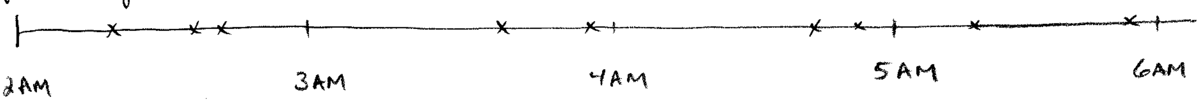
11-7-2000

Poisson process

Situation

You answer telephones for a mail-order company.
You work the 2 AM – 6 AM shift; there are very few calls, and no time tends to be busier than other times. 3 per hour, on average.
You pay attention to the times at which the phone rings, and you have lots of time to think!

A typical night:



Modeling

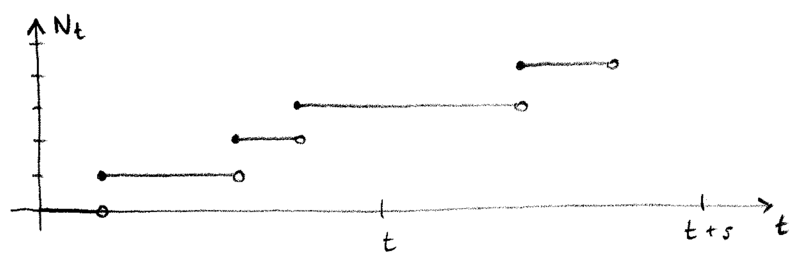
Think about the number of calls received over each hour:

Are these random variables independent?
identically distributed?

Both seem reasonable.

Number of calls over each half hour, $B_1 \dots B_8$? Different distribution, but still iid.

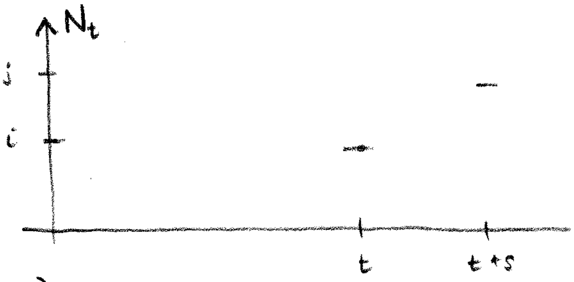
Call 2 AM time 0, imagine your shift going on forever.
Let N_t be the number of calls that arrive in $(0, t]$.



Then $N_{t+s} - N_t$ is the number of calls that arrive in $(t, t+s]$.

N is a stochastic process with index set $[0, \infty)$ and state space $\{0, 1, 2, \dots\}$.

Is N a Markov process?



$$\mathbb{P}(N_{t+s} = j \mid N_t = i,\ N_r,\ r \le t)$$

$$= \mathbb{P}(N_{t+s} - N_t = j - i \mid N_t = i,\ N_r,\ r \le t)$$

$$= \mathbb{P}(\text{ j-i calls arrive in } (t, t+s] \mid N_t = i,\ N_r,\ r \le t)$$

$$= \mathbb{P}(\text{ j-i calls arrive in } (t, t+s]) \qquad * \text{ reasonable modeling assumption}$$

$$= \mathbb{P}(\text{ j-i calls arrive in } (t, t+s] \mid N_t = i)$$

$$= \mathbb{P}(N_{t+s} = j \mid N_t = i), \qquad \text{so yes, it is a Markov process}$$

$$= \mathbb{P}(\text{ j-i calls arrive in } (0, s]) \qquad * \text{ reasonable modeling assumption}$$

$$= \mathbb{P}(N_s = j \mid N_0 = i) \qquad \text{time homogeneous}$$

So far, so good.

By the Decomposition Theorem, the successive states visited by N form a Markov chain $Y_0, Y_1, Y_2, \ldots$

The value of Y always increases.

Can it increase by more than one?

Modeling assumption: no, cannot have two calls arrive at <u>exactly</u> the same time.

Only if people conspired to call at the same time.

Then Q is very easy!

$$Q(i, i+1) = 1, \quad \text{all others } 0.$$

$$Q = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & & \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

From the theory we've done, we know that the length of time spent in state i satisfies

$$\mathbb{P}(\, W_t > u \;\mid\; N_t = i\,) \;=\; e^{-\lambda(i)\, u}, \qquad u \ge 0,$$

where the parameter λ may depend on i.

But look closer:

$$\mathbb{P}(\, W_t > u \mid N_t = i\,) = \mathbb{P}(\text{ no arrivals in } [t, t+u] \mid N_t = i)$$
$$= \mathbb{P}(\text{ no arrivals in } [t, t+u]),$$

irrespective of i, the number of arrivals up until time t.

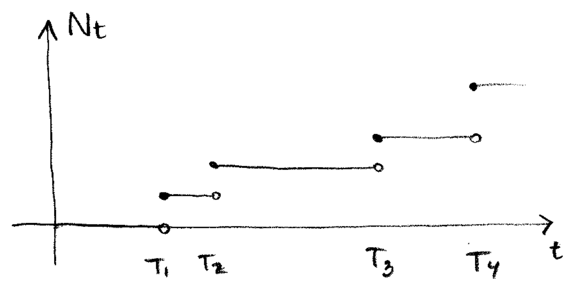
Thus, $\lambda(i)$ is the same for all i; we'll just write λ.

Poisson process

assumptions:
- N_t, $t \ge 0$ takes values in $\{0, 1, 2, \dots\}$
- N increases by jumps of size one only.
- $N_{t+s} - N_t$ is independent of N_r, $r \le t$.
- The distribution of $N_{t+s} - N_t$ does not depend on t

Then there is a number $\lambda \ge 0$, called the <u>arrival rate</u> for which the length of time N spends in each state is Exponential with mean $\frac{1}{\lambda}$.

Successive times between jumps are independent and identically distributed.



Example $\lambda = 3$ calls per hour.

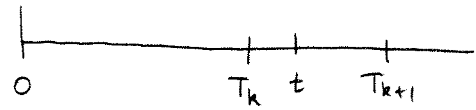
Suppose you arrive $\frac{1}{2}$ hour late for your shift.
What is the probability that you have not missed a call?

$$\mathbb{P}(T_1 > \tfrac{1}{2}) = e^{-\lambda \cdot \frac{1}{2}} = e^{-3 \cdot \frac{1}{2}} = 0.2231.$$

Pretty low. Better not be that late!
5 minutes late? $\mathbb{P}(T_1 > \frac{1}{12}) = e^{-3 \cdot \frac{1}{12}} = 0.7788.$

Distribution of N_t

First, relate N_t to the times of arrivals:

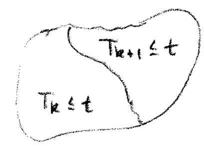


$$\{N_t = k\} = \{T_k \le t \text{ and } T_{k+1} > t\},$$

because arrivals occur singly.

So

$$\begin{aligned} \mathbb{P}(N_t = k) &= \mathbb{P}(T_k \le t \text{ and } T_{k+1} > t) \\ &= \mathbb{P}(T_k \le t \text{ and not } T_{k+1} \le t) \\ &= \mathbb{P}(T_k \le t) - \mathbb{P}(T_{k+1} \le t) \end{aligned}$$



$\mathbb{P}(T_k \le t) = \mathbb{P}(T_{k+1} \le t)$
$+ \mathbb{P}(T_k \le t \text{ and not } T_{k+1} \le$

What is $\mathbb{P}(T_k \le t)$?

This is the cdf of T_k, which is the sum of k independent Exponential random variables.

These are Gamma$(1, \frac{1}{\lambda})$ variables, and the sum of k iid ones is a Gamma$(n, \frac{1}{\lambda})$ random variable,

so the density of T_k is $\quad f_k(t) = \dfrac{\lambda (\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}, \quad t \ge 0.$

The cdf of T_k is $\quad F_k(t) = 1 - e^{-\lambda t} \displaystyle\sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!},$

which you can check by differentiating $(F_k' = f_k)$.

Finally,

$$\begin{aligned} \mathbb{P}(N_t = k) &= F_k(t) - F_{k+1}(t) \\ &= \left(1 - e^{-\lambda t} \sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!}\right) - \left(1 - e^{-\lambda t} \sum_{n=0}^{k} \frac{(\lambda t)^n}{n!}\right) \\ &= e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \end{aligned}$$

so that $N_t \sim$ Poisson(λt).

Example Telephone center for mail-order catalog.

What we know:

- qualitative: calls arrive according to a Poisson process with some parameter λ (same for all states)
- quantitative: on average, there are 3 calls per hour.

Now we have done enough theory to know that $N_1 \sim \text{Poisson}(\lambda)$,

so $\qquad \mathbb{E} N_1 = \lambda = 3,$

so in fact the parameter λ is equal to the average number of calls per hour:

$$\lambda = 3 \, \frac{\text{calls}}{\text{hour}} \qquad \underline{\text{Arrival rate}}.$$

Now we can do some specific numerical calculations.

- Expected time of first phone call (after time 0):

$$\mathbb{E} T_1 = \frac{1}{\lambda} = \frac{1}{3 \frac{1}{\text{hour}}} = \frac{1}{3} \text{ hour} = 20 \text{ minutes}$$

That makes sense.

- Expected time of 5^{th} phone call:

$$\begin{aligned} \mathbb{E}\, T_5 &= \mathbb{E}\; T_5 - T_4 \quad + T_4 - T_3 \quad + \quad \cdots \quad T_1 \\ &= 5 \cdot \frac{1}{\lambda} \\ &= 5 \cdot \frac{1}{3} \text{ hour} \\ &= 100 \text{ minutes} \end{aligned}$$

- Average number of calls per 4-hour shift:

$$\mathbb{E} N_4 = \lambda \cdot 4 = 12.$$

- Probability of more than 15 calls in a shift

$$\begin{aligned} \mathbb{P}(N_4 > 15) &= 1 - \mathbb{P}(N_4 \leq 15) \\ &= 1 - \text{poisscdf}(15, 12) \qquad \text{Matlab} \\ &= 0.1556 \end{aligned}$$

- $$\mathbb{P}(N_1 = 4,\ N_2 = 8,\ N_3 = 12)$$
$$= \mathbb{P}(N_3 - N_2 = 4,\ N_2 - N_1 = 4,\ N_1 = 4)$$
$$= \mathbb{P}(N_3 - N_2 = 4)\ \mathbb{P}(N_2 - N_1 = 4)\ \mathbb{P}(N_1 = 4)$$
$$= \mathbb{P}(N_1 = 4)^3$$
$$= \left(\frac{e^{-\lambda}\lambda^4}{4!}\right)^3$$
$$= 0.0047442874$$

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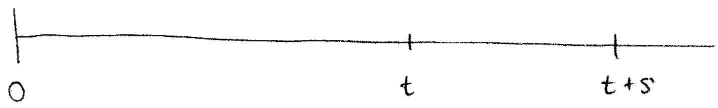
Theoretical results

- Poisson distribution



$$N_{t+s} - N_t \sim \text{Poisson}(\lambda s)$$

N_t and $N_{t+s} - N_t$ are Poisson (λt) and Poisson (λs) and indep.

Their sum, N_{t+s} is Poisson $(\lambda(t+s))$.

One <u>adds</u> the parameters.

In fact, if $X \sim \text{Poisson}(\mu)$, $Y \sim \text{Poisson}(\nu)$, independent, then

$$X + Y \sim \text{Poisson}(\mu + \nu).$$

<u>Proof</u> ① Let $\lambda t = \mu$, $\lambda s = \nu$, use Poisson processes!

② $\mathbb{P}(X+Y=k) = \sum_{l=0}^{k} \mathbb{P}(X=l, X+Y=k)$ ....

- Transition matrix

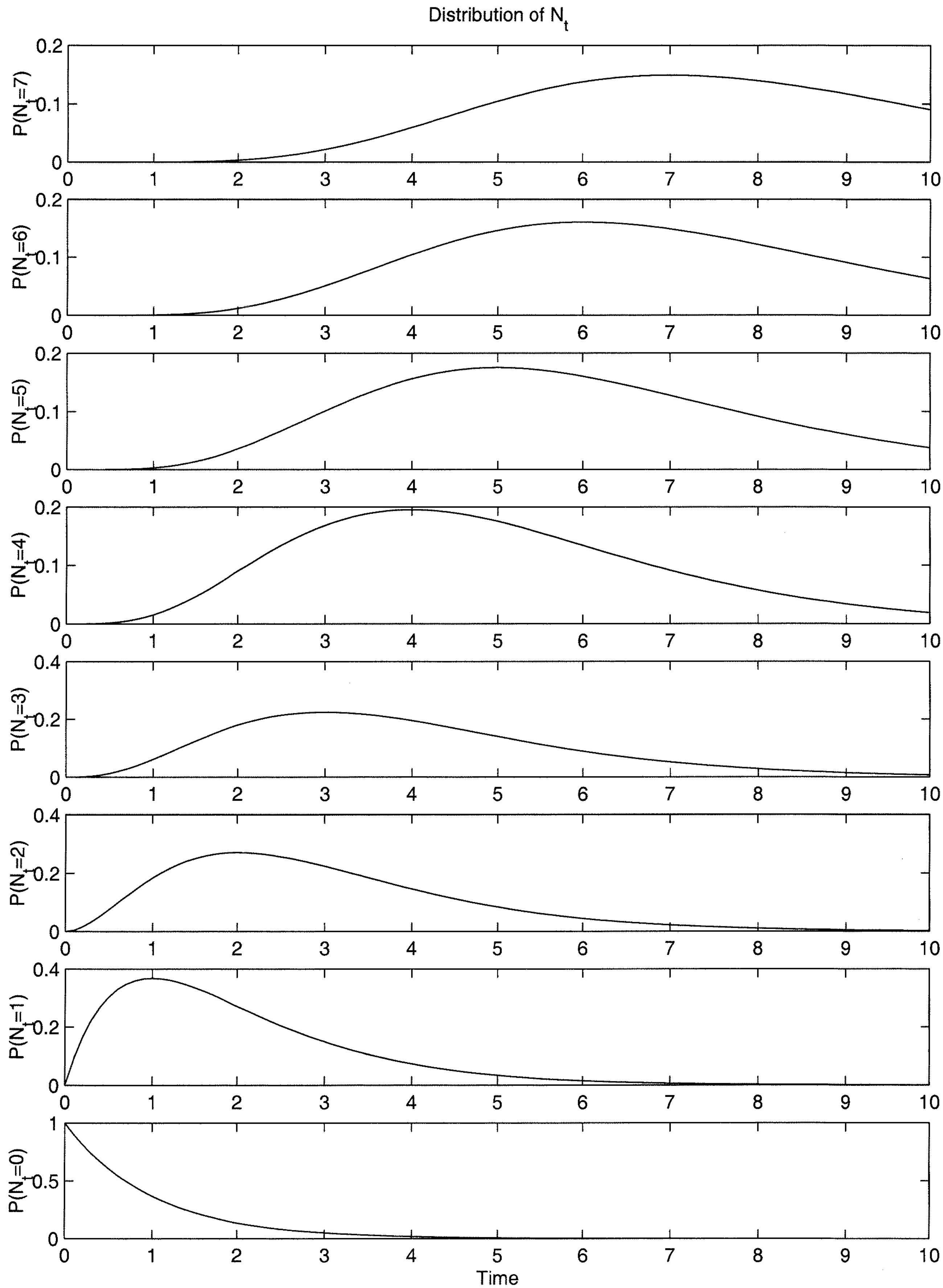
$$P_{ij}(s) = \mathbb{P}(N_{t+s} = j \mid N_t = i)$$

$$= \mathbb{P}(N_{t+s} - N_t = j - i)$$

$$= \begin{cases} \dfrac{e^{-\lambda s} (\lambda s)^{j-i}}{(j-i)!} & j \geq i \\ 0 & \text{otherwise} \end{cases}$$

$$P(s) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ \end{array} \begin{bmatrix} \overset{0}{e^{-\lambda s}} & e^{-\lambda s} \overset{1}{\cdot} \lambda s & \dfrac{e^{-\lambda s}(\lambda s)^2}{2} & \dfrac{e^{-\lambda s}(\lambda s)^3}{3!} & \cdots \\ 0 & e^{-\lambda s} & e^{-\lambda s} \lambda s & & \\ 0 & 0 & e^{-\lambda s} & & \\ & & & & \end{bmatrix}$$

This is one of the only cases in which you can write down $P(s)$ easily/explicitly.



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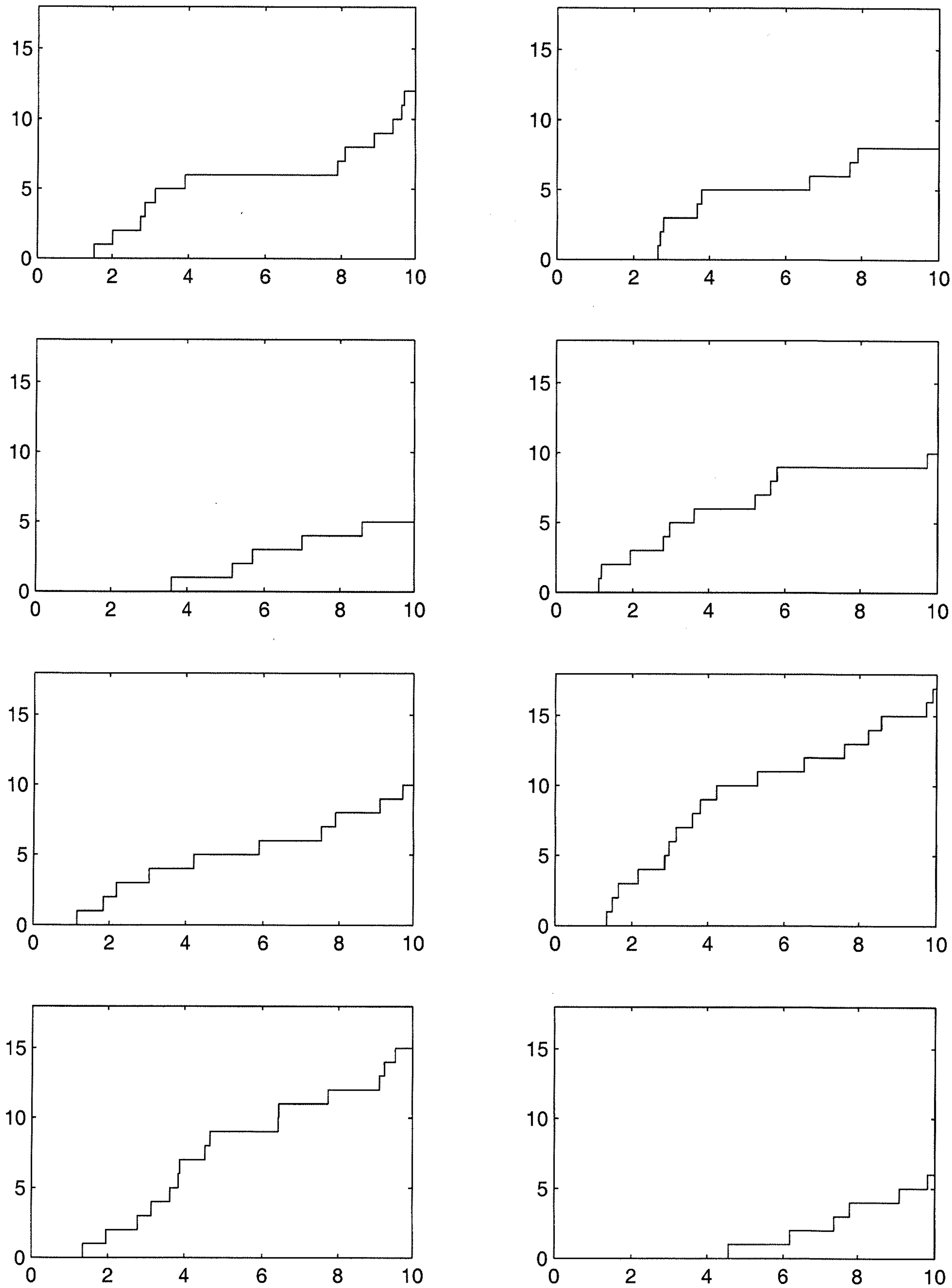
Outcomes of the Poisson process, rate 1



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