COMMUNICATIONS IN ALGEBRA, 25(3), 905–921 (1997)

THE GLOBAL DIMENSION OF AN f-RING VIA ITS SPACE OF MINIMAL PRIME IDEALS

Robert T. Finn, Jorge Martinez & Warren W. McGovern
Department of Mathematics
University of Florida
Gainesville, FL 32611-8105

ABSTRACT. This article examines hereditary f-rings, after first characterizing the hereditary von Neumann regular commutative rings as those for which the space of maximal ideals is hereditarily paracompact. It is shown that if $C(X,\mathbb{Z})$, the ring of integer-valued continuous functions on a zero-dimensional space X, is hereditary, then X is finite. This is shown two ways; once as a consequence of the following: if A is any singular archimedean f-ring, then $A/2A$ is a boolean ring, and $gd(A) \geq gd(A/2A) + 1$, where $gd(A)$ stands for the global dimension of A. As a consequence of this it is also shown that if A is a singular archimedean f-ring and $gd(A) \leq 2$, then $Min(A)$, the space of minimal prime ideals is hereditarily paracompact. The paper concludes with a calculation of the global dimension of a semihereditary singular archimedean f-ring A, in which the cellularity of $Min(A)$ is "much less" than $|A|$; if finite, it is $k+2$, where $\aleph_k = |A|$.

1. INTRODUCTION

In this article all rings will be commutative, with identity, and frequently also be *semiprime*; that is, having no nonzero nilpotent elements. For later use, let us record that an *f-ring* is a lattice-ordered ring A for which $a \wedge b = 0$ implies that, for each $c \geq 0$, $a \wedge bc = 0$. The best reference for general background on f-rings remains [BKW].

All topological spaces in this article are assumed to be Hausdorff and, nearly always, zero-dimensional; that is, possessed of a base of clopen sets. $C(X, \mathbf{Z})$ stands for the ring of all continuous integer-valued functions on X, with pointwise operations, making it an archimedean semiprime f-ring.

DEFINITION & COMMENTS 1.1 (a) A ring is *hereditary* if every ideal is projective. For basic references about hereditary rings the reader is referred to [R]. However, [Gl] will also be cited. Recall that an integral domain A is hereditary if and only if it is a Dedekind domain; i.e., every ideal of A is invertible, or, equivalently, A is Noetherian, integrally closed and has Krull dimension 1. There are zillions of characterizations of Dedekind domains; we refer the reader to [AM].

If A is hereditary, then so is its classical ring of quotients, qA (4.2.20, [Gl]). From 1.3.13 in [Gl], we also have that, for each maximal ideal M, A_M is either a field or a hereditary valuation domain, and therefore a discrete valuation domain. Thus, if A is hereditary, its Krull dimension is at most 1.

For later use, let us remind the reader that a ring A is *semihereditary* if every finitely generated ideal of A is projective.

(b) $Spec(A)$ denotes the set of all prime ideals of A. Unless the contrary is indicated, we shall always regard it, as well as a number of canonical subsets, as a topological space under the hull-kernel topology.

(c) Recall that an ideal I of a ring A is *pure* if, for each $a \in A$, $aA \cap I = aI$. Equivalently, I is pure if for each $a \in I$, there is an $e \in I$ such that $ae = a$. Although we shall not use it here, observe that I is a pure ideal of A if and only if A/I is a flat A-module. (For amplification, see Theorem 1.2.15 in [Gl].) Recall that A is *von Neumann regular* if for each $a \in A$ there is a $b \in A$ such that $a^2b = a$. It is well known (see [DM]) that A is von Neumann regular if and only if every ideal of A is pure.

(d) Independently, Brookshear (in [Br]) and De Marco (in [DM]) have shown that if $C(X)$ is hereditary, then X is finite. We shall prove (Theorem 3.7) that, if X is a zero-dimensional space and $C(X, \mathbf{Z})$ is hereditary, then X is finite. (We actually prove more than this.)

(e) For later use we also record the following definitions. If M is an A-module, then the *projective dimension* of M, denoted $pd(M)$ is defined as follows. Consider a projective resolution of M:

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M = P_{-1} \longrightarrow 0 = P_{-2}.$$

We say that the *length* of the resolution is $n+1$ $(-1 \leq n < \omega)$ if n is the smallest number for which the kernel of $P_n \longrightarrow P_{n-1}$ is projective. $pd(M) = n+1$ means that M has a projective resolution of length $n+1$, but none of shorter length. Otherwise, (that is, if M has no projective resolutions of finite length), $pd(M) = \infty$.

The *global dimension*, denoted $gd(A)$, is the supremum of the projective dimensions of all the A-modules. Thus, to say that A is hereditary is to say that $gd(A) \leq 1$.

Similarly, one defines the *flat dimension* of a A-module M, using resolutions by flat modules, and denotes it $fd(M)$. The *weak dimension* of A, written $wd(A)$, is the supremum of the flat dimensions of all A-modules. It is well known that every projective module is flat; consequently, $wd(A) \leq gd(A)$. In particular, a hereditary ring has weak dimension at most 1.

It is shown in [FMMc] that, for any zero-dimensional space X, $wd(C(X,\mathbf{Z})) = 1$.

(f) We shall also have occasion to use 2.47 in [Os2]: if every ideal of A is generated by no more than $\aleph_k$ elements (with $k < \omega$), then $gd(A) \leq wd(A) + k + 1$.

2. Von Neumann Regular Rings

Our first goal is to characterize the von Neumann regular rings which are hereditary, in terms of the prime spectrum; as such, this section is quite general. Recall that a ring A is von Neumann regular if and only if it is semiprime and every prime ideal is maximal. Since every hereditary ring is semihereditary, we conclude from 4.2.19 and also 4.2.2 of [Gl]:

PROPOSITION 2.1 *Suppose A is a hereditary ring; then we have the following.*

(a) *qA is von Neumann regular and hereditary. In particular, A is a semiprime ring.*

(b) *$Min(A)$ is compact.*

(c) *Each maximal ideal of A contains a unique minimal prime ideal.*

PROPOSITION 2.2 *If A is hereditary then each minimal prime ideal of A is pure.*

PROOF: Recall some terminology at the outset. A subset $\mathcal{S}$ of prime ideals is *closed under specialization* if whenever $P \leq Q$, where Q is a prime ideal and $P \in \mathcal{S}$, then $Q \in \mathcal{S}$. We use 1.7 of [DM]: if $\mathcal{S}$ is a set of prime ideals, closed under specialization, and open in the spectral topology on $Spec(A)$, then $P = \cap\mathcal{S}^c$ is a pure ideal; ($\mathcal{S}^c$ denotes the complements of $\mathcal{S}$ in $Spec(A)$.)

Now, if $P \in Min(A)$, then $\{ Q \in Spec(A) \; : \; P \not\leq Q \}$ is open and (in view of 2.1(c)) closed under specialization. Thus, by the comment in the preceding paragraph, $P = \cap\{ Q \in Spec(A) \; : \; P \leq Q \}$ is a pure ideal. □

REMARKS 2.2.1 (a) Observe, for later use, that Proposition 2.2 works because every maximal ideal contains a unique minimal prime. The full force of heredity is not used. So 2.2 is valid for any singular f-ring. (We define this notion shortly.)

(b) In [DM], Proposition 1.14, it is shown that, for each projective ideal I of A, there is a "pure projective" hull; that is to say, there is an ideal τI, which contains I, is pure and projective, and is the smallest pure ideal of A containing I. Note as well that I is finitely generated if and only if τI is finitely generated, and, if so, then τI is, in fact, generated by an idempotent. If A is semiprime then every pure ideal is an intersection of minimal prime ideals (1.6, [DM]). For hereditary rings things are much nicer.

PROPOSITION 2.3 *Suppose that A is hereditary.*

(a) *For each ideal I there is a least pure ideal τI containing I, which is the intersection of all the minimal prime ideals that contain I.*

(b) *An ideal of A is pure if and only if it is an intersection of minimal prime ideals.*

PROOF: (a) Use Proposition 2.2 and the remarks in 2.2.1(b).

(b) The necessity is accounted for in 2.2.1(b). But, clearly, if I is an intersection of minimal prime ideals, Proposition 2.2 implies that $\tau I = I$. □

We record the following as a curiosity; the proof is left to the reader. Recall that an ideal I of A is a *d-ideal* if, for each $a \in I$, $a^{\perp\perp} \leq I$. (We use $I^{\perp}$ to denote the annihilator ideal of I.) In the context of f-rings, this concept is much used in the work of Huijsmans & de Pagter ([HP]).

COROLLARY 2.3.1 *If A is hereditary, then the ideal I is pure if and only if it is a d-ideal. In particular, every annihilator ideal is pure.*

Recall that a ring A is projectable if, for each $a \in A$, $A = a^{\perp\perp} \oplus a^{\perp}$.

COROLLARY 2.3.2 *If A is hereditary then it is projectable.*

PROOF: $a^{\perp\perp} + a^{\perp}$ is pure. However, as is well known, in a semiprime ring, this sum is contained in no minimal prime ideals. Therefore it is A. □

REMARK 2.3.3 We need a notion, adapted from [DM], to suit our purposes. An ideal I is said to be *star-finite* if there is a set of generators $\{ a_\lambda \,:\, \lambda \in \Lambda \}$ of I, so that, for each $x \in I$,

$$|\{ \lambda \in \Lambda \,:\, xa_\lambda \neq 0 \}| < \aleph_0,$$

and $x = \sum_\lambda xa_\lambda$. Observe, in addition, that if A is von Neumann regular and if I has such a generating set, then one may assume that the generators are idempotent, and, in view of the star-finiteness, that they are pairwise disjoint; i.e., that $e_\lambda e_\mu = 0$, for all $\lambda \neq \mu$. Well order the original star-finite set of idempotents, and disjointify; owing to the star-finiteness, at each stage each idempotent need only be made disjoint from finitely many.

We now have the following result:

PROPOSITION 2.4 *Suppose that A is a von Neumann regular ring. Then A is hereditary if and only if every ideal of A is star-finite.*

PROOF: Necessity is a consequence of Theorem 1.13 in [DM]. As to sufficiency, the comment in 2.3.3 implies that if I has a star-finite generating system then it is a direct sum of ideals generated by idempotents, hence projective. □

All of which leads to the following corollary. We shall let the reader go review the definition of paracompactness at leisure; we suggest [En]. "Hereditarily paracompact" simply means that every subspace is paracompact.

COROLLARY 2.4.1 (a) *Suppose that A is von Neumann regular. Then A is hereditary if and only if $Max(A)$ is hereditarily paracompact.*

(b) *If A is hereditary, then $Min(A)$ is hereditarily paracompact.*

PROOF: (a) follows directly from 2.4(a) in [DM]. (b) is a consequence of (a) and Proposition 2.1, along with the observation that $Min(A)$ and $Min(qA)$ are always homeomorphic. □

We believe the following two examples to be illustrative.

EXAMPLES 2.5 (a) Consider the ring $S(\beta\omega)$, of all real sequences having finite range. ($\beta\omega$ stands for the Stone-Čech compactification of the discrete naturals ω.) $S(\beta\omega)$ is von Neumann regular, but its maximal ideal space, that is, $\beta\omega$, is not hereditarily paracompact. To see this note that a separable space is hereditarily paracompact if and only if it is hereditarily Lindelöf (that is, every subspace is Lindelöf) (5.1.F(c), [En]), and hence perfectly normal (meaning, that every closed subset is a G_δ-set) (3.8.A(b), [En]). However, no point of $\beta\omega \setminus \omega$ is a G_δ-point (9.6, [GJ]). Thus $S(\beta\omega)$ is not hereditary.

We realize that there are other ways to see that $S(\beta\omega)$ is not hereditary. One can, for example invoke the techniques of Osofsky, from [Os2], and, in particular, 2.50 of that reference.

(b) All metric spaces are hereditarily paracompact (5.1.3, [En]). Consider now a compact, zero-dimensional metric space X and $C(X, \mathbf{Z})$. Let $\mathfrak{B}(X)$ denote the boolean algebra of all clopen subsets. It is well known that compact metric spaces have a countable base, and by 2N(3)(c) in [PW], $\mathfrak{B}(X)$ is also countable. Invoking 2.47 of [Os2], we conclude that $gd(C(X, \mathbf{Z})) \leq 2$, since $wd(C(X, \mathbf{Z})) \leq 1$. However, as we shall presently see (Corollary 3.7.1), $C(X, \mathbf{Z})$ is not hereditary; $gd(C(X, \mathbf{Z}))$ is *precisely* 2. Thus, the converse of Corollary 2.4.1(b) is false.

We now turn to f-rings.

3. HEREDITARY f-RINGS

Recall that one of the goals is to show that if $C(X, \mathbf{Z})$ is hereditary, then X is finite. The following lemma is one of the ingredients; it is well known, but we'll supply a proof, for completeness.

LEMMA 3.1 *Suppose that A is a semihereditary ring. Then every ideal I of A which is not contained in any minimal prime ideal contains a regular element.*

PROOF: If I contains no regular elements, then there is a prime ideal P, containing I, which also contains no regular elements. Since qA is von Neumann regular, and the contraction $Q \mapsto Q \cap A$ is an order-preserving bijection from $Spec(qA)$ onto the set of prime ideals which contain no regular elements, it follows that $P = Q \cap A$, for a suitable *minimal* prime ideal Q of qA. Thus, P too is minimal, and the proof is complete. □

Now for the second piece of the puzzle; this result appears in [Va], Proposition 1.6. From our point of view of pure ideals the proof takes two lines; we give that proof.

LEMMA 3.2 *Suppose that A is a hereditary ring. Then every ideal I of A which is not contained in any minimal prime is finitely generated.*

PROOF: If I is such an ideal then its pure hull τI is A, which is finitely generated, whence I is as well, by the remark in 2.2.1(b). □

The lemma which follows is known; however, since the claim is false in general (see the example due to Quentel on pp. 118-119 in [Gl]), it does not seem out of place to provide at least a sketch of the proof.

LEMMA 3.3 *If A is a semiprime f-ring for which $Min(A)$ is compact, then qA is von Neumann regular.* (Note: the converse is true for any ring.)

PROOF: By Proposition 0.2 of [M] it suffices to show that for each positive $a \in A$, there exists a $b > 0$ in A, so that $ab = 0$ and $a + b$ is regular.

Consider the set $C_a = \{0 < b \in A : a \wedge b = 0\}$. The set $U(a)$ of all minimal prime ideals which do not contain a, which is clopen, and therefore compact, is covered by the $U(b)$ $(b \in C_a)$. Thus, as in the preceding proof, one reduced to a finite subcover of $U(a)$ by $U(b_1), \cdots, U(b_k)$. Letting $b = b_1 + \cdots + b_k$ we get the element we wanted. □

DEFINITION & REMARKS 3.4 We come now to some review of certain aspects of f-rings:

(a) Recall first that the f-ring A satisfies the *bounded inversion property* if each $x \geq 1$ is invertible. This concept was invented in [HIJ], where it is shown that A has the bounded inversion property precisely when each maximal ideal is (order) convex. Note that, for each Hausdorff space X, the ring $C(X)$ of all real-valued continuous functions on X, is a semiprime f-ring with the bounded inversion property.

An f-ring A is said to be *1-convex* if $0 \leq b \leq a \in A$ imply that $b = ac$, for some $c \in A$ (and $0 \leq c \leq 1$, without loss of generality). Clearly, a 1-convex f-ring satisfies the bounded inversion property. (1-convexity was invented by Suzanne Larson.) It is well known (see [GJ], Chapter 14) that $C(X)$ is 1-convex precisely when X is an *F-space*; that is to say, when every cozeroset of X is C^*-embedded.

Suppose that A is a semiprime f-ring with the bounded inversion property. By [MW], Theorem 1, A is 1-convex precisely when every localization of A at a maximal ideal is a valuation domain. In addition, A is 1-convex if and only if it is a Bézout ring (defined in 3.6). By Theorem 1.3 in [FM], A is 1-convex if and only if $wd(A) \leq 1$.

(b) An f-ring A is said to be *singular* if the identity is a singular element. (In any lattice-ordered group G, $0 < s \in G$ is *singular* if $0 \leq x \leq s$ implies that $x \wedge (s - x) = 0$.) For the basics on singular elements we refer the reader to [CMc]. Many of the facts proved in the latter paper are summarized in [HM], and again in [FMMc]. We should like to highlight that if A is a singular f-ring, then $Min(A)$ is a compact zero-dimensional space.

(c) Recall the following terminology from ideal theory: a *fractional ideal* F is an A-submodule of qA of the form $F = x^{-1}J$, where J is an ideal of A, and $x \in A$ is regular. Under ordinary complex multiplication of sets, we say that a fractional ideal F is *invertible* if $FG = A$, for a suitable fractional ideal G. (Note: the product FG is a product of ideals, consisting of finite sums of products xy, where $x \in F$ and $y \in G$.) Recall that every invertible fractional ideal is finitely generated. In view of what we are about to state and prove, recall also the famous characterization of Dedekind domains: those for which every nonzero ideal is invertible.

The following observation is well known, though typically written down in the literature for domains only. *A finitely generated ideal I which contains a regular element is projective if and only if it is an invertible ideal.*

We are now prepared to prove the following theorem.

THEOREM 3.5 *Suppose that A is a semiprime f-ring. Then A is hereditary if and only if the following are satisfied.*

(a) *$Min(A)$ is compact and hereditarily paracompact;*

(b) *every ideal of A which is not contained in any minimal prime ideal is invertible.*

PROOF: That (a) follows is Corollary 2.4.1(b); (b) comes from Lemmas 3.1 and 3.2.

Conversely, suppose that (a) and (b) hold. By Lemma 3.3, qA is von Neumann regular. Since $Min(A)$ and $Min(qA)$ are homeomorphic, Corollary 2.4.1 implies that qA is hereditary. Now any invertible ideal is finitely generated and contains a regular element; see [LMc], 6.3(1), p. 125. By the remarks in 3.4(c) and (b) in this theorem, each ideal of A which is not contained in any minimal prime ideal is projective. Now invoke Corollary 4.2.20 of [Gl], to conclude that A itself is hereditary. □

For semiprime f-rings with bounded inversion we have the following corollary.

COROLLARY 3.5.1 *Suppose that A is a semiprime f-ring which satisfies the bounded inversion property. Then A is hereditary if and only if these conditions hold:*

(a) *$Min(A)$ is compact and hereditarily paracompact;*

(b) *every ideal of A which is not contained in any minimal prime ideal is principal, and generated by a regular element.*

REMARK 3.5.2 (a) Theorem 3.5 implies the result of Brookshear and De Marco mentioned in 1.1(d): *If $C(X)$ is hereditary then X is finite.*

For suppose that $C(X)$ is hereditary. Owing to the remarks in 3.4(a) and 4.2.19 in [Gl], $C(X)$ is 1-convex, whence we conclude that X is an F-space (see 3.4(a)), and therefore βX is also an F-space ([GJ], Theorem 14.25). If X is infinite, then βX contains a copy of $\beta\omega$ (14N.5, [GJ]). But, as we saw in Example 2.5(a), $\beta\omega$ is not hereditarily paracompact.

(b) It is also a consequence of Theorem 3.5 that if A is a hereditary semiprime f-ring, then, for each minimal prime ideal P of A, A/P is a Dedekind domain.

DEFINITION 3.6 Recall that a ring A is said to be *Bézout* if every finitely generated ideal is principal. Alling (in [Al]) showed that $C(X, \mathbf{Z})$ is always a Bézout ring.

THEOREM 3.7 *Suppose that A is a Bézout singular f-ring. Then if A is hereditary it follows that $Min(A)$ is finite.*

PROOF: We show that each $P \in Min(A)$ is isolated. Note that for each minimal prime ideal P, A/P is a totally ordered Dedekind domain in which the identity is the least positive element (and, in particular, A/P is not a field). So we may choose a maximal ideal M which contains P properly, and for some regular $g \in A$ we have $M = Ag$, by virtue of Theorem 3.5, and since A is Bézout. Without loss of generality, since $g^+ \wedge g^- = 0$, we may assume that $g > 0$.

Now, suppose that P is not isolated. Let e be the idempotent defined by $e^\perp = (g-1)^\perp$. If $e \in P$ then $h \equiv (1-e)g + e$ is idempotent; this is easily verified by checking that it is either 1 or 0 modulo every minimal prime ideal. Moreover, $h \in M$, and therefore $h = ag$, for some $a \in A$, whence $1 = ag$ modulo P, which is impossible, since either $g > 1$ mod

P. Thus, $1-e \in P$, and since P is not isolated, there is an idempotent $e' \in P$ such that $1-e < e'$. But then $h' \equiv e' + (1-e')g \in M$; under these circumstances $h' = bg$, for some $b \in A$. Now, pick any minimal prime ideal Q such that $e'e \notin Q$, and observe that $1 = bg$ mod Q, which, once again, is impossible.

This proves that every point of the compact space $Min(A)$ is isolated, which shows that $Min(A)$ is finite. □

Finally the much-announced result, which follows immediately from Theorem 3.7.

COROLLARY 3.7.1 *Suppose that X is a zero-dimensional space, and $C(X, \mathbf{Z})$ is hereditary. Then X is finite.*

The last claim in this section is a comment, which is a direct paraphrase of 3.1, in [DM]. Say that an f-ring A is *square root closed* if for each positive $a \in A$ there is a (necessarily unique positive) $b \in A$ such that $b^2 = a$.

PROPOSITION 3.8 *Suppose that A is an f-ring which is square root closed. Then if A is hereditary it is von Neumann regular.*

PROOF: The square root closure implies that $P = P^2$, for each prime ideal P. Then apply the argument in [DM], 3.1, which shows that $\tau P = P$, and hence that every prime ideal must be minimal. □

To conclude this section, some remarks and an example.

QUESTIONS & REMARKS 3.9 (a) The most obvious question, perhaps, is whether Theorem 3.7 holds without the assumption that the ring be Bézout. In the next section we show, via a different argument, that it does for all archimedean singular f-rings. However, we do not know the outcome in general.

(b) In view of the remarks in 3.4(a), we have that if A is a hereditary f-ring with bounded inversion, then $wd(A) \leq 1$, and A is 1-convex. Applying Theorem 1 of [MW] and Corollary 3.5.1, this means that

(i) $Min(A)$ is compact and hereditarily paracompact and

(ii) for each maximal ideal M, A_M is either a field or else a discrete valuation domain.

However, if A is a semiprime f-ring with the bounded inversion property, and (i) and (ii) are satisfied, then A need not be hereditary. It should be noted that (i) and (ii) imply that A is 1-convex and $Min(A) \cong Max(A)$. As already observed, $wd(A) \leq 1$, and, owing to Lemma 3.3, qA is von Neumann regular. Applying 4.2.19 of [Gl], we see that A is semihereditary.

Now, let $\Sigma = \mathbf{R}[[T]]$ denote the ring of formal power series over the reals in one variable T. This is ordered so that $1 >> T >> \cdots >> T^n >> \cdots$. Then Σ is a totally ordered discrete valuation domain which is 1-convex. Let A stand for the ring of sequences of power series $(f_1, f_2, \cdots)$ such that, for a suitable $k < \omega$, $f_k = f_{k+1} = \cdots$. The ordering of A is coordinatewise. It is easy to verify the following: (1) A is a 1-convex semiprime f-ring; (2) $Min(A) = \alpha\omega$, the one-point compactification of the discrete space ω (and hence compact and hereditarily paracompact); (3) for each maximal ideal M of A, $A_M = \Sigma$. Note also that the minimal prime ideals of A are as follows: for each natural number n, let P_n be the set of all $f \in A$ which vanish in the n-th coordinate, and P_∞ the ideal of sequences which are eventually zero.

However, the ideal I, consisting of all sequences $(f_1, f_2, \cdots) \in A$ for which the constant term is eventually zero, is not finitely generated, yet I is not contained in any minimal prime ideal. Thus, condition (b) of Corollary 3.5.1 is violated, and A is not hereditary.

Elsewhere we will show that the global dimension of A is 2.

Proposition 3.8 suggests that if an archimedean hereditary f-ring is "sufficiently" like $C(X)$ then it is von Neumann regular. Here is an example showing that such f-rings may fail to be von Neumann regular.

EXAMPLE 3.10 *An example of an archimedean hereditary f-ring with bounded inversion, which is not von Neumann regular.*

Let $A_0 = \mathbf{Q}[T]_T$, the polynomial ring over the rationals in the indeterminate T, localized at the maximal ideal generated by T. Now, we consider functions f from the set $\omega = \{ \frac{1}{n} : n < \omega \}$, with values in $\mathbf{Q}$, which we call *eventually in* A_0 if there exists a $g(T) \in A_0$ such that $f(\frac{1}{n}) = g(\frac{1}{n})$, for all but finitely many n. Denote by A the ring of all $f \in \mathbf{Q}^\omega$ which are eventually in A_0. Under pointwise ordering A is an archimedean f-ring.

Observe that if $1 < f \in A$, and $f'(T) \in A_0$ is chosen so that f is eventually f', then $f'(0) \geq 1$, which means that f' is a unit in A_0. Thus, the function f^{-1} is also eventually in A_0, proving that A satisfies the bounded inversion property. Notice, incidentally that the identity function j is eventually a nonunit in A_0, whence j is not invertible in A, which shows that A is not von Neumann regular.

Finally, note that A is projectable, and that $Max(A) \cong Min(A) = \alpha\omega$, which is compact and hereditarily paracompact. Moreover, the minimal prime ideals of A are the $M_n = \{ f \in A : f(\frac{1}{n}) = 0 \}$, for each $n < \omega$, and $P_\infty = \{ f \in A : f \text{ is eventually } 0 \}$. Then, the only ideals of A which are not contained in any minimal prime are (as the reader will easily check), for each $n < \omega$, the ideals $I_n = Aj^n$, each of which is projective. Thus, A is hereditary.

4. Global Dimension Under a Change of Rings

The goal of this section is to prove that if A is a singular archimedean f-ring and $gd(A) \leq 2$, then $Min(A)$ is hereditarily paracompact. We will also examine the relationship between the global dimension of certain f-rings and that of its subring of bounded elements.

REMARKS 4.1 We begin by recalling some results concerning global dimension under a change of rings. Let $\phi : A \longrightarrow B$ be a ring homomorphism; in general, not much can be said about the relationship between $gd(A)$ and $gd(B)$. There are two special cases in which some useful information can be obtained.

(a) *Suppose that B is a flat A-module.* Then the functor $(\cdot) \otimes B$ preserves short exact sequences and direct sums. Since every projective module is a summand of a free one, and $F \otimes B$ is B-free when F is an A-free module, it follows that if

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is a projective resolution of the A-module M, then

$$\cdots \longrightarrow P_n \otimes B \longrightarrow P_{n-1} \otimes B \cdots \longrightarrow P_0 \otimes B \longrightarrow M \otimes B \longrightarrow 0$$

is a projective resolution of $M \otimes B$ (of B-modules). It should therefore be clear that $pd_B(M \otimes B) \leq pd_A(M)$.

It is well known that the embedding of a ring in its ring of quotients is a flat embedding (see Theorem 3.73, [R]). In this context, each ideal J of qA is of the form $I \otimes qA$, for a suitable ideal I of A. (Note: $I \otimes qA = \{ \frac{a}{d} : a \in I, d \text{ regular} \}$.) Applying the comment of the preceding paragraph we have that $pd_{qA}(J) \leq pd_A(I)$, whence it follows that $gd(qA) \leq gd(A)$.

(b) *Suppose that $B = A/Ax$, where x is regular, but not invertible.* In this event, we have a theorem of Kaplansky (Theorem 1.3.3, [Gl]), which implies that $gd(A) \geq gd(A/Ax) + 1$. (See also Theorem 9.33 in [R].)

We wish to apply the preceding comments to f-rings. First, a reminder: if A is an f-ring, then A^* denotes the convex subring generated by the identity, the so-called subring of *bounded* elements of A. More precisely, $A^* = \{ a \in A : \exists n < \omega, |a| \leq n1 \}$.

THEOREM 4.2 *Suppose that A is a 1-convex semiprime f-ring. Then $gd(A^*) \geq gd(A) \geq gd(qA)$.*

PROOF: First observe that since A has the bounded inversion property, A is a subring of $q(A^*) = qA$: each $a \in A^+$ can be written in the form $a = (a \wedge 1)(a \vee 1) = (a \wedge 1)b^{-1}$, where $b = (a \vee 1)^{-1}$ (see [BKW]; the identity $a = (a \wedge 1)(a \vee 1)$ is obvious for totally ordered rings, and follows immediately for f-rings). Now, by Theorem 1 in [MW], A is a Bézout ring, and therefore Prüfer. Although there are many characterizations of Prüfer rings (see Chapter 10 of [LMc]), the relevant one here is that all the overrings of A^* in $q(A^*)$ are flat over A^*. Thus Remark 4.1(a) applies here. But observe in addition that, if I is an ideal of A^*, then $I \otimes_{A^*} A$ is the ideal of A generated by I. We show that every ideal of A has this form.

If J is an ideal of A we let $I = J \cap A^*$. Employing Theorem 1 of [MW] once more, we observe that every ideal of A is order convex. So if $a \in J^+$, $a \wedge 1 \in I$, and since $a = (a \wedge 1)(a \vee 1)$, we conclude that $a \in I \otimes_{A^*} A$, proving that $J \leq I \otimes_{A^*} A$. Since the reverse inclusion is evident, we've established the claim.

By 4.1(a), $gd(A^*) \geq gd(A) \geq gd(qA)$. □

We have two corollaries:

COROLLARY 4.2.1 *For any F-space X, $gd(C^*(X)) \geq gd(C(X))$.*

COROLLARY 4.2.2 *For any 1-convex semiprime f-ring A, if A^* is hereditary, then so is A.*

REMARK 4.2.3 (a) We don't know whether Corollary 4.2.1 is valid more generally.

(b) The converse of Corollary 4.2.2 is false. Consider $\beta\omega$ and any nonisolated point $p \in \beta\omega$. This is an F-space, and the maximal ideal M_p of all continuous functions that vanish at p is not a minimal prime ideal. Let $A = A^* = C(\beta\omega)/O_p$, where O_p is the unique minimal prime ideal contained in M_p. Since $\beta\omega$ is an F-space, A is a valuation domain and 1-convex. Then, following [FM], or using techniques of Osofsky from [Os1], we obtain that $gd(A) \geq 3$. But consider now the field of fractions $F = qA$; it follows from [CD] that $F^* = A$. Obviously, F is hereditary, but F^* is not.

This example also points out that, in the passage from a 1-convex f-ring to its quotient ring, the global dimension can decrease by as much as 3. We have no examples for which the difference exceeds 3.

Next, we wish to use Remark 4.1(b) on singular f-rings. First, we have to present a brief sketch of how one represents singular f-rings as rings of almost integer-valued functions on a compact space, via the Yosida representation. We refer the reader to [HR] for a general account of Yosida representations, and to [HM] for the singular case.

REMARKS 4.3 (a) We begin with a review of the "almost integer-valued" functions. Let X be a Hausdorff space. $D(X, \mathbf{Z})$ stands for the set of all continuous functions f defined on X with values in the extended integers, so that f is finite on a dense subset of X. It is well known that, under pointwise supremum and infimum, $D(X, \mathbf{Z})$ is a lattice. In general it is not a group or ring under pointwise operations. However, we may speak of an *ℓ-group G in $D(X, \mathbf{Z})$*, if G is a subset which is closed under addition. More precisely (and formally), if f, g & $h \in D(X, \mathbf{Z})$, say that $h = f + g$ if there is a dense open subset U of X so that $h(x) = f(x) + g(x)$, for each $x \in U$. G is an ℓ-group in $D(X, \mathbf{Z})$ if for each $f, g \in G$ there is a (unique) $h \in G$ such that $h = f + g$. Similar remarks apply to multiplication and the concept "f-ring" in $D(X, \mathbf{Z})$.

(b) We assume now that A is a singular *archimedean* f-ring; that is, with $a, b \geq 0$ in A, $na \leq b$, for each $n < \omega$, implies that $a = 0$. It is shown in [HM] that there is a compact, Hausdorff, zero-dimensional space X and an additive ℓ-isomorphism of A onto an ℓ-group A' in $D(X, \mathbf{Z})$. It is not hard to show that the representation also preserves multiplication.

Note (see [HM]) that in the above representation, $Min(A)$ may be used for X.

THEOREM 4.4 *Suppose that A is a singular archimedean f-ring. Then*

(a) $gd(A) \geq gd(A/2A) + 1$.

(b) *$A/2A$ is a boolean ring.*

(c) *The canonical map $A \longrightarrow A/2A$ induces a boolean isomorphism from the algebra of idempotents $E(A)$ of A onto $A/2A$. In particular $Min(A)$ and $Max(A/2A)$ are homeomorphic.*

PROOF: (a) Since 1 is a singular element, 2 is not invertible; but it is regular. Apply 4.1(b); note that this part of the theorem does not require A to be archimedean.

(b) To show that $A/2A$ is boolean amounts to showing that 2 divides $f^2 - f$, for each $f \in A$. For this we embed A in $D(X, \mathbf{Z})$, and identify it with its image. We assume as well that $X = Min(A)$. Now f is integer-valued on a dense open set U, and for each $x \in U$, either $f(x)$ or $f(x) - 1$ is even. This suffices to show that $A/2A$ is boolean.

(c) If e_1 and e_2 are idempotents and $e_1 - e_2$ is divisible by 2, then, clearly, $e_1 = e_2$. So the canonical map clearly induces a bijection from $E(A)$ onto $A/2A$. Since this map preserves products, to show it is a boolean isomorphism, it suffices to show that it preserves complements; but this is obvious. The final part of (c) follows from Stone Duality, and the observation that the spaces in question are Stone duals of $E(A)$ and $A/2A$. □

Theorem 4.4 immediately delivers the following corollaries. The first one harkens back to Remark 3.9(a).

COROLLARY 4.4.1 *If A is a hereditary archimedean singular f-ring, then $Min(A)$ is finite.*

PROOF: By Theorem 4.4(a), $gd(A/2A) = 0$, which implies that it has finitely many maximal ideals. 4.4(c) then does the rest. □

COROLLARY 4.4.2 *Suppose that A is a singular archimedean f-ring of global dimension two. Then $Min(A)$ is hereditarily paracompact.*

PROOF: Apply Theorem 4.4: $A/2A$ is hereditary, by (a); von Neumann regular, by (b); which implies that $Max(A/2A)$ is hereditarily paracompact, according to Corollary 2.4.1(a). By (c) of 4.4, $Min(A)$ is hereditarily paracompact. □

We conclude this section with a remark.

REMARK 4.4.3 (a) In §5 we use Theorem 4.4 to compute the global dimension of a semihereditary singular archimedean f-ring A, assuming that the cellularity of $Min(A)$ is *much less* than $|A|$. The precise meaning of the phrase "much less" will be explained.

(b) In view of Theorem 4.4(c) it seems fair to ask whether, for a singular archimedean f-ring A, qA and $A/2A$, which are both von Neumann regular rings, have the same global dimension. Corollary 2.4.1(a) also suggests that the global dimension of a von Neumann regular ring is completely determined by the space of maximal ideals. On the other hand, thanks to recent conversation with Barbara Osofsky, the authors have also been made aware that this is close to an old problem of R. S. Pierce, which seems to yield hardly at all.

(c) Note that Theorem 4.4(b) is false without the assumption that A is archimedean. Consider $\mathbf{Z}[T]$, the polynomial ring in one indeterminate, totally ordered so that $1 << T << T^2 << \cdots$. This is a singular domain, in which 2 does not divide $T^2 - T$. In fact, modulo the ideal generated by 2, one gets $\mathbf{Z}_2[T]$.

(d) We do not know whether the converse of Corollary 4.4.2 is true. We suspect that if X is a compact, hereditarily paracompact, zero-dimensional space, then $gd(C(X, \mathbf{Z})) \leq 2$, but have not been able to prove it. Example 2.5(b) does it for compact, zero-dimensional metric spaces. At the conclusion of the paper we give some more evidence, in the form of one example (Example 5.4), that this might be true. We expect to deal with this matter elsewhere.

5. When the Cellularity is Much Less than the Weight

We begin with a review of two cardinal functions in topology.

DEFINITION & REMARKS 5.1 (a) Let X be a Hausdorff space (and not necessarily zero-dimensional). A family $\mathcal{V}$ of pairwise disjoint open sets is called a *cell*; the *cellularity* of X, denoted $c(X)$ is the supremum of all cardinals of cells of X. A family $\mathcal{V}$ of open sets is a *base* for the topology if every open set is a union of members of $\mathcal{V}$, and for each pair $U, V \in \mathcal{V}$ and $p \in U \cap V$ there is a $W \in \mathcal{V}$ so that $p \in W \subseteq U \cap V$. The *weight* of X, written $w(X)$, is the least cardinal of any base for the topology on X.

It is well known that $c(X) \leq w(X)$, and that they are the same for metrizable spaces. If X is compact and zero-dimensional then $w(X) = |\mathfrak{B}(X)|$, the size of the boolean algebra of clopen subsets of X ([PW], 2N(3)(c)). In general though these cardinal invariants are different; for example, $c(\beta\omega) = \aleph_0$, but $w(\beta\omega) = \mathfrak{c}$.

We make one further comment: the notion of cellularity can be defined *mutatis mutandis* in any boolean algebra. We will speak of the *cellularity* of a boolean algebra A with this understanding. It is also denoted $c(A)$.

(b) We say that a cardinal κ is *much less than* τ if, for each cardinal $\mu < \tau$, $\mu^{<\kappa} < \tau$, where $\mu^{<\kappa}$ denotes the supremum of all the cardinals μ^λ, for $\lambda < \kappa$.

A disclaimer is probably in order, straight away: presumably, an infinite cardinal τ can be much less than itself. Such a cardinal could legitimately be called *super-strongly inaccessible*. The reader is referred to [Ku], p. 34, for definitions of *inaccessible cardinals*, both weak and strong. There it is noted that the existence of such cardinals is very much dependent on the model of set theory. Using Lemma 10.42 of [Ku] one can easily show that, under the Generalized Continuum Hypothesis (GCH), every strongly inaccessible cardinal is super-strongly inaccessible. This is as far as we wish to carry this discussion.

(c) A subset T of a boolean algebra A is said to be *independent* if for each pair of finite subsets F and G of T, with $F \cap G = \emptyset$, $(\inf F) \wedge (\inf G') \neq 0$, where G' denotes the set of complements of the elements of G.

(d) We will employ a theorem of Shelah, which can be found in [Ko], 10.1:
Suppose that A is a boolean algebra. Assume that $\lambda \leq |A|$, $c(A) < \kappa$, and κ is much less than λ, where both of these cardinals are regular. If X is any subset of A of size λ, then it contains an independent subset of size λ.

(e) With regard to independent sets, we should point out the following about the calculation of global dimension. Suppose that A is any commutative ring, and E is its boolean algebra of idempotents. In [Os2], 2.50, Osofsky shows that if T is any independent set of idempotents, and $T = \aleph_n$, with $n < \omega$, then the ideal $\langle T \rangle$ generated by T has projective dimension at least n. So if one can produce independent sets of idempotents of cardinality $\aleph_n$, where $n < \omega$, but arbitrarily large n, then $gd(A)$ is infinite.

(f) Recall (see §4.2 of [Gl]) that every semihereditary ring has weak dimension no greater than 1. In particular, $C(X, \mathbf{Z})$ is always semihereditary. In [FMMc] we characterized the semihereditary singular f-rings, and we refer the reader to Corollary 2.10.2 in that article. It is worth underscoring that for singular f-rings "semihereditary" and "Prüfer" are equivalent conditions.

We now have the following theorem. The reader should reflect in advance of it that, if A is any singular f-ring, then $Min(A)$ is the Stone dual of $E(A)$, the boolean algebra of idempotents of A.

In the context of cardinal numbers, κ^+ stands for the immediate successor of κ.

THEOREM 5.2 *Suppose that A is a semihereditary singular archimedean f-ring and $c(Min(A))^+$ is much less than $|A|$, then, if $gd(A) < \infty$,*

$$gd(A) = k + 2, \text{ where } |A| = \aleph_k, \text{ with } k < \omega.$$

PROOF: We note at the outset that, in light of 5.1(e), the assumption that $|A| = \aleph_k$, with $k < \omega$, is probably justified. As we note in the third paragraph of this proof, it is.

Next observe, using Proposition 4.5 in [HM], that each element of A is a countable supremum of integer multiples of idempotents, whence $|A| = |E(A)|$. Moreover, the following identities should be noted: $c(A/2A) = c(Min(A)) = c(E(A))$ and $|E(A)| = |A/2A| = w(Min(A))$ (from Theorem 4.4(c)).

Now, on the one hand, since A is semihereditary, by 2.47 in [Os2], $gd(A) \leq k+2$. By Shelah's Theorem (see 5.1(d)), there is an independent subset F of $A/2A$ such that $|F| = |A|$. From 2.50 in [Os2] we conclude that $pd(I) \geq k$, where I is the ideal of $A/2A$ generated by F. Thus, $k+2 \leq gd(A/2A)+1 \leq gd(A)$, from Theorem 4.4(a), which proves this theorem. □

We close the proceedings with some remarks, an example, and then another application of Shelah's Theorem, assuming the Continuum Hypothesis (CH).

COMMENT 5.3 For βD, with D discrete, one can compute the global dimension of $C(\beta D, \mathbf{Z})$ directly. Denote $2^{|D|} = \aleph_k$, where $k < \omega$ (see 5.1(e)). We are able to produce an independent family of subsets (and therefore of clopen subsets of βD) of cardinality $\aleph_k$, thanks to a result of Hausdorff ([H]). Then, exactly as in the proof of Theorem 5.2, one shows that $gd(C(\beta D, \mathbf{Z})) = k+2$.

By contrast, an example of a compact, zero-dimensional space, for which $gd(C(X, \mathbf{Z})) \neq k+2$, where $\aleph_k = w(X)$.

EXAMPLE 5.4 We consider an uncountable discrete space D, and αD, the one-point compactification of D. This is a hereditarily paracompact space for which the cellularity and weight coincide with $|D|$. We show, however, that $gd(C(\alpha D, \mathbf{Z})) = 2$. This is newsworthy, since this ring has ideals which are not countably generated!

Now, let I be an ideal of $C = C(\alpha D, \mathbf{Z})$. For each point $x \in D$, let e_x denote the characteristic function of $\{x\}$. Observe that each function in C is constant, except possibly on a finite set. Then $I = J + neC$, where $J = \Sigma_{x \in D} n_x e_x C$ and e is the characteristic function of a cofinite subset G of D, while n and the n_x are nonnegative integers. If $n = 0$ then $I = J$ which is a direct sum of principal ideals, and therefore projective. So we assume now that $n > 0$. Rewrite J as $J = \Sigma_{x \in H} n_x e_x C$, with $H \subseteq D$ and each $n_x > 0$. Next, $I/neC \cong J/(J \cap neC)$, and the latter is of the form $(\Sigma_{x \in H} n_x e_x C)/(\Sigma_{x \in H \cap G} m_x e_x C)$, where m_x is the least common multiple of n and n_x. As a quotient of two projective ideals, $pd(J/(J \cap neC)) \leq 1$. It follows that the projective dimension of I itself does not exceed 1. This proves our claim, and shows that the "conjecture" of the comments in 5.3(c) cannot be true.

The argument in this example can be generalized, and we will do that elsewhere.

Finally, a curious application of Shelah's Theorem for spaces with countable cellularity. We will review, in brief, the concept of the absolute of a compact space and its relationship to the completion of a boolean algebra. The reader is referred to [Wa] for further discussion of the absolute, and also to [PW], in which there is a richer context of this discussion.

We begin with a review of some aspects of Stone duality.

DEFINITION & REMARKS 5.5 (a) First, concerning Stone duality, recall that the dual of the free boolean algebra on κ generators is the Cantor space 2^κ, which is the product of κ copies of the discrete two-element space with the Tychonoff topology. (We refer the reader to Chapter 3 of [PW], and, in particular, to the discussion in 3K.) The Cantor space 2^κ has countable cellularity, for $\kappa > \mathfrak{c}$, but its weight is κ (3R(6) & 3O(1), [PW]).

(b) Assume that X is a compact, Hausdorff (but not necessarily zero-dimensional) space. One constructs the *Gleason cover* or the *absolute* of X, denoted EX, as the Stone dual of the boolean algebra of regular closed subsets of X. There is a continuous surjection $e_X : EX \longrightarrow X$, which maps an ultrafilter $\mathcal{U}$ of regular closed subsets to its unique point of intersection. (See Chapter 10 of [Wa].) Abstractly, the absolute of X can be characterized as the projective cover of X.

More precisely, recall that a continuous surjection $f : Y \longrightarrow X$ between compact spaces is said to be *irreducible* if $f(A) = X$, with $A \subseteq Y$ closed, implies that $A = Y$. If the spaces in question are zero-dimensional, then f is irreducible if and only if the Stone dual of X is densely embedded in that of Y.

Now, in the construction of the absolute of X it turns out that EX is extremally disconnected (meaning, that the closure of any open set is open, or, equivalently that the Stone dual is a complete boolean algebra) and the map e_X is irreducible. Conversely, if Y is a compact extremally disconnected space and $f : Y \longrightarrow X$ is an irreducible surjection, then there is a homeomorphism $g : EX \longrightarrow Y$ such that $f \cdot g = e_X$.

It is shown in Chapter 10 of [Wa] that in the category of compact Hausdorff spaces the extremally disconnected ones are precisely the projective ones, and that the absolute of X is the projective cover in the sense that if $h : Z \longrightarrow X$ is a continuous surjection, with Z extremally disconnected, then there is a continuous surjection $h' : Z \longrightarrow EX$ such that $e_X \cdot h' = h$.

If X is also zero-dimensional, then the Stone dual of e_X is the embedding of the dual of X in its completion.

(c) Note that if Y and X are compact Hausdorff spaces, and $f : Y \longrightarrow X$ is an irreducible surjection, then (see [PW], 6B(4)) that $c(Y) = c(X)$. This is not true, in general, for the weights. For example, $\beta\omega$ is the absolute of $\alpha\omega$, but while $w(\alpha\omega) = \aleph_0$, $w(\beta\omega) = \mathfrak{c}$. In any event, if $f : Y \longrightarrow X$ is an irreducible surjection, then $w(X) = |\mathfrak{B}(X)| \leq |\mathfrak{B}(Y)| = w(Y)$.

Note as well that, if $f : Y \longrightarrow X$ is an irreducible surjection between compact Hausdorff spaces, then EX and EY are homeomorphic.

We close, as promised, with an application of Shelah's Theorem, under the assumption of CH.

THEOREM 5.6 *Suppose that X is a compact, Hausdorff and zero-dimensional space, with an irreducible surjection onto the Cantor space 2^κ, with $\kappa \geq \mathfrak{c}$. Assume that κ is the largest possible cardinal with this property, and that $gd(C(X, \mathbf{Z})) < \infty$. Assume CH. Then, $\kappa = w(X) = \aleph_k$, for suitable $k < \omega$, and $gd(C(X, \mathbf{Z})) = k + 2$.*

PROOF: The thrust of our assumption is that there exist independent families of clopen sets of size κ, but none of larger cardinality. In view of 5.1(e) (again) this makes sense, and we have that $\kappa = \aleph_k$, for some $k < \omega$. Suppose that $\kappa < w(X)$, by way of contradiction. We claim that $\aleph_1$ (the successor of the cellularity!) is much less than $\aleph_{k+1}$. If this is correct then by Shelah's Theorem there is an independent family of clopen sets of size $\aleph_{k+1}$, which contradicts our choice of κ. It then follows that $\kappa = w(X)$, and the assertion about global dimension does as well.

Suppose therefore that $\kappa < \aleph_{k+1}$ and $\lambda \leq c(X)$. If $\kappa \leq \lambda$, then (10.26, Chapter 1, [Ku]) $\kappa^\lambda = 2^\lambda \leq \mathfrak{c} \leq \kappa < \aleph_{k+1}$. If $\kappa > \lambda$, then by Exercise 16, p. 45, [Ku], since $\kappa = \aleph_i$, for some $i < \omega$, $\kappa^\lambda \leq \kappa^{\aleph_0} = \kappa < \aleph_{k+1}$. This is where CH is used. This proves that $c(X)^+$ is much less than $\aleph_{k+1}$, as promised. □

There is an analogous result about boolean algebras, but we shall deal with that elsewhere. The use of CH in Theorem 5.6 is a convenience, masking a deeper and more intriguing problem, which involves the so-called "omega powers" – cardinals κ for which $\kappa^{\aleph_0} = \kappa$ - of which $\mathfrak{c}$ is the least one. This too is postponed until that "elsewhere".

REFERENCES

[Al] N. L. Alling, *Rings of continuous integer-valued functions and non-standard arithmetic.* Trans. AMS **June 1965**, 498-525.

[AM] M. F. Atiyah & I. G. MacDonald, *Introduction to Commutative Algebra.* Addison-Wesley (1969), Amsterdam-Reading-London.

[BKW] A. Bigard, K. Keimel & S. Wolfenstein, *Groupes et Anneaux Réticulés.* Lecture Notes in Math.(1977), Springer Verlag, Berlin-Heidelberg-New York.

[Br] J. G. Brookshear, *On projective prime ideals in $C(X)$.* Proc. AMS **69** (1978), 203-204.

[CD] G. Cherlin & M. Dickmann, *Real closed rings, I.* Fund. Math. **126** (1986), 147-183.

[CMc] P. F. Conrad & D. McAlister, *The completion of a lattice-ordered group.* Jour. Austral. Math. Soc. **9** (1969), 182-208.

[DM] G. De Marco, *Projectivity of pure ideals.* Rend. Sem. Mat. Univ. Padova, **68** (1983), 289-304.

[En] R. Engelking, *General Topology.* Mon. Mat. **60** (1977), Polish Scientific Publishers, Warsaw.

[FM] Robert T. Finn & Jorge Martinez, *Global dimensions of rings of continuous functions and related f-rings.* Work in progress.

[FMMc] Robert T. Finn, Jorge Martinez & Warren W. McGovern, *Commutative singular f-rings.* Submitted.

[GJ] L. Gillman & M. Jerison, *Rings of Continuous Functions.* GTM **43** (1976), Springer Verlag, New York-Heidelberg- Berlin.

[Gl] S. Glaz, *Commutative Coherent Rings.* Lec. Notes in Math. **1371**, (1989) Springer Verlag, Berlin-Heidelberg-New York.

[HM] A. W. Hager & J. Martinez, *Singular archimedean ℓ-groups.* preprint.

[HR] A. W. Hager & L. C. Robertson, *Representing and ringifying a Riesz space.* Symposia Math. **21** (1977), 411-431.

[H] F. Hausdorff, *Über zwei Sätze von G. Fichtenholz und L. Kantorovich.* Studia Math. **6** (1936), 18-19.

[HIJ] M. Henriksen, J. R. Isbell & D. G. Johnson, *Residue class fields of lattice-ordered algebras*. Fund. Math. **50** (1961), 107-117.

[HP] C. B. Huijsmans & B. de Pagter, *On z-ideals and d-ideals in Riesz spaces, I.* Indag. Math. Vol. **242** (Fasc. 2) (1980), 183-195.

[Ko] S. Koppelberg, *Handbook of Boolean Algebra, I.* North-Holland.

[Ku] K. Kunen, *Set Theory; an Introduction to Independence Proofs*. Studies in Logic **102** (1980), North Holland, Amsterdam-New York-Oxford-Tokyo.

[LMc] M. D. Larsen & P. J. McCarthy, *Multiplicative Theory of Ideals*. Pure & Appl. Math. **43** (1971), Academic Press, Boston-San Diego-London.

[M] J. Martinez, *The maximal ring of quotients of an f-ring*. Alg. Univ. **33** (1995), 355-369.

[MW] J. Martinez & S. Woodward, *Bézout and Prüfer f-rings*. Comm. in Alg. **20** (10) (1992), 2975-2989.

[Os1] B. L. Osofsky, *Global dimension of valuation rings*. Trans. AMS **127** (1967), 136-149.

[Os2] B. L. Osofsky, *Homological Dimensions of Modules*. CBMS Reg. Conf. Series in Math. **12** (1973), American Mathematical Society, Providence, RI.

[PW] J. R. Porter & R. G. Woods, *Extensions and Absolutes of Hausdorff Spaces.* (1989) Springer Verlag, New York- Berlin-Heidelberg.

[R] J. J. Rotman, *An Introduction to Homological Algebra*. Pure & Appl. Math. **85** (1979), Academic Press, Boston- San Diego-London.

[Va] W. V. Vasconcelos, *The Rings of Dimension Two.* Lecture Notes in Pure & Appl. Math. **42** (1976), Marcel Dekker, New York.

[Wa] R. C. Walker, *The Stone-Čech Compactification*; Ergebn. Math. **83** (1974), Springer Verlag, Berlin-Heidelberg-New York.

Received: April 1996