Markov chains

A sequence of discrete random variables $X_0, X_1, \dots$ taking values in a set S is called a Markov chain if for all $n, j, i_0 \dots i_n$,

$$\mathbb{P}(X_{n+1} = j \mid X_0 = i_0, \dots, X_n = i_n) = \mathbb{P}(X_{n+1} = j \mid X_n = i_n)$$

The distribution $\mu^{(0)}$ of X_0 is called the initial distribution of the chain.

If the probabilites $\mathbb{P}(X_{n+1} = j \mid X_n = i)$ do not depend on n, the chain is said to be time-homogeneous. This is the usual case.

We set

$$P_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i),$$

and call these numbers transition probabilitres and the matrix P the transition matrix

Note: $\sum_{j \in S} P_{ij} = \sum_{j \in S} \mathbb{P}(X_{n+1} = j \mid X_n = i) = 1$ – rows sum to 1.

As in the case of gambler's wealth, the distribution $\mu^{(n)}$ of X_n is given by

$$\mu^{(n)} = \mu^{(0)} P^n.$$

Interpretation of P^n

Consider $\mathbb{P}(X_n = j \mid X_0 = i)$.

We can compute this by making the initial distribution be

$$\mu^{(0)} = [0 \dots 1 \; 0 \dots 0]$$

i^{th} place.

Then
$$\begin{aligned}\mathbb{P}(X_n = j \mid X_0 = i) &= \mu^{(n)}_j \\ &= (\mu^{(0)} P^n)_j \\ &= ([0 \dots 1 \dots 0 \dots] P^n)_j \\ &= (\text{row } i \text{ of } P^n)_j \\ &= (P^n)_{ij}, \quad \text{the } ij \text{ entry of } P^n.\end{aligned}$$

Thus, P^n tells the transition probabilities over exactly n steps.

Look back at P^{225} for gambler's wealth.

Think about what happens after we run a M. chain for a long time.

Example $S = \{1, 2, 3\}$. p. 98

Let X be the Markov chain with state space S and

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

rows sum to 1

columns sum to 1.3, 0.7, 1.0

$$\mu_0 = [1 \quad 0 \quad 0]$$

Simulating the chain

Generate X_0 randomly according to μ_0.

Repeat this:

starting with X_n, use the probabilities in row X_n of P to generate the value of X_{n+1}.

chain.m does this

manychains.m puts 6 on each page

Notice that there is no absorbing state.

Notice that the chain spends most time in state 1, least in 2.

Question:

Does the chain spend more time in state 1 because it started in state 1?

Transition matrices

On the handout, P, P^2, P^5, P^{10}.

P^{10} is interesting: no matter what X_0 is, the distribution of X_{10} is the same!

$$P^{10} = \begin{bmatrix} \pi \\ \pi \\ \pi \end{bmatrix} \quad \text{for a row vector } \pi.$$

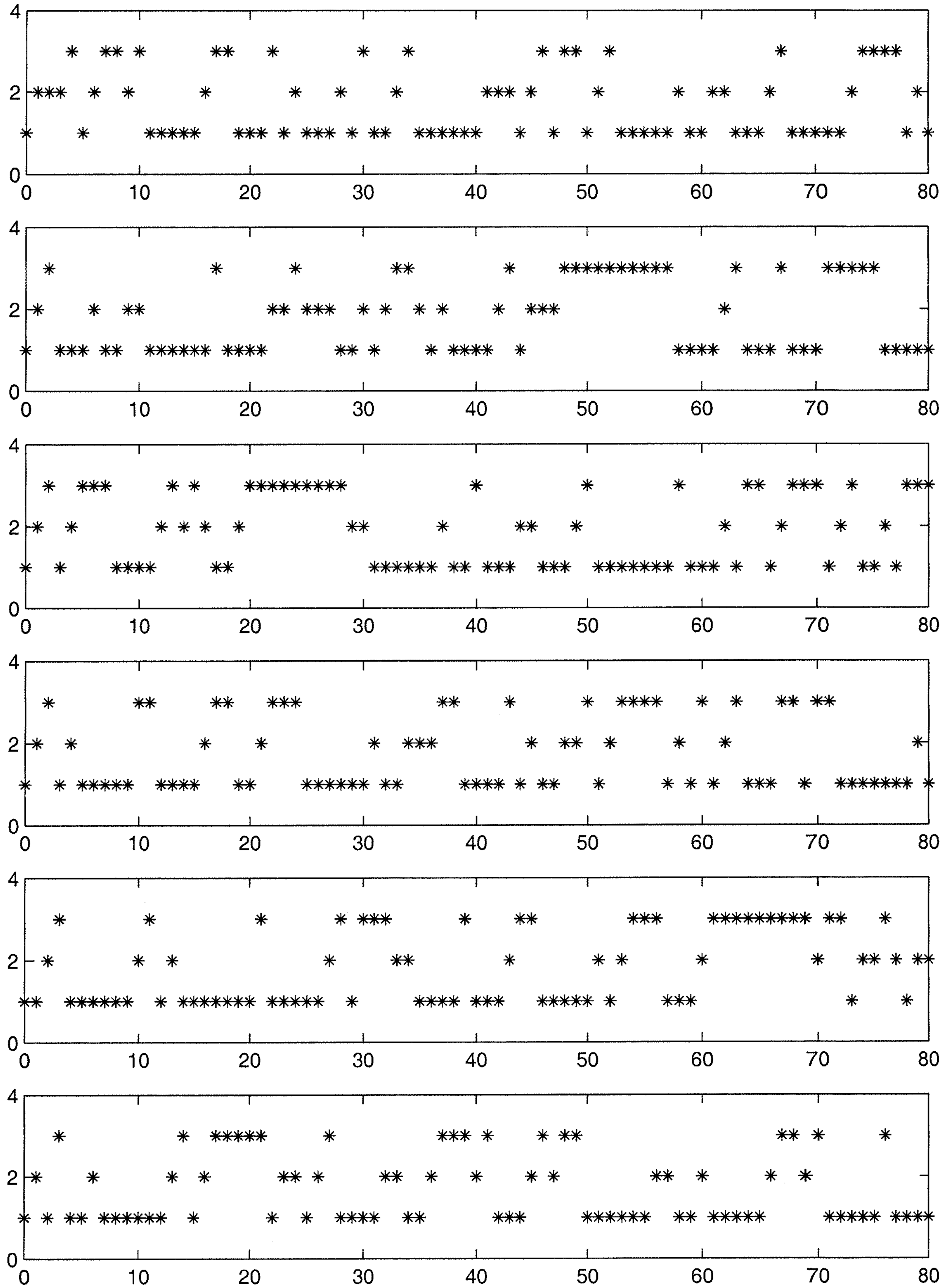
Check again: for different initial distributions μ, $\mu P^{10} = \pi$.

We can see that $P(X_{10} = j \mid X_0 = i) \approx \mathbb{P}(X_{10} = j)$,

so that X_0 and X_{10} are practically independent!

Answer: No. The chain quickly forgets that it started in state 1.

Simulations of a three-state chain using manychains.m



Example: Loss of memory in a Markov chain

```
>> P

P =
    0.6000    0.3000    0.1000
    0.3000    0.3000    0.4000
    0.4000    0.1000    0.5000

>> P^2

ans =
    0.4900    0.2800    0.2300
    0.4300    0.2200    0.3500
    0.4700    0.2000    0.3300

>> P^5

ans =
    0.4692    0.2427    0.2881
    0.4701    0.2417    0.2882
    0.4701    0.2426    0.2873

>> P^10

ans =
    0.4697    0.2424    0.2879
    0.4697    0.2424    0.2879
    0.4697    0.2424    0.2879
```

Try various initial distributions:

```
>> mu=[1 0 0]

mu =
     1     0     0

>> mu*P^10

ans =
    0.4697    0.2424    0.2879

>> mu=[0 1 0]

mu =
     0     1     0

>> mu*P^10

ans =
    0.4697    0.2424    0.2879

>> mu=[0 0 1]

mu =
     0     0     1

>> mu*P^10

ans =
    0.4697    0.2424    0.2879

>> mu=[0.2 0.3 0.5]

mu =
    0.2000    0.3000    0.5000

>> mu*P^10

ans =
    0.4697    0.2424    0.2879
```

Note that $\mu P^{10} = \mu \begin{bmatrix} \pi \\ \pi \\ \pi \end{bmatrix}$ = linear combination of the rows of P^{10} using the numbers in μ as the weights

$= \pi$.

Now $\mu P^{10+n} = (\mu P^n) P^{10} = \pi$ in the same way, so

$\mu_n = \pi$ for $n \geq 10$ (to four decimal places at least)

From $n = 10$ onwards, the distribution of X_n does not change.

Limiting distribution

Suppose $X_0 \sim \mu_0$.

Then X_n has distribution $\mu_n = \mu_0 P^n$.

If $\lim_{n\to\infty} \mu_n$ exists, call it π, then π is said to be the limiting distribution of X.

Supposing $\pi = \lim_{n\to\infty} \mu_n$ exists, we have

$$\mu_{n+1} = \mu_n P,$$

taking the limit as $n \to \infty$, this becomes

$$\pi = \pi P.$$

We say that π is an invariant distribution.

This is because if $X_0 \sim \pi$, then

$$X_1 \sim \pi P = \pi,$$

$$X_2 \sim \pi, \quad \text{etc.},$$

and we see that X_n has distribution π for all n.

The distribution doesn't change over time.

Note: The existence of an invariant distribution π does not imply that $\lim_{n\to\infty} \mu_n$ will exist and equal π.

Solving $\pi = \pi P$

You can solve this system of equations to find an invariant distribution.

Example $S = \{1, 2\}$.

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}, \qquad 0 \le a, b \le 1.$$

$$\pi = \pi P$$

$$[\pi_1 \ \ \pi_2] = [\pi_1 \ \ \pi_2] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$\begin{cases} \pi_1 = (1-a)\pi_1 + b\pi_2 \\ \pi_2 = a\pi_1 + (1-b)\pi_2 \end{cases}$$

These do not uniquely specify a solution; homogeneous linear system.

π must also satisfy: $\pi_1 + \pi_2 = 1$

$0 \le \pi_1, \pi_2 \le 1$.

Instead, solve the system:

$$\begin{cases} \pi_1 = (1-a)\pi_1 + b\pi_2 \\ \pi_1 + \pi_2 = 1 \\ 0 \le \pi_1, \pi_2 \le 1 \end{cases}$$

$$\begin{cases} a\pi_1 - b\pi_2 = 0 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

$-\frac{1}{a}R_1 + R_2$

$$\begin{cases} a\pi_1 - b\pi_2 = 0 \\ (1 + \frac{b}{a})\pi_2 = 1 \end{cases}$$

$$\pi_2 = \frac{1}{1 + \frac{b}{a}} = \frac{a}{a+b}$$

$$\pi_1 = \frac{b}{a} \cdot \pi_2 = \frac{b}{a+b}.$$

Thus, $\pi = \left[\frac{b}{a+b} \quad \frac{a}{a+b} \right]$.

Note: when $a = 1$, $b = 1$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

If $\mu_0 = [1 \quad 0]$,

then $\mu_1 = [0 \quad 1]$

$\mu_2 = [1 \quad 0]$

$\vdots$

and no limiting distribution exists.

In this case, there *is* an *invariant* distribution, $\pi = [\frac{1}{2} \quad \frac{1}{2}]$,

but not always a *limiting* distribution.

Discrete time queueing model

Situation: An ATM is located at a certain location, which is busy 24 hrs per day, with a reasonably constant arrival rate of customers. One person uses the machine, while others wait in line (queue). Once a customer is in line, he stays in line until he is served. We are interested in understanding the number of customers waiting in line and the fluctuations in this number.

Model: Discrete time.

C_n = # customers (at machine or in line) at time n.

possible values 0, 1, 2, ...

Are the C_n independent?

Are the C_n Markov?

Under what possible conditions?

A_n = # arrivals between $(n-1, n]$. iid

Given that $C_n = 0$, does the past matter? No.

$C_n = 1$, " ? Yes, if each transaction takes exactly 5 minutes, say.

Just write down a transition matrix for a Markov model.

Break into two steps:

- completion of transaction and departure.
- arrival of new customers.

First, start with some number of people in line, assume no new arrivals.

C_n, try to predict change in C_n due to completion of transactions.

$$D = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & \cdots \; N \\ \hline 0 & 1 & 0 & 0 & 0 & \\ 1 & p & q & 0 & 0 & \\ 2 & 0 & p & q & 0 & \\ 3 & & & & & \\ \vdots & & & & & \\ N & & & & & 1 \end{array}$$

Simple rule. no transaction takes less than a minute.

Choices: Each transaction takes exactly 1 minute

Transactions can take longer, following a M. model.

Transaction length has a geometric distribution, a memoryless distribution.

Arrival matrix A

The number of arrivals in each minute can be 0, 1, 2, 3, ...

There are good reasons for assuming that the numbers that arrive in different minutes are independent and have a Poisson distribution.

All that needs to be specified is the mean number of new arrivals per minute. λ.

At any rate, write $a_i = \mathbb{P}(i \text{ new arrivals in one minute})$, $i = 0, 1, 2, \ldots$

$$A = \begin{array}{c|l} & \begin{array}{ccccc} 0 & 1 & 2 & \cdots & \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \left[\begin{array}{cccccc} a_0 & a_1 & a_2 & & & \\ 0 & a_0 & a_1 & a_2 & \cdots & \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ & & & & & \end{array}\right. \end{array}$$

The distribution of the number of new arrivals does not depend on how many people are currently in line.

In principle, the number in the system could be any number.

Then A would be infinitely large, and we couldn't do numerical computations.

For Matlab,

$$A = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} & \left[\begin{array}{cccccc} a_0 & a_1 & a_2 & & & b_0 \\ 0 & a_0 & a_1 & \cdots & & b_1 \\ & & & & & \\ & & & & & \\ & & & & a_0 & b_{N-1} \\ & & & & & 1 \end{array}\right] \end{array},$$

where $b_0, b_1, \ldots, b_{N-1}$ are chosen to make each row sum to 1.

Transition matrix P

Suppose μ_0 is the initial distribution of the number in the queue.

Get to μ_1 in two steps.

First, allow the departures for the first minute to occur:

$$\mu_{\frac{1}{2}} = \mu_0 D \qquad \text{(by the same reasoning as earlier)}$$

Next, allow the arrivals to occur:

$$\begin{aligned} \mu_1 &= \mu_{\frac{1}{2}} A \\ &= (\mu_0 D) A \\ &= \mu_0 (DA) \\ &= \mu_0 P, \end{aligned}$$

so we see that $P = DA$.

Illustration

$p = 0.8$

$\lambda = 0.78$

$N = 30$

queue.m

manychains.m

Analysis

How do you feel about this situation?

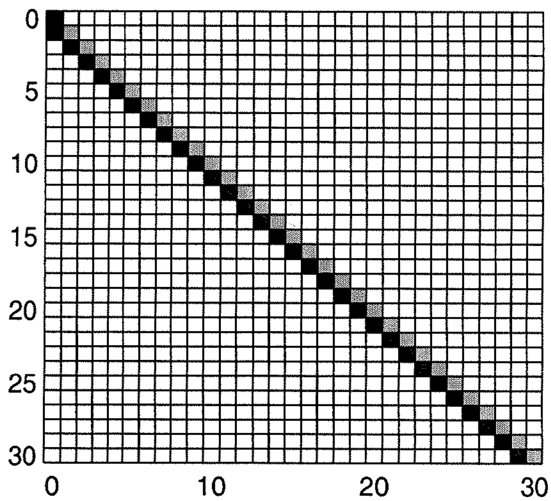
If you ran the bank that operated the ATM, would you be happy?

What is the distn of the number in the queue after it has been operating a long time?

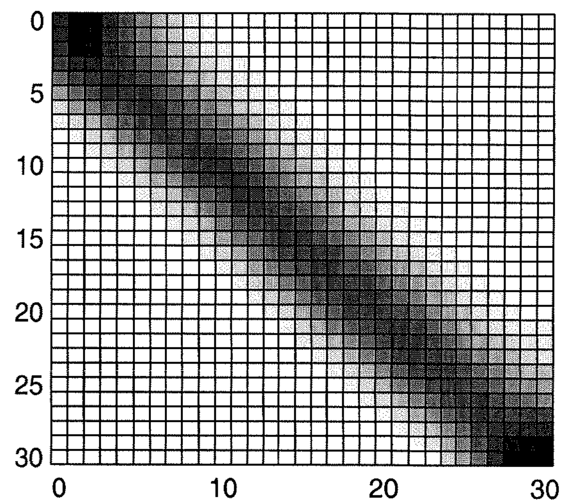
0.78 customers arrive each minute, 0.8 can be served each minute.

A heavily-loaded queue.

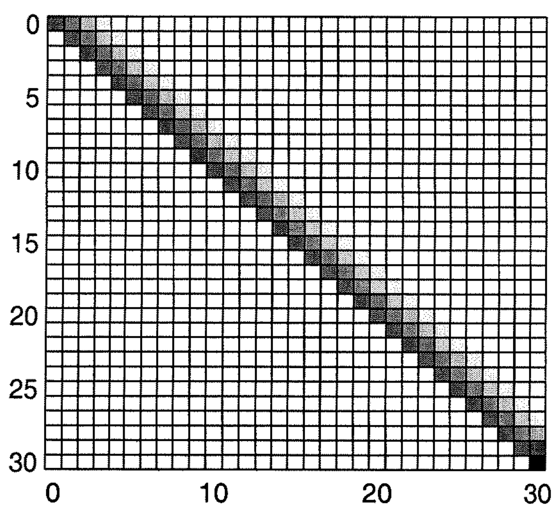
Departure matrix D



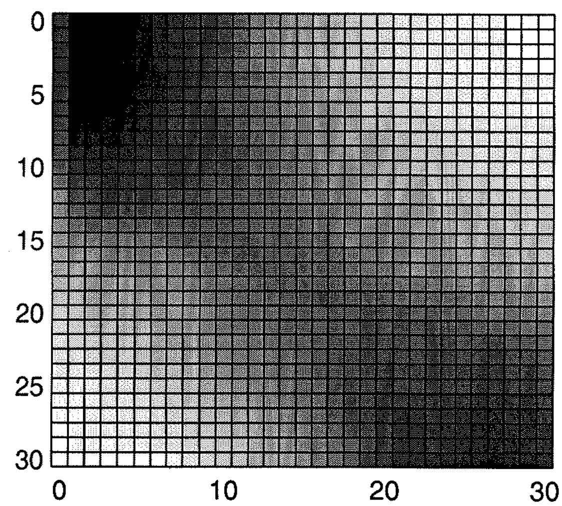
P^{10}



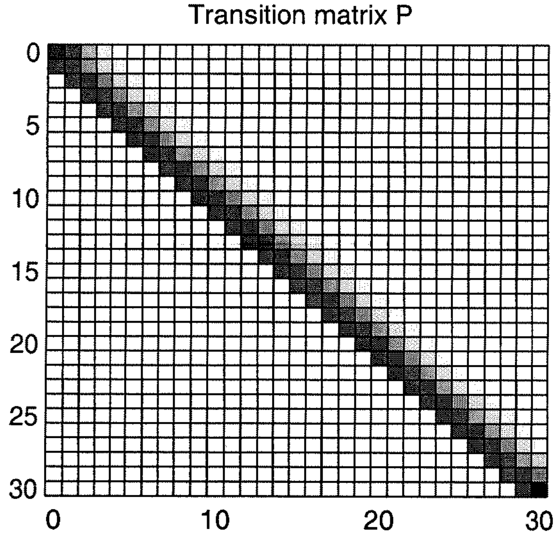
Arrival matrix A



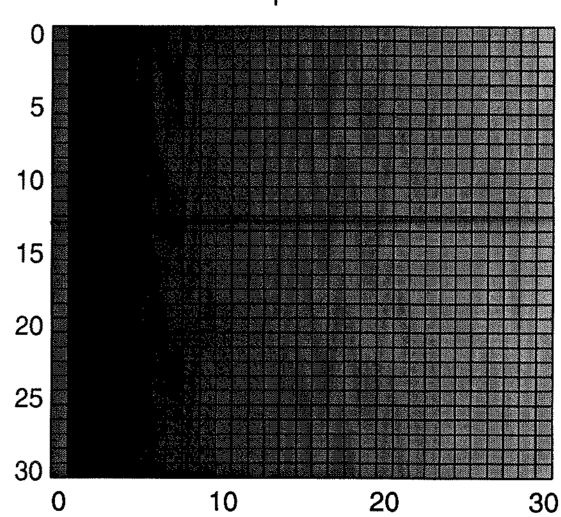
P^{100}



Transition matrix P

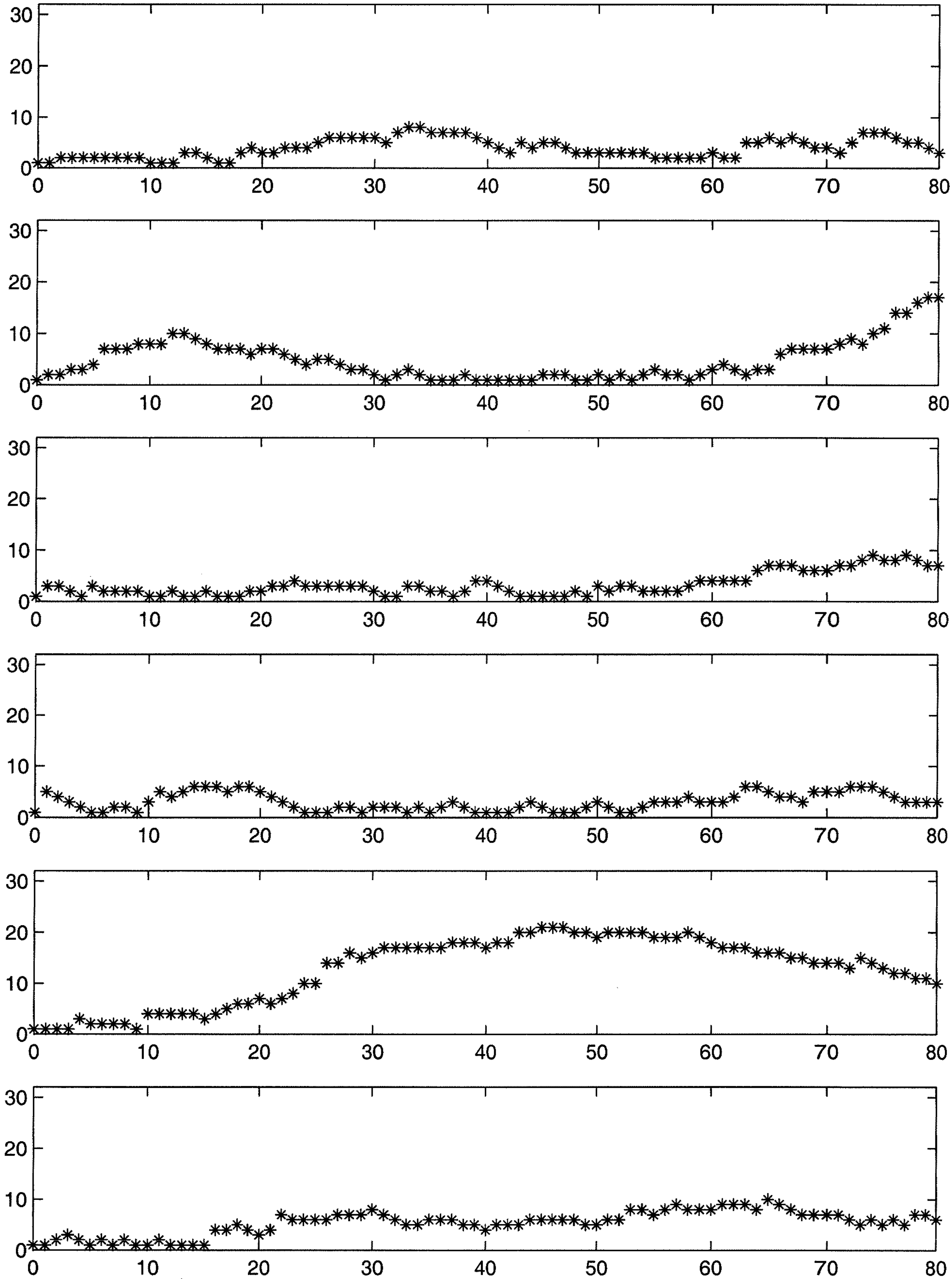


P^{1000}



queue.m

Simulation of queue with p=0.8 and λ=0.78



$$P = DA$$

$$= \begin{array}{c} 0 \\ 1 \\ \\ \\ \\ \end{array}\begin{bmatrix} 1 & 0 & 0 & & & \\ p & q & 0 & \cdots & & \\ 0 & p & q & 0 & \cdots & \\ & & & & & \\ & & & & p & q \end{bmatrix} \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_N \\ 0 & a_0 & a_1 & a_2 \cdots & b_1 \\ & 0 & a_0 & & b_2 \\ & & 0 & & \vdots \\ & & & & b_{N-1} \\ & & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & \cdots & a_N \\ a_0 p & a_1 p + a_0 q & a_2 p + a_1 q & & & & a_N p + b_1 q \\ 0 & a_0 p & a_1 p + a_0 q & & & & b_1 p + b_2 q \\ \vdots & 0 & & & & & \vdots \\ \vdots & \vdots & & & & & \\ & & & & & & b_{N-1} p + q \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_N \\ c_0 & c_1 & c_2 & \cdots & \\ 0 & c_0 & c_1 & & \\ 0 & 0 & c_0 & & \\ & & & \ddots & \end{bmatrix}$$

$$\pi = \pi P$$

$$= [\pi_0 \quad \pi_1 \quad \cdots \quad \pi_N] \; P$$

$$= \left[(\pi_0 a_0 + \pi_1 c_0) \quad (\pi_0 a_1 + \pi_1 c_1 + \pi_2 c_0) \quad (\pi_0 a_2 + \pi_1 c_2 + \pi_2 c_1 + \pi_3 c_0) \quad \cdots \right]$$

Equations

$$\pi_0 = \pi_0 a_0 + \pi_1 c_0$$
$$\pi_1 = \pi_0 a_1 + \pi_1 c_1 + \pi_2 c_0$$
$$\pi_2 = \pi_0 a_2 + \pi_1 c_2 + \pi_2 c_1 + \pi_3 c_0$$
$$\vdots$$
$$\pi_j = \pi_0 a_j + \pi_1 c_j + \pi_2 c_{j-1} \qquad + \pi_{j+1} c_0$$

Add these $j+1$ equations, solve for $\pi_{j+1} c_0$.

$$\pi_1 c_0 = \pi_0 (1 - a_0)$$
$$\pi_2 c_0 = \pi_0 (1 - a_0 - a_1) + \pi_1 (1 - c_0 - c_1)$$
$$\pi_3 c_0 = \pi_0 (1 - a_0 - a_1 - a_2) + \pi_1 (1 - c_0 - c_1 - c_2) + \pi_2 (1 - c_0 - c_1)$$
$$\vdots$$
$$\pi_{j+1} c_0 = \pi_0 (1 - a_0 - \cdots - a_j) + \pi_1 (1 - c_0 - \cdots - c_j) + \pi_2 (1 - c_0 - \cdots - c_{j-1}) + \cdots + \pi_j (1 - c_0 - c_1)$$

These just rattle on until $\pi_N c_0$, which differs from the rest.

Queueing – invariant distn

$$P = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \\ c_0 & c_1 & c_2 & \cdots & \\ 0 & c_0 & c_1 & c_2 & \cdots \\ \vdots & & & & \\ & & & & \end{bmatrix}$$

$$c_0 = a_0 p$$
$$c_j = a_j p + a_{j-1} q$$

$\pi = \pi P$ becomes, for π_j,

$$\pi_j = \pi_0 a_j + \pi_1 c_j + \pi_2 c_{j-1} + \cdots + \pi_{j+1} c_0.$$

Adding the equations for π_0 to π_j and solving,

$$c_0 \pi_{j+1} = \pi_0 (1 - a_0 - a_1 - \cdots - a_j) + \pi_1 (1 - c_0 - \cdots - c_j)$$
$$+ \pi_2 (1 - c_0 - \cdots - c_{j-1}) + \cdots + \pi_j (1 - c_0 - c_1).$$

$$a_0 p \pi_{j+1} = \pi_0 (1 - a_0 - a_1 - \cdots - a_j) + \pi_1 (1 - a_0 p - a_1 p - a_0 q - a_2 p - a_1 q - \cdots - c_j)$$
$$+ \cdots$$
$$= \pi_0 (1 - a_0 - a_1 - \cdots - a_j) + \pi_1 (1 - a_0 - a_1 - \cdots - a_{j-1} - p a_j)$$
$$+ \cdots + \pi_j (1 - a_0 - p a_1)$$
$$= \beta_0 \pi_j + \beta_1 \pi_{j-1} + \cdots + \beta_{j-1} \pi_1 + \delta_j \pi_0$$

When j is large, the coefficient $(1 - a_0 - a_1 - \cdots - a_j)$ of π_0 will be small, maybe even exactly 0, if there is, say, 0 possibility of more than 10 arrivals in any one minute. We may then essentially write

$$a_0 p \pi_{j+1} = \beta_0 \pi_j + \beta_1 \pi_{j-1} + \cdots + \beta_m \pi_{j-m},$$

so that π_{j+1} depends only on the previous $m+1$ values of π.
In fact, let us assume that this is strictly true, so that $A \le m+1$.

$\beta_m = 1 - a_0 - a_1 - \cdots - a_m - p a_{m+1}$.
$\beta_{m+1} = 0$ means $a_0 + a_1 + \cdots + a_{m+1} = 1$, so $A \le m+1$.

In such a situation, it is common for the solution to be of the form

(* don't use γ; looks too much like r)

$$\pi_j = c\,\gamma^j$$

for some constant γ. This is geometric growth ($\gamma > 1$) or decay ($\gamma < 1$).

Let us substitute this in and see what we find.

$$a_0 p \gamma^{j+1} = \beta_0 \gamma^j + \cdots + \beta_m \gamma^{j-m},$$

where we have already factored out the constant c.

Multiply through by γ^{m-j} to make clear that the equation does not depend on j:

$$a_0 p \gamma^{m+1} = \beta_0 \gamma^m + \cdots + \beta_m \gamma^0.$$

Rather than try to solve for γ in terms of p and $a_0, a_1, a_2, \ldots$ (better suited to a numerical procedure), ask what must happen to obtain $\gamma = 1$, the boundary case between growth and decay.

When $\gamma = 1$,

$$\begin{aligned} a_0 p &= \beta_0 + \beta_1 + \cdots + \beta_m \\ &= \sum_{i=0}^{m} \beta_i \\ &= \sum_{i=0}^{m} \left(1 - \sum_{j=0}^{i} a_j - p a_{i+1}\right) \\ &= \sum_{i=0}^{m} \left(\mathbb{P}(A > i) - p a_{i+1}\right) \qquad A = \#\text{ arrivals} \\ &= \mathbb{E}A - p \sum_{i=0}^{m} a_{i+1} \\ &= \mathbb{E}A - p(1 - a_0). \end{aligned}$$

Subtract $a_0 p$ from both sides:

$$0 = \mathbb{E}A - p$$

$$p = \mathbb{E}A = \lambda.$$

Let $r = \frac{\lambda}{\rho} = \frac{\text{expected number of arrivals per minute}}{\text{expected number of departures per minute}}$ traffic intensity

When $r = 1$, so that the values of π_j neither increase nor decrease with j, the traffic intensity will equal 1.

The converse is more interesting, and can also be shown to be true (I'm sure):

When $r < 1$, π_j decreases to 0 geometrically fast and there is an invariant distribution for the number in the system, which is concentrated near low numbers of customers.

If the system size N is finite, the same is true.

When $r = 1$, π_j is constant for large j.

The "invariant distribution" π has one problem: it cannot be normalized.

In this case there is no invariant distribution.

In fact, the number in the system will become large occasionally, but will occasionally return to 0 as well.

If the system size N is finite, the invariant distribution π will be spread out roughly uniformly between $0, 1, 2, \ldots, N$.

When $r > 1$, π_j increases geometrically with j.

The "invariant" distribution" is not normalizable; there is no inv. dist'n.

The number of people in the system will eventually exceed 100, 1000, 1 million, etc. and stay larger (converging to ∞).

When N is finite, π will be normalizable, but will concentrate on the states very near N.

This analysis is correct whether or not A is technically bounded.

445/545 9-26-2000

We've studied stability, now let's look at performance characteristics

Little's Law

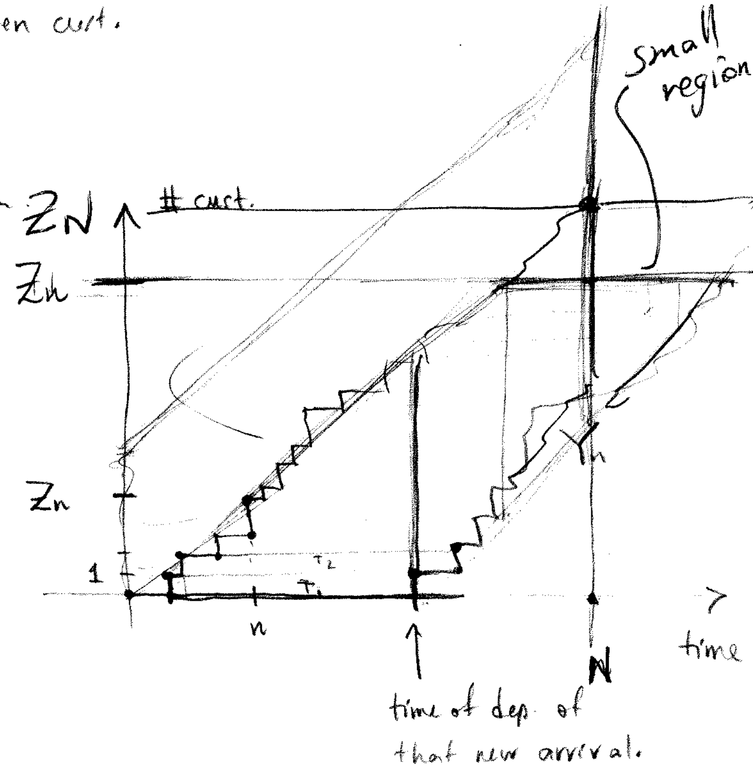
$L = \mathbb{E}\, C$ = exp. length of queue
= exp. # in syst.

λ = arrival rate = $\mathbb{E}$ # per minute

W = $\mathbb{E}$ time in system for a given cust.

Let π be the inv. dist'n for C_n

Start the system empty.



Z_n = cumulative # new arrivals to time n.

Y_n = cumul. # departures

After all old cust. are out, $Z_n - Y_n = C_n$ = # cust. in system.

$Z_n - Y_n$ has dist'n π, or conv. to it anyway.

Go out to time n, very large.

If you add up $T_1, T_2, T_3, \dots,$

(2) $\text{Area} = T_1 + T_2 + T_3 + \dots + T_{Z_n}$ — area of small region.

You also get the area of the region, which is also:

(1) $$\text{Area} = \sum_{n=0}^{N} C_n .$$

$$\frac{\text{Area} = \sum_{n=1}^{N} C_n}{N} = \frac{T_1 + T_2 + \dots T_{Z_N}}{Z_N} \quad \frac{Z_N}{N}$$

ergodic theory

$\downarrow$ $\mathbb{E}C = \mathbb{E}T \cdot \lambda.$ $L = \lambda W.$

1 Mean waiting time

When a customer arrives, the waiting time until the cust. leaves the system is :

One server

If C is the # cust. One new cust. arrives.

$$W = \mathbb{E}\,\text{time insystem} = \mathbb{E} \sum_{i=1}^{C+1} T_i ,$$

wait

the trans. times T_i being iid.

needs some steps

$$= (\mathbb{E}\, C+1)(\mathbb{E}\, T)$$

$$= (1 + \mathbb{E}C)(\mathbb{E}\,T)$$

$$= \frac{1}{p} \cdot (1 + \pi_0 \cdot 0 + \pi_1 \cdot 1 + \cdots)$$

$$= \frac{1}{p}(1 + \mathbb{E}C)$$

Matlab

```
π = invariant (P).
W = (1 + (0:N)*π')/p;
```

— see queue.m

Two servers

$$\sum \pi_i \; \mathbb{E}\,\text{time to serve } i$$

Need to calc. exp. time to finish serving i customers.

OK, but only useful if you can write down a formula.

o/w, use Little, ~~W~~

$$L = \lambda W,$$
$$W = \frac{L}{\lambda}.$$

$u = 1 + Du$ as before

$u_0 = 0$

$$\begin{bmatrix} 1 & 0 & 0 & - \\ p & q & 0 & - \\ p^2 & 2pq & q^2 \end{bmatrix}$$

~~u_0~~ $= u_0$

$u_1 = 1 + pu_0 + qu_1$

$u_2 = 1 + p^2u_0 + 2pq\,u_1 + q^2u_2$

$u_3 = 1 + p^2u_1 + 2pq\,u_2 + q^2u_3$

$$u_1 = \frac{1}{1-q} = \frac{1}{p}$$

$$u_2 = \frac{2q+1}{1-q^2}$$

$$u_3 = \frac{p + \frac{4pq^2}{1-q^2} + 1}{1-q^2}$$

$u_4 = \cdots$

$$u_1 = \frac{1}{p}$$

$$u_2 = \frac{1 + 2pq\,u_1}{1-q^2}$$

$$u_3 = \frac{1 + p^2u_1 + 2pq\,u_2}{1-q^2}$$

$\vdots$

9-26-2000

Mean waiting time

C customers in the system, in equil, C has some dist'n π.

Consider ~~last~~ next cust, just arriving.

How long until that cust. is done?

$$W = \frac{1}{p}(1 + \mathbb{E}C)$$

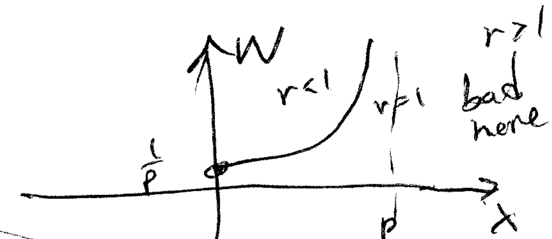
$$= \frac{1}{p}(1 + L)$$

$L = \lambda W.$

$L = \mathbb{E}C$

$$pW = 1 + \lambda W$$

$$\frac{1}{p - \lambda} = W$$



$\boxed{\text{reasonable, but not for finite length queues.}}$

$$L = \lambda W$$

$$= \frac{\lambda}{p - \lambda} \qquad (?)$$

$$= \frac{1}{\frac{1}{r} - 1}$$

$$= \frac{r}{1 - r}$$

$$r = \frac{\lambda}{p}$$

Last time:

Queue in equil. One new customer, so now $C+1$.

$$\mathbb{E} \text{ time spent in system} = \mathbb{E}\left[\sum_{i=1}^{C+1} S_i\right]$$

$$= \mathbb{E}(C+1) \quad \mathbb{E}(S_1)$$

$$= \frac{1}{p}\cdot\left(\mathbb{E}[C]+1\right)$$

Uses a result which is HW problem 1.

This is easy because the new customer waits exactly $\sum_{i=1}^{C+1} S_i$ before departing.

Another way:

Suppose there are n people in the system.
How long until all of them are served, on average?

Ignore new arrivals; just use the departure matrix D.

Let T = time 0 is first reached.

$$u_k = \mathbb{E}\left[T \mid C_0 = k\right]. \quad \text{(using } D \text{ only)}$$

We did a case like this before w/ gambler's wealth. First-step

$$\begin{cases} u_0 = 0 \\ \vec{u} = 1 + D\vec{u} & \text{otherwise} \end{cases}$$

Write these out:

$$\begin{bmatrix} 1 & 0 & \cdots & & \\ p & q & 0 & \cdots & \\ 0 & p & q & 0 & \cdots \end{bmatrix}\begin{bmatrix} 0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix}$$

$$u_1 = 1 + q u_1$$

$$u_2 = 1 + p u_1 + q u_2$$

$$\vdots$$

$$u_1 = \frac{1}{1-q} = \frac{1}{p}$$

$$u_2 =$$

$$u_{n+1} = \frac{1+p u_n}{1-q} = \frac{1}{p} + u_n$$

$$u_n = \frac{n}{p}$$

$$\mathbb{E} \text{ time spent} =$$

$$= \sum_{i=0}^{\infty} \mathbb{P}(C = i) \; \mathbb{E}[T \mid C_0 = i+1]$$

$$= \sum \pi_i \frac{i+1}{p}$$

$$= \frac{1}{p}\left(\sum \pi_i (i+1)\right)$$

$$= \frac{1}{p}\left(\sum i\pi_i + 1\right)$$

$$= \frac{1}{p}(1 + \mathbb{E}C), \qquad \text{as before.}$$

Two servers.

Departure matrix

$$D2 = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & & & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \\ N \end{array} & \begin{bmatrix} 1 & 0 & 0 & \cdots & & \\ p & q & 0 & \cdots & & \\ p^2 & 2pq & q^2 & 0 & \cdots & \\ 0 & p^2 & 2pq & q^2 & 0 & \cdots \\ & & & & \ddots & \\ & & & & & q^2 \end{bmatrix} \end{array}$$

$p($

$$pq + pq = 2pq$$

$$1 - p^2 - q^2$$

$$p^2 + q^2 + 2pq = (p+q)^2 = 1.$$

Arrival matrix is the same.

Mean waiting time

Harder now.

Idea: Starting $\Big($ w/ i cust. in syst, ~~how~~ what is exp. time until served? Until $C_n = 0$?

Make 0 absorbing, do the comp. as before.

$\vdots$

Vector u. of exp. waiting times.

π = inv (P2);

$$W = \pi_0 \cdot u_1 + \pi_1 \cdot u_2 + \cdots$$

$= \mathbb{E}$ time for new cust. to be finished

$= \mathbb{E} \quad u(C+1)$

$= \sum_{i}^{N} \pi_i \, u(i+1).$