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Bézout rings with almost stable range 1

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Abstract

Elementary divisor domains were defined by Kaplansky [I. Kaplansky, Elementary divisors and modules, Trans. Amer. Math. Soc. 66 (1949) 464–491] and generalized to rings with zero-divisors by Gillman and Henriksen [L. Gillman, M. Henriksen, Some remarks about elementary divisor rings, Trans. Amer. Math. Soc. 82 (1956) 362–365]. In [M.D. Larsen, W.J. Lewis, T.S. Shores, Elementary divisor rings and finitely presented modules, Trans. Amer. Math. Soc. 187 (1) (1974) 231–248], it was also proved that if a Hermite ring satisfies (N), then it is an elementary divisor ring. The aim of this article is to generalize this result (as well as others) to a much wider class of rings. Our main result is that Bézout rings whose proper homomorphic images all have stable range 1 (in particular, neat rings) are elementary divisor rings.

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1. Introduction

The study of elementary divisor rings and Hermite rings has a rich history. Kaplansky [14] defined the ring R to be an *elementary divisor ring* if every matrix M over R (not necessarily square) admits diagonal reduction, that is, there exist invertible square matrices P and Q such that PMQ is a diagonal matrix, say (d_{ij}), for which d_{ii} is a divisor of $d_{i+1,i+1}$ for each i. He defined a ring R to be a *right Hermite ring* if every 1×2 matrix over R admits diagonal reduction and showed that a right Hermite ring is a right Bézout ring, i.e., a ring for which every finitely generated right ideal is principally generated. For integral domains the notions of right Hermite and right Bézout are equivalent.

A few years after Kaplansky's work on elementary divisor rings, Gillman and Henriksen [7] examined commutative Hermite and elementary divisor rings with zero-divisors. They discovered equivalent conditions for a ring to be Hermite and an elementary divisor ring, respectively. They showed that the commutative ring A is a Hermite ring if and only if for all $a, b \in A$ there exists $a_1, b_1, d \in A$ such that $a = a_1 d, b = b_1 d$, and $a_1 A + a_2 A = A$. Furthermore, they proved the following result which we state formally.

Theorem 1.1 (*Theorem 6, [7]*). *Suppose that A is a commutative ring with identity. A is elementary divisor ring if and only if A is a Hermite ring that satisfies the extra condition that for all $a, b, c, \in A$ with $aA + bA + cA = A$ there exist $p, q \in A$ such that $paA + (pb + qc)A = A$.*

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They used the conditions to construct examples of commutative rings with zero-divisors that are Bézout but not Hermite, and Hermite yet not elementary divisor rings. They also raised the question of whether the three properties are equivalent for domains.

Kaplansky showed that for the class of adequate domains being a Hermite ring was equivalent to being an elementary divisor ring. Gillman and Henriksen showed that this was also true for rings with zero-divisors. Later, Henriksen [11] proved that the hypothesis of adequacy could be weakened to a property that we call Henriksen's Hypothesis. We include those definitions now. We let $n(A)$ and $\mathfrak{J}(A)$ denote the nil radical and Jacobson radical of A, respectively.

Definition 1.2. A commutative ring A is called an *adequate ring* if for $a, b \in A$ with $a \neq 0$, there exist $r, s \in A$ such that $a = rs$, $rA + bA = A$, and if s' is a non-unit divisor of s, then $s'A + bA \neq A$.

The notion of an adequate domain was originally defined by Helmer [10]. The ring of entire functions on the complex plane is an adequate Bézout domain.

Definition 1.3. We say the commutative ring A satisfies *Henriksen's hypothesis* if for every $a, b \in A$ with $a \notin \mathfrak{J}(A)$ there is an $r \in A$ such that the set of maximal ideals of A containing r is precisely the set of maximal ideals of A containing a that do not contain b.

An adequate ring satisfies Henriksen's hypothesis (see [11]). In the last section we will supply an example of a ring satisfying Henriksen's hypothesis which is not an adequate ring.

Larsen, Lewis, and Shores [15] also investigated elementary divisor rings. They defined the following important property.

Definition 1.4. The commutative ring A is said to *satisfy (N)* if for $a, b \in A$ with $a \notin \mathfrak{J}(A)$, (1) there exists $m \in A$ such that $bA + mA = A$, and (2) if whenever $n \in A$ for which $nA + aA \neq A$ and $nA + bA = A$, then $nA + mA \neq A$.

They showed that if a commutative ring satisfies Henriksen's hypothesis, then it satisfies (N). Furthermore, Corollary 2.6 of [15] states that a Hermite ring satisfying (N) is an elementary divisor ring. In principle, this result strengthened Henriksen's result. Interestingly, the authors did not provide an example of a Hermite ring satisfying (N) but not satisfying Henriksen's hypothesis.

At the end of Section 2, after a description of some tools used in the article, we show that no such example exists. On the way, we give some interesting results regarding the maximal ideal space of a commutative ring with identity. Section 3 contains the main theorem of the paper, stating that a Bézout ring which has almost stable range 1 is an EDR. This theorem generalizes many results previously proved on the subject. We will also introduce the new class of almost semi-clean rings, showing that it includes the class of rings satisfying Henriksen's hypothesis and is included in the class of rings which have almost stable range 1. In Section 4 we provide new characterizations of Bézout domains satisfying Henriksen's hypothesis (or which are almost semi-clean) in terms of their groups of divisibility viewed as ℓ-groups. These characterizations enable us to furnish examples of almost semi-clean Bézout domains neither satisfying Henriksen's hypothesis, nor neat.

2. Preliminaries

Throughout the rest of this article all rings are assumed to be commutative and possess an identity.

We denote the set of maximal ideals of A by $\mathrm{Max}(A)$. For $a \in A$,

$$V(a) = \{M \in \mathrm{Max}(A) : a \in M\}$$

and $U(a) = \mathrm{Max}(A) \smallsetminus V(a)$. The collection of sets of the form $U(a)$ forms an open base for a topology on $\mathrm{Max}(A)$ called the *Zariski* (or hull-kernel) topology. In particular observe that $U(a) \cap U(b) = U(ab)$ for all $a, b \in A$. The Zariski topology makes $\mathrm{Max}(A)$ a compact T_1-space. Notice that Henriksen's hypothesis can be restated as follows. For every $a, b \in A$ with $a \notin \mathfrak{J}(A)$, there is an $r \in A$ such that $V(r) = V(a) \smallsetminus V(b)$.

The maximal ideal space of A can also be equipped with another natural topology. Since for each pair of finitely generated ideals $I, J \leq A$, $V(I) \cap V(J) = V(I + J)$ it follows that the collection of sets

$$\{V(I) : I \text{ is a finitely generated ideal of } A\}$$

forms a base for a topology on $\mathrm{Max}(A)$. We call this topology the *inverse topology*. In general, the Zariski topology and the inverse topology are different as the inverse topology is not commonly compact. Recall that a topological space is said to be *zero-dimensional* if it has a base of clopen subsets.

Lemma 2.1. *Suppose A is a commutative ring with identity. The inverse topology on* $\mathrm{Max}(A)$ *is a zero-dimensional Hausdorff topology. Moreover, the inverse topology on* $\mathrm{Max}(A)$ *is finer than the Zariski topology.*

Proof. Let $M, N \in \mathrm{Max}(A)$ be distinct maximal ideals and let $a \in M \setminus N$. Then there are $x \in A$ and $n \in N$ such that $1 = xa + n$. It follows that $V(a) \cap V(n) = \emptyset$. Since $M \in V(a)$ and $N \in V(n)$ the inverse topology on $\mathrm{Max}(A)$ is Hausdorff. Next, let I be a finitely generated ideal of A and define

$$R_I = \{x \in A : I + xA = A\}.$$

Observe that R_I is the set of elements of A which are comaximal to I. It is straightforward to check that

$$V(I) = \bigcap_{x \in R_I} U(x).$$

Since each $U(x)$ is a basic closed set in the inverse topology it follows that $V(I)$ is closed, and hence clopen. Thus, the inverse topology has a base of clopen sets. Moreover, since each $U(a)$ is clopen it follows that the inverse topology is finer than the Zariski topology. ∎

Proposition 2.2. *Let A be a commutative ring with identity. The inverse topology on* $\mathrm{Max}(A)$ *is compact if and only if the inverse topology on* $\mathrm{Max}(A)$ *equals the Zariski topology.*

Proof. Since the Zariski topology is compact the sufficiency follows. So suppose that the inverse topology on $\mathrm{Max}(A)$ is compact, and thus by the previous proposition, makes $\mathrm{Max}(A)$ into a compact, zero-dimensional Hausdorff space. Notice that for each $a \in A$ we have

$$U(a) = \bigcup_{b \in R_a} V(b).$$

Since $U(a)$ is a closed subset of a compact space it is also compact and therefore,

$$U(a) = V(b_1) \cup \cdots \cup V(b_n) = V(b_1 \cdots b_n)$$

for some finite collection of elements $b_1, \cdots, b_n \in R_a$. Hence, for every $a \in A$ there is some $b \in A$ such that $U(a) = V(b)$. From here it is straightforward to show that for every finitely generated ideal I of A, there is some $b \in A$ such that $V(I) = U(b)$ and so the Zariski topology equals the inverse topology. ∎

Theorem 2.3. *Let A be a Bézout ring. A satisfies (N) if and only if A satisfies Henriksen's hypothesis.*

Proof. The sufficiency is shown in the remarks after Corollary 2.6 of [15].

Let $a \in A$ with $a \notin \mathfrak{J}(A)$ and $b \in A$. Since $V(a) \setminus V(b) = V(a) \setminus V(d)$ where d is a gcd of a and b, we assume without loss of generality that $b|a$. Since A satisfies (N) there is an $m \in A$ for which $bA + mA = A$, and if whenever $n \in A$ for which $nA + aA \neq A$ and $nA + bA = A$, then $nA + mA \neq A$. Once again without loss of generality we assume that $m|a$. It follows that $V(b), V(m) \subseteq V(a)$. We claim that $V(m) = V(a) \setminus V(b)$. Since $V(m) \cap V(b) = \emptyset$ one containment is clear. As for the reverse, suppose by way of contradiction that there is an $M \in V(a) \setminus V(b)$ and $M \notin V(m)$. Since $V(a) \setminus V(b)$ is a clopen subset of $\mathrm{Max}(A)$ with respect to the inverse topology it follows that there is a basic open set of the form $V(e)$ for which $M \in V(e) \subseteq V(a) \setminus V(b)$ and $V(e) \cap V(m) = \emptyset$. Since $V(e) \subseteq V(a)$ we assume without loss of generality that $e|a$, thus $eA + aA = eA < A$. Since $V(e) \cap V(b)$ it follows that $eA + bA = A$. Therefore, by choice of m we have that $mA + eA < A$. But this contradicts that $V(m) \cap V(e) = \emptyset$. ∎

In the next section we show that Corollary 2.6 of [15] can be generalized to a wider class of rings.

3. Almost stable range 1

Definition 3.1. If whenever $aA + bA = A$ there is an $x \in A$ such that $(a + bx)A = A$, then we say A has *stable range* 1. We say that A has *stable range 2* if whenever $aA + bA + cA = A$ there are $x, y \in A$ such that

$(a + cx)A + (b + cy)A = A$. We say that A has *almost stable range* 1 if every proper homomorphic image of A has stable range 1.

Observe that A has stable range 1 if and only if $A/\mathfrak{J}(A)$ has stable range 1, and similarly for stable range 2. It is a well-known result of Vaserstein (Theorem 1 of [18]) that if A has stable range 1, then it has stable range 2. For more information on Bézout domains with stable range 1 we encourage the reader check out [17].

Proposition 3.2. *Suppose A is a Bézout ring. If A has stable range* 1*, then A has almost stable range* 1.

Proof. Let I be a nonzero, proper ideal of A. Let $R = A/I$ and for any $x \in A$, denote $\overline{x} = x + I$. To show that R has stable range 1, suppose $\overline{r}R + \overline{s}R = R$. It is straightforward to check that there is some $i \in I$ for which

$$rA + sA + iA = A.$$

Since A is a Bézout ring we let d be a gcd of s and i, and write $d = sp + iq$. Therefore, $rA + dA = A$ and so by hypothesis there is an $x \in A$ such that $(r + xd)A = A$. Another straightforward check gives us that $\overline{r + s(px)}R = R$, whence R has stable range 1. ■

Remark 3.3. We are unable to determine whether Proposition 3.2 can be generalized to arbitrary rings.

In a recent paper by Zabavs'kyi [19], the stable range of a commutative ring is used to characterize the class of Hermite rings within the class of Bézout rings. This result will be pivotal in our work.

Theorem 3.4 (*Theorem 1, [19]*). *Let A be a Bézout ring. Then A is Hermite if and only if A has stable range 2.*

The following consequence of Theorem 3.4 is actually a generalization of Theorem 3 of [11] from Hermite rings to Bézout rings.

Proposition 3.5. *Suppose A is a Bézout ring. A is an elementary divisor ring if and only if $A/\mathfrak{J}(A)$ is an elementary divisor ring.*

Proof. For the ease of the reader let $R = A/\mathfrak{J}(A)$ and for any $x \in A$, set $\overline{x} = x + \mathfrak{J}(A)$.

The necessity is clear since the homomorphic image of an elementary divisor ring is again an elementary divisor ring. As for the sufficiency suppose $A/\mathfrak{J}(A)$ is an elementary divisor ring (and hence a Hermite ring). Let $aA + bA + cA = A$ and notice that

$$\overline{a}R + \overline{b}R + \overline{c}R = R.$$

Therefore, by Theorem 3.4, R has stable range 2 and so there are $\overline{x}, \overline{y} \in R$ such that

$$\overline{a + cx}R + \overline{b + cy}R = R.$$

This means that the elements $a + cx$ and $b + cy$ are comaximal in A, whence $(a + cx)A + (b + cy)A = A$ and so A has stable range 2. Applying Theorem 3.4 once again we conclude that A is a Hermite ring. Finally, from Theorem 3 of [11] we conclude that A is an elementary divisor ring. ■

Theorem 3.6. *Suppose A has almost stable range* 1*. Whenever $a \notin \mathfrak{J}(A)$ and $aA + bA + cA = A$ there is a $y \in A$ such that $aA + (b + cy)A = A$. In particular, A has stable range* 2*.*

Proof. Suppose A has almost stable range 1 and that $aA + bA + cA = A$. First, suppose that $a \notin \mathfrak{J}(A)$. If a is a unit of A, then $aA + (b + yc)A = A$ for every $y \in A$. Thus, we can further assume that a is not a unit of A, whence $V(a) \neq \emptyset$. Define

$$I = \bigcap_{M \in V(a)} M$$

and set $R = A/I$. Since $a \in I$ it follows that $0 \neq I$ and so by hypothesis R has stable range 1. For any $x \in A$, let $\overline{x} = x + I \in R$. Since $\overline{b}R + \overline{c}R = R$ there is a $y \in A$ such that $\overline{b + yc}R = R$.

We claim that $aA + (b + yc)A = A$. If not, then there would be a maximal ideal M for which $a, b + yc \in M$. But this is impossible since this would imply that M/I is a maximal ideal of R containing $\overline{b + yc}$. Consequently, $aA + (b + yc)A = A$; proving the first statement of the proposition.

Next, suppose $a \in \mathfrak{J}(A)$, then $bA+cA = A$ and so letting $x = 1$ and $y = 0$ gives us that $(a+cx)A+(b+cy)A = cA + bA = A$. Since we have considered both case we have shown that A has stable range 2. ■

Returning to Bézout rings we now give our promised result generalizing and strengthening Corollary 2.6 of [15].

Theorem 3.7. *Suppose A has almost stable range* 1. *Then A is a Bézout ring if and only if A is an elementary divisor ring.*

Proof. Suppose A is a Bézout ring that has almost stable range 1. By Theorem 3.6 A has stable range 2, and so by Theorem 3.4 A is a Hermite ring. Finally, Theorem 1.1 and Theorem 3.6 allow us to conclude that A is an elementary divisor ring. ■

We conclude this section by showing that there is a nice source of rings which have almost stable range 1. The notion of a clean ring plays a pivotal role.

Definition 3.8. Let A be a ring and $a \in A$. If a can be written as the sum of a unit and an idempotent, then a is called a *clean* element of A. If every element of A is clean, then we call A a *clean ring*. We call a ring for which every proper homomorphic image is clean a *neat ring*. For a historical account of clean rings and information on neat rings please see the article [16].

We shall need the following two characterizations of commutative clean rings. We let $Id(A)$ denote the collection of idempotents of A.

Theorem 3.9 (*Theorem 1.6, [16]*). *The ring A is clean if and only if the collection of idempotent clopen subsets* $\mathcal{E} = \{U(e) : e \in Id(A)\}$ *is a base for the Zariski topology on* Max(A). *In this case* Max(A) *is a boolean space, that is,* Max(A) *is a compact, zero-dimensional, Hausdorff space.*

Theorem 3.10 (*Theorem 12, [2]*). *Let A be a commutative ring with identity. A is a clean ring if and only if whenever* $aA + bA = A$ *there is an idempotent* $e \in A$ *such that* $(a + eb)A = A$.

It follows that a clean ring has stable range 1 (see [2] where the above condition is called *idempotent stable range* 1). Furthermore, a neat ring has almost stable range 1. Therefore, from Theorem 3.7 we gather the affirmative answer to a personal question raised by M. Henriksen [13] of whether every neat Bézout domain is an elementary divisor domain.

Definition 3.11. We call a commutative ring A a *semi-clean ring* if whenever $a, b \in A$ such that $aA + bA = A$ there are $x, y \in A$ such that (1) $xA + yA = A$, (2) $xy \in \mathfrak{J}(A)$, (3) $a|x$ and $b|y$. If every proper homomorphic image of A is a semi-clean ring, then we say A is an *almost semi-clean ring*. We shall see later why the name of this class of rings makes sense.

Proposition 3.12. *Let A be a commutative ring with identity. A is a semi-clean ring if and only if* Max(A) *is a zero-dimensional topological space with respect to the Zariski topology. Therefore, A is a semi-clean ring if and only if* $A/fJ(A)$ *is a semi-clean ring.*

Proof. Suppose A is a semi-clean ring and let $M \in U(a)$ for $a \in A$. This means that there is an $m \in M$ and $r \in A$ such that $1 = ra + m$. Thus, $aA + mA = A$ and so by hypothesis there are $x, y \in A$ such that $xA + yA = A$, $xy \in \mathfrak{J}(A)$, $a|x$, $m|y$.

Consider $U(x)$. Since $U(x) \cap U(y) = U(xy) = \emptyset$ where the last equality follows from the fact that $xy \in \mathfrak{J}(A)$. Since $U(x) \cup U(y) = U(xA + yA) = $ Max(A) we gather that $U(x)$ is a clopen subset of Max(A). That a divides x means that whenever x does not belong to a maximal ideal, then neither does a. Equivalently, this means that $U(x) \subseteq U(a)$. Finally, $m \in M$ implies $y \in M$, and so $x \notin M$. Therefore, $M \in U(x) \subseteq$ which shows that Max(A) has a base of clopen subsets, i.e., Max(A) is zero-dimensional.

Conversely, suppose Max(A) is zero-dimensional. Notice that a zero-dimensional T_1-space is in fact Hausdorff. Therefore, Max(A) is a compact, zero-dimensional Hausdorff space, and so disjoint closed sets can be separated by a clopen subset. Let $a, b \in A$ satisfy $aA + bA = A$. This translates to $V(a) \cup V(b) = \emptyset$ and so there is a clopen subset of Max(A), say K, such that $V(a) \subseteq K$ and $V(b) \cap K = \emptyset$. Since K is open it can be written as the union of sets of the form $U(x)$. Since K is also compact it can be written as the finite union of sets of that form, yielding $K = U(I)$

for some finitely generated ideal I of A. Similarly, $\mathrm{Max}(A) \smallsetminus K = U(J)$ for some finitely generated ideal J of A. It follows that $U(I) = V(J)$ and $U(J) = V(I)$, whence $I + J = A$ and $IJ \leq \mathfrak{J}(A)$. Since $I + J = A$ we can write $1 = y + x$ for some $y \in I$ and $x \in J$. Notice that $U(x) \cap U(y) = \emptyset$ and $U(x) \cup U(y) = \mathrm{Max}(A)$ so that in fact $U(I) = U(y)$ and $U(J) = U(x)$.

Next, notice that $V(a) \subseteq V(x)$ means that $V(ax) = V(a) \cup V(x) = V(x)$ so without loss of generality, we assume that $a|x$ and $b|y$. Finally,

$$U(xA + yA) = U(x) \cup U(y) = \mathrm{Max}(A) \quad \text{and} \quad U(x) \cap U(y) = \emptyset$$

translates to $xA + yA = A$ and $xy \in \mathfrak{J}(A)$, whence A is a semi-clean ring.

Finally, since $\mathrm{Max}(A)$ and $\mathrm{Max}(A/\mathfrak{J}(A))$ are homeomorphic topological spaces with respect to the Zariski topology it follows that A is a semi-clean ring if and only if $A/\mathfrak{J}(A)$ is a semi-clean ring. ■

Theorem 3.13. *A commutative clean ring is a semi-clean ring, and a neat ring is an almost semi-clean ring. Furthermore, A is a semi-clean ring if and only if $A/\mathfrak{J}(A)$ is a clean ring.*

Proof. If A is a clean ring, then by Theorem 3.9 $\mathrm{Max}(A)$ is a zero-dimensional topological space and so by Proposition 3.12 A is a semi-clean ring. It then follows that if every proper homomorphic image of A is clean, i.e. A is a neat ring, then A is an almost semi-clean ring.

As for the second statement the sufficiency is clear. So suppose that A is a semi-clean ring. We aim to show that $A/\mathfrak{J}(A)$ is a clean ring. Because we already know that $A/\mathfrak{J}(A)$ is a semi-clean ring it suffices to show that a semi-clean ring A for which $\mathfrak{J}(A) = 0$ is a clean ring. To that end let $a \in A$ and $M \in U(a)$. Choose $m \in M$ such that $aA + mA = A$ and by hypothesis there are $x, y \in A$ such that (1) $xA + yA = A$, (2) $xy = 0$, and (3) $a|x$, $m|y$. By (1) and (2) we assume, without loss of generality, that $x, y \in Id(A)$. By (3) $x \notin M$, i.e., $M \in U(x)$. Also by (3) if $N \in U(x)$, then $N \in U(a)$. It follows that the collection of clopen idempotents subsets of $\mathrm{Max}(A)$ forms a base for the Zariski topology, whence A is a clean ring by Theorem 3.9. ■

Proposition 3.14. *If A is an almost semi-clean ring, then A has almost stable range* 1.

Proof. Notice that it suffices to show that a semi-clean ring has stable range 1. Recall that A has stable range 1 if and only if $A/\mathfrak{J}(A)$ has stable range 1. Furthermore, A being a semi-clean ring implies that $A/\mathfrak{J}(A)$ is a clean ring, which has stable range 1 by Theorem 3.10. Therefore, a semi-clean ring has stable range 1. ■

The next result shows that almost semi-clean rings generalize rings satisfying Henriksen's hypothesis. In the last section we will supply an example of an almost semi-clean ring that does not satisfy Henriksen's hypothesis.

Proposition 3.15. *Suppose A is a ring satisfying Henriksen's hypothesis. Then A is an almost semi-clean ring.*

Proof. Notice that if A satisfies Henriksen's hypothesis, then for each nonzero $a \in A$, we have that $\mathrm{Max}(A/aA)$ is a zero-dimensional topological space and thus A/aA is a semi-clean ring. Since a homomorphic image of a semi-clean ring is again a semi-clean ring it follows that every proper homomorphic image of A is a semi-clean ring, whence A is an almost semi-clean ring. ■

Remark 3.16. In Corollary I.2 of [4] the author shows that a semi-clean ring is local–global, that is, each polynomial over A in finitely many indeterminates which admits unit values locally admits unit values globally. Then (Theorem II.1 [4]) it is shown that a Bézout local–global ring is an elementary divisor ring. An *almost local–global ring* is a ring whose proper homomorphic images are all local–global. It follows that an almost semi-clean ring is an almost local–global ring. From Proposition 4.6 and Theorem 4.7 in Chapter V of [6] one deduces that an almost local–global Bézout domain is an elementary divisor domain. Since local–global rings have stable range 1 it follows that an almost local–global ring has almost stable range 1. Therefore, Theorem 3.7 of this article generalizes the above results in [6] to the wider class of commutative rings with almost stable range 1 and zero-divisors.

4. Lattice-ordered groups of divisibility

In this section we revisit the Jaffard–Ohm–Kaplansky Theorem. We will apply the theorem to construct Bézout domains demonstrating that the classes defined in the previous section are in fact different. For an integral domain D, we let $G(D)$ denote its group of divisibility. For $d \in D$ we let $\hat{d}$ be the associated element of $G(D)$. For standard notation and fundamental information on lattice-ordered groups (e.g. convex ℓ-subgroups, prime subgroups, polars, etc.) we urge the readers to check [1,5,9] and [6]. Recall the following theorem.

Theorem 4.1 (*Jaffard–Ohm–Kaplansky*). *Let G be an abelian ℓ-group. There exists a Bézout domain D for which $G(D) \cong G$.*

Let G be an ℓ-group (written multiplicatively) and $u \in G^+$. We say u is a *(weak-)order unit* if whenever $u \wedge x = 1$, then it follows that $x = 1$. It is known that a positive element of G is an order unit precisely when it does not belong to any minimal prime subgroup of G. Not every ℓ-group possesses an order unit, e.g. the group of divisibility of the integers.

For $x \in G^+$ if there exists a $y \in G^+$ such that $x \wedge y = 1$ and $x \vee y$ is an order unit, then x is said to be a *complemented element of* G. If every positive element of G is complemented, then G is said to be a *complemented* ℓ-group. G is called *locally complemented* if for each $g \in G^+$ the convex ℓ-subgroup generated by g (denoted $G(g)$) is complemented. (For more information about complemented ℓ-groups see [3].)

If G has the property that whenever $a \wedge b = 1$ there are $x, y \in G^+$ such that (1) $x \wedge y = 1$, (2) $x \vee y$ is a weak-order unit, and (3) $a \in G(x), b \in G(y)$, then G is called *weakly complemented*. Whenever G has the property that $G(g)$ is weakly complemented for every $g \in G$, we say G is *locally weakly complemented*. A complemented ℓ-group is weakly complemented. In Theorem 5.5 [16] it is shown that an ℓ-group G is weakly complemented if and only if the collection of minimal prime subgroups of G, denoted $\mathrm{Min}(G)$, has a zero-dimensional inverse topology.

Definition 4.2. Let G be an ℓ-group. G is called *projectable* if for every $g \in G$, $G = g^{\perp} \oplus g^{\perp\perp}$. A projectable ℓ-group has *stranded primes*, that is, every prime subgroup contains a unique minimal prime subgroup. A Bézout domain is adequate if and only if its group of divisibility is a projectable ℓ-group. We classify Bézout domains satisfying some of the properties examined in the article via their group of divisibility.

Proposition 4.3 (*Theorem 5.7, [16]*). *Suppose D is a Bézout domain. D is a neat ring if and only if $G(D)$ is locally weakly complemented and has stranded primes.*

Proposition 4.4. *Suppose D is a Bézout domain. D satisfies Henriksen's hypothesis if and only if $G(D)$ is a locally complemented ℓ-group.*

Proof. The main point of the proof is the use of the bijection between the maximal ideals of D and the minimal prime subgroups of $G(D)$. For each $d \in D$, this bijection restricts to a bijection of the maximal ideals of D containing d, namely $V(d)$, and the minimal prime subgroups of the convex ℓ-subgroup of $G(D)$ generated by $\hat{d}$, which we shall denote $\mathrm{Min}(G(\hat{d}))$. These results and others can be found in [16].

So suppose D satisfies Henriksen's hypothesis and let $\hat{d} \in G(D)^+$. For $\hat{b} \in G(\hat{d})^+$ we know there is an $r \in D$ such that

$$V(r) = V(d) \smallsetminus V(b). \tag{1}$$

Without loss of generality we can assume that r divides d and so $\hat{r} \in G(\hat{d})$. (1) translates to $\mathrm{Min}(G(\hat{d}))$ in that $\hat{r} \wedge \hat{b} = 0$ and $\hat{r} \vee \hat{b}$ is not in any minimal prime subgroup of $G(\hat{d})$. This last statement means that $\hat{r} \vee \hat{b}$ is a weak-order unit of $G(\hat{d})$. It follows that $G(\hat{d})$ is a complemented ℓ-group, whence G is a locally complemented ℓ-group. The converse is similar. ■

Proposition 4.5. *Suppose D is a Bézout domain. D is an almost semi-clean ring if and only if $G(D)$ is a locally weakly complemented ℓ-group.*

Proof. Once again we use the bijection between $V(d)$ and $\mathrm{Min}(G(\hat{d}))$ for every $d \in D$. Moreover, this bijection is a homeomorphism between the Zariski topology on $V(d)$ and the inverse topology on $\mathrm{Min}(G(\hat{d}))$. Therefore, the Zariski topology on each $V(d)$ is zero-dimensional if and only if the inverse topology on $\mathrm{Min}(G(\hat{d}))$ is zero-dimensional. It follows that D is almost semi-clean if and only if $G(D)$ is locally weakly complemented. ■

Before we provide examples showing that the properties above are different we need to recall some basic information on $C(X)$, the set of all real-valued continuous functions on X. $C(X)$ is an abelian ℓ-group under the pointwise operations. The constant function **1** is a weak-order unit. The Ref. [8] still is the ultimate source for rings and groups of continuous real-valued functions.

Suppose X is a topological space. A subset $Z \subseteq X$ is called a *zeroset of* X if there is an $f \in C(X)$ such that $Z = \{x \in X : f(x) = 0\}$. A *cozeroset* is the (set-theoretic) complement of a zeroset. We shall assume that all of our topological spaces are Tychonoff, that is, completely regular and Hausdorff. A Hausdorff space is Tychonoff if and only if the collection of cozerosets form a base for the open sets.

A space X is called *cozero complemented* if for each cozero set C there is a disjoint cozero set C' so that $C \cup C'$ is a dense subset of X. It is well-known that X is cozero complemented if and only if $C(X)$ is a complemented ℓ-group (see [12]). (Metric spaces and basically disconnected spaces are examples of cozero complemented spaces.) We call a space X *weakly cozero complemented* if for each pair of disjoint cozero sets C_1, C_2 there exists a pair of disjoint cozerosets T_1, T_2 such that $C_i \subseteq T_i$ and the union of T_1 and T_2 is a dense subset of X (see [16] for more information on weakly cozero complemented spaces.) X is a weakly cozero complemented space if and only if $C(X)$ is a weakly complemented ℓ-group.

The spaces for which every prime subgroup of $C(X)$ contains a unique minimal prime subgroup are called *F-spaces*. It is is shown in [12] that an F-space X is cozero complemented precisely when X is a basically disconnected space. In [16] one can find the result that states an F-space is weakly cozero complemented precisely when X is strongly zero-dimensional.

Example 4.6. (1) We construct an example of an almost semi-clean Bézout domain that does not satisfy Henriksen's hypothesis. Let $G_1 = C(\beta\mathbb{N} \setminus \mathbb{N})$ where $\beta\mathbb{N}$ denotes the Stone–Čech compactification of the natural numbers. Since the space $\beta\mathbb{N} \setminus \mathbb{N}$ is a strongly zero-dimensional F-space which is not basically disconnected it follows that G_1 is weakly complemented and has stranded primes but is not complemented. Therefore, if D is a Bézout domain whose group of divisibility is isomorphic to G_1, then D is a neat Bézout domain, and hence almost semi-clean by Theorem 3.13. Since G_1 is not (locally) complemented D does not satisfy Henriksen's hypothesis.

(2) Let $G_2 = C(\mathbb{R})$. Then G_2 is a complemented ℓ-group that does not have stranded primes, and so if D is a Bézout domain for which $G(D) \cong G_2$ then D is an almost semi-clean ring which is not neat.

(3) Let $G_3 = G_1 \oplus G_2$, then G_3 is a weakly complemented ℓ-group which is neither complemented nor has stranded primes, and thus there is an almost semi-clean Bézout domain which neither satisfies Henriksen's condition nor is neat.

Remark 4.7. In line with the question of whether every Bézout domain is an elementary divisor domain we have been unable to construct a Bézout domain (let alone an elementary divisor domain) which does not have almost stable range 1.

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References

[1] M. Anderson, T. Feil, Lattice Ordered Groups. An Introduction, D. Reidel Publ. Co., Dordrecht, 1988.
[2] H. Chen, Rings with many idempotents, Int. J. Math. Math. Sci. 22 (3) (1999) 547–558.
[3] P. Conrad, J. Martinez, Complemented lattice-ordered groups, Indag. Math. 1 (3) (1990) 281–298.
[4] F. Couchot, Indecomposable modules and Gelfand rings, Comm. Algebra 35 (1) (2007) 231–241.
[5] M. Darnel, Theory of Lattice-Ordered Groups, in: Monographs and Textbooks in Pure and Applied Mathematics, vol. 187, Marcel Dekker, Inc., New York, 1995.
[6] L. Fuchs, L. Salce, Modules Over Non-Noetherian Domains, in: Math. Surv. Monographs, vol. 84, Amer. Math. Soc., Providence, 2001.
[7] L. Gillman, M. Henriksen, Some remarks about elementary divisor rings, Trans. Amer. Math. Soc. 82 (1956) 362–365.
[8] L. Gillman, M. Jerison, Rings of Continuous Functions, in: Springer-Verlag Grad. Texts. Math., vol. 43, 1960.
[9] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York, 1972.
[10] O. Helmer, The elementary divisor theorem for certain rings without chain conditions, Bull. Amer. Math. Soc. 49 (1943) 225–236.
[11] M. Henriksen, Some remarks on elementary divisor rings, II, Michigan Math. J. 3 (1955–1956) 159–163.

[12] M. Henriksen, R.G. Woods, Cozero complemented spaces; when the space of minimal prime ideals of a $C(X)$ is compact, Topology Appl. 141 (1–3) (2004) 147–170.
[13] M. Henriksen, Personal communication during the conference on Ordered Algebraic Structures, Gaionesville, FL, 2004.
[14] I. Kaplansky, Elementary divisors and modules, Trans. Amer. Math. Soc. 66 (1949) 464–491.
[15] M.D. Larsen, W.J. Lewis, T.S. Shores, Elementary divisor rings and finitely presented modules, Trans. Amer. Math. Soc. 187 (1) (1974) 231–248.
[16] W.Wm. McGovern, Neat rings, J. Pure Appl. Algebra 205 (2) (2006) 243–265.
[17] D.E. Rush, Bézout domains with stable range 1, J. Pure Appl. Algebra 158 (2001) 309–324.
[18] L.N. Vaserstein, Stable rank of rings and dimensionality of topological space, Funct. Anal. Appl. 5 (1971) 102–110.
[19] B.V. Zabavs'kyi, Reduction of matrices over Bézout rings of stable rank not higher than 2, Ukrainian Math. J. 55 (4) (2003) 665–670.