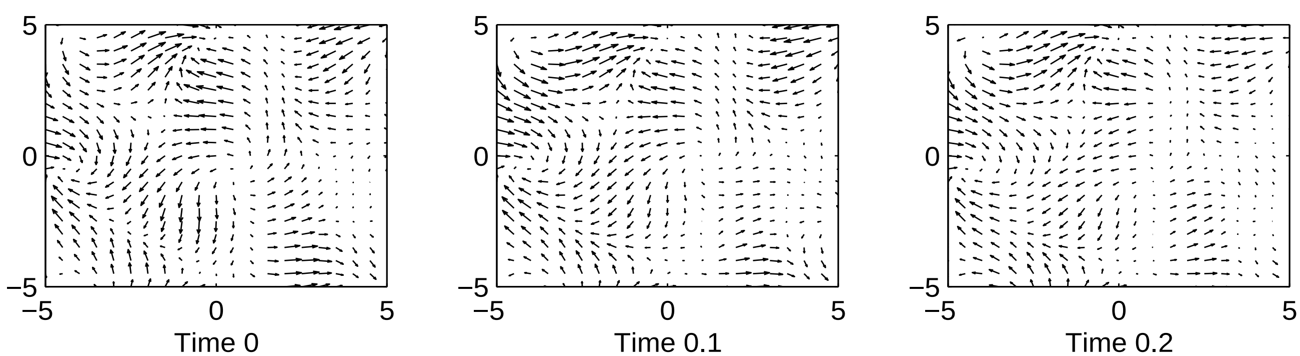
Articles and this poster are available electronically at the URL above.

Title pages of two articles are available from the presenter.

1. Particle motion in random velocity fields

Random velocity fields are used to model turbulent or randomly–forced fluid flows. The field U has some initial state (including variables you don't see) and a way of evolving, possibly subject to random influences. For example:

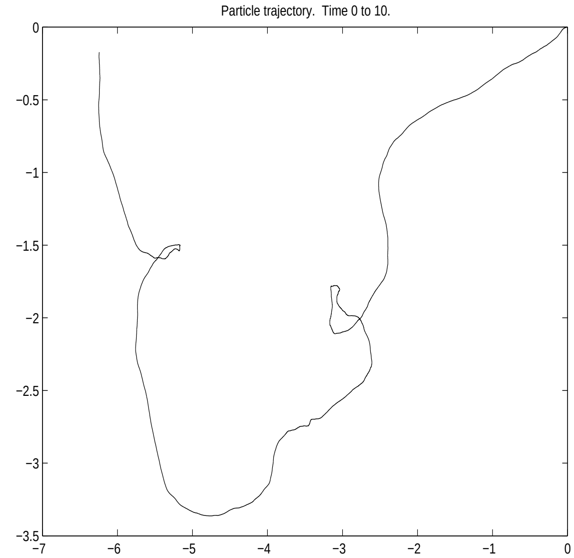


Or, imagine a vortex wandering in the ocean.

A particle moves in the field according to

$$dX_t = U(X_t, t)dt + \sigma dW_t,$$

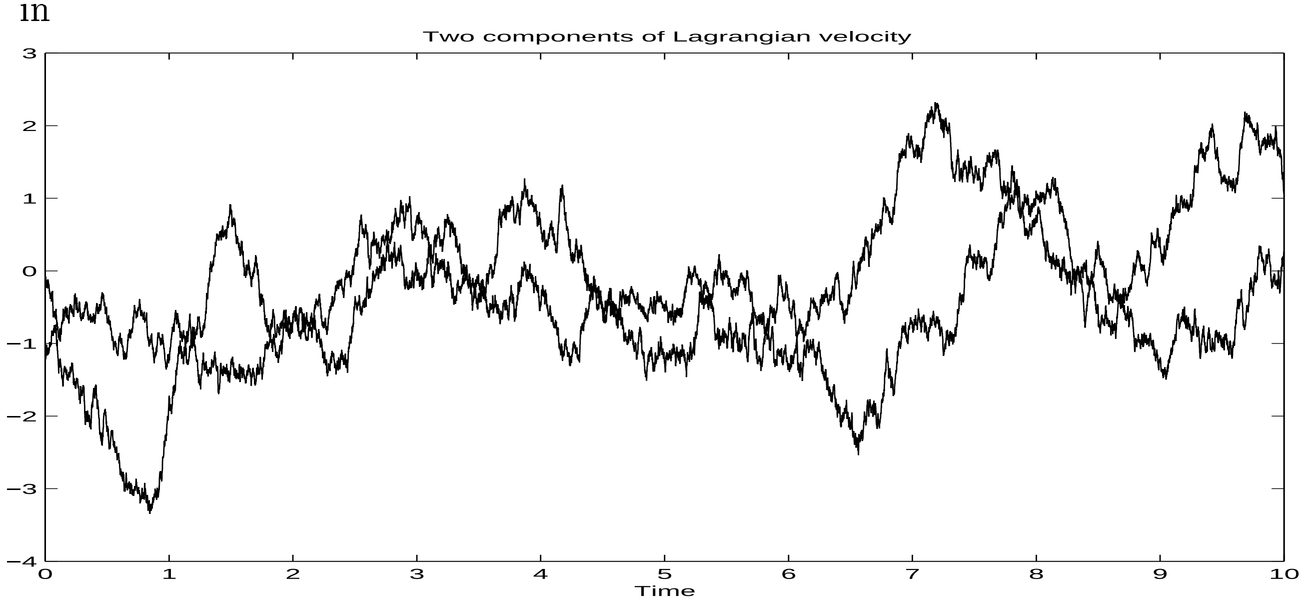
where dW_t represents molecular diffusion. For example:



We would like to know all about the process $X_t,\ t \geq 0$; its mean, variance, distribution at time t, etc.

The Lagrangian velocity process is $U(X_t, t),\ t \geq 0.$ For this example, its two components look like:

in



(They are very rough because the velocity field U is an Ornstein–Uhlenbeck process in time.)

In general it is very difficult to say what the law of the Lagrangian velocity process is. It is important to find cases in which one can write down its law exactly.

2. Markov velocity fields

Look at all of $U(\cdot, t)$ relative to location X_t:

$$V(x,t) = U(x + X_t, t), \qquad x \in \mathbb{R}^d.$$

Of course, $V(0,t) = U(X_t, t)$. The field V alone may contain enough information to tell how the velocity will evolve in the future.

Theorem If U is homogeneous and Markov, then V is Markov as well. Moreover, one can write down the transition matrix or generator of V, and so find its law.

Ask for the article "Markov velocity fields and the generalized Lagrangian velocity" by C. L. Zirbel or see "Turbulent diffusion in Markovian flows" by A. Fannjiang and T. Komorowski.

3. Discrete Markov velocity fields

Suppose the "ocean" consists of a grid with periodic boundary conditions.

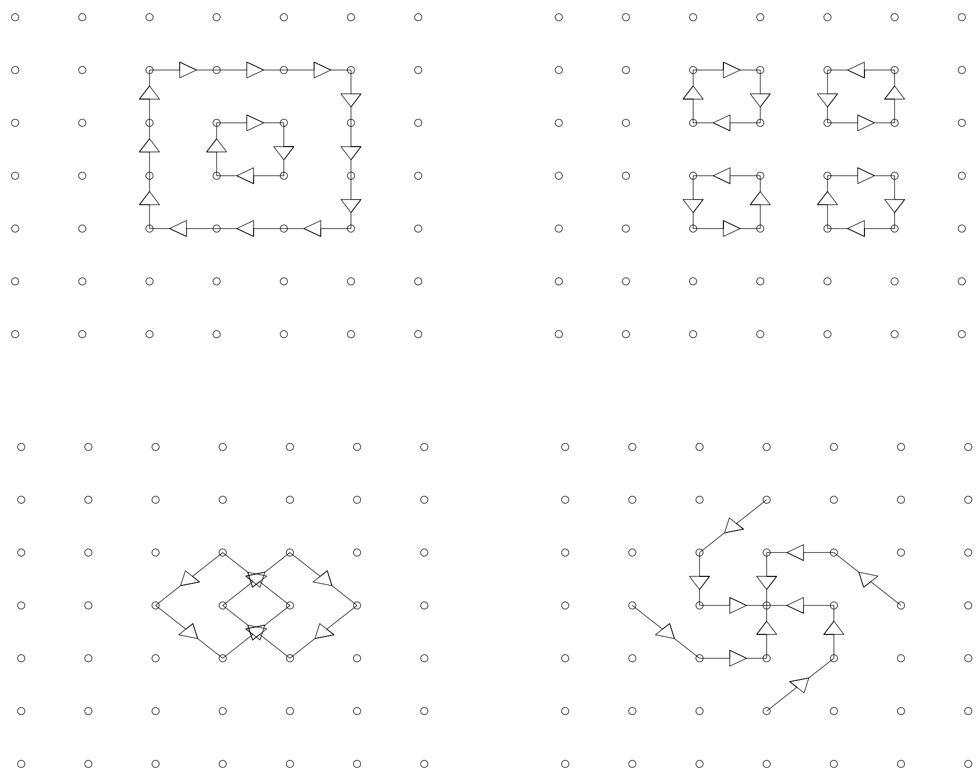
Proposition If U is homogeneous and Markov, then U satisfies $U(x,n) = u(I_n, x - L_n), n = 0, 1, 2, ...$ for some *vortex types* $u(1,\cdot), u(2,\cdot), \ldots$, a *type process* I and a *location process* L.

The velocity field consists of **vortices** which change **type** and **location**.

The type process I is Markov, and the location process L is an additive random walk.

4. Discrete hidden Markov velocity fields

Arbitrary vortex types $u(1,\cdot), u(2,\cdot), \ldots\ldots$ Sample types:



Type changes according to a Markov chain I. Location process L is an additive random walk whose characteristics depend on the current vortex type:

$$L_{n+1} = L_n + A(I_n, n)$$

The random velocity field is $U(x,n) = u(I_n, x - L_n)$.
The vortex is repeated periodically throughout $\mathbb{Z}^2$.
The particle trajectory obeys

$$X_{n+1} = X_n + U(X_n, n) + \Delta_n,$$

where Δ_n represents molecular diffusion.

5. Lagrangian velocity process

Let $M_n = L_n - X_n$. Then M is the location of the vortex from the point of view of the moving particle.

Note that $U(X_n, n) = u(I_n, X_n - L_n) = u(I_n, -M_n)$ and that M evolves according to

$$M_{n+1} = \sigma(I_n, M_n) + A(I_n, n) + \Delta_n,$$

where σ is a mapping on the grid generated by u. This is almost the same as how L evolves.

The point is this: the Lagrangian velocity depends on a Markov chain (I, M), where I is Markov and M evolves in a simple way. The state space of (I, M) is the same size as that of (I, L), the chain which determines the velocity field. The transition matrix Q of (I, M) is easy to get from the transition matrix P of (I, L), in this way:

$$Q = \Sigma D P$$

where Σ is a matrix built from the vortex types and D corresponds to molecular diffusion. When the vortices are incompressible, Σ will be a permutation matrix.

For two–particle motion, there are two Lagrangian location processes M^1 and M^2. They evolve in the same way as above, but with different diffusion:

$$\begin{aligned} M^1_{n+1} &= \sigma(I_n, M^1_n) + A(I_n, n) + \Delta^1_n \\ M^2_{n+1} &= \sigma(I_n, M^2_n) + A(I_n, n) + \Delta^2_n \end{aligned}$$

Then $(U(X^1_n, n), U(X^2_n, n)) = (u(I_n, -M^1_n), u(I_n, -M^2_n))$.

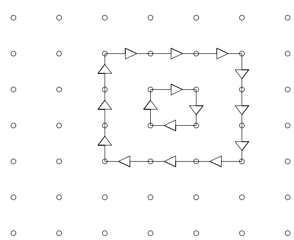
6. One–type examples

The velocity field has only one vortex type, which is repeated periodically throughout $\mathbb{Z}^2$. The vortex moves up and to the right, with probabilities 0.3 (right) 0.3 (up) 0.2 (up and right) 0.2 (don't move).

The vortices are standardized to have the same "energy" $\mathbb{E}|U(0,0)|^2$.

The Lagrangian drift is $\mu = \lim_{t\to\infty} \mathbb{E}X_t/t$.

The diffusivity matrix is $\beta = \lim_{t\to\infty} \mathrm{Cov}(X_t)/t$.

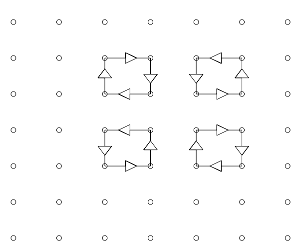


Eulerian spectral gap 0.0990,

Lagrangian 0.1456 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.2499 & 0.0945 \\ 0.0945 & 0.2499 \end{bmatrix}$.

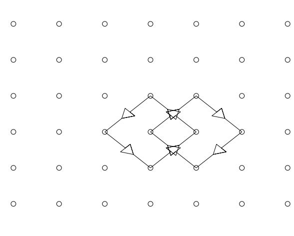


Eulerian spectral gap 0.0990,

Lagrangian 0.1565 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.1841 & 0.0484 \\ 0.0484 & 0.1841 \end{bmatrix}$.

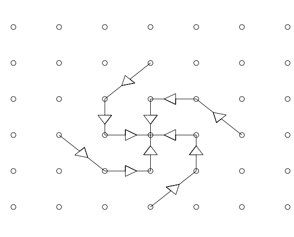


Eulerian spectral gap 0.0990,

Lagrangian 0.1281 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.0929 & 0.0545 \\ 0.0545 & 0.1724 \end{bmatrix}$.



Eulerian spectral gap 0.0990,

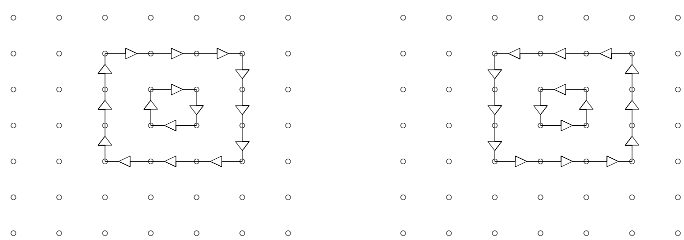
Lagrangian 0.1820 (mixes faster).

Lagrangian drift $\mu = [0.2415 \;\; 0.1934]$.

Diffusivity matrix $\beta = \begin{bmatrix} 0.6925 & 0.5639 \\ 0.5639 & 0.8110 \end{bmatrix}$.

7. Two–type examples

The velocity field has two vortex types, which alternate according to the transition matrix $\begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$. The current vortex is repeated periodically throughout $\mathbb{Z}^2$. Vortex 1 moves up and to the right, with probabilities 0.3 (right) 0.3 (up) 0.2 (up and right) 0.2 (don't move), while vortex 2 moves down and to the right, with probabilities 0.3 (right) 0.3 (down) 0.2 (down and right) 0.2 (don't move).

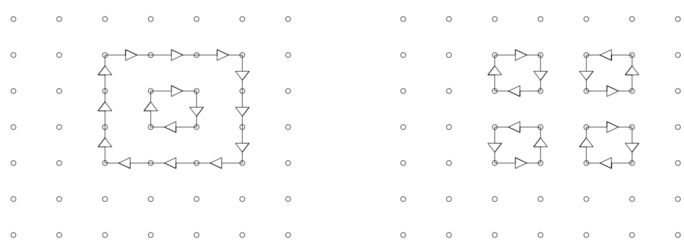


Eulerian spectral gap 0.0949,

Lagrangian 0.1468 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.1473 & 0.0000 \\ 0.0000 & 0.1550 \end{bmatrix}$.

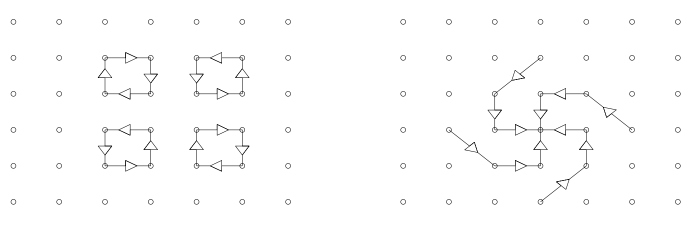


Eulerian spectral gap 0.0949,

Lagrangian 0.1583 (mixes faster).

Lagrangian drift μ is zero.

Diffusivity matrix $\beta = \begin{bmatrix} 0.2177 & -0.0089 \\ -0.0089 & 0.1885 \end{bmatrix}$.

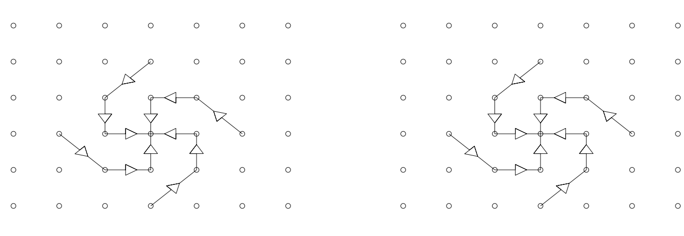


Eulerian spectral gap 0.0949,

Lagrangian 0.1613 (mixes faster).

Lagrangian drift μ is $[0.0394 \quad -0.0610]$.

Diffusivity matrix $\beta = \begin{bmatrix} 0.3209 & -0.1360 \\ -0.1360 & 0.3077 \end{bmatrix}$.



Eulerian spectral gap 0.0949,

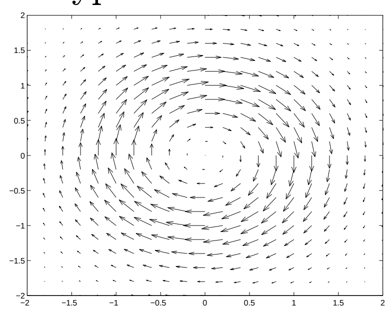
Lagrangian 0.1442 (mixes faster).

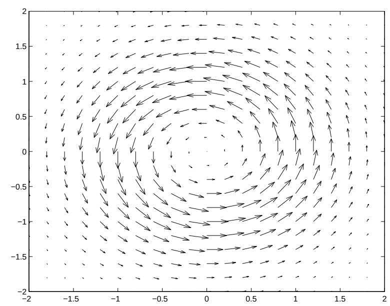
Lagrangian drift μ is $[0.2109 \quad -0.0169]$.

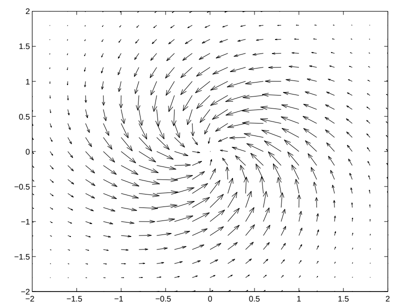
Diffusivity matrix $\beta = \begin{bmatrix} 0.6871 & 0.0595 \\ 0.0595 & 0.5622 \end{bmatrix}$.

8. Extensions to continuous space and time

The most obvious extension to continuous space and time is to envision a finite number of localized vortices which change type according to a Markov transition matrix and move about in the plane according to a Brownian motion whose drift and diffusion coefficient may depend on the current vortex type.







The form of the velocity will be the same as above:

$$U(x,t) = u(I_t, x - L_t)$$

However, it is also possible to make a velocity field by superimposing multiple vortices of varying types. One could make these vortices interact in an interesting way.