Least Integer Closed Groups[1]

Anthony W. Hager, Chawne M. Kimber [2] and Warren Wm. McGovern

This paper is dedicated to Paul Conrad
on the occasion of his 80th birthday.

ABSTRACT. An a-closure of a lattice-ordered group is an extension which is maximal with respect to preserving the lattice of convex ℓ-subgroups under contraction. We describe the a-closures of some local singular archimedean lattice-ordered groups with designated weak unit. In particular, we provide explicit descriptions of all of the a-closures of groups that are singularly convex, such as the group $C(X, \mathbb{Z})$ of continuous integer-valued functions on a zero-dimensional space.

1 Preliminaries

An (abelian) *lattice-ordered group (ℓ-group)* is a group $(G, +, \leq, \vee, \wedge)$ with a partial ordering such that $g \leq h \Rightarrow g + k \leq h + k$ for all $g, h, k \in G$, and with respect to which G is a lattice (that is, for all $g, h \in G$, the supremum $g \vee h$ and infimum $g \wedge h$ exist). An *ℓ-subgroup* is a subgroup that is also a sublattice. G^+ is the positive cone of the group and $|g| = g \vee 0 + (-g) \vee 0$. The ℓ-group G is a *(real) vector lattice* if for every $g \in G^+$, we have that $rg \in G$ for every $0 \leq r \in \mathbb{R}$. G is *archimedean* if for every $g, h \in G^+$ there exists a natural number n such that ng is not less than or equal to h. See Chapter 10 of [9] for properties of archimedean ℓ-groups. For instance, Theorem 53.3 is the result that all archimedean ℓ-groups are abelian.

An ℓ-subgroup $H \leq G$ is *convex* if $0 \leq g \leq h \in H$ implies that $g \in H$. Let $\mathfrak{C}(G)$ denote the lattice of all convex ℓ-subgroups of G. The convex ℓ-subgroup generated by an element $g \in G$ is denoted $G(g)$. If $P \in \mathfrak{C}(G)$ and for every $g, h \in G^+$, we have that

[1] **AMS Subject Classifications.** 06F25, 20F60
Keywords. lattice-ordered groups, archimedean groups, singular groups, hyperarchimedean, rings of continuous functions, groups of continuous functions

[2] This author was partially supported by a doctoral fellowship from the Florida Education Fund and was Van Vleck Visiting Assistant Professor of Mathematics at Wesleyan University during the development of this work. She thanks both institutions for their generosity.

J. Martínez (ed.), Ordered Algebraic Structures, 245–260.

$g \wedge h \in P \Rightarrow g \in P$ or $h \in P$, then we call P a *prime* subgroup. By Zorn's Lemma, minimal prime subgroups exist.

A positive element $u \in G^+$ is a *weak unit* if $\{g \in G \mid |g| \wedge u\} = \{0\}$. We work in the category $\mathbf{W}$ consisting of archimedean ℓ-groups with designated weak unit (G, u) and lattice-preserving group homomorphisms that also preserve the unit. The ℓ-group $(C(X), \mathbf{1})$ of real-valued continuous functions on a completely regular space is a $\mathbf{W}$-object under pointwise operations. In the main, we are interested in particular subgroups: $C(X, \mathbb{Z})$ is the group of all the integer-valued continuous functions; $S(X, \mathbb{R})$ consists of continuous functions with finite range; and $S(X, \mathbb{Z})$ contains the integer-valued continuous functions with finite range. Note that for any $r \in \mathbb{R}$, we use $\mathbf{r}$ to represent the function on X that is constantly equal to r.

Let (G, u) be in $\mathbf{W}$. By Zorn's Lemma, there exist convex ℓ-subgroups of G that are maximal with respect to not containing u. We call such subgroups *values* of u. Let YG be the set of values of the designated unit u. Then YG is a compact Hausdorff space in the hull-kernel topology.

Define

$$D(X) = \{ f : X \to \mathbb{R} \cup \{\pm\infty\} \mid f \text{ continuous and } f^{-1}\mathbb{R} \text{ dense } \}.$$

$D(X)$ is a lattice under pointwise operations. However the pointwise sum of two elements need not exist: for $f, g \in D(X)$, the sum $f + g$ is defined on the dense set $f^{-1}\mathbb{R} \cap g^{-1}\mathbb{R}$, though it may not extend to a continuous function on X. Recall that a completely regular space X is *quasi-F* if for every dense cozero set $A \subseteq X$, every bounded $\mathbb{R}$-valued continuous function on A has a continuous extension to X. It is shown in [17] that $D(X)$ is a group under pointwise addition (thus an ℓ-group) precisely when X is a quasi-F space. However, $D(X)$ may contain a sublattice H that is a group under the operation: $(f+g)(x) = f(x)+g(x)$ for all x in some dense subset of X. When this is the case, we call H an *ℓ-group in $D(X)$*. The following representation theorem has many incarnations, among them [20] and Theorem 2.7 of [15].

Theorem 1.1. [Yosida Embedding Theorem] *Let (G, u) be in $\mathbf{W}$. Then there is an ℓ-isomorphism of G onto $\widehat{G}$, an ℓ-group in $D(YG)$ such that $\widehat{G}$ separates the points of YG and $u \mapsto \mathbf{1}$.*

Henceforth, we identify any object (G, u) of $\mathbf{W}$ with its image in $D(YG)$.

An element $s \in G^+$ is called *singular* if $0 \leq g \leq s \Rightarrow g \wedge (s - g) = 0$. We call an ℓ-group G *singular* if for each $g \in G^+$, there exists a singular s such that $s \leq g$. In particular, (G, u) is in $\mathbf{W}$ and u is singular, then G is called singular. Define

$$\mathbf{W_s}G = \{g \in G \mid p \in g^{-1}\mathbb{R} \Rightarrow g(p) \in \mathbb{Z}\}.$$

In [13], the authors demonstrate that $\mathbf{W_s}$ is a monocoreflection of $\mathbf{W}$ into the (full) subcategory of singular groups in $\mathbf{W}$; $\mathbf{W_s}G \leq G$ is the maximum subgroup of G that is singular in $\mathbf{W}$. We will call $\mathbf{W_s}G$ the *singular part* of G.

G is a *large* ℓ-subgroup of H if for every $h \in H^+$ there is $g \in G^+$ and a natural number n such that $g \leq nh$. Let $g, h \in G^+$. We say that g and h are *a-equivalent* and write $g \sim_a h$ if there exist natural numbers n, m such that $g \leq nh$ and $h \leq mg$. If G is an ℓ-subgroup of H, then H is an *a-extension* of G if every positive element of H is a-equivalent to a positive element of G. In this case, we write $G \leq_a H$. By Theorem 2.1 of [6], $G \leq_a H$ if and only if the map $K \mapsto K \cap G$ gives a lattice isomorphism $\mathfrak{C}(H) \to \mathfrak{C}(G)$. If G has no proper a-extensions, then we say G is *a-closed.* In Example 6.4 of [6], Conrad produces two nonisomorphic abelian a-closures of an ℓ-group. Therefore, a-closures are not unique. For many other examples of this, see [11].

The first of the following properties is a generalization of Theorem 1.1 of [7]; the second property follows from Theorem 2.1 of [6].

Proposition 1.2. *Let G be an ℓ-group.*

(i) *Let $G \leq_a H$. If G is archimedean then H is archimedean.*

(ii) *If (G, u) is an object of* **W** *and $G \leq_a H$, then (G, u) is a* **W***-subobject of (H, u), $YH = YG$ and $G^* \leq_a H^*$.*

Proof. 1. Let G be an ℓ-group. Assume that H is a non-archimedean a-extension of G. Let $h_1, h_2 \in H^+$ such that $0 \leq h_2 - nh_1$ for every natural number n. There exist $g_1, g_2 \in G$ and $m_1, m_2 \in \mathbb{N}$ such that $g_1 \leq m_1 h_1$ and $h_2 \leq m_2 g_2$. Then $0 \leq m_1 m_2 g_2 - n g_1$ for every natural number n and hence G is not archimedean.

2. First, we show that u is a weak unit in H. Let $h \in H^+$. There exists $g \in G^+$ and natural numbers m, n such that $h \leq mg$ and $g \leq nh$. If $u \wedge h = 0$, then $0 \leq u \wedge g \leq u \wedge (nh) = 0$. Thus, $g = 0 = h$, as desired.

Second, we must show that $YH = YG$. The contraction mapping $\varphi : YH \to YG$ is surjective by Proposition 12.11 of [9]. By Theorem 2.1 of [7], φ is one-to-one and therefore a homeomorphism.

Let $h \in (H^*)^+$ and let $g \in G$ be such that $g \sim_a h$. Then $0 \leq g \leq mh \leq nu$ for some $m, n \in \mathbb{N}$. Therefore, $g \in G^*$ and hence $G^* \leq_a H^*$. ∎

The task we undertake is to describe some of the a-closures of ℓ-groups that are singular in **W**. First, we focus on the subclass of **W** in which these a-closures must lie. Second, we describe those ℓ-groups that are least integer closed (defined below). Third, we discuss those ℓ-groups that are strongly least integer closed (defined below) and show that all singularly convex groups (defined below) are in this class. As an example, we explicitly compute all of the a-closures of the singularly c^3 groups and, in particular, of the group $C(X, \mathbb{Z})$.

Unless otherwise stated, all groups are in W and are identified with their Yosida representations.

2 Bounded away ℓ-groups

In [7], Conrad introduced the class of *hyperarchimedean* ℓ-groups (there called *epi-archimedean*): the ℓ-groups for which every ℓ-homomorphic image is archimedean. In $\mathbf{W}$, these are precisely the groups in which $G = G(u)$ and for every $g \in G^+$, there exists $r \in \mathbb{R}$ such that $0 < g(p) \Rightarrow 0 < r \leq g(p)$; see Theorem 2.3 below and [11]. Note that with Proposition 1.2 it is evident that

Proposition 2.1. [[7], Theorem 1.1] *An a-extension of a hyperarchimedean group is also hyperarchimedean.*

Removing the $G = G(u)$ condition in the definition of hyperarchimedean ℓ-groups allows for some interesting results.

Definition 2.2. From [18]: An element $g \in D(X)$ is *bounded away from* 0 if there exists $r \in \mathbb{R}$ such that $0 < g(p) < \infty \Rightarrow 0 < r \leq g(p)$. Call a $\mathbf{W}$-object (G, u) *bounded away* if in $D(YG)$ each of the elements of G is bounded away from 0. Let S be a subgroup of $\mathbb{R}$ containing 1. We say that G is *S-bounded away* if every $g \in G$ and for each $s \in S$ there exists $r \in \mathbb{R}$ (depending on g and s) such that if $0 < s < g(p) < \infty$ then $s < r \leq g(p)$.

Since elements of singular groups are integer-valued, all singular groups are bounded away. To each element of other kinds of groups, we associate an integer-valued function by composing elements $g \in G$ with the "least integer function" on $\mathbb{R}$. That is, for $g \in G$ and $n \in \mathbb{Z}$, define $[g](p) = [g(p)] = n$ if $p \in g^{-1}(n-1, n]$ and $[g](p) = g(p)$, if $g(p) = \pm\infty$. The theorem below specifies when one can expect $[g]$ to be continuous.

Theorem 2.3. *Let (G, u) be in $\mathbf{W}$ and let $G^* = G(u)$. The following are equivalent:*

(i) *G is bounded away.*

(ii) *G is $\mathbb{Z}$-bounded away.*

(iii) *G is $\mathbb{Q}$-bounded away.*

(iv) *For every $g \in G$, the set $g(YG) \cap \mathbb{Q}$ is a closed discrete subset of $\mathbb{Q}$.*

(v) *For every $g \in G$ and every $q \in \mathbb{Q}$, then set $g^{-1}\{q\}$ is open.*

(vi) *For every $g \in G$, the zeroset $Z(g) = \{p \in YG \mid g(p) = 0\}$ is clopen in YG.*

(vii) *For every $g \in G$, the function $[g]$ is continuous on YG.*

(viii) *Every $\mathbf{W}$-homomorphic image of G is bounded away.*

(ix) *G^* is hyperarchimedean.*

(x) *G^* is the maximum hyperarchimedean convex ℓ-subgroup of G.*

(xi) *Every proper prime subgroup of G not containing G^* is a value of u.*

(xii) *Every value of u is a minimal prime subgroup of G.*

(xiii) $G^* = \{ g \in G \mid \forall h \in G, \exists m \in \mathbb{N}, |g| \wedge m|h| = |g| \wedge (m+1)|h| \}$

(xiv) $G^* = \cap_{g \in G^+} G(g) \oplus g^\perp$, *where* $g^\perp = \{h \in G \mid |g| \wedge |h| = 0\}$.

(xv) *For every $g \in G^+$, there is a natural number n such that*

$$\mathbf{1} \wedge ng \wedge (\mathbf{1} - ng) \leq 0 \quad \text{or} \quad \mathbf{1} \wedge ng \wedge (\mathbf{1} - ng)^+ = 0.$$

(xvi) *For every $g \in G^+$ there is a natural number n such that $(ng - \mathbf{1})^{+\perp} = g^\perp$.*

Proof. The equivalence of (5), (11), (12), (15) and (16) is by Theorem 1.1 of [7]; the eqivalence of (9), (13) and (14) is by Theorem 2.1 from [19]. It is evident that (5) $\Rightarrow$ (6), (8) $\Rightarrow$ (1) and (9) $\Leftrightarrow$ (10).

(1) $\Leftrightarrow$ (2): Proceed via translation. That is, G is bounded away from 0 if and only if for every $g \in G$ and for every $n \in \mathbb{N}$, $g - \mathbf{n}$ is bounded away from 0. This is equivalent to the statement that for every $g \in G$ and for every $n \in \mathbb{N}$ there exists $r \in \mathbb{R}$ such that $n < g(p) \Rightarrow 0 < n < n + r \leq g(p)$, which is exactly the definition of $\mathbb{Z}$-bounded away.

(1) $\Leftrightarrow$ (3): It is easy to verify that G is bounded away if and only if the divisible hull dG is bounded away; see Section 4 of [7] and the following. Every $g \in G$ is bounded away from 0 if and only if for every $g \in G$ and for every $0 < q \in \mathbb{Q}$ there exists $r \in \mathbb{R}$ such that $0 < g(p) \Rightarrow 0 < qr \leq qg(p)$, which is the statement that dG is bounded away.

Using this, we verify that (1) is equivalent to (3), as above. G is bounded away from 0 if and only if for every $g \in G$ and for every $0 \leq q \in \mathbb{Q}$, $g - \mathbf{q}$ is bounded away from 0 in dG. This is equivalent to the statement that for every $g \in G$ and for every $0 \leq q \in \mathbb{Q}$ there exists $r \in \mathbb{R}$ such that $q < g(p) \Rightarrow 0 < q < q + r \leq g(p)$, which is exactly the definition of $\mathbb{Q}$-bounded away.

(1) $\Leftrightarrow$ (4): Let $\{q_j\}_{j=1}^{\infty} \subseteq g(YG) \cap \mathbb{Q}$ be a convergent sequence. If $q_j \to q = \frac{m}{n} \in \mathbb{Q}$, then $ng - mu$ is not bounded away from zero. Thus, $g \in G$ is bounded away if and only if $g(YG) \cap \mathbb{Q}$ contains no convergent sequence and its limit. This is equivalent to saying that the set is closed and discrete.

(1) $\Leftrightarrow$ (9): By the definitions and Theorem 1.1 of [6].

(6) $\Rightarrow$ (5): Let $q = \frac{m}{n} \in \mathbb{Q}$, then $Z(ng - \mathbf{m}) = g^{-1}\{q\}$ is clopen by condition (6).

(6) $\Leftrightarrow$ (7): $Z(g - \mathbf{n})$ is open for every integer n and every $g \in G$ if and only if $g^{-1}\{n\}$ is open for every integer n and every $g \in G$. This is the same as the statement that $g^{-1}(n-1, n]$ is clopen for every n and every g, which precisely says that every $[g]$ is continuous.

(9) $\Leftrightarrow$ (8): Let $\varphi : G \to H$ be a surjective $\mathbf{W}$-morphism. Since G^* is hyperarchimedean, we know that $\varphi(G^*) = H^*$ is hyperarchimedean. Therefore, H is bounded away by (1) $\Leftrightarrow$ (9).

(6) $\Leftrightarrow$ (2): Condition (6) is equivalent to the property that for every $g \in G$ and for every integer n the set $g^{-1}\{n\}$ is open. This, in turn, is equivalent to: for each $g \in G$ and for each $n \in \mathbb{Z}$ there is an open neighborhood $U_n \subseteq \mathbb{R}$ such that $g(YG) \cap U_n = \{n\}$. The latter property is equivalent to (2). ■

Corollary 2.4. *If G is bounded away, then every a-extension of G is bounded away.*

Proof. Let G be bounded away. Assume that $G \leq_a H$, let $0 < h \in H^*$ and obtain $g \in G^*$ such that $g \sim_a h$. Then $Z(g) = Z(h)$. Since $Z(g)$ is clopen by Theorem 2.3, so is $Z(h)$. Therefore, H is bounded away. ■

Proposition 2.5. *The following are equivalent.*

(i) *G is $\mathbb{R}$-bounded away.*

(ii) *For each $g \in G$ and $r \in \mathbb{R}$, there is an open neighborhood $r \in V_r \subset \mathbb{R}$ such that $|g(YG) \cap V_r| < \omega$.*

(iii) *For each $g \in G$ and $r \in \mathbb{R}$, the set $g^{-1}\{r\}$ is open.*

(iv) *For each $g \in G$ and $p \in YG$ such that $g(p) \in \mathbb{R}$, there is an open neighborhood $p \in U_p \subseteq YG$ such that $|g(U_p)| = 1$.*

(v) *For each $g \in G$ and $p \in YG$ such that $g(p) \in \mathbb{R}$, there is an open neighborhood $p \in U_p \subseteq YG$ such that $|g(U_p)| < \omega$.*

(vi) *For each $g \in G$, the set $g(YG) \cap \mathbb{R}$ is closed, countable and discrete in $\mathbb{R}$.*

If G is a vector lattice, then G is bounded away if and only if G satisfies the above conditions.

Proof. The proof of the equivalence of (1) and (6) is the same as the proof of (1) $\Leftrightarrow$ (4) in Theorem 2.3.

(1) $\Leftrightarrow$ (2): An element $g \in G$ is bounded away from every $r \in \mathbb{R}$ if and only if no $r \in \mathbb{R}$ is a cluster point of the range $g(YG)$, which is precisely condition (2).

(4) $\Leftrightarrow$ (5): Clearly (4) $\Rightarrow$ (5). Conversely, let $g \in G$ and $p \in YG$. Then there is a neighborhood U'_p such that $g(U'_p) = \{r_1, \ldots r_n\}$, where $g(p) = r_1$. Then there exists $\varepsilon > 0$ such that $g(U'_p) \cap (r_1 - \varepsilon, r_1 + \varepsilon) = \{r_1\}$. Let $U_p = g^{-1}(r_1 - \varepsilon, r_1 + \varepsilon) \cap U'_p$. Then $g(U_p) = \{r_1\}$.

(3) $\Leftrightarrow$ (4): The set $g^{-1}\{r\}$ is open if and only if for every $p \in g^{-1}\{r\}$ there exists a neighborhood $U_p \subseteq g^{-1}\{r\}$.

(1) $\Leftrightarrow$ (3): An element $g \in G$ is bounded away from $r \in \mathbb{R}$ if and only if $g - \mathbf{r} \in D(YG)$ is bounded away from 0. We have already seen that this is the same as saying that $g^{-1}\{r\} = Z(g - \mathbf{r})$ is open.

It is always the case that $\mathbb{R}$-bounded away implies bounded away. If G is a vector lattice, then for all $r \in \mathbb{R}$, the constant function $\mathbf{r}$ is in G. Hence, if G is a bounded away vector lattice, $g - \mathbf{r} \in G$ is bounded away from 0 and, therefore, G must be $\mathbb{R}$-bounded away. ∎

Example 2.6. A hyperarchimedean ℓ-group that is not $\mathbb{R}$-bounded away: This example is 6.4 of [6]. Let $\alpha\mathbb{N}$ denote the one-point compactification of the discrete space $\mathbb{N}$, in which α is the point at infinity. Then $S(\alpha\mathbb{N}, \mathbb{Z})$ is the ℓ-group of eventually constant sequences of integers, under pointwise addition and the pointwise ordering. Define a new function by $b(n) = \pi + \frac{1}{n}$ and $b(\alpha) = \pi$. Then the ℓ-group $S(\alpha\mathbb{N}, \mathbb{Z}) + b\mathbb{Q}$ is hyperarchimedean. The element b fails condition (2) of Proposition 2.5, so the group is not $\mathbb{R}$-bounded away. See [11] for some generalizations of this example.

Example 2.7. A vector lattice that is not bounded away: Let $C(\mathbb{N})$ be the vector lattice of continuous real-valued functions on the discrete space $\mathbb{N}$, under pointwise addition and the pointwise ordering. In the Yosida representation $C(\mathbb{N}) \hookrightarrow D(\beta\mathbb{N})$, the Stone-extension of the function $f(n) = \frac{1}{n}$ fails to be bounded away from zero. Therefore, $C(\mathbb{N})$ is not bounded away. In fact, this example indicates how one shows that $C(X)$ is bounded away if and only if X is finite.

We note a fairly close connection with a class of ℓ-groups studied by Marlow Anderson. Let G be an ℓ-group (not necessarily in $\mathbf{W}$), and let $\mathcal{M}(G)$ be the set of prime subgroups of G that are both maximal and minimal. Then G is called *locally flat* if $\cap\mathcal{M}(G) = \{0\}$; see [1].

Theorem 2.8. [[1], Theorem 2.2] *Let G be an ℓ-group. Then G is locally flat if and only if G can be embedded as a large ℓ-subgroup of $C(X)$, for some space X such that the zerosets of G form a clopen base for the closed sets of X.*

A value V of u is called *real* if the totally ordered group G/V is archimedean, thus embeddable in $\mathbb{R}$ by Hölder's Theorem.

Proposition 2.9. *A $\mathbf{W}$-object (G, u) is locally flat if G is bounded away and the set of real values of u is dense in YG.*

Proof. Let X be the set of real values of u. Since X is dense in YG, the canonical embedding, i.e., the Yosida representation restricted to X, given by

$$G \hookrightarrow \prod_{V \in X} G/V \subseteq C(X)$$

is large. Since G is bounded away, the zerosets of G form a clopen base for the closed sets of YG and hence restrict to a clopen base for X. Thus, G is locally flat by Theorem 2.8. ∎

The converse of Proposition 2.9 is false: If Y is an infinite zero-dimensional space with a dense set X of P-points, then $C(Y)$ is not bounded away, but is locally flat since $C(Y)|_X$ is a representation as in Theorem 2.8. (See [1], 4.4.)

3 Least integer closed ℓ-groups

Definition 3.1. We call G *least integer closed* (resp., *weakly least integer closed*) if for all $g \in G$ we have $[g] \in G$ (resp., if there exists $g' \in G$ and a dense set $U \subseteq YG$ such that $[g]|_U = g'|_U$).

Proposition 3.2. *Hyperarchimedean groups are least integer closed and least integer closed groups are bounded away.*

Proof. If G is hyperarchimedean, then for every $g \in G$, we know that $[g] \in S(YG, \mathbb{Z}) \leq G$; thus, every hyperarchimedean group is least integer closed. In view of Theorem 2.3, a least integer closed group is bounded away. ∎

The examples following Theorem 3.4 illustrate that the converses of the above are not true.

Lemma 3.3. *Let G be bounded away. Then for each $g \in G^+$, we have that $[g]$ is continuous, $g \leq [g]$ and there is a positive integer m such that $[g] \leq mg$. Thus, every least integer closed group is an a-extension of its singular part.*

Proof. By Theorem 2.3, all of the functions $[g]$ are continuous since G is bounded away. Since $g \in G^+$ is bounded away from 0, there exists $r = \min(g(YG) \setminus \{0\})$. Let $m \in \mathbb{N}$ such that $mr > 1$. Then $g \leq [g]$ and $[g] \leq mg$. Finally, if G is least integer closed, then for each $g \in G^+$, we have $g \sim_a [g] \in \mathbf{W_s}G$, as desired. ∎

Theorem 3.4. *The following are equivalent.*

(i) *G is least integer closed.*

(ii) *G is weakly least integer closed and bounded away.*

(iii) *G is weakly least integer closed and $\mathbf{W_s}G \leq_a G$.*

(iv) *$G = \mathbf{W_s}G + G^*$ and G is bounded away.*

(v) *$G = \mathbf{W_s}G + G^*$ and $\mathbf{W_s}G \leq_a G$.*

Proof. That (1) $\Rightarrow$ (2) is clear; (2) $\Rightarrow$ (1) is a consequence of the equivalence of conditions (1) and (7) of Theorem 2.3. Lemma 3.3 is the interesting part of the proof of (1) $\Rightarrow$ (3). The following finishes the proof.

(3) $\Rightarrow$ (5) : By the preceding argument, $G = \mathbf{W_s}G + G^*$.

(5) $\Rightarrow$ (4) : By Corollary 2.4, G is bounded away.

(4) $\Rightarrow$ (1) : Since G is bounded away, G^* is hyperarchimedean and, hence, G^* is also least integer closed. Then for $g = f + h \in \mathbf{W_s}G + G^*$, we have $[g] = f + [h] \in G$. ∎

Each of the properties (weakly least integer closed, $\mathbf{W_s}G \leq_a G$, and $G = \mathbf{W_s}G + G^*$) appearing in Theorem 3.4 merits and will receive some concentrated attention from the authors in future papers. In particular, we have observed (as did a referee) that there is a nice connection between weakly least integer closed groups and projectable groups and this is addressed in [12]. Examples 3.7–3.10 demonstrate that the three classes introduced in the preceeding theorem are distinct. The next observation is motivating, both for the introduction of the class of least integer closed groups, and for the material of sections 4 and 5 below.

Proposition 3.5. *Let X be a zero-dimensional Tychonoff space. If $C(X,\mathbb{Z}) \leq_a H$, then H is least integer closed and thus $H = C(X,\mathbb{Z}) + H^*$ where $S(X,\mathbb{Z}) \leq_a H^*$.*

Proof. First note that $Y = YC(X,\mathbb{Z})$ is the maximal zero-dimensional compactification of X (see [13]), so $X \subseteq Y$. Now suppose that $C(X,\mathbb{Z}) \leq_a H$. By part (2) of Proposition 1.2, $YH = Y$ and $S(X,\mathbb{Z}) \leq_a H^*$ since $C^*(X,\mathbb{Z}) = S(X,\mathbb{Z})$. Let $h \in H$. By Corollary 2.4, H is bounded away, so $[h]$ is continuous on Y. Thus $[h]|_X$ is continuous. Since $h \sim_a f \in C(X,\mathbb{Z})$, the element h is real-valued on X and so is $[h]$. This shows that $[h] \in C(X,\mathbb{Z})$, so H is least integer closed and $\mathbf{W_s}H = C(X,\mathbb{Z})$. Now apply Theorem 3.4. ∎

In 5.4 below, we prove a converse to Proposition 3.5: If $S(X,\mathbb{Z}) \leq_a K$, then $C(X,\mathbb{Z}) + K$ is an ℓ-group a-extending $C(X,\mathbb{Z})$. We turn now to examples distinguishing the properties in Theorem 3.4.

Example 3.6. The following is an example of a vector lattice that is bounded away but is not an a-extension of its singular part. By Theorem 3.4, the group also is not weakly least integer closed. Let $X = \alpha\mathbb{N}$ and define $f, g \in D(X)$ by $f(n) = n^3$, $g(n) = n^{2-\frac{1}{n}}$ and $f(\alpha) = g(\alpha) = \infty$. Let G be the vector lattice spanned by $S(X,\mathbb{R})$, f and g. Then $\mathbf{W_s}G = S(X,\mathbb{Z}) + f\mathbb{Z}$. If $\mathbf{W_s}G \leq_a G$ then since g is unbounded, we must have $g \sim_a k + mf \in S(X,\mathbb{Z}) + f\mathbb{Z}$, for some nonzero integer m. Note that $k + mf \sim_a f$. Yet, if $f \sim_a g$ then there exists an integer s such that $f \leq sg$, whence $s \geq n^{1+\frac{1}{n}}$ for all n. This is a contradiction. Thus, G is not an a-extension of $\mathbf{W_s}G$.

Example 3.7. Recall that every least integer closed group is bounded away and an a-extension of its singular part. This example shows that the converse does not hold. Let $X = \alpha\mathbb{N}$ and define $f \in D(X)$ by $f(n) = 2n + \chi_E(n)$ and $f(\alpha) = \infty$, where χ_E is the characteristic function of the set of even numbers in $\mathbb{N}$. If H is the divisible hull of the ℓ-group generated by $S(X,\mathbb{R})$ and f, then H is bounded away and

$$\mathbf{W_s}H = S(X,\mathbb{Z}) + f\mathbb{Z} \leq_a H.$$

Since $[\frac{1}{2}f] \notin \mathbf{W_s}H$, the group H is not least integer closed. Moreover, there is no $f' \in H$ such that $f'|_U = [\frac{1}{2}f]|_U$, for any dense set U; so H is not weakly least integer closed either.

Example 3.8. The group $C(\mathbb{N})$ is weakly least integer closed by Theorem 3.10, but, as noted in Example 2.7, it is not bounded away. Therefore $C(\mathbb{N})$ is not least integer closed by Proposition 3.2. To see this explicitly, note that if $h(n) = \frac{1}{n}$ then $[h]$ is the characteristic function of $\mathbb{N}$, which is not continuous.

Example 3.9. Note that every element of the group $G = C(\alpha\mathbb{N})$ is bounded, so $G = \mathbf{W_s}G + G^*$, but G is not bounded away.

The following result from a paper in preparation by the authors gives a hint of the interesting properties of weakly least integer closed groups. Recall that a space X is *basically disconnected* if $cl_X \operatorname{coz}(f)$ is open for every $f \in C(X)$.

Theorem 3.10. *Let X be a Tychonoff space. The following are equivalent.*

(i) *$C(X)$ is weakly least integer closed.*

(ii) *$D(X)$ is an ℓ-group which, as a* **W***-object, is weakly least integer closed.*

(iii) *X is basically disconnected.*

4 Strongly least integer closed ℓ-groups

Let X be zero-dimensional. In Proposition 3.5, we showed that any a-extension of $C(X,\mathbb{Z})$ has the form $C(X,\mathbb{Z})+K$, where K is an a-extension of $S(X,\mathbb{Z})$. The goal of this section and the next is to generalize this result.

Theorem 4.1. *The following are equivalent:*

(i) *G is bounded away and*

$$G \leq_a H \Rightarrow \mathbf{W_s}G \leq_a \mathbf{W_s}H, \;\; G^* \leq_a H^* \text{ and } H = \mathbf{W_s}H + H^*.$$

(ii) *G is bounded away and ($G \leq_a H \Rightarrow H = \mathbf{W_s}H + H^*$).*

(iii) *Every a-extension of G is least integer closed.*

(iv) *Every a-extension of G is bounded away and*

$$G \leq_a H \Rightarrow \mathbf{W_s}G \leq_a \mathbf{W_s}H, \;\; G^* \leq_a H^* \text{ and } H = \mathbf{W_s}H + H^*.$$

(v) *G is least integer closed and ($G \leq_a H \Rightarrow H = \mathbf{W_s}H + H^*$).*

Proof. (1) $\Rightarrow$ (2): clear.

(2) $\Rightarrow$ (3): Follows from Corollary 2.4 and Theorem 3.4.

(3) $\Rightarrow$ (4): Let $G \leq_a H$. By Proposition 3.2, H is bounded away; it follows from Proposition 1.2 that $G^* \leq_a H^*$. We see that $H = \mathbf{W_s}H + H^*$ by Theorem 3.4. Lemma 3.3 shows that $\mathbf{W_s}G \leq_a G$. Therefore, $\mathbf{W_s}G \leq_a H$ and we conclude that $\mathbf{W_s}G \leq_a \mathbf{W_s}H$.

(4) $\Rightarrow$ (5): This follows from Theorem 3.4.

(5) $\Rightarrow$ (1): By Proposition 3.2, G is bounded away; it follows that H is bounded away by Corollary 2.4 and, therefore, least integer closed by Theorem 3.4. Lemma 3.3 shows that $\mathbf{W_s}G \leq_a \mathbf{W_s}H$. ∎

Definition 4.2. An ℓ-group G is *strongly least integer closed* (or *SLIC*) if it satisfies one, hence all, of the conditions of Theorem 4.1.

In section 5, we identify a fairly wide class of ℓ-groups which are stongly least integer closed. For now, we note the following.

Proposition 4.3. *Hyperarchimedean ℓ-groups are strongly least integer closed.*

Proof. Since an a-extension of a hyperarchimedean group must be hyperarchimedean, this follows from Proposition 3.2 and Theorem 3.4. ∎

Corollary 4.4. *If $\mathbf{W_s}G$ is strongly least integer closed and $\mathbf{W_s}G \leq_a G$, then G is strongly least integer closed.*

Proof. By condition (4) of Theorem 4.1. ∎

Corollary 4.5. *If G is strongly least integer closed, then $\mathbf{W_s}G \leq_a G$.*

Proof. G is least integer closed, by Theorem 4.1. Therefore, $\mathbf{W_s}G \leq_a G$ by Theorem 3.4. ∎

5 Singularly convex ℓ-groups

We now present the desired generalization of Proposition 3.5.

Definition 5.1. A singular ℓ-group G is *singularly a-closed* if it has no proper a-extension which is singular.

Example 5.2. As mentioned before, the group $\mathbf{W_s}H$ in Example 3.7 is singular, and singularly a-closed, yet it is not strongly least integer closed since $\mathbf{W_s}H \leq_a H \neq \mathbf{W_s}H + H^*$.

Example 5.3. Let G be the ℓ-group generated by $S(\alpha\mathbb{N},\mathbb{Z})$ and the function f defined by $f(x) = 2n + \pi\chi_E(n)$ if $x = n$ and $f(\alpha) = \infty$, where χ_E is the characteristic function of the set of even numbers. Then G is not strongly least integer closed, since no a-extension can contain $[f]$ and G is not an a-extension of $\mathbf{W_s}G$. Yet, $\mathbf{W_s}G = S(\alpha\mathbb{N},\mathbb{Z})$ is singularly a-closed and strongly least integer closed by Corollary 5.8 and Corollary 5.9.

Theorem 4.1 indicates that finding the a-closures of a strongly least integer closed group G should split into two steps. We must first find the maximal hyperarchimedean extensions of G^* and, secondly, find the singular a-closures of $\mathbf{W_s}G$. These are relatively difficult tasks to carry out. The first step is demonstrated in great detail in [11] for $G = S(\alpha\mathbb{N},\mathbb{Z})$. Below, we show that for certain other, (possibly unbounded) singular groups, this first step suffices also: we hold the singular part of a singularly a-closed group fixed so that we need only a-extend the bounded part of the group to compute the a-closures.

Recall that $f \in D(YG)$ is *locally in G* if for every $p \in YG$ there is a neighborhood U_p and an element $g \in G$ such that $f|_{U_p} = g|_{U_p}$. If G contains all the elements of $D(YG)$ that are locally in G, then we call G *local*. For further discussion of local groups, see [14] and [13].

Theorem 5.4. *Let G be singular and local and let $G^* \leq_a K$. Then $G + K$ is a least integer closed ℓ-group and $G \leq_a G + K$.*

Proof. Since both G and K are groups, $G + K$ is a group. We must show that it is a lattice. Let $f = g + k \in G + K$. The set $\operatorname{pos}(f) \equiv \{p \mid f(p) \geq 0\} = f^{-1}[0,\infty]$ is always closed. In this situation, it is also open: Since K is bounded, there exists an integer M such that $|k| < M$. Then

$$\operatorname{pos}(f) = \cup_{|n|\leq M}(\{p \mid k(p) \geq -n\} \cap g^{-1}\{n\}) \cup g^{-1}[M,\infty],$$

where each set appearing in this expression is open since G is singular and K is hyperarchimedean. Thus, $\operatorname{pos}(f)$ is clopen.

Let $A = \operatorname{pos}(f)$ and let $\chi_A \in S(YG,\mathbb{Z}) \subseteq G^*$ be the corresponding characteristic function. Since the unit of K is strong,

$$k\chi_A = (k \vee 0) \wedge (M\chi_A) - ((-k) \vee 0) \wedge (M\chi_A) \in K;$$

note that, in general, a $\mathbf{W}$-object will be local if its unit is strong. Since G is local, $g\chi_A \in G$. Thus $f \vee 0 = f\chi_A = g\chi_A + k\chi_A \in G + K$ and, therefore, $G + K$ is a lattice.

$G + K$ is least integer closed since for $f = g + k \in G + K$, we have that $[f] = g + [k] \in G + S(YG,\mathbb{Z}) \subseteq G$. By Lemma 3.3, we have that $G \leq_a G + K$ since $G + K$ is least integer closed and $\mathbf{W_s}(G + K) = G$. ∎

Theorem 5.5. *Let G be local and singularly a-closed. If $G^* \leq_a K$ is an a-closure, then $G + K$ is an a-extension maximal with respect to being least integer closed. In particular, if G is also strongly least integer closed, then $G + K$ is an a-closure of G.*

Proof. $G \leq_a G+K$ by Theorem 5.4. Assume that $G+K \leq_a H$ where H is least integer closed. Let $h \in \mathbf{W_s}H^+$ and obtain $g+k \in G+K$ such that $h \sim_a g+k$. Since G is local, $g+[k] \in G$ and hence, $h \sim_a g+k \sim_a g+[k] \in G$. Thus, $G \leq_a \mathbf{W_s}H$. We then conclude that $\mathbf{W_s}H = G$ since G is singularly a-closed. In addition, $H^* = K$ since K is a-closed and $K = (G+K)^* \leq_a H^*$. Thus $H = \mathbf{W_s}H + H^* = G+K$, as desired. The final statement of the theorem follows from the above since every a-extension of a strongly least integer closed group is least integer closed by Theorem 4.1. ∎

We now consider modifications to our purposes of the classes of c^3, and convex, $\mathbf{W}$-objects discussed in [16], [2], [3], [4] and [5], among other places. (See these papers for remarks on the efficacy of the classes.)

[13] defines G to be *singularly* c^3 if G is singular and for each $g_1, g_2, \ldots \in G$, each $f \in C(\cap_{n\in\omega} g_n^{-1}\mathbb{R}, \mathbb{Z})$ extends over YG to an element of G. This means that G is the direct limit of the system

$$\{\, C(U,\mathbb{Z}) \mid U = \cap_{n\in\omega} g_n^{-1}\mathbb{R}, \text{ and } g_n \in G\,\}$$

with bonding maps given by restriction of functions. This direct limit is the union of the $C(U,\mathbb{Z})$ modulo the equivalence $f_1 \sim f_2$ if $f_1 = f_2$ on the intersection of the domains. So we abbreviate the condition to $G = \cup_U C(U,\mathbb{Z})$.

We define G to be *singularly convex* if G is singular and a convex subset of $D(YG,\mathbb{Z})$. One can easily verify the following.

Proposition 5.6.

1. *If X is zero-dimensional (not necessarily compact), then $C(X,\mathbb{Z})$ is singularly c^3.*
2. *Singularly c^3 implies singularly convex.*

Theorem 5.7. *If G is singularly convex, then G is local, singularly a-closed and strongly least integer closed.*

Proof. Let G be singularly convex. If $f \in D(YG)$ is locally in G, then by compactness, $YG = \cup_{i=1}^n U_i$ for open U_i, with $g_i \in G$ for which $f|_{U_i} = g|_{U_i}$. Then $f \in D(YG,\mathbb{Z})$ and $|f| \leq \vee|g_i| \in G$, whence $f \in G$. Now suppose $G \leq_a H$ with H singular. Using (2) of Proposition 1.2, $H \subseteq D(YG,\mathbb{Z})$, and since $h \in H$ implies that $|h| \leq mg$ for some m and g, $h \in G$ by convexity. Once again let $G \leq_a H$. If $h \in H^+$ and $h \sim_a g \in G^+$ then by Theorem 2.3 we know that $[h] \in D(YG)$, $[h] \sim_a g$ and for some integer m we have $0 \leq [h] \leq mg$. Since G is singularly convex, $[h] \in G$. Thus, H is least integer closed. ∎

Corollary 5.8. *Every singularly c^3 group is strongly least integer closed. In fact, if $G = \cup_U C(U,\mathbb{Z})$, then $G \leq_a H$, if and only if $H = G + \cup_U K_U$ for groups $K_U = H^* \cap S(U,\mathbb{Z})$, where $S(U,\mathbb{Z}) \leq_a K_U$ for each U.*

Corollary 5.9.

1. *Let G be singularly convex and $G \leq H$. Then $G \leq_a H$ if and only if $H = G + K$ where $G^* \leq_a K$. Such an H is a-closed if and only if K is a-closed.*

2. *If $\mathbf{W}_s H$ is singularly convex and if H is an a-extension of $\mathbf{W}_s H$, then H is a-closed if and only if H^* is a-closed.*

Proof. These are merely restatements of Theorem 5.5 in the light of Theorem 5.7. ∎

Open questions:

(i) Must every singularly a-closed, local group that is also strongly least integer closed be singularly convex?

(ii) For what groups is G a-closed if and only if $\mathbf{W}_s G$ is singularly a-closed and G^* is a-closed?

(iii) For what class of groups is the vector lattice hull an a-closure?

(iv) Is an a-extension of a local group necessarily local?

(v) What conditions on an archimedean f-ring will guarantee that every a-closure is also an f-ring?

References

[1] M. Anderson, *Locally flat vector lattices.* Canad. J. Math. **32** no. 4, (1980), 924-936.

[2] E. Aron, *Embedding lattice-ordered algebras in uniformly closed algebras.* (1971) University of Rochester Dissertation.

[3] E. Aron and A. Hager, *Convex vector lattices and ℓ-algebras.* Topology and Appl. **12** (1981), 1-10.

[4] R. Ball and A. Hager, *One the localic Yosida representation of an archimedean lattice-ordered group with weak order unit.* J. Pure and Applied Algebra, **70** (1991), 17-43.

[5] R. Ball and A. Hager, *Algebraic extensions of an archimedean lattice-ordered group, I.* J. Pure and Applied Algebra, **85** (1993), 1-20.

[6] P. Conrad, *Archimedean extensions of lattice-ordered groups.* J. Indian Math. Soc. **30** (1966), 131-160.

[7] P. Conrad, *Epi-archimedean groups*, Czech. Math. J. **24** (99) (1974), 192-218.

[8] P. Conrad, M. Darnel and D. G. Nelson, *Valuations of lattice-ordered groups.* J. Algebra **192** no. 1, (1997), 380-411.

[9] M. Darnel, *Theory of Lattice-Ordered Groups.* Pure and Appl. Math. **187** (1995) Marcel Dekker, New York.

[10] L. Gillman and M. Jerison, *Rings of Continuous Functions.* (1960) D. Van Nostrand Publ. Co.

[11] A. Hager and C. Kimber, *Some examples of hyperarchimedean lattice-ordered groups.* Submitted.

[12] A. Hager, C. Kimber and W. McGovern *Pseudo-least integer closed groups.* In progress.

[13] A. Hager and J. Martinez, *Archimedean singular lattice-ordered groups.* Alg. Univer. **40** (1998), 119-147.

[14] A. Hager and L. Robertson, *Extremal units in an archimedean Riesz space.* Rend. Sem. Mat. Univ. Padova **59** (1978), 97-115.

[15] A. Hager and L. Robertson, *Representing and ringifying a Riesz space.* Symp. Math. **21** (1977), 411-431.

[16] M. Henriksen, J. Isbell, and D. Johnson, *Residue class fields of lattice-ordered algebras.* Fund. Math. **50** (1961), 107-117.

[17] M. Henriksen and D. Johnson, *On the structure of a class of Archimedean lattice-ordered algebras.* Fund. Math. **50** (1961), 73-94.

[18] C. Kimber and W. McGovern, *Bounded away lattice-ordered groups.* (1998) Manuscript.

[19] J. Martinez, *The hyper-archimedean kernel sequence of a lattice-ordered group.* Bull. Austral. Math. Soc. **10** (1974), 337-340.

[20] K. Yosida, *On the representation of the vector lattice.* Proc. Imp. Acad. Tokyo **18** (1942), 339-343.

Department of Mathematics, Wesleyan University, Middletown, CT 06459-0128, USA
ahager@wesleyan.edu

Department of Mathematics, Lafayette College, Easton, PA 18042, USA
kimberc@lafayette.edu

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, USA
warrenb@bgnet.bgsu.edu