Understanding the generator A

$A = \mathrm{diag}(\lambda)(Q - I)$, so it can be written this way:

$$A = \begin{bmatrix} -\lambda(1) & \lambda(1)Q_{12} & \lambda(1)Q_{13} & \cdots \\ \lambda(2)Q_{21} & -\lambda(2) & \lambda(2)Q_{23} & \\ & & -\lambda(3) & \\ & & & \ddots \end{bmatrix}$$

Note:

- $-\lambda(i)$ on the diagonal is the rate of leaving state i
- $\lambda(i)Q_{ij}$ is the rate of leaving state i times the probability of jumping to state j. It is best thought of as the rate of jumping from state i to state j.
  In applied problems, this can often be figured out directly.
- If state i is absorbing, then $\lambda(i) = 0$ and row i of A is zero, no matter what row i of Q is.
- Each row of A sums to 0.
- Knowing the matrix A, one can recover $\lambda(1), \lambda(2), \ldots,$ then divide each row by these to recover Q.
  So knowing A is equivalent to knowing both λ and Q.

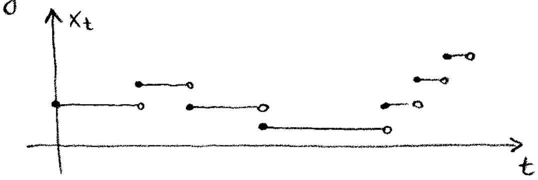
Matlab

```
lambda = - diag(A);
Q = inv(diag(lambda)) * (A + diag(lambda));
```

(assuming no states are absorbing)

Example Population with birth and death

Population size X_t is Markov, can go up or down by 1 for birth and death.



Get the parameters by writing down the generator first.

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & \cdots & \\ \delta & -(\beta+\delta) & \beta & & & \\ 0 & 2\delta & -(2\beta+2\delta) & 2\beta & 0 & \cdots \\ & & & & & \\ & & & & & \end{array} \right] \end{array}$$ ← rate of jumping is zero.

Rate of jumping from 1 to 2 is β, the birth rate.
1 to 0 is δ, the death rate.

Make row 1 sum to 0.

Rate of jumping from 2 to 3 is 2β, twice the individual birth rate.
2 to 1 is 2δ, ...

Rate from i to $i+1$ is $i\beta$, i to $i-1$ is $i\delta$.

Read off parameters from A:

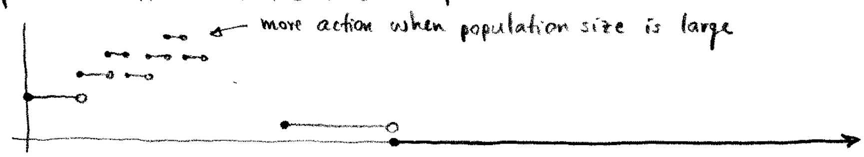
$$\lambda(i) = i(\beta+\delta)$$ rate of leaving state i

$$Q_{i,i+1} = \frac{A_{i,i+1}}{\lambda(i)} = \frac{i\beta}{i(\beta+\delta)} = \frac{\beta}{\beta+\delta}.$$

$$Q_{i,i-1} = \frac{\delta}{\beta+\delta}.$$

Drift depends on the relative size of β and δ.

Outcome:



0 is <u>absorbing</u>.

If you don't like that, allow <u>immigration</u>: new individuals arrive at rate α.

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccccc} -\alpha & \alpha & 0 & \cdots & \\ & & \beta+\alpha & & \\ & & & 2\beta+\alpha & \end{array} \right] \end{array}$$

Still, no fundamental limit on population growth.

Kolmogorov's forward equation

We know that
$$\frac{dP(t)}{dt} = AP(t) = A\cdot e^{At} = A\sum_{n=0}^{\infty}\frac{A^n t^n}{n!}$$
$$= \left(\sum_{n=0}^{\infty}\frac{A^n t^n}{n!}\right)A$$
$$= e^{At}A$$
$$= P(t)A,$$

so P satisfies $\frac{dP(t)}{dt} = P(t)A$ as well, which is called <u>Kolmogorov's forward equation.</u>

Example Yule process

We did this earlier.

$$\frac{dP_{ij}(t)}{dt} = (P(t)A)_{ij} = \begin{array}{c} \\ i \\ \\ \\ \end{array}\begin{bmatrix} \cdots\ P_{i,j-1}(t)\ P_{ij}(t)\ \cdots \\ \\ P(t) \\ \\ \end{bmatrix} \begin{array}{c} \\ \\ \\ j \\ \\ \end{array}\begin{bmatrix} \overset{j}{0} \\ \vdots \\ 0 \\ \lambda(j-1)\cdot 1 \\ \cdots\cdots -\lambda(j) \cdots \\ 0 \\ 0 \end{bmatrix}$$

$$= \lambda(j-1)P_{i,j-1}(t) - \lambda(j)P_{ij}(t)$$

Interpretation:

rate of change of probability of being in state j is:

(rate of leaving $j-1$)(prob. of being in $j-1$)

$-$ (rate of leaving j)(prob. of being in j)

General interpretation

$$\frac{dP_{ij}(t)}{dt} = (P(t)A)_{ij}$$
$$= -\lambda(j)P_{ij}(t) + \sum_{k\neq j}\lambda(k)Q_{jk}P_{ik}(t)$$

= decrease due to leaving state j + increase due to the rate of jumping into state j from various states k.

This is called the <u>forward equation</u> because it is based on where the process is at <u>time t</u>, which is looking forward in time.

Kolmogorov's backward equation

We derived $\frac{dP(t)}{dt} = AP(t)$ rigorously, but here is a way of thinking of this equation in terms of rates of change.

$$\frac{dP_{ij}(t)}{dt} = \left(AP(t)\right)_{ij}$$

$$= \left(\begin{array}{cc} & \\ i & \\ & \end{array} \!\!\begin{bmatrix} & & & \\ \cdots \lambda(i)Q_{ik} & \cdots & -\lambda(i) & \cdots \\ & & & \end{bmatrix} \begin{array}{c} j \\ \begin{bmatrix} \vdots \\ P_{ij}(t) \\ \vdots \end{bmatrix} \end{array} \right)_{ij}$$

$$= -\lambda(i)\, P_{ij}(t) + \sum_{k \neq i} \lambda(i)\, Q_{ik}\, P_{kj}(t)$$

= decrease due to the rate of leaving state i at time 0 + rate of jumping from i to k at time 0, then getting from k to j in time t.

This is harder to understand intuitively than the forward equation, I think.

 11-30-2000

Reading Taylor & Karlin, Chapter VI.

They write things like (p. 339, Yule process)

$$\mathbb{P}(X_{t+h} - X_t = 1 \mid X_t = n) = n \cdot \beta \cdot h + o(h) \quad \text{as } h \to 0.$$

(smaller than h

Interpretation:

$$\mathbb{P}(X_{t+h} = n+1 \mid X_t = n) = n\beta h + o(h)$$

$$P_{n,n+1}(h) = n\beta h + o(h)$$

$$\frac{P_{n,n+1}(h)}{h} = n\beta + \frac{o(h)}{h}$$

$$\lim_{h \to 0} \frac{P_{n,n+1}(h)}{h} = n\beta$$

$$\frac{d}{dh} P_{n,n+1}(h)\Big|_{h=0} = n\beta$$

For us,

$$\frac{d}{dt} P(t) = AP(t)$$

$$\frac{d}{dt} P(t)\Big|_{t=0} = AP(0) = A$$

$$\frac{d}{dt} P_{n,n+1}(t)\Big|_{t=0} = A_{n,n+1} = \lambda(n)\, Q_{n,n+1},$$

the rate of jumping from n to $n+1$.

Easy translator:

$$\mathbb{P}(X_{t+h} = j \mid X_t = i) = \rho h + o(h) \quad \text{as } h \to 0$$

means that X jumps from i to j at rate ρ, so $A_{ij} = \rho$.

Examples: pure death $\quad \mathbb{P}(X_{t+h} = k-1 \mid X_t = k) = \mu_k h + o(h)$ $\quad$ p. 345

birth & death $\quad$ p. 356

Numerical calculation of $P(t)$

Let $A = \Lambda(-I + Q)$.

Then $P(t) = e^{At}$.

For example, $P(1) = e^{A}$,

$$P(2) = e^{2A} = (e^{A})^2 = P(1) \cdot P(1)$$

$$P(3) = e^{3A} = (e^{A})^3$$

Or, use a different time increment:

$$P(n \cdot \Delta t) = e^{n \cdot \Delta t \cdot A} = \left(e^{\Delta t A}\right)^n.$$

The upshot is this:

1st compute $e^{\Delta t A}$. Matlab expm $(\Delta t A)$

2nd simply raise this to higher powers.

Distribution of X_t

$$\mu(t) = \mu(0) \cdot P(t)$$
$$= \mu(0) \cdot e^{tA}$$

$$\mu(n \cdot \Delta t) = \mu(0)\, e^{n \cdot \Delta t A}$$
$$= \mu(0) \cdot \left(e^{\Delta t A}\right)^n.$$

Begin with $\mu(0)$, repeatedly multiply on the right by $e^{\Delta t A}$.

Examples

Limiting distribution if $e^{\Delta t A}$ has one, as a M. chain transition matrix.

Invariant distribution of X_t

In the examples we have seen so far, the distribution of X_t has converged to a limiting distribution π very quickly as $t \to \infty$.

This is not surprising, since $P(n) = (e^A)^n$, just like raising the transition matrix of a Markov chain to a high power. Besides, in many cases e^A has all positive entries, so not even periodicity is a problem.

The limiting distribution is an invariant distribution, as before.

Numerical calculation

There are several methods:

$\pi = e^{At}$ for large t, e.g. expm(A * 1000)

$\pi = (e^A)^{1000}$, using expm(A) ^ 1000

PI = invariant(expm(A)) to find the invariant distribution for $P(1) = e^A$.

Analytic calculation of the invariant distribution

An invariant distribution π would satisfy $\pi P(t) = \pi$ for all t.
Differentiating component by component would give $\frac{d}{dt}(\pi P(t)) = 0$.
But

$$\frac{d}{dt}\left(\pi P(t)\right) = \pi \frac{dP(t)}{dt} = \pi\left(A P(t)\right) = \pi A P(t).$$

This equals 0 for all t precisely when $\boxed{\pi A = 0.}$
This is more useful than $\pi P(t) = \pi$ since A is easier to obtain, either directly or from λ and Q.

Caution: solving $\pi A = 0$ will only give the solution up to a multiplicative constant. Be sure to normalize π to add up to 1, and make sure it has no negative entries.

Relationship to λ and Q

This is neat:

$$\pi A = \pi \Lambda (Q - I) = 0$$
$$\pi \Lambda Q - \pi \Lambda = 0$$
$$\pi \Lambda Q = \pi \Lambda .$$

The equation may be written more simply as $\eta Q = \eta$.

Thus, if η is the invariant distribution for the jump matrix Q, then π equals $\eta \Lambda^{-1}$.

That is, η is what the invariant distribution would be if you considered jumps only, and Λ^{-1} weights each state according to the mean length of time spent there.
Caution: π still needs to be normalized!

Example Machine repair (Karlin & Taylor p. 369 is similar)

Situation: The Bowling Green Post Office has 40 delivery trucks.
From time to time, each truck fails and goes in for repair.
The repair shop can only work on 3 trucks simultaneously.
On a daily basis, only 30 trucks are used at most,
but if fewer trucks are available, then trucks are rented
to bring the number up to 30.

Model Let X_t be the number of working trucks at time t.
Possible states $\{0, \ldots, 40\}$.
Assume that X is a Markov process.

Then: time between failures is exponential
time to complete a repair is exponential

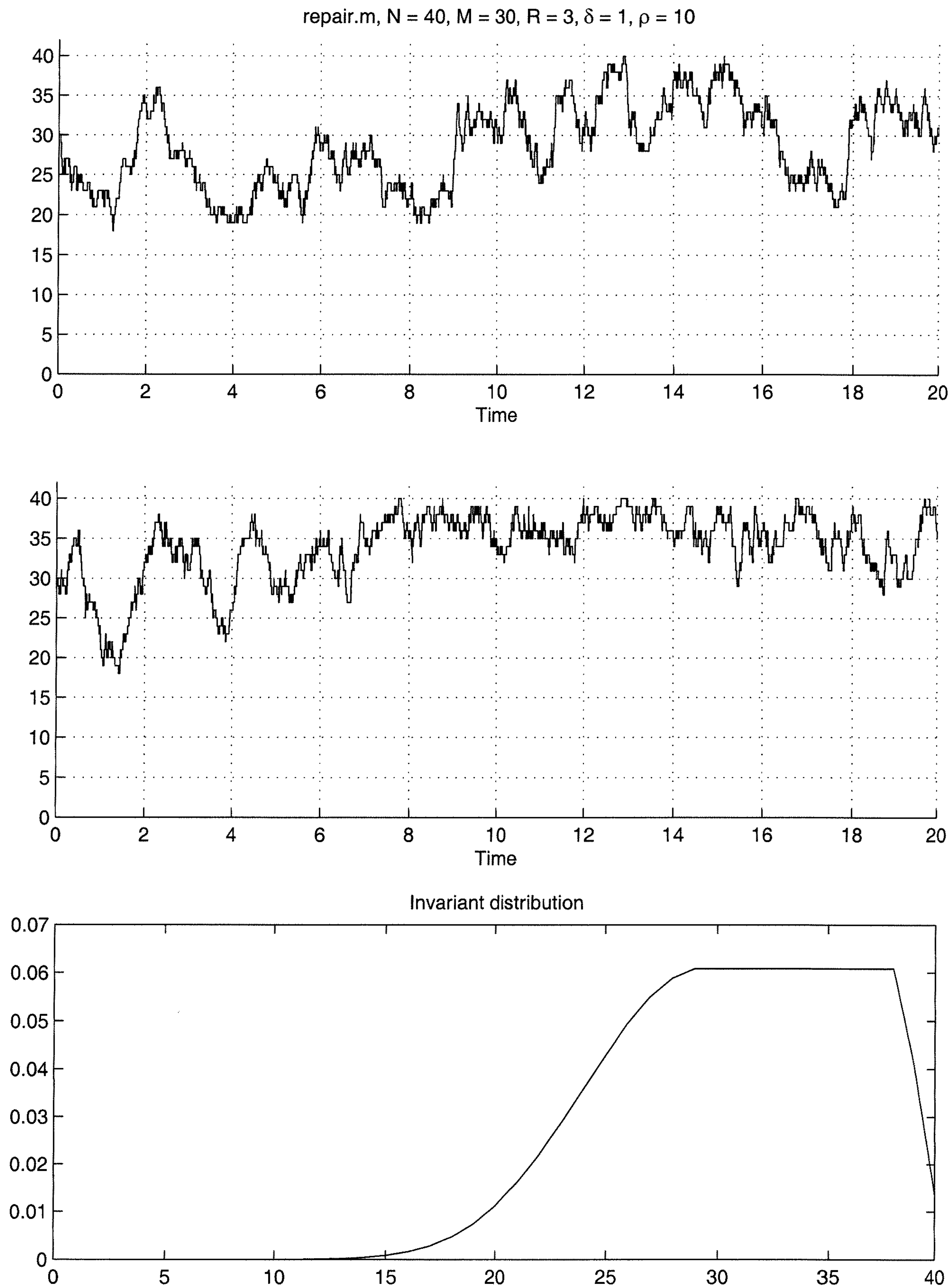
This is not perfect, but it's OK for a first model.

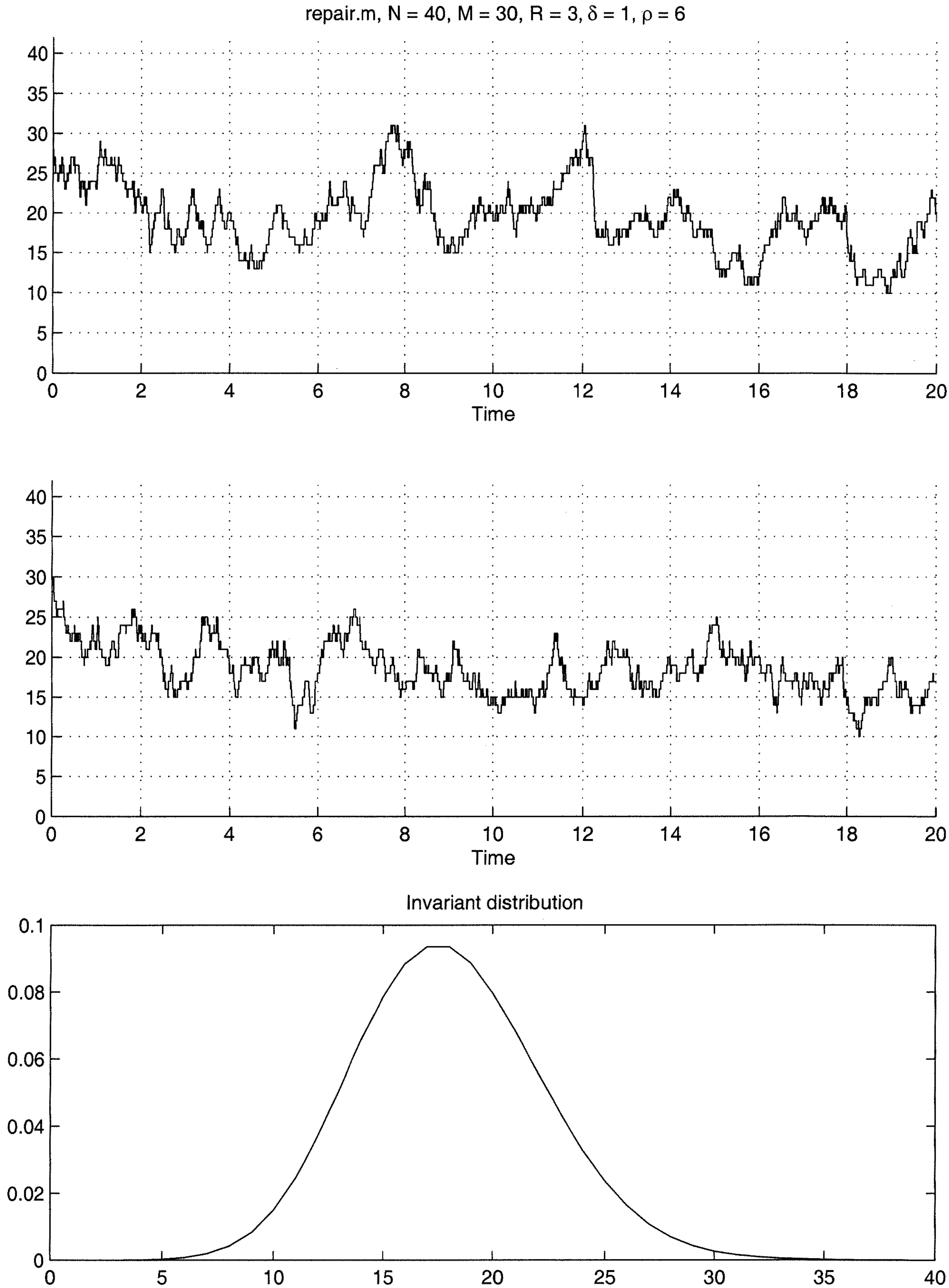
Write μ for the rate at which failures occur for one truck
ρ for the rate at which repairs are completed.
(book uses λ)

Generator

$$
A = \begin{array}{c|cccccccccccccc}
 & 0 & 1 & 2 & & \cdots 29 & 30 & 31 & & \cdots & 37 & 38 & 39 & 40 \\
\hline
0 & -3\rho & 3\rho & 0 & 0 & & & & & & & & & \\
1 & \mu & -(\mu+3\rho) & 3\rho & 0 & & & & & & & & & \\
2 & 0 & 2\mu & -(2\mu+3\rho) & 3\rho & & & & & & & & & \\
3 & & & & & & & & & & & & & \\
\vdots & & & & & & & & & & & & & \\
30 & & & & & 30\mu & -(\ ) & 3\rho & & & & & & \\
31 & & & & & & 30\mu & -(\ ) & 3\rho & & & & & \\
 & & & & & & & & & \ddots & & & & \\
37 & & & & & & & & & & \ddots & 3\rho & & \\
38 & & & & & & & & & & 30\mu & -(\ ) & 2\rho & \\
39 & & & & & & & & & & & 30\mu & -(30\mu+\rho) & \rho \\
40 & & & & & & & & & & & & 30\mu & -30\mu
\end{array}
$$

In general,
- N machines
- M used at a time
- R is maximum number in repair


repair.m, N = 40, M = 30, R = 3, δ = 1, ρ = 10
Time
Time
Invariant distribution



repair.m, N = 40, M = 30, R = 3, δ = 1, ρ = 6
Time
Time
Invariant distribution

12-6-2000

Inv. dist'n:

pi = invariant (expm (A)) ;

Cost

$$c(X_n) \qquad c = \begin{array}{c} 0 \\ 1 \\ \vdots \\ \vdots \\ \\ 30 \\ \\ \vdots \\ 40 \end{array} \begin{bmatrix} 30r \\ 29r \\ \vdots \\ 2r \\ 1r \\ 0 + 3p \\ 0 \\ \vdots \\ \ \end{bmatrix}$$

per unit time

plus cost of ~~mai~~ having a repair shop.

Expected # working

Ways to improve the system

More spares

larger machine shop

faster repairs

better trucks