A POINTFREE STUDY OF BASES FOR SPACES OF MINIMAL PRIME IDEALS

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Abstract. Let $C(X)$ denote the ring of all real-valued continuous functions on a topological space $X$, and $mX$ its space of minimal prime ideals with the hull-kernel topology. It has a base consisting of clopen sets and which contains the closure of the union of any sequence of its members. In the paper *Spaces with a pretty base* (J. Pure and Appl. Algebra 70 (1991), 81-87), spaces with a base with these properties are studied and are shown to have almost all of the known properties of the space $mX$ including the fact that if it is a weakly Lindel"of space, then it is basically disconnected. In the present paper, analogous results are derived in a pointfree context in which topological spaces are replaced by frames. In some cases, we are able to obtain more general results with simpler proofs.

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1. Introduction. In the paper *Spaces with a pretty base* (see [7]), M. Henriksen attempted to axiomatize a natural base for the space of minimal prime ideals of the ring of real-valued continuous functions on a topological space, and to place it in a more general context. If $R$ is a commutative ring, $mR$ will denote its space of minimal prime ideals in the hull-kernel topology, which has as a base $\mathfrak{B} = \{ \{ P \in mR : a \notin P \} : a \in R \}$. If $X$ is a Tychonoff space, $C(X)$ will denote the ring of all continuous real-valued functions on $X$, and $mX$ will abbreviate $mC(X)$. (Throughout all rings considered will be commutative and semiprime, and all topological spaces will be Tychonoff.) In [8], it was shown that $mX$ is always countably compact and zero-dimensional, and that the base $\mathfrak{B}$ described above for the topology of $mX$ has the properties (i) each member of $\mathfrak{B}$ is clopen, and (ii) the closure of a sequence of members of $\mathfrak{B}$ is in $\mathfrak{B}$. In [7], a space with a
base for its topology satisfying (i) and (ii) is said to have a \textit{pretty base}. Such spaces share almost all of the known properties of \( m(X) \). For example, if a space with a pretty base is weakly Lindelöf then it is basically disconnected (that is, closures of cozero sets are open). Many open questions remain. It is asked also if a space with a pretty base must be homeomorphic to \( mR \) for some lattice-ordered ring \( R \) in which the family of annihilators of elements is closed under countable intersection (as is the case in \( C(X) \)).

The present paper is devoted to placing these results in a pointfree context. It is part of a larger program started by Ball and Walters-Wayland in [1] of creating a pointfree formulation of the classical theory of rings of continuous functions beginning with the analogs of \( C^- \) and \( C^*^- \)-embeddings of subspaces. Pointfree topology is part of the study of complete lattices in which the meet of an element distributes over arbitrary joins. This latter kind of lattice is called a \textit{frame}. This notion generalizes both the lattice of open sets of a topological space and that of a Boolean algebra. In the present paper, we verify the correctness of certain definitions and results concerning \( C^*^- \)-quotients (the frame analog of a \( C^*^- \)-embedding) and basic disconnectedness by applying them in a concrete context. In some instances, results are stronger and proofs more transparent than their topological counterparts. \textit{Pseudocomplements} in frames are the analogs of complements of elements in a Boolean algebra. In the frame of open sets of a topological space, the pseudocomplement of an open set is the interior of its complement. The double pseudocomplement is used carefully below (Section 6) to formulate the notion of a frame with a pretty base; thereby replacing an incorrect one in Section 4 of [7]. In the frame of open sets of a topological space, the double pseudocomplement of an open set is the interior of its closure, or the smallest regular open set containing the given one. The abundance of non-spatial frames satisfying this definition gives rise to the, as yet, unfulfilled hope that some counterexamples will exist to questions about pretty bases.

The standard reference for frames is [10]. For more information about pretty bases, see [4] and [9].

2. \textbf{Some basics on frames.} A \textit{frame} is a complete lattice \( L \) with greatest element \( e \), called \textit{top}, and least element \( 0 \), called \textit{bottom}, satisfying the frame distributivity law:

\[ a \land \bigvee_i b_i = \bigvee_i (a \land b_i) \]

for all \( \{b_i : i \in I\} \subseteq L \). For any topological space \( X \), the lattice of open sets, denoted \( \mathcal{O}(X) \), is a frame. The \textit{pseudocomplement} of an element \( a \in L \) is

\[ a^* = \bigvee \{ b : a \land b = 0 \} \]

For any \( \{a_i : i \in I\} \subseteq L \), the de Morgan law holds, namely that \( (\bigvee a_i)^* = \bigwedge a_i^* \).

An element \( a \in L \) is \textit{dense} if \( a^* = 0 \), and \( x \leq a \) is \textit{dense in} \( a \) if \( x^* = a^* \). A frame map \( f : L \rightarrow M \) is a lattice morphism which preserves the top, the bottom, binary meets and arbitrary joins. It is said to be \textit{dense} if \( f(a) = 0 \) implies \( a = 0 \), and

\textit{codense} if \( f(a) \) is a \textit{cozero}, if \( a = \text{the frame of real numbers} \).

The notion \( \circ \) denote two fra

\textit{isomorphism} \( i : \) is equivalent to refer a \textit{part}

\textit{element} \( a \) \textit{of a fra}

\textit{regarded as fram}

A quotient is \( b \in L \), regarded \( \varepsilon \) equivalent to one map from \( L \) onto \( A \). closed quotient \( A \) nucleus on \( a \) \textit{meet preserving} \( \imath \) between the nuclei \( m : L \rightarrow M \) largest element \( m \)

The closure of a \( L \rightarrow \uparrow j \in \mathcal{M}(0_L) \). The lattice in the refin is given by the id consisting of one \( e \).

For elements \( a \leq b \), if \( a^* \lor b = t \) is a \textit{scale} from \( a \) t

such that \( q_0 = a \in \varepsilon \).

A frame \( L \) is \textit{refin} if \( a = \bigvee \{ b : a \land b \leq \}


Such spaces form a space with a unique structure, closures of elements, if a space with a vector space structure. If a space with a ring structure $R$, the closure or locale intersection defines a frame.

In a free context, it is possible to generalise the concept of functions beginning with 'a' to 'f', and the free topology is based on a free algebra of certain definitions, $\mathcal{F}$-embedding, etc. In some instances, the topological properties of elements of an algebraic space, the algebra, are of interest. The double dual of a space, the notion of a space, is defined in section 4 of [7]. In [8], a copy of an algebra of elements containing the topological algebra, it gives rise to questions about pretty objects, with greatest interest being the frame.

The closure of open sets, $\overline{V(a)} = \bigwedge a_i$. A frame free algebra contains $a = 0$, and codense if $f(a) = e$ implies $a = e$. An element $a \in L$ is a cozero element, or simply a cozero, if $a = h(\mathbb{R} \setminus 0)$ for some frame surjection $h : \mathbb{OR} \to L$, where $\mathbb{OR}$ denotes the frame of real numbers. See Banaschewski [2] for details on the frame of real numbers.

The notion of subspace is captured by the concept of a quotient frame. We declare two frame surjections $f_1, f_2 : L \to S_i$ to be equivalent if there is a frame isomorphism $i : S_1 \to S_2$ such that $if_1 = f_2$. Formally, a quotient of a frame $L$ is an equivalence class of frame surjections $f : L \to S$, however it is customary to refer to a particular member of the equivalence class as the quotient. For an element $a$ of a frame $L$ we let

$$\uparrow a \equiv \{ b \in L : b \geq a \} \quad \text{and} \quad \downarrow a \equiv \{ b \in L : b \leq a \},$$

regarded as frames in the order inherited from $L$, with

$$c_{\uparrow a} = 0_{\uparrow a} = a,$$
$$c_{\downarrow a} = c_L,$$
$$0_{\downarrow a} = 0_L.$$

A quotient is open if it is equivalent to one of the form $a \mapsto a \wedge b$ for some fixed $b \in L$, regarded as a frame surjection from $L$ onto $\downarrow b$. A quotient is closed if it is equivalent to one of the form $a \mapsto a \vee b$ for some fixed $b \in L$, regarded as a frame map from $L$ to $\uparrow b$. An open quotient is dense if and only if $b$ is a dense element. A closed quotient is dense if and only if it is improper, i.e., $f$ is an isomorphism. A nucleus on a frame $L$ is a closure operator $j : L \to L$, that is, an order and meet preserving map satisfying $j(j(a)) \leq j(a)$ for all $a \in L$. There is a bijection between the nuclei on $L$ and the quotients of $L$. In particular, for any quotient frame $m : L \to M$ there is a corresponding nucleus $j_M = m_*m$ where $m_*(x)$ is the largest element mapped to $x$. Thus

$$j_M(a) = \bigvee \{ x \in L | m(x) \leq m(a) \}.$$

The closure of a quotient $m : L \to M$ can thus be thought of as the quotient $L \to \lim_{\to} j_M(0_L)$. The collection of all quotients of a given frame forms a complete lattice in the refinement ordering, in fact, a frame (see [10]). Note that the bottom is given by the identity map, and the top is the unique map to the trivial frame consisting of one element.

For elements $a$ and $b$ of a frame $L$ we say that $a$ is rather below $b$, and write $a \prec b$, if $a \wedge b = 0$. We say that $a$ is completely below $b$, and write $a \ll b$, if there is a scale from $a$ to $b$, i.e., a subset

$$\{ a_q : q \in [0,1] \} \subseteq L$$

such that $a_0 = a$ and $a_1 = b$, and such that $a_p \prec a_q$ whenever $p < q$ in $[0,1]$. A frame $L$ is regular if $a = \bigvee \{ b : b \ll a \}$ for all $a \in L$, and is completely regular if $a = \bigvee \{ b : b \ll a \}$ for all $a \in L$. 

**Definition 1.** Quotients \( m_1 : L \to M_1 \) and \( m_2 : L \to M_2 \) are **separated** if

\[
j_1 j_2(0) = e = j_2 j_1(0)
\]

where \( j_i \) is the nucleus corresponding to \( m_i : L \to M_i \), for \( i = 1, 2 \).

We recall that two subsets of a topological space are separated if each of them is disjoint from the closure of the other. The following lemma shows that the frame definition is compatible with the spatial one.

**Lemma 2.** Subsets \( S, T \) of a topological space \( X \) are separated in \( X \) iff they are separated as quotients of the frame \( \mathcal{D}X \).

**Proof.** Given a topological space \( X \) with subsets \( S \) and \( T \). Suppose \( S \) and \( T \) are separated, that is \( S \cap clT = \emptyset = clS \cap T \). Now consider the quotient \( \mathcal{D}X \to \mathcal{D}S \): the associated nucleus \( j_S : \mathcal{D}X \to \mathcal{D}S \) is given by

\[
j_S(U) = \bigcup \{ V \in \mathcal{D}X | V \cap S = U \cap S \},
\]

that is, \( j_S(U) \) is the largest open set of \( X \) whose restriction to \( S \) is the same as \( U \) restricted to \( S \). Then \( j_S(\emptyset) = X \setminus clS \) and \( j_T(X \setminus clS) \) is the largest open set whose restriction to \( T \) is the same as \( T \cap (X \setminus clS) \). Thus \( T \cap clS = \emptyset \) so \( T \cap (X \setminus clS) = T \) and so \( j_T(X \setminus clS) = X \). By symmetry \( j_S j_T(\emptyset) = X \). Conversely, suppose \( j_T j_S(\emptyset) = X \) then \( X \cap T = j_S(\emptyset) \cap T \) so \( T \subseteq j_S(\emptyset) \). But \( j_S(\emptyset) = X \setminus clS \) so \( T \subseteq X \setminus clS \). That is \( T \cap clS = \emptyset \). Similarly \( S \cap clT = \emptyset \).

As one would expect, the notion of separated quotients is indeed weaker than that of completely separated quotients as defined in [1]: quotients \( m_1 \) and \( m_2 \) of \( L \) are completely separated by \( f \in CL \) (where \( CL \) denotes all the frame maps \( h : \mathcal{D} \to L \)) if

\[
m_1 f(\emptyset \setminus 0) = m_2 f(\emptyset \setminus 1) = 0.
\]

In particular, this means that there exists a binary cozero cover of \( L \) whose elements are mapped respectively to 0 by the \( m_i \).

**4. Weakly Lindelöf frames.**

**Proposition 3.** If quotient \( m \) is separated then they are separated.

**Proof.** Let \( j_1 \) be the nucleus of \( m \) and \( j_2j_1(0) = e \). Since the \( m_i \) are \( a \setminus b = e \) and \( m_1(\emptyset) \) is the largest element mapped to \( 0 \),

\[
m_2, m_2(\emptyset) = m_2, m_2(e) = m_2, m_2(\emptyset).
\]

**Lemma 5.** If \( x \) is dense in \( L \), then \( z \land x \land a = 0 \).

**Proof.** Given that \( x^* = 0 \), le

\[
z \land x \land a = 0.
\]

**Lemma 6.** If \( x_n \) is dense in \( e \), then \( \vee a_n \neq 0 \).

**Proof.** Since \( x_n^* = a_n^* \), apply \( (\vee a_n)^* \).

Similar to the result that we obtain the following:

**Proposition 7.** Every cozero cover of \( L \).

**Proof.** Let \( L \) be weakly Lindelöf, \( S \subseteq L \). Since \( \vee a_n \neq 0 \), \( T_a \subseteq S \) with \( a_n \neq \emptyset \). Thus, for each \( n \), \( a_n \lor T_n \) are dense in \( a_n \) (by Lemma 5). Therefore, now let \( L = \bigcup T_n \), then

\[
\bigvee (a_n \land T_n).
\]
the existence of zero-sets (equivalence classes) of sets in $\mathcal{F}_{\mathbb{R}}$

The analogous result to Proposition 2 is:

**Proposition 3**. If quotients $m_1 : L \to M_1$ and $m_2 : L \to M_2$ are completely separated then they are separated.

**Proof**. Let $j_1$ be the nucleus associated with $m_i (i = 1, 2)$. We will show that $j_i m_1 (0) = e$. Since the $m_i$ are completely separated there exist cozeroes $a, b$ of $L$ with $a \lor b = e$ and $m_1 (a) = 0 = m_2 (b)$. Now $j_1 (0) = m_1 \circ m_1 (0) = m_1 (0)$, the largest element mapped to 0 by $m_1$, and so $a \leq j_1 (0)$. But $j_2 (a) = m_2 \circ m_2 (a) = m_2 \circ m_2 (a \lor b) = m_2 \circ m_2 (e) = e$ so $j_2 j_1 (0) = e$. □

## 4. Weakly Lindelöf frames

We now need to consider a slightly weaker notion than that of Lindelöf.

**Definition 4**. A frame $L$ is weakly Lindelöf if whenever $\forall S = e$ for some $S \subseteq L$, there exists a countable subset $T \subseteq S$ with $\forall T'$ dense in $L$, that is $(\forall T)' = e$. An element $a \in L$ is weakly Lindelöf if whenever $\forall S = a$ for some $S \subseteq L$, there exists a countable subset $T \subseteq S$ with $\forall T$ dense in $a$, that is $(\forall T) = a$.

Note that weakly Lindelöf frames are discussed in [12] in connection with quasi-metric covers.

**Lemma 5**. If $x$ is dense in $L$ then $x \leq a$ is dense in $\downarrow a$.

**Proof**. Given that $x' = 0$, let us show that $(x \land a)' \leq a'$, that is, $z \leq a'$ whenever $z \land x \land a = 0$. But $z \land x \land a = 0$ implies $z \land a \leq x'$ and since $x' = 0$ we have $z \leq a'$. □

**Lemma 6**. If $x_n$ is dense in $a_n$ then $\lor x_n$ is dense in $\downarrow \lor a_n$.

**Proof**. Since $x_n = a_n$, applying de Morgan’s law gives $(\lor x_n)' = \land x_n = \land a_n = (\lor a_n)'$. □

Similar to the result that every cozero element of a Lindelöf frame is Lindelöf, we obtain the following:

**Proposition 7**. Every cozero element of a weakly Lindelöf frame is weakly Lindelöf.

**Proof**. Let $L$ be weakly Lindelöf and take $a \in Coz L$. Suppose $a = \lor S$ for some $S \subseteq L$. Since $a$ is a cozero, $a = \lor a_n$ with $a_n < a$. That is $a_n' \lor a = e$ for each $n$. Thus, for each $n$, $a_n' \lor S = e$. Since $L$ is weakly Lindelöf, there exists a countable $T_n \subseteq S$ with $a_n' \lor T_n$ dense in $L$. And thus $a_n \land (a_n' \lor T_n) = a_n \land T_n$ is dense in $a_n$ (by Lemma 5). Therefore $(a_n \land T_n)$ is dense in $\lor a_n = a$ (by Lemma 6). Now let $T = \bigcup T_n$ then

\[
\lor (a_n \land T_n) \leq \lor a_n \land \lor T_n = a \land \lor T,
\]
and so $a \wedge \bigvee T$ is dense in $a$. But $a \wedge \bigvee T = \bigvee T$, so $\bigvee T$ is dense in $a$. This proves that $a$ is weakly Lindelöf.

**Corollary 8.** In a weakly Lindelöf frame, an element is weakly Lindelöf iff it has a dense cozero below it.

**Proof.** In a completely regular frame, every element is the join of cozeros. Hence if an element is weakly Lindelöf then a countable join of cozeros is dense in the element. The forward implication follows since a countable join of cozeros is cozero. The reverse implication follows from the proposition above.

This corresponds to the result for Lindelöf frames that the Lindelöf elements are precisely the cozero ones.

**Proposition 9.** For any dense surjection $m : L \twoheadrightarrow M$, if $M$ is weakly Lindelöf then so is $L$.

**Proof.** It follows easily from the fact that $m(x^*) = (m(x))^*$ for a dense surjection $m$.

**Corollary 10.** The closure of a weakly Lindelöf quotient is weakly Lindelöf. That is, if $m : L \twoheadrightarrow M$ with $M$ weakly Lindelöf then $\uparrow m_0(0_L)$ is weakly Lindelöf.

5. **Basic disconnectedness and P-frames.** Recall that a space is basically disconnected if every cozero set has an open closure (see Gillman and Jerison [5]).

**Definition 11.** A frame $L$ is basically disconnected if for each $a \in Coz L$, $b \in L$, if $a \wedge b = 0$ then $a^* \vee b^* = e$.

This is compatible with the spatial definition (see [1]). The following results all appear in [1], nevertheless many of the details are repeated here.

**Lemma 12.** $L$ is basically disconnected iff for each $a \in Coz L$, $b \in L$, whenever $a$ and $b$ are disjoint, they are completely separated.

**Proof.** Suppose $L$ is basically disconnected and take $a \in Coz L$, $b \in L$ with $a \wedge b = 0$ then $a^* \vee b^* = e$ so $b \prec a^*$. But $a \wedge a^* = 0$ so $a^* \vee a^{**} = e$, hence $a^* \prec \prec a^*$ and so $b \prec \prec a^*$. Conversely, if $a \in Coz L$, $b \in L$ with $a \wedge b = 0$ then $a$ and $b$ are completely separated, so $b \prec \prec a^*$ and thus $a^* \vee b^* = e$.

**Lemma 13.** $L$ is basically disconnected iff for all $a \in Coz L$, $a^* \vee a^{**} = e$.

**Proof.** The forward implicat then $b \leq a^*$ so $a^{**} \leq b^*$. But thus the double pseudo that complemented elements sequence of elements comple rather below itself, and hence below itself.

**Lemma 14.** Every basically connected.

**Proof.** Recall that $x < a$ implies regular frames, we have that $a \prec \prec a^*$.

and the result follows using $L$.

**Lemma 15.** For fixed $b \in L$, (a)

**Proof.** If $x \in b^*$ then $x \wedge (b \wedge c^*) \subseteq (b \wedge c^*)$.

**Corollary 16.** For fixed $b \in L$, (a)

The following proposition is [8]. However it is interesting to basic elements are not require spatial setting.

**Proposition 17.** A frame with base of basically disconnected.

**Proof.** By Lemma 13 we need for any basically disconnected basically disconnected.
Bases for spaces of minimal prime ideals

Theorem. A frame which is locally basically disconnected, that is, has a base of basically disconnected elements, is basically disconnected.

Proof. By Lemma 13 we need only show that, for every \( c \in \text{Coz} L \), \( c^* \vee c^{**} = e_L \).

For any basically disconnected element \( b \in L \), \( b \wedge c \in \downarrow b \), and since \( \downarrow b \) is basically disconnected,

\[
\begin{align*}
b = e_L b &= (b \wedge c)^* \vee (b \wedge c)^{**} \\
&= (b \wedge c^*) \vee (b \wedge c^{**}) \\
&= b \wedge (c^* \vee c^{**}).
\end{align*}
\]
Since basically disconnected elements form a base for \( L \), the top element \( e_L \) is a join of such elements, say \( e_L = \bigvee b_i \), then

\[
e_L = \bigvee b_i = \bigvee (b_i \wedge (c^* \vee c^{**}))
\]
\[
= (\bigvee b_i) \wedge (c^* \vee c^{**})
\]
\[
= e_L \wedge (c^* \vee c^{**})
\]

and so \( c^* \vee c^{**} = e_L \).

\[ \square \]

**Lemma 18.** If a frame \( L \) is basically disconnected then so is \( \downarrow a \) for each \( a \in \text{Coz} \ L \).

**Proof.** We first recall that \( L \twoheadrightarrow \downarrow a \) is a coz surjection \([1]\). Take any \( x \in \text{Coz} \ \downarrow a \), then there exists \( y \in \text{Coz} \ L \) with \( x = y \wedge a \). Since \( L \) is basically disconnected, \( y^* \vee y^{**} = e_L \). Now, in \( \downarrow a \)

\[
x^* \vee x^{**} = (y \wedge a)^* \vee (y \wedge a)^{**}
\]
\[
= (y^* \wedge a) \vee (y^{**} \wedge a)
\]
\[
= (y^* \vee y^{**}) \wedge a
\]
\[
= e_L \wedge a
\]
\[
= a = e_{\downarrow a}.
\]

and so \( \downarrow a \) is basically disconnected.

**Recall that a P-space (or pseudo-discrete space) is a space in which every prime ideal is maximal, or equivalently, each cozero set is closed, or \( \text{coz} \ C(X) \) is closed under complementation (see \([6]\)).

**Definition 19.** \( L \) is a **P-frame** if \( \text{Coz} \ L \) is complemented.

**Proposition 20.** Every P-frame is basically disconnected.

**Proof.** Suppose \( a \in \text{Coz} \ L \) where \( L \) is a P-frame. Then \( a \vee a^* = e \) so \( a^* \vee a^{**} = e \). Hence by Lemma 13, \( L \) is basically disconnected.

The converse is not true (see \([5]\) for an example of a space which is basically disconnected but not a P-space).

Recall that a space is an almost P-space if every zero set is the closure of its interior (see \([11]\)).

**Definition 21.** A frame \( L \) is an **almost P-frame** if for \( a \in \text{Coz} \ L \), \( a = a^{**} \). Equivalently for \( a \in \text{Coz} \ L \), if \( a^* = 0 \) then \( a = e \).

**Lemma 22.** Every P-frame is an almost P-frame.

**Proof.** If \( a \in C \) so \( a = e \).

**Lemma 23.** Every complemented A is complemented.

**Proof.** Recall that for \( A \) complemented.

**6. Pretty base**

**Proposition 25.** If \( B \subseteq BL \) (i.e.

1. \( B \subseteq BL \) (i.e.
2. whenever \( \{b_i\} \)

If \( B \) is a base \( \mathcal{B} \) can show that \( \mathcal{B}(\mathcal{B}) \)

It should be noted that \( \mathcal{B} \) bases (simply take any base \( \mathcal{B} \) for any sequence \( \{b_i\} \).

**Proposition 26.** A basis disconnectivity.

**Proof.** By Lemma 12 a pretty base \( a \) is "weakly Lindelöf, an"
Proof. If $a \in \text{Coz} \ L$ with $a^* = 0$ then since $a$ is complemented, $a \lor a^* = e$, and so $a = e$. □

**Lemma 23.** Every basically disconnected almost $P$-frame is a $P$-frame.

**Proof.** Recall that in a basically disconnected frame, for each $a \in \text{Coz} \ L$, $a^{**}$ is complemented. And by definition of an almost $P$-frame $a = a^{**}$, and so $\text{Coz} \ L$ is complemented. (Clearly $a^* \in \text{Coz} \ L$, since complemented elements are cozeroes.) □

### 6. Pretty bases for frames

We would like to make clear that the definition proposed in the last section of [7] is not correct and what should have appeared is the following:

**Definition 24.** A base $B$ for a frame $L$ is pretty if

1. $B \subseteq BL$ (i.e. every element is complemented), and
2. whenever $\{b_n\} \subseteq B, (\lor b_n)^{**} \in B$.

If $\mathfrak{B}$ is a base for a topological space $X$, and if $\mathfrak{B}$ is pretty, then $cl(\bigcup B_n) \in \mathfrak{B}$ for any sequence $\{B_n\} \subseteq \mathfrak{B}$, and so $cl(\bigcup B_n)$ is open and closed, and therefore one can show that $cl(\bigcup B_n) = (\lor B_n)^{**}$. Conversely if $\mathfrak{B}$ is a pretty base for the frame $\Omega X$, then $(\lor B_n)^{**}$ is complemented and so $(\lor B_n)^{**} \in BL$. It should be noted that all complete Boolean algebras are frames with pretty bases (simply take the entire algebra as the base) and thus non-spatial frames with pretty bases are ubiquitous.

**Proposition 25.** If $L$ is a basically disconnected frame then the complemented elements, $BL$, form a pretty base.

**Proof.** By Lemma 14, we have that $BL$ is a base. The first condition of being a pretty base is obvious. For the second: take countable $\{b_n\} \subseteq BL$. Each $b_n \in \text{Coz} \ L$ (complemented implies cozero) and so $\lor b_n \in \text{Coz} \ L$ (since $\text{Coz} \ L$ is a $\sigma$-frame). Since the frame is basically disconnected, $(\lor b_n)^{**} \in BL$. Thus $BL$ forms a pretty base. □

**Proposition 26.** (cf. 2.4 of [7]) A weakly Lindelöf frame with a pretty base is basically disconnected.

**Proof.** Let $B$ be a pretty base for a weakly Lindelöf frame $L$. Take $a \in \text{Coz} \ L$, we need to show that $a^* \lor a^{**} = e$. Now $a = \lor b_i$ with $b_i \in B$. Since $a$ is a cozero, it is weakly Lindelöf, and so there are countably many $b_i$, say $\{b_n\}$ with $(\lor b_n)^* = a^*$.
But $(\bigvee b_n)^* = (\bigvee b_n)^{**}$ is complemented, that is, $a^{**} = e$. 

We call a frame locally weakly Lindelöf if it has a base of weakly Lindelöf elements.

**Proposition 27.** (cf. 2.9(b) of [7]) A locally weakly Lindelöf frame with a pretty base is basically disconnected.

**Proof.** Let the base of weakly Lindelöf elements be denoted by $W$. Clearly if $B$ is a pretty base for a frame $L$, then for each $a \in L$, $\{a \wedge b \mid b \in B\}$ is a pretty base for $\downarrow a$. Now consider $a \in W$; by the prior proposition, $\downarrow a$ is basically disconnected, an hence by Proposition 17, so is $L$. 

We note that the above proposition is stronger than that proven in [7], as there it was also assumed that the base was cocompact (see Section 7 below).

**Lemma 28.** (cf. 2.2 of [7]) Suppose $L$ is a frame with a pretty base $B$ and two quotients $m : L \to M$ and $n : L \to N$. If $M$ and $N$ are separated and weakly Lindelöf, then there exists $a \in B$ with $m(a) = 0_M$ and $n(a) = 0_N$.

**Proof.** Recall that for every quotient $m : L \to M$ there is an associated nucleus, say $j_M$, defined by $j_M(x) = \bigvee \{y \in L \mid m(y) \leq m(x)\}$. Since $M$ and $N$ are separated we have that

$$j_M j_N(0_L) = e_L = j_N j_M(0_L).$$

Let $z = j_M(0_L)$; then $\uparrow z$ is the closure of $M$. Now $j_N(z) = e_L$ and so $n(z) = e_N$. Since $B$ is a base, $z = \bigvee b_i$ with $b_i \in B$, and so $n(z) = \bigvee n(b_i) = e_N$ and $m(\bigvee b_i) = 0_M$. Since $N$ is weakly Lindelöf, we can obtain a countable family, say $\{a_n\}$, dense in $N$. That is, $(\bigvee n(a_n))^* = 0_N$. Then $y = \bigvee n(b_n)$ is a dense cozero element of $N$. Similarly we can obtain a dense cozero element $x$ of $M$, with $x = \bigvee c_i$, where $c_i \in B$ and $n(\bigvee c_i) = 0_N$. By Proposition 7, $x$ and $y$ are weakly Lindelöf.

Now for each $n$, let

$$p_n = b_n \wedge (\bigvee_{m \leq n} c_m)^* \text{ and } q_n = c_n \wedge (\bigvee_{m \leq n} b_m)^*$$

then $p_m \wedge q_n = 0_L$ for all $m, n \in \mathbb{N}$. Let $p = \bigvee p_n$ and $q = \bigvee q_n$. Using the fact that $B$ is a base of complemented elements, we can obtain that $m(q) = x$ and $n(p) = y$. Also $m(p) = 0_M$, $n(q) = 0_N$, and $p \wedge q = 0_L$.

Since $p \in L$, $p = \bigvee a_i$ for $a_i \in B$ and thus $y = n(p) = \bigvee n(a_i)$. Since $y$ is weakly Lindelöf, we obtain a countable family, say $\{a_n\}$, whose join is dense in $y$. That is, $(\bigvee n(a_n))^* = y^* = 0_N$ (the latter equality follows since $y$ is dense in $N$). Since $\{a_n\} \subseteq B$, and $B$ is a pretty base, $a = (\bigvee a_n)^* \in B$. It remains to show that $m(a) = 0_M$ and $n(a) = e_N$. Since $a \in B$, $a$ is complemented and hence so is $n(a)$. 

But $\bigvee n(a_n) \leq n(0_L)$ (the bottom and since $\bigvee a_n \leq \uparrow a \wedge q = 0_L$, and so $x$ is dense in $M$).

**Corollary 29.** $S$ are completely sep.

If $CL$ denotes a maps from $\mathcal{D}$ into $f \in CL$ which fact for $r \in \mathbb{Q}^+$, then $w$ of $h$ over $m$ if $m$ 

We say that $M$ 

$C^*$-quotient map if $M$

**Theorem 30.** (cf. base are $C^*$-quotient.

**Proof.** Suppose $I$ Using the frame for $b$ and $a$ are complete and $c, d \in M$ with $a \leq c$ frame and therefore the following diagram:
But $\bigvee n(a_n) \leq n(a)$, so $n(a)^* \leq (\bigvee n(a_n))^* = 0_N$ and thus the complement of $n(a)$ is the bottom making $n(a)$ the top, that is $n(a) = e_N$. Recall that $p \land q = 0_L$ and since $\bigvee a_n \leq p$, $\bigvee a_n \leq q^*$. Therefore $(\bigvee a_n)^{**} \leq q^{***} = q^*$. This means that $a \land q = 0_L$, and so $m(a) \land x = m(a) \land m(q) = 0_M$, giving $m(a) \leq x^* = 0_M$ (since $x$ is dense in $M$). □

**Corollary 29.** Separated weakly Lindelöf quotients of a frame with a pretty base are completely separated.

If $CL$ denotes all the frame maps from $\Omega R$ into $L$, and $C^*L$ the bounded frame maps from $\Omega R$ into $L$, (that is the elements of $C^*L$ are precisely those frame maps $f \in CL$ which factor through some closed quotient $a \mapsto a \lor a_r$, where $a_r = \mathbb{R}[r,]-r$, for $r \in \mathbb{Q}^+$), then we say of frame maps $h \in CM$ and $g \in CL$ that $g$ is an extension of $h$ over $m$ if $mg = h$.

\[
\begin{align*}
L & \twoheadrightarrow M \\
g & \downarrow h \\
\Omega R & 
\end{align*}
\]

We say that $M$ is a $C$-quotient with $m : L \rightarrow M$ a $C$-quotient map if every $h \in CM$ has an extension over $m$ and that $M$ is a $C^*$-quotient with $m : L \rightarrow M$ a $C^*$-quotient map if every $h \in C^*M$ has an extension over $m$.

**Theorem 30.** (cf. 2.3 of [7]) Weakly Lindelöf quotients of a frame with a pretty base are $C^*$-quotients.

**Proof.** Suppose $L$ has a pretty base $B$, $M$ is weakly Lindelöf and $m : L \twoheadrightarrow M$. Using the frame formulation of Urysohn’s theorem [1], it suffices to show that if $a$ and $b$ are completely separated in $M$ then they are $m$-completely separated: take $a$ and $b$ completely separated in $M$, then there exist completely separated cozeroes $c, d \in M$ with $a \leq c$ and $b \leq d$. Note that $c$ and $d$ are cozeroes of a weakly Lindelöf frame and therefore, by Proposition 7, themselves weakly Lindelöf. Consider the following diagram:

\[
\begin{align*}
L & \twoheadrightarrow M \\
& \downarrow c \\
d & 
\end{align*}
\]
We will show that \( \downarrow c \) and \( \downarrow d \) are separated quotients of \( L \). It is clear that they are quotients and hence have corresponding nuclei, say \( j_c \) and \( j_d \). Now \( j_d(0_L) \) is the largest \( x \in L \) satisfying \( d \land m(x) = 0_M \) (equivalently \( m(x) \leq d^* \)) and so \( j_d(0_L) = m_*(d^*) \). But \( m_*(d^*) \) maps \( c \land d^* \) to \( c \land d^* \). Since \( c \) and \( d \) are completely separated, they are disjoint so \( c \leq d^* \). Hence \( m_*(d^*) \) maps to \( c \), the top of \( \downarrow c \), and thus \( j_c(m_*(d^*)) = e_L \). So we have shown that \( j_cj_d(0_L) = e_L \). By symmetry \( j_dj_c(0_L) = e_L \), and so \( \downarrow c \) and \( \downarrow d \) are separated. By Corollary 29, they are therefore completely separated in \( L \), and hence \( m \)-completely separated quotients of \( M \). Clearly then \( a \) and \( b \) are also \( m \)-completely separated.

**Theorem 31.** (cf. 2.6(a) of [7]) Weakly Lindelöf quotients of an almost P-frame with a pretty base are C-quotients.

**Proof.** By Theorem 30, \( L \to M \) is a C*-quotient map. We show that for any \( c \in Coz L \) with \( m(c) = e_M \) there exists \( d \in Coz L \) with \( m(d) = 0_M \) and \( c \lor d = e_L \) (see [1]). So take \( c \in Coz L \) with \( m(c) = e_M \). Since \( L \) has a pretty base \( B \), \( c = \bigvee b_i \) where \( b_i \in B \). Thus \( \bigvee m(b_i) = e_M \) and since \( M \) is weakly Lindelöf, there is a countable subfamily, say \( \{ m(b_i) \} \), which is dense, that is \( \bigvee m(b_i) = 0_M \). Now \( \bigvee (\bigvee m(b_i))^* = \bigvee m(b_i) \), thus \( \bigvee m(b_i) \in B \) since \( B \) is pretty. But \( \bigvee b_i \in Coz L \), so since \( L \) is an almost P-frame, \( \bigvee b_i = (\bigvee m(b_i))^* \), and thus \( \bigvee b_i \in B \). Let \( d = (\bigvee b_i)^* \), then \( c \lor d = e \) and it remains to show that \( m(d) = 0_M \). Since

\[
m(d) \land m(b_i) = m(d \land \bigvee b_i) = m \left( (\bigvee b_i)^* \land \bigvee b_i \right) = m(0_L) = 0_M,
\]

we have that \( m(d) \leq (\bigvee m(b_i))^* \), and the latter is the bottom of \( M \).

**Corollary 32.** (cf. 2.6(b) of [7]) If the quotient above is Lindelöf, then it is closed.

**Proof.** Every Lindelöf C-quotient is closed (see [1]), thus the required result follows from the theorem above.

**Theorem 36.** (cf. 2.9(a) of [7]) Weakly Lindelöf with \( m : L \to L^* \) compactification of \( M \).

**Proof.** Recall that the closure \( C^* \)-quotient map (Theorem 36) is compact and by \( t_l \) and hence itself compact.

**Corollary 37.** (cf. 2.9(b) of [7]) Locally compact.

**Proof.** Suppose \( x \) is a wet is weakly Lindelöf by Prop. 33 gives that the closure of \( \downarrow c \) is basically disconnected and \( s \) is compact. Recall that if \( c \) has a base of weakly Lindelöf rather below it.

**Definition 33.** If \( B \) is a base for a frame \( L \) then the subframe \( L^* \) of \( L \) generated by \( \{ b^* | b \in B \} \) is called the quasi-complemented frame of \( L \) generated by \( B \). If \( B \) is a pretty base and \( L^* \) is compact then \( B \) is called a cocompact pretty base.
It is clear that \( j_\partial \) is the top of \( \downarrow c \), so \( m(\partial_\partial) = d^* \) are completely \( \downarrow c \) and \( \uparrow \downarrow c \) the top of \( \downarrow c \), respectively. By symmetry \( \uparrow \downarrow c \), they are therefore the quotients of \( M \).

\[ \square \]

**Lemma 34.** If \( L \) is a frame with a pretty base and \( \uparrow \downarrow s \) is weakly Lindelöf for \( s \in L \), then \( s \in L^* \). Moreover, \( \uparrow \downarrow s \) is a quotient of \( L^* \).

**Proof.** Take any \( c \in B \) with \( c \leq s \). Since \( c^* = \bigvee b_i \) for \( b_i \in B \). Clearly \( c^* \leq s \). Now \( c^* \leq s \). It is clear that \( \bigvee b_i \leq s \) and \( c^* \leq s \). We have \( \bigvee b_i \leq s \). Clearly \( c^* \leq s \). Now \( c^* \leq s \). It is clear that \( \bigvee b_i \leq s \) and hence weakly Lindelöf property of \( \bigvee b_i \), we may obtain a countable dense subfamily of \( \bigvee b_i \), say \( \{b_n \} \). That is \( \bigvee b_n = s \). Also \( b = (\bigvee b_n)^* = B \). We will show that \( c \leq b^* \leq s \). Thus \( c \leq b^* \leq s \). Clearly, for each \( c \in B \), \( c \leq s \) is a cozero element of \( \downarrow s \) and hence is itself weakly Lindelöf (Proposition 7). So by the above result, \( c \leq s \in L^* \). Using this one can show that any \( t \geq s \), and thus \( \uparrow \downarrow s \) is a quotient of \( L^* \).

\[ \square \]

**Corollary 35.** (cf. 2.8 of [7]) For a frame \( L \) with a pretty base, if \( m : L \to M \) and \( M \) is weakly Lindelöf then \( \uparrow j_m(0_L) \) is a closed quotient of \( L^* \).

**Theorem 36.** (cf. 2.9(a) of [7]) If \( L \) has a cocompact pretty base and \( M \) is a weakly Lindelöf with \( m : L \to M \), then the closure of \( M \) is \( \beta M \), the Stone-Cech compactification of \( M \).

**Proof.** Recall that the closure of \( M \) is given by \( \uparrow j_m(0_L) \). Since \( \uparrow j_m(0_L) \equiv M \) is \( C^* \)-quotient map (Theorem 30), by [1] it suffices to show that \( \uparrow j_m(0_L) \) is compact. But \( L^* \) is compact and by the above corollary, \( \uparrow j_m(0_L) \) is a closed quotient of \( L^* \), and hence itself compact.

\[ \square \]

**Corollary 37.** (cf. 2.9(b) of [7]) If \( L \) is also locally weakly Lindelöf then it is locally compact.

**Proof.** Suppose \( x \) is a weakly Lindelöf element of \( L \). If \( c \in Coz (\downarrow x) \) then \( \downarrow c \) is weakly Lindelöf by Proposition 7. Applying the previous theorem to \( L \to \downarrow c \), we find that the closure of \( \downarrow c \), namely \( \uparrow c^* \), is compact. By Proposition 26, \( \downarrow x \) is basically disconnected and so \( \uparrow c^* \equiv \downarrow c^* \). For each cozero element \( e \) of \( \downarrow x \), \( c^* \) is compact. Recall that if \( c < x \), then \( c^* \leq x \), and then the result follows since \( L \) has a base of weakly Lindelöf elements each of whom is a join of cozero elements rather below it.

\[ \square \]
We note that the third parts of both Theorem 2.6 and Theorem 2.9 of [7] do not make obvious sense in a frame setting and hope that some of the results given above can be used to resolve some of the unsolved problems listed in [7].

REFERENCES


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