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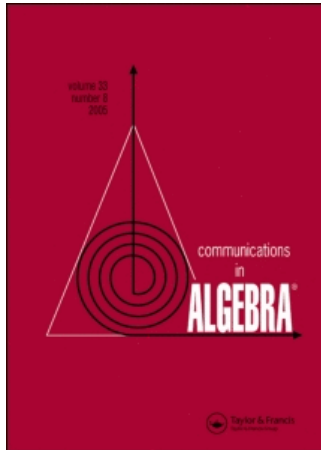
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BÉZOUT SP-DOMAINS

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SP-domains were first introduced by Vaughan and Yeagy (1978) as generalizations of Dedekind domains. In particular, they showed that an SP-domain is an almost Dedekind domain. Later, Olberding (2005) gave a complete characterization of SP-domains within the class of almost Dedekind domains. This article characterizes Bézout SP-domains using the order structure of the group of divisibility of an integral domain. We use this information to construct a Bézout SP-domain with zero Jacobson radical.

Key Words: Bézout domain; Lattice-ordered group of divisibility; Specker group.

Mathematics Subject Classification: Primary 13F05; Secondary 06F20.

1. SP-DOMAINS

Throughout, all rings are commutative and possess an identity.

Recall that an ideal of a ring is said to be *semiprime* if it is the intersection of a collection of prime ideals. (These ideals are also known as *radical* ideals.) Vaughan and Yeagy (1978) defined an *SP-domain* as an integral domain R for which every nontrivial ideal of R is a product of semiprime ideals. This is a natural generalization of a Dedekind domain. In fact, they showed that an SP-domain is an almost Dedekind domain, that is, every localization at a maximal ideal is a Dedekind domain. Later, Yeagy (1979) showed that if R is a union of a tower of Dedekind domains, then R is an SP-domain if and only if R has no critical maximal ideals. Recall that for an integral domain R , a maximal ideal M is said to be *critical* if every finite subset of M is contained in the square of a maximal ideal of R . Finally, this last result was generalized to arbitrary almost Dedekind domains by Olberding (2005). In particular, the following theorem and corollary were demonstrated.

Theorem 1.1 (Olberding, 2005, Theorem 2.1). *The following statements are equivalent for the almost Dedekind domain R .*

- (i) R is an SP-domain;
- (ii) R has no critical maximal ideals;

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- (iii) If I is a finitely generated ideal R , then \sqrt{I} is also finitely generated;
 (iv) Every principal ideal of R is a product of semiprime ideals.

Corollary 1.2. *Suppose R is an integral domain. R is an SP-domain if and only if R is a Prüfer domain of Krull dimension 1 and R has no critical maximal ideals.*

We wish to add another item to Theorem 1.1; one which we shall have cause to use later.

Proposition 1.3. *Suppose R is an almost Dedekind domain. R is an SP-domain if and only if every nonzero maximal ideal contains a nonzero finitely generated semiprime ideal.*

Proof. The necessity follows from (iii). As for the sufficiency we show that R has no critical maximal ideals. Let M be a maximal ideal of R . By hypothesis, let I be a nonzero finitely generated semiprime ideal contained in M . We claim that I is not contained in the square of any maximal ideal. Let N be a maximal ideal of R and suppose that $I \leq N$. Since I is semiprime and R is an almost Dedekind domain it follows that $IR_N = NR_N$, and since R_N is a Dedekind domain it follows that I is not contained in N^2 . Therefore, M is not critical, whence R is an SP-domain. \square

Throughout the rest of this article we assume that A is a Bézout domain. We let $G(A)$ denote the group of divisibility of A . Since A is Bézout we know that $G(A)$ is an abelian lattice-ordered group (or ℓ -group for short). We assume familiar knowledge of the group of divisibility of a Bézout domain, and elementary knowledge of abelian lattice-ordered groups. For more information on ℓ -groups we urge the readers to consult Gilmer (1992) or Darnel (1995). The article by Heinzer and Ohm (1971), in particular, Section 2 might also be useful. Since the ℓ -groups in question are all abelian, we will use additive notation for the operation.

The *positive cone* of G is the set $G^+ = \{g \in G : g \geq 0\}$. An ℓ -subgroup of G is a subgroup which is also a sublattice. We say the ℓ -subgroup H of G is *convex* if whenever $h_1 \leq g \leq h_2$ with $h_1, h_2 \in H$, then it follows that $g \in H$. A common example of a convex ℓ -subgroup is one of the form

$$u^\perp = \{g \in G : |u| \wedge |g| = 0\}.$$

This set is called the *polar* of u . Another example is given by first observing that given a subset of G there is a least convex ℓ -subgroup containing said set. In particular, we let $G(u)$ denote the convex ℓ -subgroup generated by u and observe that

$$G(u) = \{g \in G : |g| \leq nu \text{ for some } n \in \mathbb{N}\}.$$

When $G = G(u)$, the element u is called a *strong order unit*. A weak order unit is an element $u \in G$ such that $u^\perp = 0$. Obviously, a strong order unit is a weak order unit. The convex ℓ -subgroup P is called a *prime subgroup* whenever $a, b \in G^+$ and $a \wedge b = 0$, then either $a \in P$ or $b \in P$. (This is equivalent to the usual property that algebraists associate with prime, namely, if $a, b \in G^+$ and $a \wedge b \in P$, then $a \in P$

or $b \in P$.) It is a straightforward Zorn's Lemma argument to show that every prime subgroup contains in a minimal prime subgroup. We denote the collection of minimal prime subgroups of the ℓ -group G by $\text{Min}(G)$. For $g \in G$, we define $U(g) = \{P \in \text{Min}(G) | g \notin P\}$.

There is an order reversing bijection between the nonzero prime ideals of the Bézout domain A and the prime subgroups of $G(A)$. Since we already know that if A is an SP-domain, then its Krull dimension is 1 which translates to $G(A)$ having the property that every prime subgroup is maximal (and hence minimal). Such ℓ -groups are well-studied in the literature.

Definition 1.4. Let G be an ℓ -group. G is called *archimedean* if whenever $a, b \in G^+$ and $a \leq nb$ for all $n \in \mathbb{N}$, then $a = 0$. Not every ℓ -homomorphic image of an *archimedean* ℓ -group is archimedean. The ℓ -group G is called *hyperarchimedean* when every homomorphic image of G is archimedean. This definition is due to Conrad (1974). (They were originally called *epi-archimedean groups* but the use of this name has faded over the years.) The next theorem can be found in Conrad (1974) or in Theorem 55.1 of Darnel (1995).

Theorem 1.5. *The following are equivalent:*

- (i) G is hyperarchimedean;
- (ii) Every prime subgroup of G is maximal (and hence minimal);
- (iii) For every $g \in G$, $G = G(g) \oplus g^\perp$;
- (iv) G is ℓ -isomorphic to an ℓ -subgroup of \mathbb{R}^I , say G^* , such that for each $0 < g, h \in G^*$ there exists an $n \in \mathbb{N}$ for which $h(i) < ng(i)$ for all $i \in \{i \in I : g(i) \neq 0\}$.

Moreover, each representation of a hyperarchimedean ℓ -group as a group of real-valued functions must satisfy (iv).

Example 1.6. It is known that the class of hyperarchimedean ℓ -groups is closed under homomorphic images and ℓ -subgroups. These properties will be useful later. Also, note that for any Tychonoff space X , the set of bounded integer-valued continuous functions on X , denoted $C^*(X, \mathbb{Z})$, is a hyperarchimedean ℓ -group.

Returning to Bézout domains, we point out that for a prime ideal M of A , the group of divisibility of the localization A_M is ℓ -isomorphic to $G(A)/P$ where P is the prime subgroup associated to M . Thus, whenever A_M is a Dedekind domain it follows that $G(A)/P$ is a copy of \mathbb{Z} . Therefore, if A is a Bézout almost Dedekind domain, then every totally-ordered homomorphic image is a copy of \mathbb{Z} . Such ℓ -groups are called *hyper- \mathbb{Z}* .

Definition 1.7. Let G be an ℓ -group. We call the element $s \in G^+$ *singular* if whenever $0 \leq t \leq s$, then $t \wedge (s - t) = 0$. If s is singular it follows that for any prime subgroup P , either $s \in P$ or $s + P$ is the least positive element of G/P . We denote the collection of singular elements by S , and let $[S]$ denote the subgroup generated by S . Theorem 4.3 of Conrad and McAlister (1969) shows that $[S]$ is an ℓ -subgroup of G . Observe that for any Tychonoff space X , the singular elements of $C(X, \mathbb{Z})$ are precisely the characteristic functions on clopen subsets of X . Furthermore, $[S]$

is the collection of bounded integer-valued continuous functions on X . Whenever it happens that $G = [S]$, then G is called an S -group (or a *Specker group*). This definition is due to Conrad (1974) where this class of groups was investigated. Some other interesting results on this class of ℓ -groups can be found in Martinez (1975). One item of note is that every S -group is a hyper- \mathbb{Z} group.

Lemma 1.8. *Suppose G is a hyperarchimedean ℓ -group. G is an S -group if and only if for each $P \in \text{Min}(G)$ there is a singular element $s \in S \setminus P$.*

Proof. Suppose G is an S -group and let $P \in \text{Min}(G)$. Choose $g \in G^+ \setminus P$. By hypothesis we can write $g = n_1 s_1 + \cdots + n_k s_k$, where $s_1, \dots, s_k \in S$. If each s_i belongs to P , then so does g , whence there is a singular element of G not belonging to P .

Conversely, if $[S] < G$, then choose $g \in G^+ \setminus [S]$. As mentioned before, a Zorn's Lemma argument assures us that there is a prime subgroup P (which is minimal since G is hyperarchimedean) for which $[S] \leq P$. By hypothesis, there is an $s \in S \setminus P$. This contradicts that $s \in [S] \leq P$. \square

Theorem 1.9. *Let A be a Bézout almost Dedekind domain and set $G = G(A)$. The following are equivalent:*

- (i) A is an SP-domain;
- (ii) For every $P \in \text{Min}(G)$, there exists a $g \in G^+ \setminus P$ such that whenever $h \in G^+ \setminus P$ and $U(h) = U(g)$, then $g \leq h$;
- (iii) G is an S -group.

Proof. Suppose that A is an SP-domain and let $P \in \text{Min}(G)$. Let $M \in \text{Max}(A)$ be its corresponding maximal ideal. Since A is an SP-domain, there is a nonzero finitely generated semiprime ideal contained in M . Since A is Bézout it follows that we can assume said semiprime ideal is principal, say $aA = \sqrt{aA} \leq M$. Let g be the element of G^+ corresponding to a . Then $g \in G^+ \setminus P$. We claim g has the desired property from (ii). Suppose $h \in G^+$ with $U(g) = U(h)$. Let $b \in A$ correspond to h . It follows that a and b belong to exactly the same maximal ideals, and hence prime ideals since A is of Krull dimension 1. Therefore, $\sqrt{bA} = \sqrt{aA} = aA$, whence $b \in aA$. This translates to the inequality $g \leq h$. Hence (i) \Rightarrow (ii).

Next, suppose (ii). We show that every nonzero maximal ideal contains a nonzero principal semiprime ideal, and thus A is an SP-domain by Proposition 1.3. To that end, let M be a (nonzero) maximal of A and let P be its corresponding minimal prime subgroup of G . Let $g \in G^+ \setminus P$ with the property given by (ii). Let $a \in A$ be any element corresponding to $g \in G^+$. First, it is clear that $a \in M$. Next, we show that $\sqrt{aA} = aA$. Let $b \in \sqrt{aA}$. Since A is a Bézout domain, we assume without any loss of generality that $aA \leq bA \leq \sqrt{aA}$. This means that b belongs to exactly the same prime ideals as a . Let $h \in G^+$ be the corresponding element to $b \in A$. Then $U(g) = U(h)$ and so $g \leq h$, i.e., $bA \leq aA$. Therefore, $aA = \sqrt{aA}$, whence (ii) \Rightarrow (i).

By the lemma (iii) \Rightarrow (ii). So suppose A is an SP-domain. We show that G is an S -group. Let $P \in \text{Min}(G)$ and let $g \in G^+ \setminus P$ have the property that if $h \in G^+$ and $U(g) = U(h)$, then $g \leq h$. We claim that $g \in S$, whence G is an S -group by Lemma 1.8. To that end, suppose that g is not a singular element of G . In particular, this

means that there is an $0 \leq h \leq g$ such that $0 \neq h \wedge (g - h)$. Let $t = h \wedge (g - h) \neq 0$. We prove that $U(g - t) = U(g)$.

First of all by properties of ℓ -groups we gather that

$$g - t = (g - h) \vee h.$$

Thus, if $g - t \notin Q$, then either $g - h \notin Q$ or $h \notin Q$. Since $h \leq g$ and $g - h \leq g$ it follows that in either case $g \notin Q$, and so $U(g - t) \subseteq U(g)$. Conversely, suppose $g \notin Q$. We consider two cases: $h \in Q$ or $h \notin Q$. In the first case, $g - h \notin Q$, and thus $g - t = (g - h) \vee h \notin Q$, by convexity of Q . Similarly, if $h \notin Q$, then also $g - t = (g - h) \vee h \notin Q$. Therefore, we obtain the reverse containment $U(g) \subseteq U(g - t)$. The property assumed on g yields that $g \leq g - t$, whence $t = 0$, a contradiction. We conclude that g is singular. \square

As we noted before a Bézout domain A is an almost Dedekind domain precisely when $G(A)$ is a hyper- \mathbb{Z} group. Since there are hyper- \mathbb{Z} groups which are not S -groups it follows that there are Bézout almost Dedekind domains which are not SP-domains. Also, we point out that since a Specker group G is ℓ -isomorphic to a $C(X, \mathbb{Z})$ if and only if G possesses a weak-order unit, we conclude that a Bézout domain A is an SP-domain with nonzero Jacobson radical if and only if $G(A) \cong C(X, \mathbb{Z})$ for some compact zero-dimensional Hausdorff space. Thus, Olberding's construction (2005) is the best possible such construction. Finally, we observe that since there are Specker groups without weak-order units (e.g., the direct sum of infinitely many copies of \mathbb{Z} ordered pointwise) it follows that there are SP-domains with zero Jacobson radical. In the literature all of the examples of SP-domains have nonzero Jacobson radicals.

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