A simple model for a gambler's wealth

Situation

A gambler has \$10.
He sits down to play a card game with someone who has \$20
They make \$1 bets on each hand, so that, for example, after the first hand he will have either \$9 or \$11.
The cards are shuffled after each hand.

★ **Questions**

What would you like to know? Probabilistic questions.
Model the amount of money you have – your wealth.

Model

Names
n is the hand number
W_n is your wealth after hand n

$W_0 = 10$
$W_1 = 9$ or 11
⋮

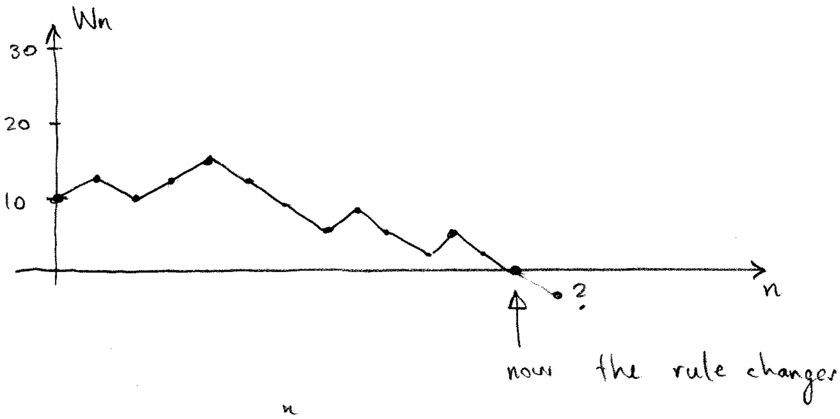
$\{W_0, W_1, \dots\}$ is a stochastic process.
The W_n are discrete random variables.

They are not independent:
W_{n+1} can equal only $W_n - 1$ or $W_n + 1$.

Omit

Let $X_n = \begin{cases} 1 & \text{if you win hand } n \\ -1 & \text{if you lose} \dots \end{cases}$

These are iid.
Then
$$W_1 = W_0 + X_1$$
$$W_2 = W_1 + X_2$$
$$= W_0 + X_1 + X_2$$
⋮
$$W_n = W_0 + X_1 + X_2 + \cdots + X_n$$
$$= W_0 + \sum_{i=1}^{n} X_i$$



$W_n = W_0 + \sum_{i=1}^{n} X_i$ unless it hits 0 or 30 along the way.

Try to describe the dependence between the W_n, how W evolves over time.

Suppose $W_n = 8$. Then W_{n+1} can be 7 or 9, probs. q and p.

$W_n = 0$. Then $W_{n+1} = 0$.

$W_n = 1$ 2 or 0

Does it matter what W_{n-1} is?

$$\mathbb{P}(W_{n+1} = j \mid W_n = 8) = \begin{cases} p & \text{if } j = 9 \\ q & \text{if } j = 7 \\ 0 & \text{otherwise} \end{cases}$$

Gambler's wealth

$W_0, W_1, W_2, W_3, \ldots$

Not independent, but dependence structure is simple:

$$\mathbb{P}(W_{n+1} = j \mid W_0 = i_0, W_1 = i_1, \ldots, W_n = i) = \begin{cases} p & \text{if } j = i+1 \text{ and } 0 < i < 30 \\ q & \text{if } j = i-1 \text{ and } 0 < i < 30 \\ 1 & \text{if } j = 30 \text{ and } i = 30 \\ 1 & \text{if } j = 0 \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \mathbb{P}(W_{n+1} = j \mid W_n = i)$$

The values of $W_0, W_1, \ldots, W_{n-1}$ do not matter once we know W_n.

We say that W_{n+1} is conditionally independent of $W_0 \ldots W_{n-1}$ given W_n.

We also say that the process $W_n, n \geq 0$ has the Markov property

omitted

The great advantage of this property is that it becomes easy to compute joint probabilities such as:

$$\mathbb{P}(W_0 = i_0, W_1 = i_1, \ldots, W_5 = i_5)$$

$$= \mathbb{P}(W_5 = i_5 \mid W_0 = i_0 \ldots W_4 = i_4) \cdot \mathbb{P}(W_0 = i_0, \ldots, W_4 = i_4)$$

$$= \mathbb{P}(W_5 = i_5 \mid W_4 = i_4) \cdot \mathbb{P}(W_4 = i_4 \mid W_3 = i_3) \cdot \ldots \cdot \mathbb{P}(W_1 = i_1 \mid W_0 = i_0) \cdot \mathbb{P}(W_0 = i_0)$$

The computation comes down to two things:

$\mathbb{P}(W_0 = i_0)$, a number from the distribution of W_0.

$\mathbb{P}(W_{n+1} = j \mid W_n = i)$ the probability of a transition from i to j.

Usually this does not depend on n:

$$\mathbb{P}(W_{n+1} = j \mid W_n = i) = \mathbb{P}(W_1 = j \mid W_0 = i) \text{ for all } n.$$

For simplicity, let $P_{ij} = \mathbb{P}(W_1 = j \mid W_0 = i)$

Then the computation above becomes

$$\mathbb{P}(W_0 = i_0, \ldots, W_5 = i_5) = \mathbb{P}(W_0 = i_0) \cdot P_{i_0 i_1} P_{i_1 i_2} \cdot P_{i_2 i_3} \cdot P_{i_3 i_4} \cdot P_{i_4 i_5} \cdot$$

Gambler's wealth – new formulation

Wealth $W_n, n \geq 0$ is a Markov chain:

Let W_0 have distribution $\mathbb{P}(W_0 = i) = \begin{cases} 1 & \text{if } i = 10 \\ 0 & \text{otherwise} \end{cases}$

Let P_{ij} be the entries in the matrix — this is easier than what we wrote earlier.

final state, initial state:

$$\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & & \cdots & 28 & 29 & 30 \\ \hline 0 & 1 & 0 & 0 & 0 & & \cdots & & 0 & 0 \\ 1 & q & 0 & p & 0 & & \cdots & & 0 & 0 \\ 2 & 0 & q & 0 & p & 0 & \cdots & & 0 & 0 \\ 3 & 0 & 0 & q & 0 & p & 0 \cdots & & 0 & 0 \\ \vdots & \vdots & & & & & & & & \\ 29 & 0 & 0 & \cdots & & & & q & 0 & p \\ 30 & 0 & 0 & \cdots & & & & 0 & 0 & 1 \end{array}$$

This organizes the information in a convenient way.

It will take a little while to answer all the questions we have raised, but some progress comes quickly.

Distribution of W_n

The distribution of W_n is comprised of the numbers $\mathbb{P}(W_n=0), \mathbb{P}(W_n=1), \ldots, \mathbb{P}(W_n=30)$.

Organize these into a row vector μ_n, where

$$\mu_n(i) = \mathbb{P}(W_n = i).$$

Now compute the distribution of W_{n+1}:

$$\begin{aligned} \mathbb{P}(W_{n+1} = j) &= \mathbb{P}(W_{n+1} = j \text{ and } (W_n = 0 \text{ or } W_n = 1 \text{ or } \ldots \text{ or } W_n = 30)) \\ &= \sum_{i=0}^{30} \mathbb{P}(W_{n+1} = j \text{ and } W_n = i) \\ &= \sum_{i=0}^{30} \mathbb{P}(W_{n+1} = j \mid W_n = i) \cdot \mathbb{P}(W_n = i) \\ &= \sum_{i=0}^{30} P_{ij} \cdot \mathbb{P}(W_n = i) \end{aligned}$$

Rewrite:

$$\mu_j^{(n+1)} = \sum_{i=0}^{30} \mu_i^{(n)} P_{ij}$$

This is vector-matrix multiplication:

$$\mu^{(n+1)} = \mu^{(n)} P$$

Now work by induction

$$\mu^{(1)} = \mu^{(0)} P, \quad \mu^{(2)} = \mu^{(1)} P = (\mu^{(0)} P) P = \mu^{(0)} P^2, \quad \mu^{(3)} = \mu^{(0)} P^3, \quad \ldots$$

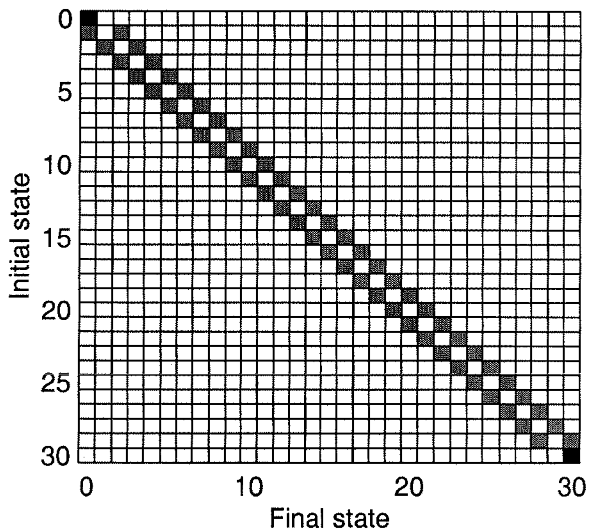
Gambler begins wit[illegible]0, oppenent with $20. Probability of gambler winning each hand is p=0.5.
Each row is the distribution of W_n, the gambler's wealth.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.250	0.000	0.500	0.000	0.250	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.125	0.000	0.375	0.000	0.375	0.000	0.125	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000	0.062	0.000	0.250	0.000	0.375	0.000	0.250	0.000	0.062	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.031	0.000	0.156	0.000	0.312	0.000	0.312	0.000	0.156	0.000	0.031	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.016	0.000	0.094	0.000	0.234	0.000	0.312	0.000	0.234	0.000	0.094	0.000	0.016	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.008	0.000	0.055	0.000	0.164	0.000	0.273	0.000	0.273	0.000	0.164	0.000	0.055	0.000	0.008	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.000	0.000	0.004	0.000	0.031	0.000	0.109	0.000	0.219	0.000	0.273	0.000	0.219	0.000	0.109	0.000	0.031	0.000	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.002	0.000	0.018	0.000	0.070	0.000	0.164	0.000	0.246	0.000	0.246	0.000	0.164	0.000	0.070	0.000	0.018	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.001	0.000	0.010	0.000	0.044	0.000	0.117	0.000	0.205	0.000	0.246	0.000	0.205	0.000	0.117	0.000	0.044	0.000	0.010	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
11	0.001	0.005	0.000	0.027	0.000	0.081	0.000	0.161	0.000	0.226	0.000	0.226	0.000	0.161	0.000	0.081	0.000	0.027	0.000	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.003	0.000	0.016	0.000	0.054	0.000	0.121	0.000	0.193	0.000	0.226	0.000	0.193	0.000	0.121	0.000	0.054	0.000	0.016	0.000	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.003	0.008	0.000	0.035	0.000	0.087	0.000	0.157	0.000	0.209	0.000	0.209	0.000	0.157	0.000	0.087	0.000	0.035	0.000	0.010	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	0.007	0.000	0.021	0.000	0.061	0.000	0.122	0.000	0.183	0.000	0.209	0.000	0.183	0.000	0.122	0.000	0.061	0.000	0.022	0.000	0.006	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.007	0.011	0.000	0.041	0.000	0.092	0.000	0.153	0.000	0.196	0.000	0.196	0.000	0.153	0.000	0.092	0.000	0.042	0.000	0.014	0.000	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
16	0.013	0.000	0.026	0.000	0.066	0.000	0.122	0.000	0.175	0.000	0.196	0.000	0.175	0.000	0.122	0.000	0.067	0.000	0.028	0.000	0.009	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
17	0.013	0.013	0.000	0.046	0.000	0.094	0.000	0.148	0.000	0.185	0.000	0.185	0.000	0.148	0.000	0.094	0.000	0.047	0.000	0.018	0.000	0.005	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
18	0.019	0.000	0.030	0.000	0.070	0.000	0.121	0.000	0.167	0.000	0.185	0.000	0.167	0.000	0.121	0.000	0.071	0.000	0.033	0.000	0.012	0.000	0.003	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000
19	0.019	0.015	0.000	0.050	0.000	0.096	0.000	0.144	0.000	0.176	0.000	0.176	0.000	0.144	0.000	0.096	0.000	0.052	0.000	0.022	0.000	0.007	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.027	0.000	0.032	0.000	0.073	0.000	0.120	0.000	0.160	0.000	0.176	0.000	0.160	0.000	0.120	0.000	0.074	0.000	0.037	0.000	0.015	0.000	0.005	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000
21	0.027	0.016	0.000	0.053	0.000	0.096	0.000	0.140	0.000	0.168	0.000	0.168	0.000	0.140	0.000	0.097	0.000	0.055	0.000	0.026	0.000	0.010	0.000	0.003	0.000	0.001	0.000	0.000	0.000	0.000	0.000
22	0.035	0.000	0.034	0.000	0.074	0.000	0.118	0.000	0.154	0.000	0.168	0.000	0.154	0.000	0.119	0.000	0.076	0.000	0.041	0.000	0.018	0.000	0.006	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000
23	0.035	0.017	0.000	0.054	0.000	0.096	0.000	0.136	0.000	0.161	0.000	0.161	0.000	0.136	0.000	0.097	0.000	0.058	0.000	0.029	0.000	0.012	0.000	0.004	0.000	0.001	0.000	0.000	0.000	0.000	0.000
24	0.043	0.000	0.036	0.000	0.075	0.000	0.116	0.000	0.149	0.000	0.161	0.000	0.149	0.000	0.117	0.000	0.078	0.000	0.044	0.000	0.021	0.000	0.008	0.000	0.003	0.000	0.001	0.000	0.000	0.000	0.000
25	0.043	0.018	0.000	0.056	0.000	0.096	0.000	0.132	0.000	0.155	0.000	0.155	0.000	0.133	0.000	0.097	0.000	0.061	0.000	0.032	0.000	0.014	0.000	0.005	0.000	0.002	0.000	0.000	0.000	0.000	0.000
26	0.052	0.000	0.037	0.000	0.076	0.000	0.114	0.000	0.144	0.000	0.155	0.000	0.144	0.000	0.115	0.000	0.079	0.000	0.047	0.000	0.023	0.000	0.010	0.000	0.003	0.000	0.001	0.000	0.000	0.000	0.000
27	0.052	0.018	0.000	0.056	0.000	0.095	0.000	0.129	0.000	0.149	0.000	0.149	0.000	0.130	0.000	0.097	0.000	0.063	0.000	0.035	0.000	0.017	0.000	0.007	0.000	0.002	0.000	0.001	0.000	0.000	0.000
28	0.061	0.000	0.037	0.000	0.076	0.000	0.112	0.000	0.139	0.000	0.149	0.000	0.139	0.000	0.113	0.000	0.080	0.000	0.049	0.000	0.026	0.000	0.012	0.000	0.004	0.000	0.001	0.000	0.000	0.000	0.000
29	0.061	0.019	0.000	0.056	0.000	0.094	0.000	0.126	0.000	0.144	0.000	0.144	0.000	0.126	0.000	0.097	0.000	0.064	0.000	0.037	0.000	0.019	0.000	0.008	0.000	0.003	0.000	0.001	0.000	0.000	0.000
50	0.161	0.000	0.033	0.000	0.063	0.000	0.087	0.000	0.104	0.000	0.110	0.000	0.107	0.000	0.096	0.000	0.079	0.000	0.060	0.000	0.042	0.000	0.027	0.000	0.016	0.000	0.008	0.000	0.004	0.000	0.005
100	0.320	0.000	0.019	0.000	0.037	0.000	0.051	0.000	0.062	0.000	0.069	0.000	0.071	0.000	0.069	0.000	0.064	0.000	0.056	0.000	0.047	0.000	0.037	0.000	0.027	0.000	0.018	0.000	0.009	0.000	0.046
500	0.631	0.000	0.002	0.000	0.003	0.000	0.004	0.000	0.006	0.000	0.006	0.000	0.007	0.000	0.007	0.000	0.007	0.000	0.007	0.000	0.006	0.000	0.006	0.000	0.004	0.000	0.003	0.000	0.002	0.000	0.298
1000	0.664	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.331
2000	0.667	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.333

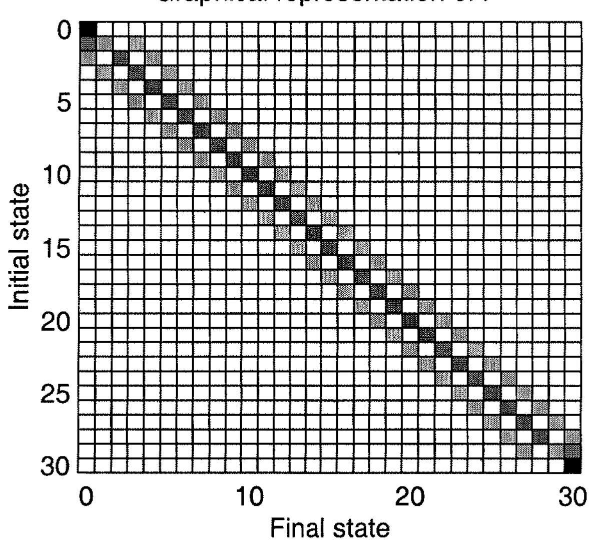
The transition matrix P

Initial state \ Final state	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
17	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
18	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
19	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
21	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
22	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000
23	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000	0.000
24	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000	0.000
25	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000	0.000
26	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000	0.000
27	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000	0.000
28	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500	0.000
29	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.500	0.000	0.500
30	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000

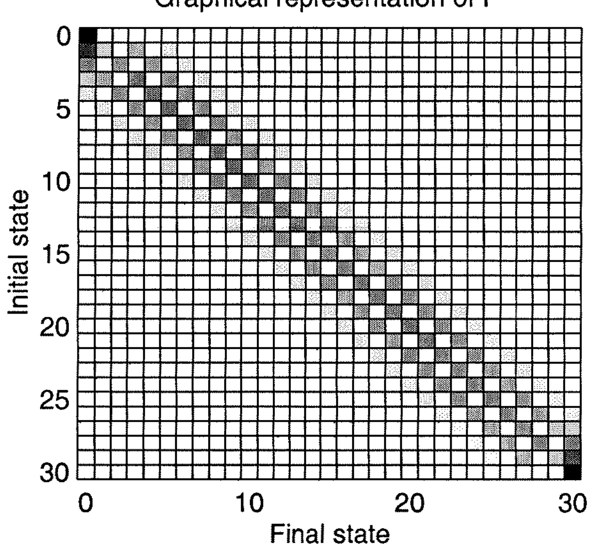
Graphical representation of transition matrix P



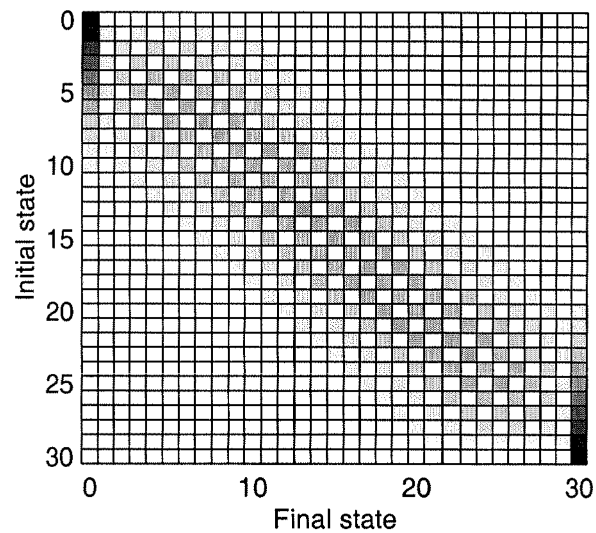
Graphical representation of P^2



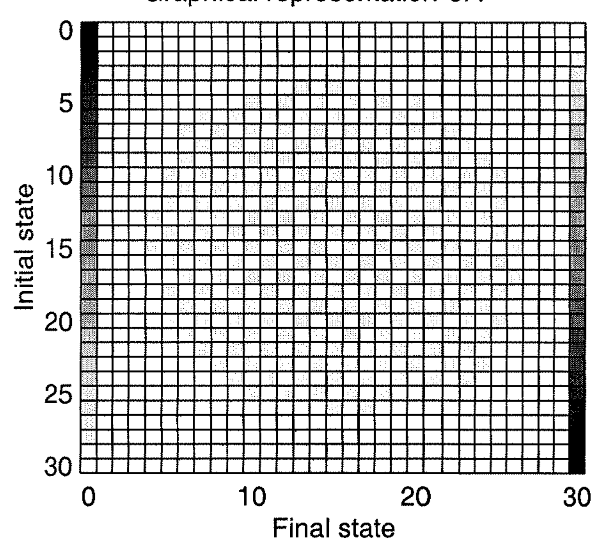
Graphical representation of P^4



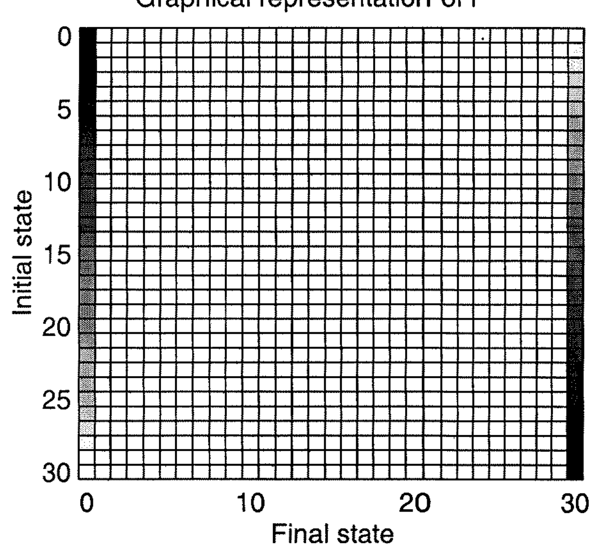
Graphical representation of P^{20}



Graphical representation of P^{225}

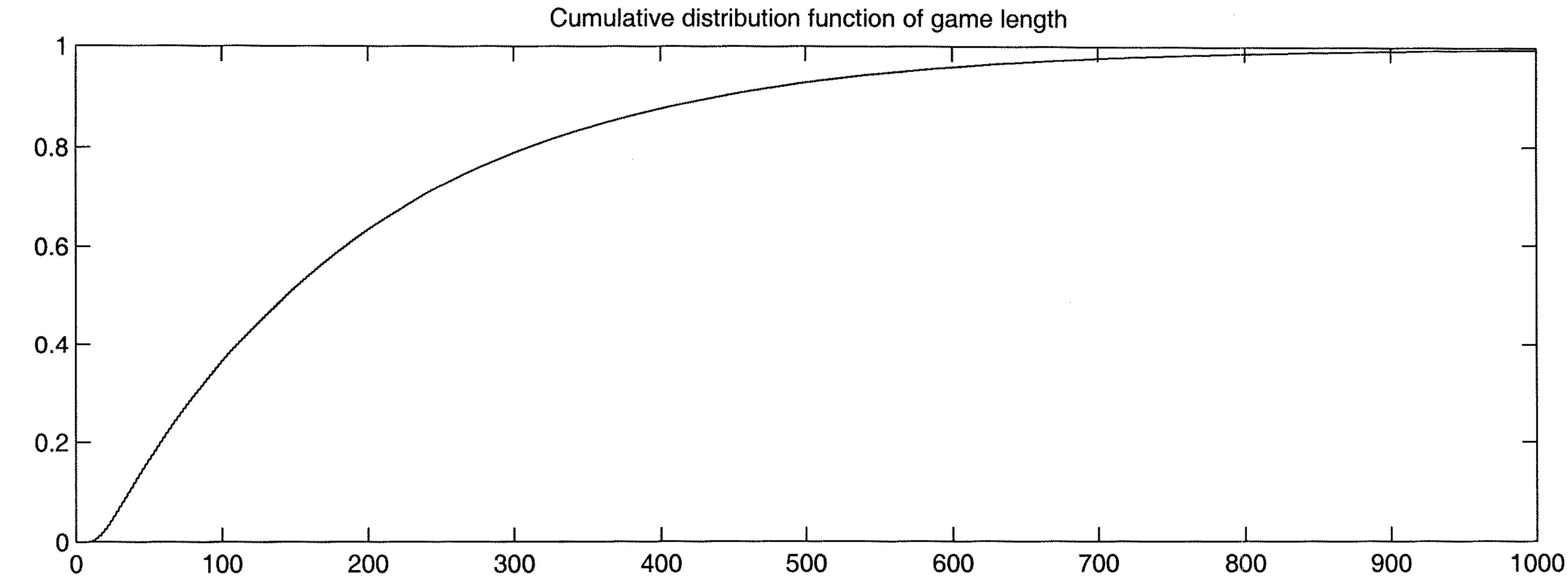


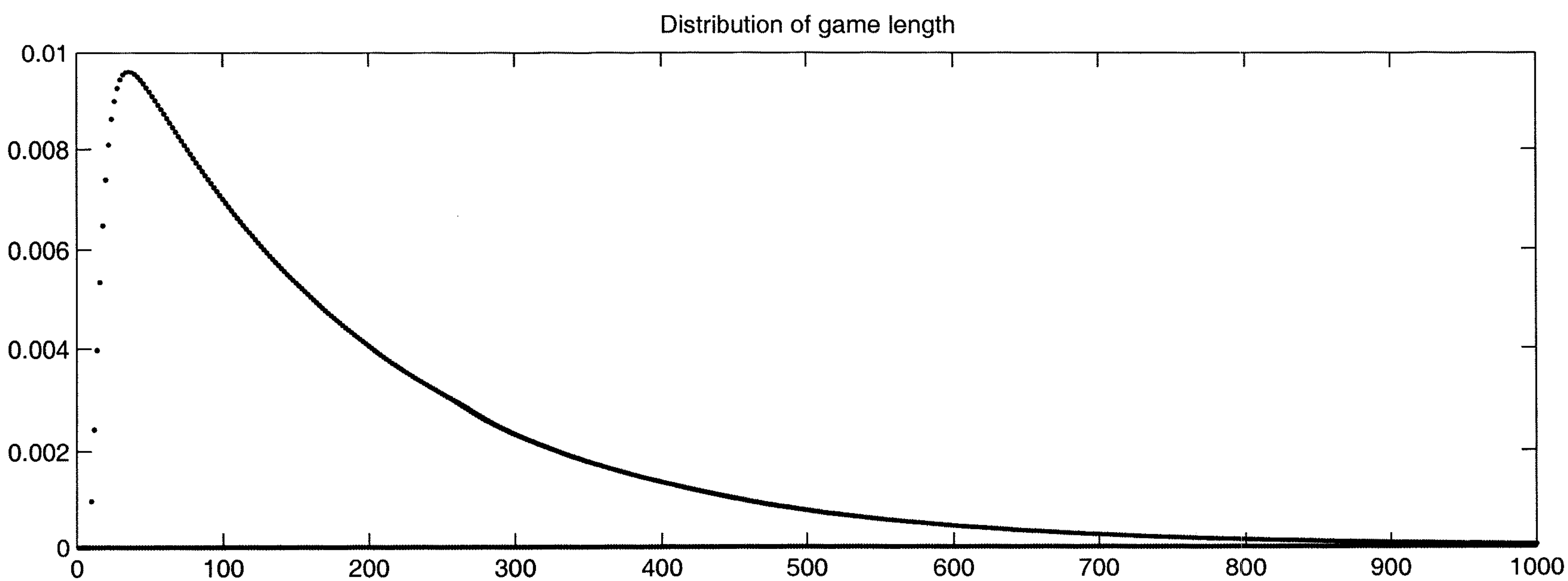
Graphical representation of P^{2000}



```
% gambler.m  sets up the transition matrix for Gambler's wealth.

N=30;              % total wealth at the table
p=0.5;             % probability of increasing wealth by 1
q=1-p;

P=zeros(N+1);      % set up a matrix of zeros, of the right size
                   % rows 1..N+1 correspond to wealths 0..N

for i=2:N,         % let i take values 2, 3, 4, ..., N
   P(i,i-1)=q;     % wealth decreases by 1
   P(i,i+1)=p;     % wealth increases by 1
end

P(1,1)=1;          % stay at 0 wealth with probability 1
P(N+1,N+1)=1;      % stay at wealth N with probability 1

mu=zeros(1,N+1);   % set up initial distribution
mu(11)=1;          % start with wealth 10 with probability 1

% the commands below display P and powers of P

x=-0.5:1:(N+0.5); % column numbers
y=x;               % row numbers

subplot(3,2,1);    % 3 by 2 array of plots, this is plot # 1

pcolor(x, y, [[P zeros(N+1,1)]' zeros(N+2,1)]');   % display matrix P
axis ij;           % number the axes as for a matrix, not a regular graph
title('Graphical representation of transition matrix P');
xlabel('Final state');
ylabel('Initial state');

(additional commands make other plots)
```

```
% gambler_length.m computes the distribution of the length of the game

% First run the program gambler.m to set up P and mu

M = mu;
n = 3000;
F = zeros(1,n+1);          % vector of zeros for CDF of game length
F(1) = 0;

for i=1:n,                 % get the distribution of W_1 to W_n
  M = M*P;
  F(i+1) = M(1) + M(N+1);
end

subplot(2,1,1)

plot(0:n, F);
title('Cumulative distribution function of game length')
axis([0 1000 0 1]);

subplot(2,1,2)

f = diff(F);               % pdf of game length
plot(1:n, f, '.');
title('Distribution of game length')
axis([0 1000 0 0.01]);

EN = sum(1-F);
fprintf('Average game length is %4.4f\n', EN);
```


Cumulative distribution function of game length
1
0.8
0.6
0.4
0.2
0
0
100
200
300
400
500
600
700
800
900
1000


Distribution of game length
0.01
0.008
0.006
0.004
0.002
0
0
100
200
300
400
500
600
700
800
900
1000

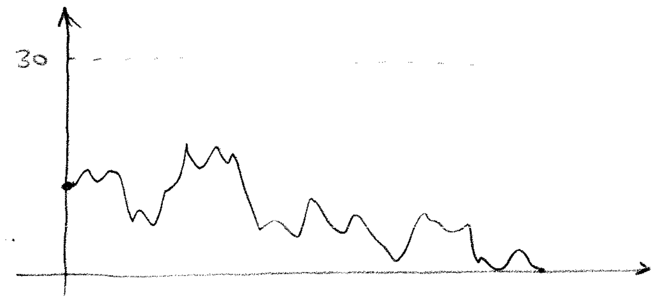
9-3-98

Simulation of gambler's wealth

It's important to have a clear visual picture in your mind of the random variables $W_0, W_1, W_2, \ldots,$ and the best way is with a graph. This is similar to tossing 10 coins at once to learn about the binomial distribution.

$W_0 = 10$ again.
$W_1 = 11$ or 9, with probs p and q.
⋮

A whole sequence of random variables.
Dependent.



Good to visualize them all together.
Usually we visualize a r.v. by displaying its density...
We know their joint distribution, but
it doesn't satisfy.

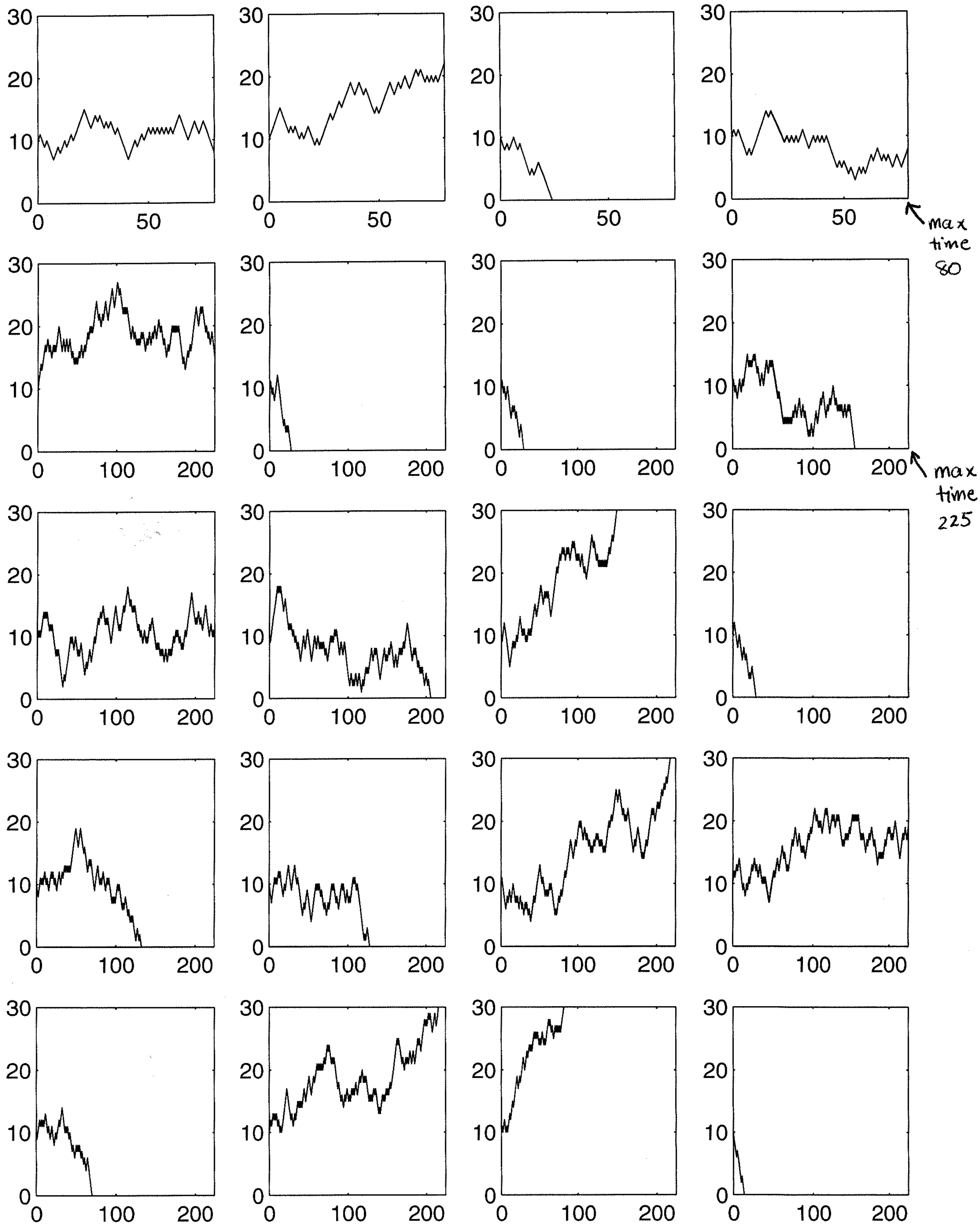
One good method is to generate and display values of $W_0 \ldots W_n$

These are only possible values, like throwing 10 coins in the air...

Bring a page-ful of graphs.

Matlab code

```
W(1)  = 10;     % 1st value, not time 1!
p     = 0.5;
N     = 30;
n     = 100;    % # steps


for i =2 : n
    if (rand <p)
        w(i) = w(i-1) + 1;
    else
        w(i) = w(i-1) - 1;
    end
end

plot (w)
title
axes ( [0  n  0  N] );
```



gambling_outcomes.m on my web page for Applied Probability

Gambler's wealth – general model and theoretical results p. 151 § III.6

Total wealth N at the table.

Probability of winning each gamble is p; set $q = 1 - p$.

W_0 is the initial wealth; we leave it unspecified.

Markov property holds:

$$\mathbb{P}(W_{n+1} = j \mid W_0 = i_0, \ldots, W_{n-1} = i_{n-1}, W_n = i) = \mathbb{P}(W_{n+1} = j \mid W_n = i).$$

In addition, this probability does not depend on n, which is called time homogeneity.

Set $P_{ij} = \mathbb{P}(W_{n+1} = j \mid W_n = i)$.

Transition matrix

$$P = \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ q & 0 & p & 0 & \cdots & 0 \\ 0 & q & 0 & p & \cdots & 0 \\ & & & & & \\ 0 & \cdots & & q & 0 & p \\ 0 & \cdots & & 0 & 0 & 1 \end{bmatrix}$$

Once the chain reaches 0 or N, it is absorbed there.

Let T be the time when this happens:

$$T = \min\{n \geq 0 : W_n = 0 \text{ or } W_n = N\}.$$

Then T is a random time.

The gambler is ruined if $W_T = 0$.

Probability of gambler's ruin

Idea: first-step analysis, § III.6

Let $u_k = \mathbb{P}(W_T = 0 \mid W_0 = k)$.
Then for $0 < k < N$,

$$u_k = \mathbb{P}(W_T = 0 \text{ and } W_1 = k+1 \mid W_0 = k) + \mathbb{P}(W_T = 0 \text{ and } W_1 = k-1 \mid W_0 = k)$$

$$= \mathbb{P}(W_T = 0 \mid W_1 = k+1, W_0 = k)\, \mathbb{P}(W_1 = k+1 \mid W_0 = k) + \mathbb{P}(W_T = 0 \mid W_1 = k-1, W_0 = k)\, \mathbb{P}(W_1 = k-1 \mid W_0 = k)$$

$$= \mathbb{P}(W_T = 0 \mid W_1 = k+1) \cdot p + \mathbb{P}(W_T = 0 \mid W_1 = k-1) \cdot q$$

$$= p\, u_{k+1} + q\, u_{k-1},$$

by time homogeneity.

Also,

$$u_0 = \mathbb{P}(W_T = 0 \mid W_0 = 0) = 1$$

$$u_N = \mathbb{P}(W_T = 0 \mid W_0 = N) = 0.$$

Think of $\vec{u} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix}$ as a column vector.

The equations above can be written as

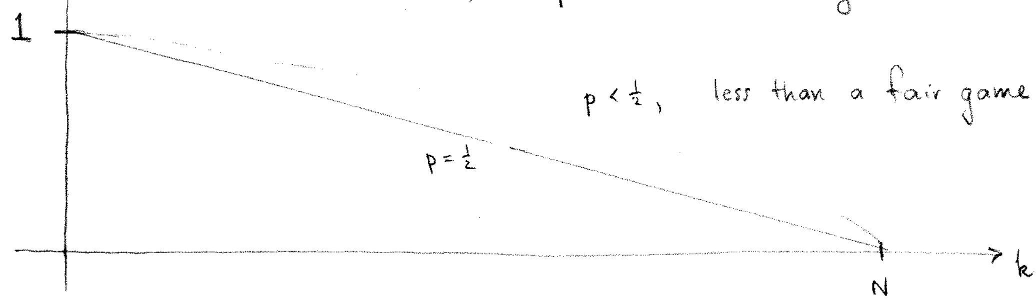
$$\vec{u} = P\vec{u} = \begin{bmatrix} 1 & 0 & \cdots & & & \\ q & 0 & p & 0 & \cdots & 0 \\ & & & & & \\ 0 & \cdots & & & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix}$$

row k is $u_k = q\, u_{k-1} + p\, u_{k+1}$.

The book shows how to solve this, finding

$$u_k = \begin{cases} 1 - \dfrac{k}{N} & \text{when } p = \frac{1}{2} \\[2ex] \dfrac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N} & \text{when } p \neq \frac{1}{2} \end{cases}$$

$u_k = \mathbb{P}(W_T = 0 \mid W_0 = k)$ = prob. of ruin starting with k



Note: If you are playing against a casino, then k is what you have, but $N-k$ is what the casino has, which is huge, and $\frac{N-k}{N} = 1 - \frac{k}{N}$ is virtually 1. You have no chance of taking all the casino's money!

Mean duration of the game

Let $m_k = \mathbb{E}[T \mid W_0 = k]$.

We can also do a first-step analysis here.
Let $0 < k < N$.

$$m_k = \sum_{t=0}^{\infty} t \, \mathbb{P}(T = t \mid W_0 = k)$$

$$= \sum_{t=0}^{\infty} t \left(\mathbb{P}(T=t, W_1 = k+1 \mid W_0 = k) + \mathbb{P}(T=t, W_1 = k-1 \mid W_0 = k) \right)$$

$$= \sum_{t=1}^{\infty} t \left(\mathbb{P}(T=t \mid W_1 = k+1, W_0 = k) \cdot p + \mathbb{P}(T=t \mid W_1 = k-1, W_0 = k) \cdot q \right)$$

$$= \sum_{t=1}^{\infty} t \left(p \cdot \mathbb{P}(T=t \mid W_1 = k+1) + q \cdot \mathbb{P}(T=t \mid W_1 = k-1) \right)$$

Markov prop.

$$= \sum_{t=1}^{\infty} t \left(p \, \mathbb{P}(T = t-1 \mid W_0 = k+1) + q \, \mathbb{P}(T = t-1 \mid W_0 = k-1) \right)$$

time homogeneity

$$= \sum_{t=0}^{\infty} (t+1) \left(p \, \mathbb{P}(T=t \mid W_0 = k+1) + q \, \mathbb{P}(T=t \mid W_0 = k-1) \right)$$

$$= p \cdot \mathbb{E}[T+1 \mid W_0 = k+1] + q \cdot \mathbb{E}[T+1 \mid W_0 = k-1]$$

$$= 1 + p \cdot \mathbb{E}[T \mid W_0 = k+1] + q \cdot \mathbb{E}[T \mid W_0 = k-1]$$

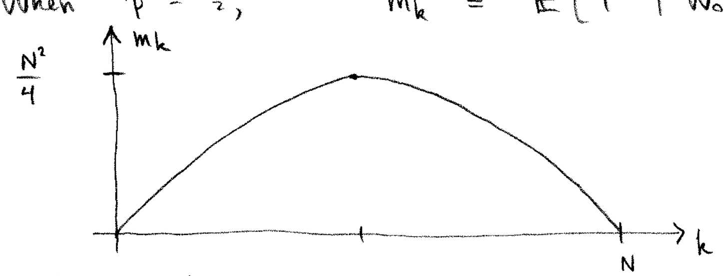
$$= 1 + p \cdot m_{k+1} + q \cdot m_{k-1}.$$

Thus, as a vector, $\vec{m} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + P\vec{m}$.

Also, $m_0 = 0$ and $m_N = 0$.

See §III.6 for the solution for $\vec{m}$.

Example: When $p = \frac{1}{2}$, $m_k = \mathbb{E}[T \mid W_0 = k] = k(N-k)$



$N = 30$

$k = 10$

$10(30-10) = 200$, as we saw.

$k = 15$, then $\mathbb{E}T = 225$.

You play the longest when each starts with the same amount of money.

Note: $\mathbb{E}T < \infty$ means $T < \infty$ with prob. 1, so the game ends after finitely many gambles!