# Stochastic processes in continuous time

A stochastic process is a collection $X_t$, $t \in T$ of random variables having common state space $S$.

For Markov chains, $T = \{0,1,2,\dots\}$, which is infinite, but countable.

Now we will consider $T = [0,\infty)$.

For many reasons, this is more natural.

## Examples

1. $X_t$ is the number of people on hold on the telephone to some company at time $t$
2. $X_t$ is the number of cars in Lot H at time $t$
3. $X_t$ is the temperature, wind speed, whatever, at time $t$
4. $X_t$ is the number of individuals in a biological population at time $t$.

In three of these examples, the state space $S$ is $\{0,1,2,\dots\}$; in the other, $S=\mathbb{R}$

It turns out that this sort of situation is the easiest to begin with.

Also, there are so many random variables in the collection $X_t$, $t \geq 0$ that it doesn't make sense to begin with the iid case.

Perhaps the simplest place to begin is with processes satisfying the Markov property.

## Markov processes (Continuous time Markov chains - Chapter VI)

Let $X_t$, $t \geq 0$ be a collection of random variables taking values in a countable set $S$.

Suppose that for all $s, t \geq 0$ and $i, j$ in $S$,

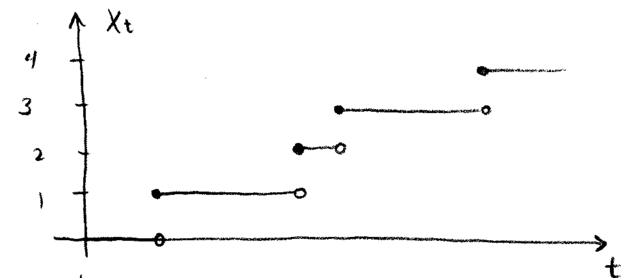
$$\mathbb{P}(X_{t+s} = j \mid X_r, r \leq t, X_t = i) = \mathbb{P}(X_{t+s} = j \mid X_t = i)$$


Then $X$ is said to be a <u>Markov process</u>.

(Some people call it a continuous time chain, but to me "chain" refers to the discreteness of time, not space)

<u>Example</u> Let $X$ be a Poisson arrival process with constant rate $\lambda$.

$X_t$ = # arrivals in $(0, t]$.




Then $X_t$, $t \geq 0$ is a collection of random variables taking values in $S = \{0, 1, 2, \ldots\}$.

Let $i \leq j$.

$$\begin{aligned}
&\mathbb{P}(X_{t+s} = j \mid X_r, r \leq t, X_t = i) \\
&= \mathbb{P}(X_{t+s} - X_t = j - i \mid X_r, r \leq t, X_t = i) \\
&= \mathbb{P}(X_{t+s} - X_t = j - i) && \text{independence} \\
&= \mathbb{P}(X_{t+s} - X_t = j - i \mid X_t = i) && \text{independence} \\
&= \mathbb{P}(X_{t+s} = j \mid X_t = i),
\end{aligned}$$

which shows that $X$ satisfies the Markov property.

Moreover,

$$\mathbb{P}(X_{t+s} = j \mid X_t = i) = \begin{cases} \dfrac{e^{-\lambda s}(\lambda s)^{j-i}}{(j-i)!} & \text{if } j \geq i \\ 0 & \text{if } j < i. \end{cases}$$

If it happens that $\mathbb{P}(X_{t+s} = j \mid X_t = i)$ does not depend on $t$ (as in the previous example), then we say $X$ is <u>time homogeneous</u>.

Then we write

$$P_{ij}(s) = \mathbb{P}(X_{t+s} = j \mid X_t = i).$$

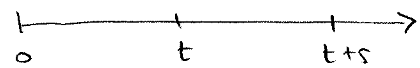
Whereas for Markov chains we had $P, P^2, P^3, P^4, \ldots, P^s$ to describe transition probabilities over $1, 2, 3, 4, \ldots, s$ time steps, now we have a matrix $P(s)$ for every $s \geq 0$ to tell the probabilities of transitions over time $s$. These are matrices parameterized by $s$.

We can still think of this as matrix multiplication:

<u>Chapman - Kolmogorov equation</u>

For $s, t \geq 0$, $\quad P(t+s) = P(t)\,P(s)$, a matrix product

<u>Proof</u>




$$\begin{aligned}
P_{ij}(t+s) &= \mathbb{P}(X_{t+s} = j \mid X_0 = i) \\
&= \sum_{k\in S} \mathbb{P}(X_{t+s} = j, X_t = k \mid X_0 = i) \\
&= \sum_{k\in S} \mathbb{P}(X_{t+s} = j \mid X_t = k, X_0 = i)\cdot \mathbb{P}(X_t = k \mid X_0 = i) \\
&= \sum_{k\in S} \mathbb{P}(X_{t+s} = j \mid X_t = k)\cdot \mathbb{P}(X_t = k \mid X_0 = i) \qquad \text{(Markov property)} \\
&= \sum_{k\in S} P_{kj}(s)\, P_{ik}(t) \\
&= (P(t)P(s))_{ij}
\end{aligned}$$

<u>Example</u> For a Poisson process,

$$P(s) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & \end{array} \\
\begin{bmatrix}
e^{-\lambda s} & \frac{e^{-\lambda s}(\lambda s)^1}{1!} & \frac{e^{-\lambda s}(\lambda s)^2}{2!} & \cdots \\
0 & e^{-\lambda s} & \frac{e^{-\lambda s}(\lambda s)^1}{1!} & \cdot \\
0 & 0 & e^{-\lambda s} & \\
0 & 0 & 0 & e^{-\lambda s} & \cdots \\
\vdots & & & &
\end{bmatrix}
\end{array}$$

For small $s$, $P(s) \approx I$

For large $s$, $P(s) \approx \begin{bmatrix} 0 & 0 & \cdots \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$

Markov property $\quad \mathbb{P}( X_{t+s} = j \mid X_t = i,\ X_r,\ r \le t) = \mathbb{P}( X_{t+s} = j \mid X_t = i)$

Time homogeneity $\quad = P_{ij}(s)$

Regularity assumption $\quad \lim_{s \to 0+} P_{ij}(s) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$

Then $P(t)$ is right continuous in $t$, for $\lim_{s\to 0+} P(t+s) = \lim_{s \to 0+} P(t)P(s) = P(t) \lim_{s\to 0+} P(s) = P(t)\cdot I = P(t)$.

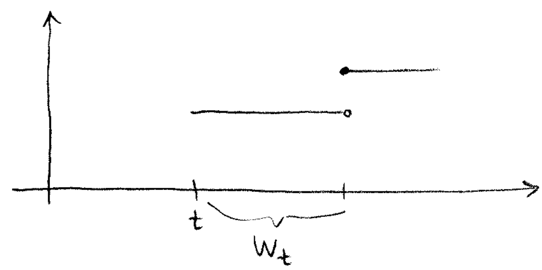
Extension of Markov property. $\quad$ Let $A$ = " $X_{t+s},\ s \ge 0$ does some particular thing ".

$$\mathbb{P}( A \mid X_t = i,\ X_r,\ r \le t) = \mathbb{P}( A \mid X_t = i)$$
$$= \mathbb{P}( A \text{ relative to } 0 \mid X_0 = i)$$

## Time spent in each state

Starting at time $t$, let $W_t$ be the amount of time the chain remains in its current state before jumping.

$$W_t = \inf ( s > 0 : X_{t+s} \ne X_t )$$



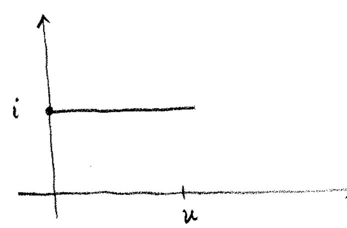

**Theorem** For each $i$ in $S$, there exists $\lambda(i)$ in $[0, \infty]$ such that, for all $t \ge 0$,

$$\mathbb{P}( W_t > u \mid X_t = i ) = e^{-\lambda(i) u} \quad \text{for all } u \ge 0.$$

**Proof** The event $A$ is that $X_{t+s},\ s \ge 0$ remains in the current state at least $u$ units of time.

By time homogeneity this probability is equal to

$$\mathbb{P}( W_0 > u \mid X_0 = i)$$




$$= \mathbb{P}( X_s = i \text{ for } s \text{ in } [0, u] \mid X_0 = i)$$

Let $f(u)$ denote this probability.

Then
$$\begin{aligned} f(u+v) &= \mathbb{P}( X_s = i \text{ for } s \text{ in } [0, u+v] \mid X_0 = i) \\ &= \mathbb{P}( X_s = i \text{ for } s \text{ in } [0,u] \text{ and } X_s = i \text{ for } s \text{ in } [u, u+v] \mid X_0 = i) \\ &= \mathbb{P}( X_s = i \text{ for } s \text{ in } [u, u+v] \mid X_s = i \text{ for } s \text{ in } [0,u],\ X_0 = i) \cdot \\ &\qquad \mathbb{P}( X_s = i \text{ for } s \text{ in } [0,u] \mid X_0 = i) \\ &= \mathbb{P}( X_s = i \text{ for } s \text{ in } [u, u+v] \mid X_u = i) \cdot \mathbb{P}( X_s = i \text{ for } s \text{ in } [0,u] \mid X_0 = i) \\ &= f(v) \cdot f(u), \end{aligned}$$

so $f(u) = e^{-\lambda(i) u}$ for some number $\lambda(i)$. $\square$

Thus, the time spent in each state is exponentially distributed.

If $\lambda(i) = 0$, then $\mathbb{P}(W_t > u \mid X_t = i) = 1$ for all $u$, so $X$ never leaves state $i$, and $i$ is called <u>absorbing</u>

If $0 < \lambda(i) < \infty$, then $i$ is called <u>stable</u>

If $\lambda(i) = \infty$, then $\mathbb{P}(W_t > u \mid X_t = i) = 0$ for all $u$, so that $X$ leaves state $i$ as soon as it enters it. $i$ is called <u>instantaneous</u>.

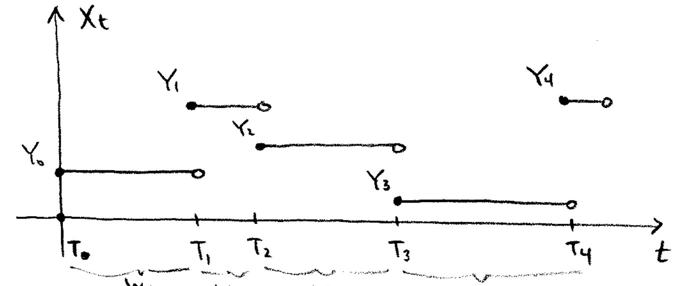
Instantaneous states are really new, particular to continuous time.

For now on, we assume there are <u>no</u> instantaneous states.

In fact, we must assume that the process is regular enough.

There are many details, all of which will be skipped, more or less.

## Jump times




Recall that

$W_t$ = length of time until next jump after $t$

Then

$$T_1 = W_0$$

$$T_2 = T_1 + W_{T_1}$$

$$T_3 = T_2 + W_{T_2}$$

$$\vdots$$

$$T_{n+1} = T_n + W_{T_n}$$

## States visited

Let $Y_n = X_{T_n}$ for $n = 0, 1, 2, \ldots$

Then $Y_n$, $n = 0, 1, 2, \ldots$ is a sequence of random variables taking values in $S$.

## Strong Markov property

Let $A$ = "$X_{T_n+s}$, $s \ge 0$ does some particular thing".

Then $\mathbb{P}(A \mid X_{T_n} = i, X_r \text{ for } r \le T_n) = \mathbb{P}(A \mid X_{T_n} = i)$

$$= \mathbb{P}(A \text{ relative to } 0 \mid X_0 = i)$$

$$= \mathbb{P}(X_s, s \ge 0 \text{ does that thing} \mid X_0 = i)$$

**Theorem** (Structure of a Markov process) (Çinlar Thm 8.3.3)

Suppose $X$ is Markov, time homogeneous, and satisfies the regularity condition. Suppose also that $X$ has no instantaneous or absorbing states.

① Below

Then for all $i, j$ in $S$, $n \geq 0$ and $u \geq 0$,

$$\mathbb{P}(Y_{n+1} = j,\ T_{n+1} - T_n > u \mid Y_n = i,\ Y_0, \dots Y_{n-1}, T_0, \dots, T_n) = Q(i,j)e^{-\lambda(i)u},$$

where

$$Q(i,j) = \mathbb{P}(Y_1 = j \mid Y_0 = i) = \mathbb{P}(X_{W_0} = j \mid X_0 = i).$$

**Proof**

$$\mathbb{P}(Y_{n+1} = j,\ T_{n+1} - T_n > u \mid Y_n = i,\ Y_0, \dots, Y_{n-1}, T_0, \dots, T_n)$$

$$= \mathbb{P}(X_{T_n + W_{T_n}} = j,\ W_{T_n} > u \mid X_{T_n} = i,\ X_r \text{ for } r \leq T_n)$$

[Recall $Y_{n+1} = X_{T_{n+1}}$, $T_{n+1} = T_n + W_{T_n}$.

Also, knowing $Y_n = i, Y_0, \dots, Y_{n-1}, T_0, \dots, T_n$ is equivalent to knowing $X_{T_n} = i$ and $X_r$ for $r \leq T_n$.]

$$= \mathbb{P}(X_{T_n + W_{T_n}} = j,\ W_{T_n} > u \mid X_{T_n} = i) \quad \text{strong Markov property,}$$

$$= \mathbb{P}(X_{W_0} = j,\ W_0 > u \mid X_0 = i) \quad \text{time homogeneity}$$

$$= \mathbb{P}(X_{W_0} = j \mid W_0 > u,\ X_0 = i) \cdot \mathbb{P}(W_0 > u \mid X_0 = i)$$

$$= \mathbb{P}(X_{W_0} = j \mid X_s = i,\ s \text{ in } [0,u]) \cdot e^{-\lambda(i)u} \quad \text{earlier result}$$

$$= \mathbb{P}(X_{u+W_u} = j \mid X_u = i) \cdot e^{-\lambda(i)u} \quad \text{Markov property}$$

$$= \mathbb{P}(X_{W_0} = j \mid X_0 = i) \cdot e^{-\lambda(i)u} \quad \text{time homogeneity}$$

$$= Q(i,j) \cdot e^{-\lambda(i)u}.$$

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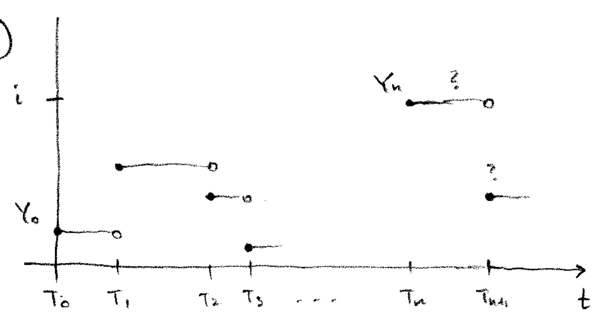


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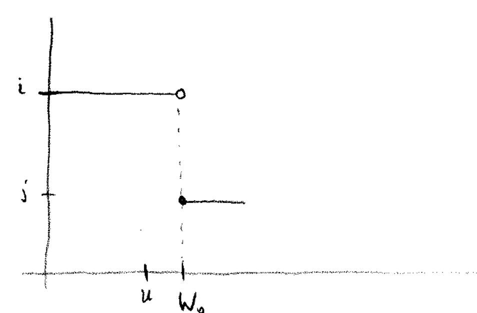

Some corollaries:

- The sequence $Y_n$, $n=0,1,\ldots$ is a Markov chain with transition matrix $Q$, for with $u=0$, and imagining the proof without conditioning on $T_0,\ldots,T_n$,

$$\mathbb{P}(Y_{n+1}=j \mid Y_n=i,\ Y_0,\ldots,Y_{n-1})$$
$$= \mathbb{P}(Y_{n+1}=j,\ T_{n+1}-T_n>0 \mid Y_n=i,\ Y_0,\ldots,Y_{n-1})$$
$$= Q(i,j)\, e^{-\lambda(i)\cdot 0}$$
$$= Q(i,j)$$
$$= \mathbb{P}(Y_{n+1}=j \mid Y_n=i).$$

Thus successive states visited forms a Markov chain.

- $Q(i,i)=0$, for

$$Q(i,i) = \mathbb{P}(X_{T_1}=i \mid X_0=i)$$

But for there to be a jump, the state must change, so this probability is 0.

- Time spent in state $i$ does not depend on previous states

$$\mathbb{P}(T_{n+1}-T_n>u \mid Y_n=i,\ Y_0,\ldots,Y_{n-1}) = e^{-\lambda(i)u},$$

by summing over $j$.

- Time spent in state $i$ does not depend on next state to be visited.

$$\mathbb{P}(Y_{n+1}=j,\ T_{n+1}-T_n>u \mid Y_n=i) = Q(i,j)\, e^{-\lambda(i)u}$$

and also equals

$$\mathbb{P}(T_{n+1}-T_n>u \mid Y_{n+1}=j,\ Y_n=i)\cdot \mathbb{P}(Y_{n+1}=j \mid Y_n=i)$$
$$= \mathbb{P}(T_{n+1}-T_n>u \mid Y_{n+1}=j,\ Y_n=i)\cdot Q(i,j),$$

so that

$$\mathbb{P}(T_{n+1}-T_n>u \mid Y_{n+1}=j,\ Y_n=i) = e^{-\lambda(i)u},$$

independent of $j$.