Component failure and replacement

Draws from III.5.4 Success runs
IV. 2.4 Age replacement policies

Situation A system has a component which fails from time to time and then needs to be replaced.

Examples: light bulb, say, in a traffic light
watch battery
belt in a car engine or industrial machine
register receipt paper — ever been in line to see it changed?
soda cans in a vending machine
employee — failure is when they quit, retire, or stop for health reasons

Failure of the component may be expensive (or dangerous), and replacing the component costs money.

Model Number successive components in service 1, 2, 3, ...

Let F_n be the amount of time in service before component n fails

$F_n = 5$ means it works for 5 units of time, then abruptly fails at the end of 5 minutes.

Are $F_1, F_2, F_3, \ldots$ independent? identically distributed?
These are reasonable.

Let F be a generic random variable with the same distribution.

Let $a_i = \mathbb{P}(F = i)$.

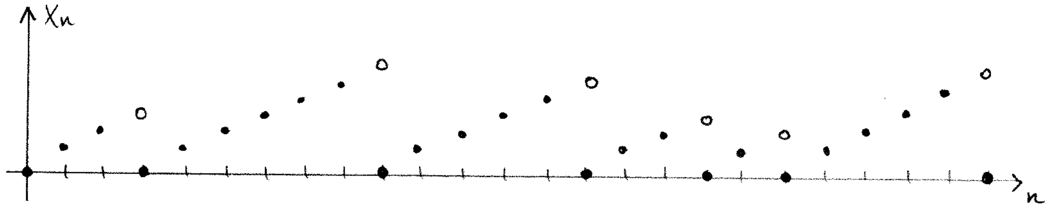
Questions

- How often must the component be replaced?
  How many times per week?
- What is the long-term cost per unit of time for this device?
- Can we save money by replacing the component <u>before</u> it fails?

Reflection at the end: what do people do for the examples above?

Age of the current component

Let X_n be the age of the current component at the beginning of time interval n.

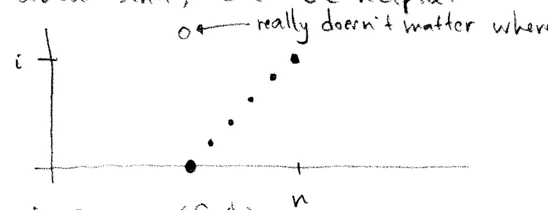


Each visit to 0 indicates a failure and a replacement.

Is X Markov?

Knowing that $X_n = i$, would information about X_{n-1}, etc. be helpful?

No. Knowing $X_n = i$ tells you a lot



Time homogeneous as well

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} p_i & \text{if } j = 0 \quad \text{(fails)} \\ q_i & \text{if } j = i+1 \quad \text{(continues)} \end{cases}$$

$$\begin{aligned} p_i &= \mathbb{P}(X_{n+1} = i+1 \mid X_n = i) \\ &= \mathbb{P}(F = i+1 \mid F > i) \\ &= \frac{\mathbb{P}(F = i+1,\ F > i)}{\mathbb{P}(F > i)} \\ &= \frac{\mathbb{P}(F = i+1)}{\mathbb{P}(F > i)} \\ &= \frac{a_{i+1}}{a_{i+1} + a_{i+2} + \cdots} = \frac{a_{i+1}}{A_{i+1}} \end{aligned}$$

"failure rate at age i"

Transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & \end{array} \\ \left[\begin{array}{cccccc} p_0 & q_0 & 0 & 0 & 0 & \cdots \\ p_1 & 0 & q_1 & 0 & 0 & \cdots \\ p_2 & 0 & 0 & q_2 & 0 & \cdots \\ p_3 & & & & & \\ \vdots & & & & & \\ & & & & & \end{array} \right] \end{array}$$

failure rates

$$\begin{aligned} q_i &= 1 - p_i \\ &= 1 - \frac{a_{i+1}}{A_{i+1}} \\ &= \frac{a_{i+2} + a_{i+3} + \cdots}{a_{i+1} + a_{i+2} + \cdots} \\ &= \frac{A_{i+2}}{A_{i+1}} \end{aligned}$$

We presume that every component will fail in finite time ($\mathbb{P}(F < \infty) = 1$), so state 0 will be recurrent for the M. chain X.
Also, every state may be reached from every other,
(if we limit the state space to the possible values of X based on a)
so in fact all states are recurrent.

It is clear that
$$\mathbb{E}_0 R_0 = \mathbb{E} F = \sum_{i=1}^{\infty} i \cdot a_i .$$

If this is finite, then 0 is positive recurrent, and by solidarity, all states are positive recurrent.
Thus, there will be a limiting distribution π. Let's find it.

$$\pi = \pi P$$
$$= [\pi_0 \ \pi_1 \ \pi_2 \ \cdots] \begin{bmatrix} p_0 & q_0 & & \\ p_1 & 0 & q_1 & \\ p_2 & 0 & 0 & q_2 \\ \vdots & & & \end{bmatrix}$$

$$[\pi_0 \ \pi_1 \ \cdots] = [(p_0\pi_0 + p_1\pi_1 + p_2\pi_2 + \cdots) \ (q_0\pi_0) \ (q_1\pi_1) \ (q_2\pi_2) \ \cdots]$$

$$\pi_0 = p_0\pi_0 + p_1\pi_1 + p_2\pi_2 + \cdots$$ but already $\pi_0 = \frac{1}{\mathbb{E}F}$.

$$\pi_1 = q_0 \pi_0$$
$$\pi_2 = q_1 \pi_1 = q_1 \cdot q_0 \cdot \pi_0$$
$$\pi_3 = q_2 \pi_2 = q_2 \cdot q_1 \cdot q_0 \cdot \pi_0$$
$$\pi_n = q_{n-1} \cdot q_{n-2} \cdot \cdots \cdot q_0 \cdot \pi_0$$
$$= \frac{A_{n+1}}{A_n} \cdot \frac{A_n}{A_{n-1}} \cdot \cdots \cdot \frac{A_2}{A_1} \cdot \pi_0$$
And $A_1 = a_1 + a_2 + \cdots = 1$
$$= A_{n+1} \cdot \pi_0$$
$$= \frac{a_{n+1} + a_{n+2} + \cdots}{\mathbb{E} F}$$

The vector π is the distribution of the age of the component in place at time n, for n large.

Examples

3 life length distributions for 3 different types of components

a is graphed on the left, π on the right

Notice that π is always decreasing:

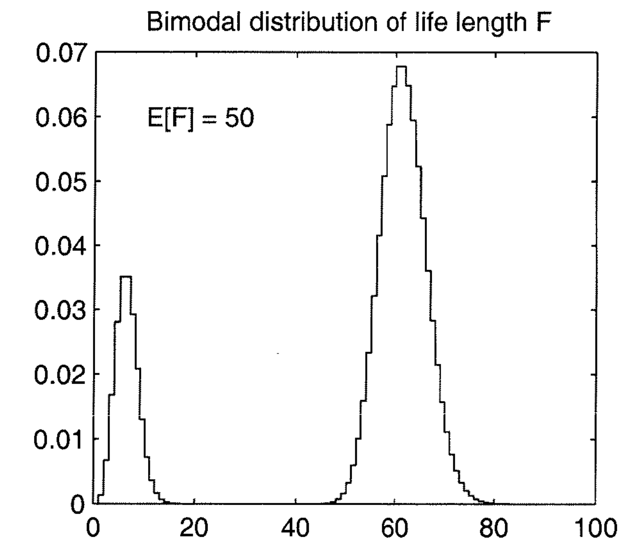
- $\pi_i = \dfrac{A_{i+1}}{\mathbb{E}F} = \dfrac{a_{i+1} + a_{i+2} + \cdots}{\mathbb{E}F}$.
- The most probable age for the current component is zero.

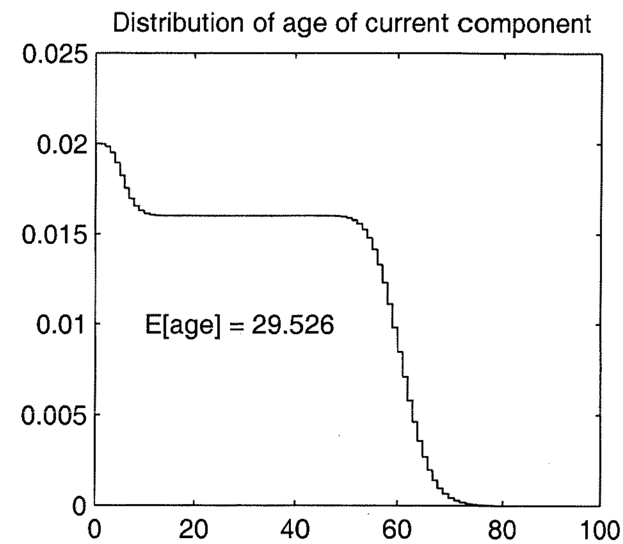
Notice that the average age of the current component is rather large compared to the average whole lifetime of a component.

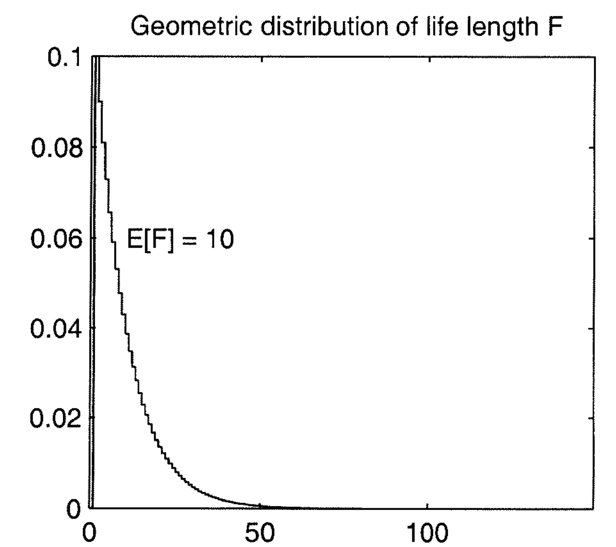
This component still has some life left in it!

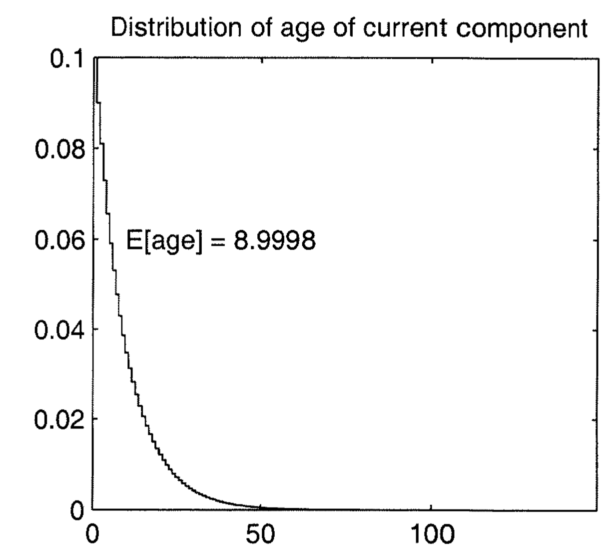
If you simulate the M. chain for dist'n 1, you see mostly long lifetimes

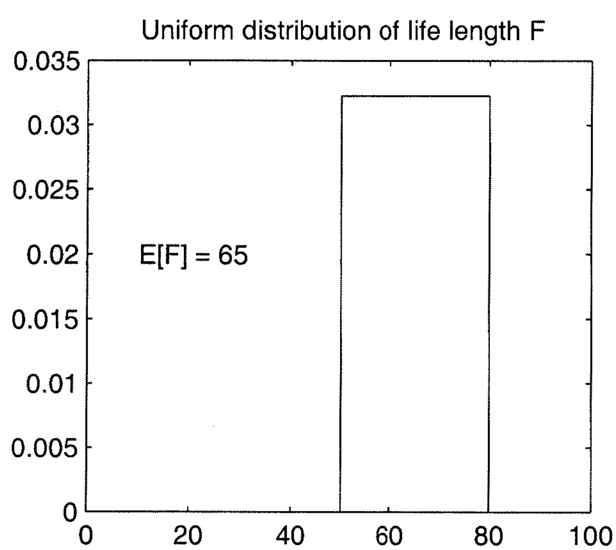
Long lifetimes occur only a little more often than short, but they occupy so much more of the time shown in the graph, they look more common.

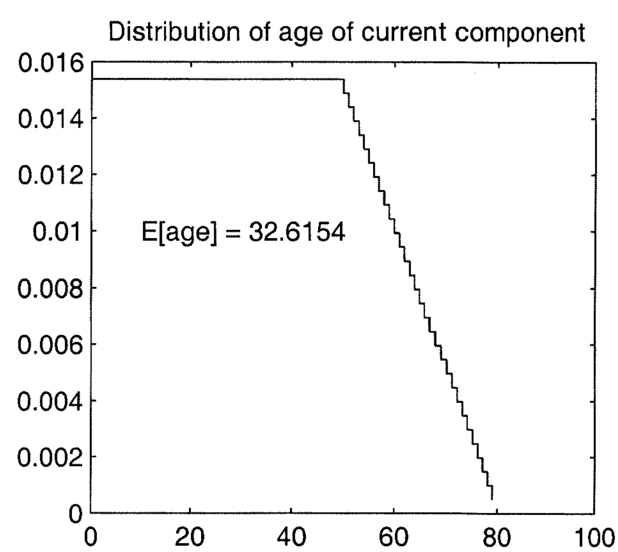












component_examples.m

Cost of the system

Every time the system fails, there is a cost C to replace the component plus a cost K due to the failure (down time, ...)

Also, in some situations, as a component ages, it costs more to operate, aside from failure. For example:

- light bulbs dim over time. reduced light is a cost
- workers get raises, and so cost more over time. new workers are usually younger and work for less

Generally speaking, when the component is in state i, there is a cost $c(i)$ associated with it.

$$c(0) = C + K \qquad \text{replacement plus failure}$$

$$\left.\begin{array}{l} c(1) = \\ c(2) = \\ \vdots \end{array}\right\} \text{whatever, maybe } 0.$$

Expected cost of the system at time n is $\mathbb{E}\, c(X_n)$, which is

$$\mathbb{E}\, c(X_n) = \sum_{i=0} c(i)\, \mu_n(i)$$

$$= \mu_n c,$$

the product of a row and column vector, if we organize the costs into a column vector $c. = \begin{bmatrix} c(0) \\ c(1) \\ \vdots \end{bmatrix}$

Then $\mathbb{E}\, c(X_n) = \mu_0 P^n c$

As $n \to \infty$, $\mu_n \to \pi$, and $\mathbb{E}\, c(X_n) \to \pi c$,

so $\boxed{\pi c \text{ is the long-run cost per unit time of this component.}}$

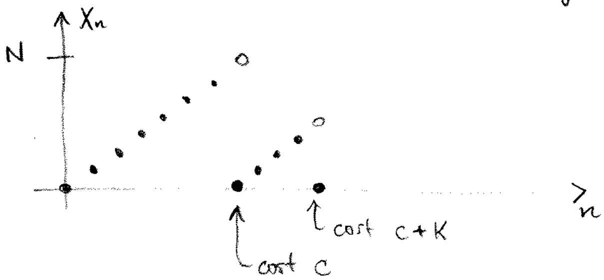
$$\pi c = \sum_{i=0} \frac{A_{i+1}}{\mathbb{E}F}\, c(i)$$

Scheduled replacement policies

Idea By replacing the component early, we may be able to avoid the cost K due to failure and pay only the cost C of replacement.

Question What is the optimal age for replacement?

Strategy: replace the component when it reaches age N, or earlier if it fails



This modifies the transition matrix:

$$P = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & & \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \\ \vdots \\ N-2 \\ N-1 \end{array} & \begin{bmatrix} p_0 & q_0 & & & \\ p_1 & 0 & q_1 & & \\ \vdots & & & \ddots & \\ p_{N-2} & & & & q_{N-2} \\ 1 & 0 & \cdots & & 0 \end{bmatrix} \end{array}$$

We can imagine that we have changed the life length distribution:

$$\begin{cases} \tilde{a}_i = a_i & \text{for } i = 0, 1, 2, \dots, N-1 \\ \tilde{a}_N = A_N = a_N + a_{N+1} + \cdots & \\ \tilde{a}_i = 0, \ i > N. & \end{cases}$$

$$p_i = \frac{\tilde{a}_{i+1}}{\tilde{A}_{i+1}}$$

$$\tilde{a}_N = A_N$$

$$\mathbb{E}\tilde{F} = \sum_{i=0}^{N} i\,\tilde{a}_i$$

This changes the invariant distribution:

$$\tilde{\pi}_0 = \frac{1}{\mathbb{E}\tilde{F}}$$

$$\tilde{\pi}_i = \frac{A_{i+1}}{\mathbb{E}\tilde{F}}, \qquad i = 0, 1, 2, \dots, N-1.$$

$$\mathbb{E}\tilde{F} = \sum_{i=0}^{N-1} A_{i+1} = \sum_{i=1}^{N} A_i$$

The cost vector is also changed:

$$\tilde{c}_0 = C + K$$

$$\tilde{c}_1 = c_1$$

$$\vdots$$

$$\tilde{c}_{N-1} = c_{N-1} - K$$

* save K by replacing after age $N-1$.

Now compute $\tilde{\pi}\tilde{c}$ for each N and find N which minimizes cost.

Examples :

$C = 2$

$K = 10$

$$c = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \\ N-1 \end{array} \begin{bmatrix} 12 \\ 0 \\ 0 \\ \vdots \\ \\ -10 \end{bmatrix}$$

simple case : ignore operating costs

save K by replacing after this state

<u>Example</u> Mixture of Poissons

$\frac{1}{5}$ of the time it fails quickly.

<u>Example</u> Geometric

Every unit of time, the component fails with prob. $a_1 = 0.1$.

Expected cost per unit of time is $(0.1)(12) + (0.9)(0) = 1.2$, w/o early replacement

With replacement at every unit of time, cost is 2.

There are the only two options :

the condition of the component is memoryless, so if you should wait at least one unit, then what will be different at the next unit? At each time step the decision must be the same, either replace now, or wait, so that you wait indefinitely.

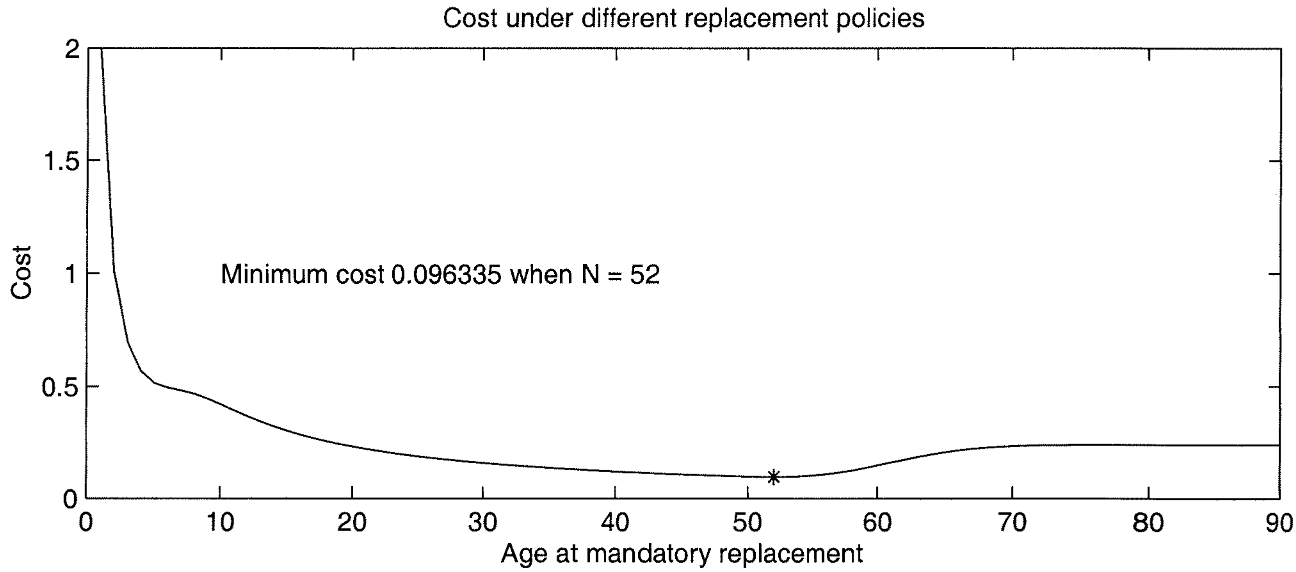
In this case, the prob. of failure is low / the cost of failure is low, so you should wait indefinitely.

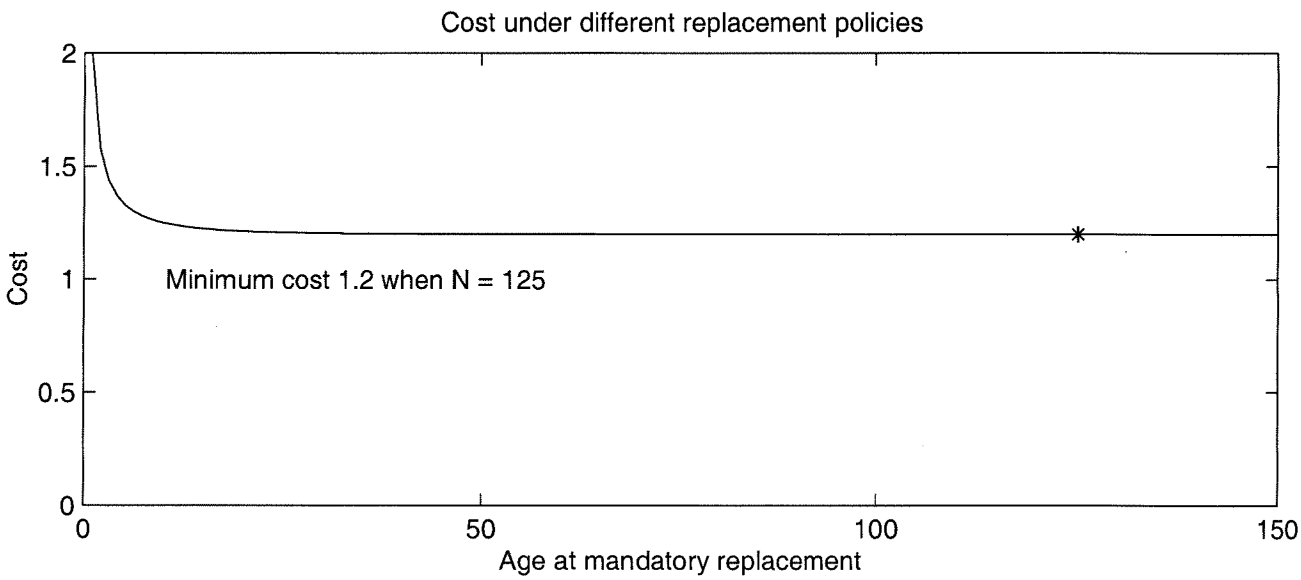
Things that don't often break are like this.

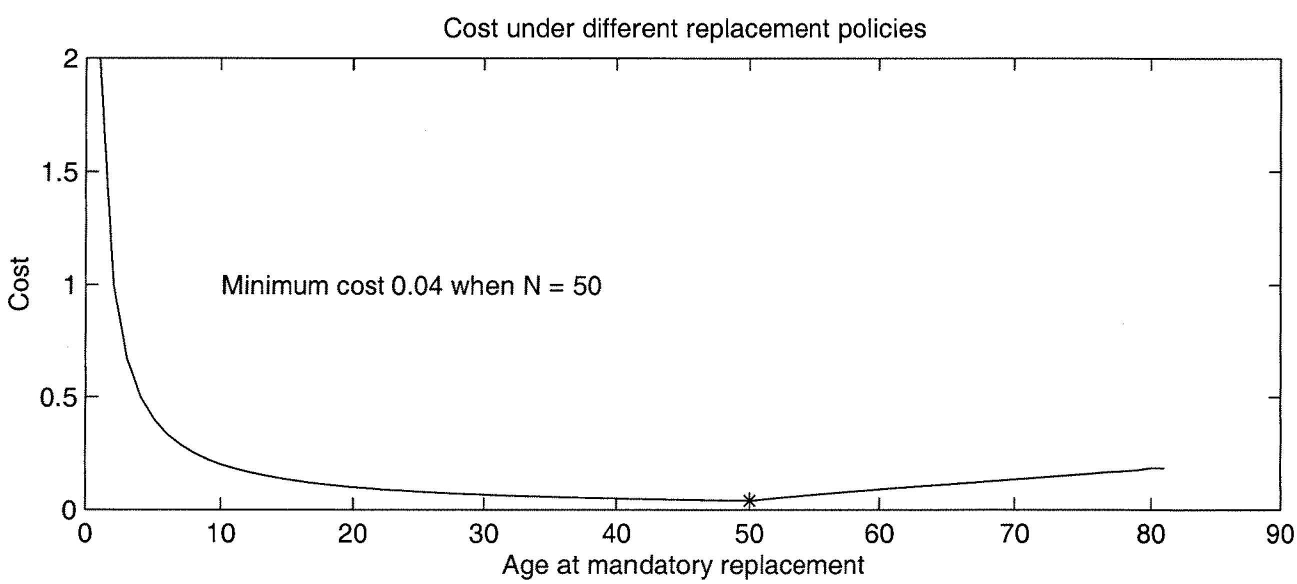
For example, a dish, an ugly lamp, a frayed lamp cord, a rear view mirror, an old fence, an old tree, over your house or not.

Either you should replace it immediately, or wait until it breaks.

<u>Example</u> Uniform


Cost under different replacement policies
Minimum cost 0.096335 when N = 52
Cost
Age at mandatory replacement
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80
90


Cost under different replacement policies
Minimum cost 1.2 when N = 125
Cost
Age at mandatory replacement
2
1.5
1
0.5
0
0
50
100
150


Cost under different replacement policies
Minimum cost 0.04 when N = 50
Cost
Age at mandatory replacement
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80
90

Example $\tilde{C}_0 = C + K$

$\tilde{C}_{N-1} = -K$

all others are 0

$$\tilde{\pi}\tilde{c} = \tilde{\pi}_0 \tilde{C}_0 + \tilde{\pi}_{N-1} \tilde{C}_{N-1}$$

$$= \frac{1}{\mathbb{E}\tilde{F}} \cdot (C + K) + \frac{A_N}{\mathbb{E}\tilde{F}} \cdot -K$$

$$= \frac{1}{\mathbb{E}\tilde{F}} \Big(C + K\,(1 - A_N) \Big)$$

$$= \frac{C + K\,(1 - A_N)}{A_1 + \cdots + A_N}$$

Matlab

$$\tilde{\pi}\tilde{c} = \frac{C + K(1 - A(N+1))}{\mathrm{sum}\big(A(2:(N+1))\big)}$$