Transience and recurrence for chains with infinitely many states

When there are infinitely many states in S, it is possible that none of them are recurrent.

Even when it is irreducible, all states may be transient.

One example will suffice.

Example Additive random walk

Let $A_0, A_1, A_2, A_3, \ldots$ be independent, identically distributed, values in $\mathbb{Z}$.
Let $a_i = \mathbb{P}(A_0 = i)$.
Let $X_0 = 0$

$$X_{n+1} = X_n + A_n, \quad n = 0, 1, 2, \ldots$$

Is X Markov?

$$\mathbb{P}(X_{n+1} = j \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i)$$

$$= \mathbb{P}(X_{n+1} - X_n = j - i \mid X_0 = i_0, \ldots, X_n = i)$$

$$= \mathbb{P}(A_n = j - i \mid \text{values of } X_0, \ldots, X_n)$$

$$= \mathbb{P}(A_n = j - i \mid \text{values of } A_0, \ldots, A_{n-1})$$

$$= \mathbb{P}(A_n = j - i) \qquad \text{by independence}$$

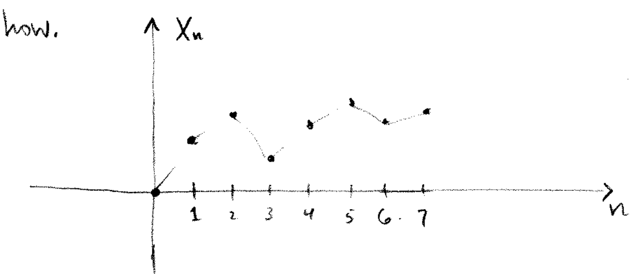
$$= a_{j-i}$$

$$= \mathbb{P}(A_n = j - i \mid X_n = i)$$

$$= \mathbb{P}(X_{n+1} = j \mid X_n = i),$$

which is what we needed to show.

Outcomes:



Transition matrix

$$P = \begin{array}{c} \\ \vdots \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \cdots \ -3 \ \ -2 \ \ -1 \ \ \ 0 \ \ \ 1 \ \ \ 2 \ \ \ 3 \ \cdots \\ \left[\begin{array}{cccccccc} & & & & & & & \\ & & & & & & & \\ & & & \ddots & & & & \\ \cdots\ a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & \cdots \\ & & & a_{-1} & a_0 & a_1 & a_2 & \\ & & & & \ddots & & & \\ & & & & & & & \end{array}\right] \end{array}$$

Special case: $a_1 = p$, $a_{-1} = q = 1 - p$, all others 0.

What is the long-term behavior of X_n?

X_n is the sum of n iid random variables.

X_n is approximately normal, most likely.

$$\lim_{n\to\infty} \frac{X_n}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} A_i = \mathbb{E} A_0,$$ the mean of the A's,

by the Strong Law of Large Numbers, (provided $\mathbb{E}|A_0| < \infty$).

Let $\mu = \mathbb{E} A_0$.
Suppose $\mu > 0$.

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_n}{n} = \mu \right) = 1$$

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = \infty \right) = 1$$

$\mathbb{P}$ (X eventually stays above 10, 100, 1000) $= 1$

$\mathbb{P}$ (X makes finitely many visits to 0) $= 1$

Then 0 is transient.

But as long as A can be positive or negative, X is irreducible,

so no other state can be recurrent.

Thus, all states are transient!

If $\mu < 0$, the story is the same

Related example – Queueing, infinite size

Probability p of serving a customer in one unit of time, equals expected # services/departures per unit time.
Arrival rate λ, equals the expected number of arrivals per unit time.

$$\mu = \text{expected change in queue size per unit time}$$
$$= \lambda - p$$

As long as the queue size stays away from 0, it is an additive random walk.

If $\mu > 0$, $\lambda > p$, $r = \frac{\lambda}{p} > 1$, traffic intensity,
then it is possible for the chain to leave state 1 and never return,
since this will eventually happen for the additive random walk.

Thus, in this case, the states of the queueing system are all transient.

Related example – absorbing boundary

Even if 0 is absorbing, it is possible for the chain to converge to ∞.

Suppose $\mu = 0$.

This is more difficult. We do not necessarily have $X_n \to \infty$.

From the Central Limit Theorem, $\frac{X_n}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1)$, $\sigma^2 = \mathrm{Var}(A)$.

That is, $\mathbb{P}\left(0 \le \frac{X_n}{\sigma\sqrt{n}} \le z\right) \to \int_0^z f(y)\,dy$, as $n \to \infty$,

where f is the normal density.

<u>Idea</u> (not vigorous)

Let $z = \frac{M}{\sigma\sqrt{n}}$, then

$$\mathbb{P}\left(0 \le \frac{X_n}{\sigma\sqrt{n}} \le \frac{M}{\sigma\sqrt{n}}\right) = \mathbb{P}(0 \le X_n \le M)$$

is approximately equal to $\int_0^{\frac{M}{\sigma\sqrt{n}}} f(y)\,dy$ as $n \to \infty$.

This is an integral over a very narrow interval, approximately equal to

$$\frac{M}{\sigma\sqrt{n}} \cdot f(0).$$

Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}(0 \le X_n \le M) \approx \sum_{n=0}^{\infty} \frac{M}{\sigma\sqrt{n}} f(0) = \infty.$$

That is,

$$\sum_{n=0}^{\infty} \sum_{j=0}^{M} P^n_{0,j} = \infty$$

$$\sum_{j=0}^{M} \left(\sum_{n=0}^{\infty} P^n_{0,j} \right) = \infty,$$

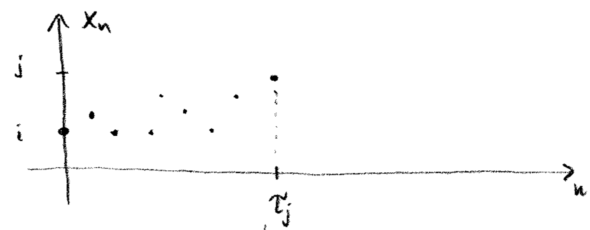
so for some $j = 0, \ldots, M$, $\sum_{n=0}^{\infty} P^n_{0j} = \infty$, so j is recurrent.

Thus, when $\mu = 0$ (and the CLT holds <u>and</u> this idea works),

all states are recurrent, and will be visited infinitely often.

<u>Specific case:</u> p. 242

Time of return



$$R_j = \min\{n \ge 1 : X_n = j\},$$

$R_j = \infty$ if no visit to j.

R_j is a random variable. We know how to compute its expected value.

Two types of recurrence

If state j is transient, $\quad \mathbb{P}_j(R_j = \infty) > 0$

so $\quad \mathbb{E}_j[R_j] = \infty$.

Suppose j is recurrent. Then $\mathbb{P}_j(R_j < \infty) = 1$.

But there are still two possibilities:

j is positive recurrent if $\quad \mathbb{E}_j[R_j] < \infty$

j is null recurrent if $\quad \mathbb{E}_j[R_j] = \infty$.

When S is finite, all states are either transient or positive recurrent.

postpone

Periodicity IV.3.2

The period of state i is $\delta = \gcd(\{n : P^n_{ii} > 0\})$

$= \gcd($ possible values of τ_i, starting in $i)$.

When $\delta = 1$, i is said to be aperiodic.

Solidarity fact: If X is irreducible, all states have the same period.

Example: Gambler's wealth.

Checkeboard pattern

Limiting distribution theorem p. 247 Çinlar p. 135

Suppose X is irreducible.

Then all states are positive recurrent

if and only if

there is a solution π of
$$\begin{cases} \pi = \pi P \\ \sum_{j \in S} \pi_j = 1 \\ \pi_j \geq 0 \end{cases}.$$

If there is a solution, it is unique, and for all i, j in S,

$$\lim_{n \to \infty} P_{ij}^n = \pi_j = \frac{1}{\mathbb{E}_j R_j}.$$

Fact When S is finite, all states are either transient or positive recurrent. No state is null recurrent.

Example Additive random walk on $\mathbb{Z}$
Suppose $\mathbb{E}A = 0$, aperiodic.
Write the equation for the invariant distribution π. $\pi = \pi P$.

By symmetry, π_i must not depend on i.

That would lead to $\pi_i = 1$ for all i.

But $\sum_{i \in \mathbb{Z}} \pi_i = \infty$, so this "distribution" is not normalizable.

In this way we see that $\pi = \pi P$, $\sum \pi_i = 1$, $\pi_i \geq 0$
does not have a solution, so all states are null recurrent.

Starting at 0, 0 will be visited infinitely many times, but the expected time between visits is infinite.

Example Queueing system, size 30.

Case 1: $p = 0.8$, $\lambda = 0.78$ $r = \frac{0.78}{0.8} = 0.975$, close to 1.
$N = 30$.

Irreducible, aperiodic.
All states are positive recurrent.

Limiting distribution π $\lim_{n \to \infty} P^n = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$

$\pi_{30} = 0.0165\ldots$

$\mathbb{E}_{30} R_{30} = \frac{1}{\pi_{30}} = 60.6641$ minutes between visits to state 30

Case 2. $p = 0.8$, $\lambda = 0.6$, $r = 0.75$

$\pi_{30} = 0.00000051718$

$\mathbb{E}_{30} R_{30} = 1{,}933{,}600$ minutes. A very long time!

```
>> queue
Mean waiting time for an arriving customer 15.08 minutes
>> p

p =

    0.8000

>> lambda

  [illegible] =

    0.7800

>> pi

pi =

  Columns 1 through 7

    0.0346    0.0511    0.0530    0.0512    0.0491    0.0471    0.0452

  Columns 8 through 14

    0.0433    0.0415    0.0398    0.0382    0.0366    0.0351    0.0337

  Columns 15 through 21

    0.0323    0.0309    0.0297    0.0285    0.0273    0.0262    0.0251

  Columns 22 through 28

    0.0241    0.0231    0.0221    0.0212    0.0203    0.0195    0.0187

  Columns 29 through 31

    0.0179    0.0172    0.0165

>> 1./pi

ans =

  Columns 1 through 7

   28.8971   19.5669   18.8751   19.5210   20.3556   21.2305   22.1413

  Columns 8 through 14

   23.0910   24.0814   25.1143   26.1914   27.3148   28.4863   29.7081

  Columns 15 through 21

   30.9822   32.3110   33.6968   35.1420   36.6492   38.2211   39.8603

  Columns 22 through 28

   41.5699   43.3529   45.2123   47.1514   49.1738   51.2829   53.4826

  [illegible]umns 29 through 31

   [illegible].7766   58.1690   60.6641

>> queue
Mean waiting time for an arriving customer  2.62 minutes
>> pi'

ans =

    0.2500
    0.2569
    0.1793
    0.1149
    0.0728
    0.0462
    0.0293
    0.0185
    0.0118
    0.0074
    0.0047
    0.0030
    0.0019
    0.0012
    0.0008
    0.0005
    0.0003
    0.0002
    0.0001
    0.0001
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    0.0000
    0.0000
    0.0000
    0.0000
    0.0000

>> pi(31)

ans =

   5.1718e-07

>> 1/pi(31)

   1.9336e+06

>> p

p =

    0.8000
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>> lambda

lambda =

    0.6000
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Basic limit theorem

Taylor & Karlin p. 246, but weaker. IV.4
Çinlar p. 126, 163

(a) If j is transient or null recurrent, then for all i in S,

$$\lim_{n\to\infty} P^n_{ij} = 0.$$

(b) If j is positive recurrent and aperiodic, then

$$\lim_{n\to\infty} P^n_{ij} = \frac{f_{ij}}{\mathbb{E}_j R_j}$$

where $f_{ij} = \mathbb{P}_i (X_n = j \text{ for some } n)$.

(c) If j is positive recurrent with period j, then

$$\lim_{n\to\infty} P^{\delta n + m}_{ij} = \frac{\delta f_{ij}}{\mathbb{E}_j R_j},$$

where m is such that i can lead to j in $\delta n + m$ steps.

The proof is based on renewal theory, see Chapter VII.

Thought problem (omitted)

Imagine all the people in the world right now. Hold that.

Choose one to start.

Among the people who that person has ever shaken hands with, choose one at random, meaning each with equal probability.

Continue this.

The sequence of people you get forms a Markov chain — right?

Is it irreducible? Are there any absorbing states? (babies!)

Any transient states?

Where does this chain spend most of its time?

How quickly will it lose its memory of where it started?

Alter the chain.

Among the people the person has ever smiled at, choose one at random.