PRÜFER DOMAINS WITH CLIFFORD CLASS SEMIGROUP

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Abstract. Bazzoni's Conjecture states that the Prüfer domain R has finite character if and only if R has the property that an ideal of R is finitely generated if and only if it is locally principal. In [4] the authors use the language and results from the theory of lattice-ordered groups to show that the conjecture is true. In this article we supply a purely ring theoretic proof.

1. Bazzoni's Conjecture

Throughout all integral domains are assumed to be commutative.

For an integral domain R, $\mathscr{F}(R)$ denotes the semigroup of fractional ideals of R (under ideal multiplication) while $\mathscr{P}(R)$ denotes the subsemigroup consisting of principal ideals. The class semigroup of R is the factor semigroup $\mathscr{F}(R)/\mathscr{P}(R)$ and is denoted $\mathscr{S}(R)$. A semigroup S is called a *Clifford semigroup* when every element is regular in the sense of von Neumann, that is, for every $a \in S$ there is an $s \in S$ for which $a^2s = a$. The domain R is called a *Clifford regular domain* when $\mathscr{S}(R)$ is Clifford regular.

In the article [1] S. Bazzoni proved that if a Prüfer domain has finite character (that is, every nonzero element belongs to a finite number of maximal ideals) then $\mathscr{S}(R)$ is a Clifford semigroup, and in turn, if $\mathscr{S}(R)$ is a Clifford semigroup, then R satisfies $(*)$ (defined below). In a later article, [2], she was able to show that if $\mathscr{S}(R)$ is a Clifford semigroup, then R has finite character. In [1] and then again in [2] she proposed the following

Conjecture: A Prüfer domain satisfies property (*) if and only if R has finite character.

Recently, the authors of [4] proved using techniques from the theory of lattice-ordered groups that the conjecture is indeed true. The main road used in their proof was to translate the concepts discussed above into the language of ℓ-groups via the lattice-ordered group of invertible ideals of a Prüfer domain. Once the translations were made they used several old and well-known results to finish the proof. In this article we give a purely ring-theoretic proof of the validity of Bazzoni's conjecture. Our proof is mainly a translation from ℓ-groups to ring theory of the proof from [4] except in one crucial place. We elaborate on this matter.

Given a Prüfer domain R and G its ℓ-group of invertible ideals any information about an ℓ-homomorphic image of G can be translated to information about an appropriate localization of R. On the other hand, and unfortunately, there is no known ring-theoretic construction that allows one to gather information about the

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kernel of an ℓ-homomorphism on G. Fortunately, we have been able to discover the correct localization (for our purposes) which allows us to mollify the situation.

We end this section with the definitions of the concepts discussed above as well lay some rules and notational devices used throughout the paper.

For a Prüfer domain R we say R satisfies $(*)$ if

$(*)$ An ideal I of R is finitely generated if and only if the localization IR_M is principally generated for every maximal ideal M of R.

Notice that if I is a finitely generated ideal, then since R_M is a valuation domain it follows that IR_M is principal ideal for every maximal ideal M of R. We let $Max(R)$ denote the set of all maximal ideals of R. For any ideal I of R we let $V_M(I)$ denote the subset of $Max(R)$ consisting of those maximal ideals which contain I. Recall that two ideals I and J are said to be *co-maximal* if $I + J = R$. If R is a Prüfer domain and I is a finitely generated ideal (and hence an invertible ideal) we use I^{-1} to denote its inverse. Recall that if $I \leq J$ are invertible ideals of R, then IJ^{-1} is also an ideal of R.

Our main references for this article are [5] for topics on localizations of rings, [6] for the theory of invertible ideals and Prüfer domains, and [3] and [4] as the main sources for some of the ideas in our proof.

2. Summits of Integral Domains

Definition 1. Let I be a finitely generated (proper) ideal of R. We say that I is a *chained ideal of* R if R/I is a chained ring; that is, the collection of ideals containing I forms a chain. We shall always assume that a chained ideal is a proper finitely generated ideal. Notice that any proper finitely generated ideal containing a chained ideal is itself a chained ideal. Moreover, when R is a Prüfer domain and I is a proper finitely generated ideal, I is a chained ideal precisely when it is contained in a unique maximal ideal.

We call a set of finitely generated ideals, say $\mathcal{S}$, a *summit* for R if each element in $\mathcal{S}$ is a chained ideal and $\mathcal{S}$ is a maximal pairwise co-maximal set of finitely generated ideals. We observe that to show a set $\mathcal{S}$ is a summit it is necessary to show that for any (proper) finitely generated ideal I not belonging to $\mathcal{S}$ there is some element of $\mathcal{S}$ which is not co-maximal to I.

Our first result characterizes when a domain possesses a summit.

Theorem 2. *The domain R has a summit if and only if every (proper) finitely generated ideal of R lies below a chained ideal.*

Proof. Necessity. Suppose R has a summit, say $\mathcal{S}$. Let I be a proper finitely generated ideal of R. We can assume that $I \notin \mathcal{S}$. Since $\mathcal{S}$ is a maximal pairwise co-maximal set it follows that for some $J \in \mathcal{S}$, $I + J < R$. Since J is a chained ideal so is $I + J$ and therefore $I \leq I + J$ lies below a chained ideal.

Sufficiency. By assumption there are chained ideals of R. The usual Zorn's Lemma argument ensures that every chained ideal can be extended to a pairwise co-maximal set of chained ideals that is maximal with respect to being a set of chained ideals which are pairwise co-maximal. So let $\mathcal{S}$ be such a set of chained ideals. Notice that there might exist an ideal J of R not belonging to $\mathcal{S}$ which is pairwise co-maximal to every element of $\mathcal{S}$; thus, J is not a chained ideal. We claim that $\mathcal{S}$ is a summit for R. Suppose otherwise. This means that there is some (proper) finitely generated ideal J of R for which $J + I = R$ for all $I \in \mathcal{S}$. Now, by hypothesis we can extend J to a chained ideal and so without loss of generality we assume that J is a chained ideal. But then $\mathcal{S} \cup \{J\}$ is a pairwise co-maximal set of chained ideals contradicting the maximality of $\mathcal{S}$.■

Definition 3. We say the integral domain R *satisfies Property (F)* if no proper finitely generated ideal lies beneath an infinite number of pairwise co-maximal finitely generated ideals of R. We observe that this is equivalent to the condition that no element of R lies in an infinite number of pairwise co-maximal finitely generated ideals of R. It is our intention to later show that if the Prüfer domain R satisfies $(*)$ then it satisfies Property (F). For now concentrate on showing that a Prüfer domain satisfying Property (F) has a summit.

Lemma 4. *Suppose R is a Prüfer domain. If J, K are incomparable finitely generated ideals of R then there are incomparable finitely generated ideals J', K' of R which are co-maximal and $J \leq J'$ and $K \leq K'$.*

Proof. It is straightforward to check that

$$J' = J(J+K)^{-1} \text{ and } K' = K(J+K)^{-1}$$

satisfy the desired properties.■

Theorem 5. *Suppose R is a Prüfer domain. If R satisfies Property (F), then R has a summit.*

Proof. Notice that by Theorem 2 it suffices to show that every proper finitely generated ideal lies below a (finitely generated) chained ideal. So suppose by way of contradiction that I is a (proper) finitely generated ideal which does not lie below a chained ideal. This means that for each finitely generated ideal $I \leq J < R$ there are incomparable finitely generated ideals, J_1, J_2 containing J. Furthermore, by Lemma 4, we can assume without loss of generality that J_1 and J_2 are co-maximal.

We recursively define a collection of ideals as follows. There are two incomparable co-maximal finitely generated ideals containing I, say $I_1, I_{1'}$. For each natural number $1 < n \in \mathbb{N}$ choose $I_n, I_{n'}$ to be two incomparable finitely generated ideals containing I_{n-1}. Then $T = \{I_{n'}\}_{n \in \mathbb{N}}$ is a set of finitely generated ideals each of which contains I. Observe that $\{I_n\}_{n \in \mathbb{N}}$ is an increasing chain of ideals.

We prove by induction that for each $n \in \mathbb{N}$, $I_{n'}$ is co-maximal to $I_{k'}$ for each $k < n$. Notice that $I_1 < I_{2'}$. Therefore, since $I_1 + I_{1'} = R$ it follows that $I_{2'} + I_{1'} = R$. Now suppose the statement is true for each $m < n$. Let $k < n$. We know that $I_k + I_{k'} = R$ and so since $I_k \leq I_{n-1} \leq I_{n'}$ and we obtain that $R = I_k + I_{k'} \leq I_{n-1} + I_{k'} \leq I_{n'} + I_{k'}$, whence $I_{n'} + I_{k'} = R$ for every $k < n$. It follows that the set T is an infinite set of pairwise co-maximal finitely generated ideals each of which contains I, contradicting the hypothesis that R satisfies Property (F).■

Definition 6. let I be a finitely generated ideal of R and set

$$S_I = \{a \in R : I + aR = R\}.$$

Observe that S_I is a saturated multiplicative subset of R and so we can discuss the localization R_{S_I}. It is the use of this localization which enables us to give a complete purely ring-theoretic proof of Bazzoni's conjecture. In [4] the authors pass from the ℓ-group G of invertible ideals of R to a specific principally generated convex ℓ-subgroup of G. Such a translation is not possible.

Proposition 7. *Suppose R is an integral domain and let I be a (proper) finitely generated ideal of R. The set of maximal ideals of R containing I is in one-to-one correspondence with the set of all maximal ideals of R_{S_I}.*

Proof. Recall the correspondence between the maximal ideals in a localization and ideals in the original ring which are maximal with respect to not meeting the multiplicative set. We leave it to the interested reader to show that this is the case.■

Proposition 8. *Suppose R is a Prüfer domain satisfying Property (F). Then for any (proper) finitely generated ideal I of R, R_{S_I} also satisfies Property (F).*

Proof. Suppose that on the contrary that there is a (proper) finitely generated ideal I for which R_{S_I} does not satisfy property (F). This means that there is a (proper) finitely generated ideal of R_{S_I}, say K, that lies below an infinite set of pairwise co-maximal finitely generated ideals. Let $\{K_\gamma\}_{\gamma\in\Gamma}$ be such a set of distinct pairwise co-maximal finitely generated ideals each lying above K. Let J be a finitely generated ideal of R for which $JR_{S_I} = K$. Also for each $\gamma \in \Gamma$ let J_γ be a finitely generated ideal of R for which $J_\gamma R_{S_I} = K_\gamma$. Without loss of generality we can choose the ideals so that $I \subseteq J_\gamma$ for each γ. We claim that $\mathcal{T} = \{J_\gamma\}_{\gamma\in\Gamma}$ is a set of pairwise co-maximal (proper) ideals each containing the finitely generated ideal J. This contradicts our assumption that R satisfies Property (F).

Let γ_1, γ_2 be distinct elements of our index set Γ. Since $K_{\gamma_1} + K_{\gamma_2} = R_{S_I}$ we can find $k_1 \in J_{\gamma_1}, k_2 \in J_{\gamma_2}$ and $s \in S_I$ for which $s = k_1 + k_2$. By the definition of S_I it follows that for some $r \in R$ and $i \in I$, $1 = sr + i$. Then $1 = k_1r + k_2r + i$. Now, $k_1r + i \in J_{\gamma_1}$ while $k_2r \in J_{\gamma_2}$. Thus, $J_{\gamma_1} + J_{\gamma_2} = R$, and so $\mathcal{T}$ is a pairwise co-maximal set of (proper) finitely generated ideals each containing J.■

Theorem 9. *Suppose R is a Prüfer domain satisfying $(*)$. Then R satisfies Property (F).*

Proof. Suppose on the contrary that R does not satisfy Property (F). Therefore, there is some (proper) finitely generated ideal I of R which lies beneath an infinite number of pairwise co-maximal (proper) finitely generated ideals. Enumerate such a set as $\mathcal{T} = \{I_\gamma\}_{\gamma\in\Gamma}$ and define

$$\overline{\mathcal{T}} = \{I_{\gamma_1} I_{\gamma_2} \cdots I_{\gamma_n} : \text{ for a distinct finite set } \gamma_1, \cdots, \gamma_n \in \Gamma\}.$$

Notice that since the ideals in $\mathcal{T}$ are pairwise co-maximal

$$I_{\gamma_1} I_{\gamma_2} \cdots I_{\gamma_n} = I_{\gamma_1} \cap I_{\gamma_2} \cap \cdots \cap I_{\gamma_n}$$

and so I lies below each element $\overline{\mathcal{T}}$. Furthermore, I lies properly below each element of $\overline{\mathcal{T}}$. To see this suppose that on the contrary that $I = I_{\gamma_1} I_{\gamma_2} \cdots I_{\gamma_n}$ for appropriate $\gamma_1, \cdots, \gamma_n \in \Gamma$. Choose $\gamma \in \Gamma$ different from each γ_i. But since the lattice of ideals of a Prüfer domain is distributive it follows that

$$I_\gamma = I + I_\gamma = (I_{\gamma_1} \cap I_{\gamma_2} \cap \cdots \cap I_{\gamma_n}) + I_\gamma = (I_{\gamma_1} + I_\gamma) \cap \cdots \cap (I_{\gamma_n} + I_\gamma) = R,$$

a contradiction.

Next, define K to be the ideal generated by the set of ideals $\{IJ^{-1}\}_{J\in\overline{\mathcal{T}}}$. We will show that K is locally principal and since R satisfies $(*)$ we can then conclude that K is a finitely generated ideal of R. First of all we point out that K is a proper ideal. This is because $\overline{\mathcal{T}}$ is a downward directed set and therefore $\{IJ^{-1}\}_{J\in\overline{\mathcal{T}}}$ is an upward directed set of finitely generated ideals. It follows that

$$K = \bigcup_{J\in\overline{\mathcal{T}}} IJ^{-1}$$

and so if $1 \in K$, then $1 \in IJ^{-1}$ for some $J \in \overline{\mathcal{T}}$, a contradiction.

Let $M \in Max(R)$ and consider KR_M. If $K \not\leq M$, then $KR_M = R_M$ so without loss of generality we assume that $K \leq M$. Suppose $\gamma \in \Gamma$ satisfies $I_\gamma \leq M$. Then since $\mathcal{T}$ is a set of pairwise co-maximal ideals it follows that for every other different $\gamma' \in \Gamma$, $I_{\gamma'} + M = R$ and so there is at most one $\gamma \in \Gamma$ for which $I_\gamma \leq M$. We consider these two cases separately.

In the first case suppose that $I_\gamma \not\leq M$ for all $\gamma \in \Gamma$. Then for each $\gamma \in \Gamma$ we know that $I_\gamma R_M = R_M$ and therefore by properties of extensions

$$(II_\gamma^{-1})R_M = IR_M(I_\gamma R_M)^{-1} = IR_M.$$

Consequently, for each $I_{\gamma_1} \cdot \dots \cdot I_{\gamma_n} = J \in \overline{\mathcal{T}}$,

$$(IJ^{-1})R_M = (II_{\gamma_1}^{-1} \cdots I_{\gamma_n}^{-1})R_M = IR_M I_{\gamma_1}^{-1}R_M \cdots I_{\gamma_n}^{-1}R_M = IR_M$$

Therefore, $KR_M = IR_M$ which is a finitely generated ideal. Since R_M is a valuation domain it follows that KR_M is a principal ideal.

In the second case denote by γ the unique element in Γ for which $I_\gamma \leq M$. By the argument above for every other $\gamma' \in \Gamma$, $II_{\gamma'}^{-1}R_M = IR_M$. Therefore, for each $I_{\gamma_1} \cdots I_{\gamma_n} = J \in \overline{\mathcal{T}}$,

$$(IJ^{-1})R_M \leq (IJ^{-1}I_\gamma^{-1})R_M = II_\gamma^{-1}R_M$$

from which gather that $KR_M = II_\gamma^{-1}$, a principal ideal of R_M.

In both cases we showed that KR_M is a principal ideal of R_M and so K is a locally principal ideal of R. As mentioned before since R satisfies $(*)$ we conclude that K is a finitely generated ideal of R. But we already pointed out that $\{IJ^{-1}\}_{J\in\overline{\mathcal{T}}}$ is an upward directed set so

$$K = I(I_{\gamma_1}I_{\gamma_2} \cdots I_{\gamma_n})^{-1}$$

for appropriate $\gamma_1, \cdots, \gamma_n \in \Gamma$. But then for any $\gamma \in \Gamma$ different than the $\gamma_1, \cdots, \gamma_n$ we have $II_\gamma^{-1} \leq K$. But once again we gather that $I_{\gamma_1}I_{\gamma_2} \cdots I_{\gamma_n} \leq I_\gamma$, a contradiction. Our only conclusion is that there is no such (proper) finitely generated ideal I lying below an infinite number of pairwise co-maximal (proper) finitely generated ideals. Whence, R satisfies Property (F).■

Proposition 10. *Suppose R is a Prüfer domain satisfying Property (F). If the Jacobson radical of R is non-zero, then R has a finite summit. Moreover, R has a finite number of maximal ideals.*

Proof. Since R satisfies Property (F) we know by Theorem 5 R has a summit, say $\mathcal{S}$. By hypothesis, we choose I to be any non-zero finitely generated ideal of R contained in the Jacobson radical of R. Let $\mathcal{T} = \{I + J : J \in \mathcal{S}\}$. Notice that each $I + J$ is a proper ideal since I lies below the Jacobson radical of R. Clearly, $\mathcal{T}$ is a pairwise co-maximal set of ideals. Consequently, $\mathcal{T}$ is a finite set since R satisfies Property (F). But then it must be true that $\mathcal{S}$ is a finite set, and so R possesses a finite summit. (Alternatively, one can simply show that $\mathcal{T}$ is a finite summit.)

Next, let $J_1, \cdots, J_n$ be the distinct elements of $\mathcal{S}$. Because each J_i is a chained ideal it follows that there is a unique maximal ideal, say M_i containing J_i. Then the set $\{M_1, \cdots, M_n\}$ is a set of distinct maximal ideals. Suppose now that R has a maximal ideal different than each M_i. Call it M. Choose $m_i \in M \smallsetminus M_i$ for each $I = 1, \cdots, n$ and let $K = m_1R + \cdots + m_nR$. Then $K \leq M$ and $K \nleq M_i$ for each $i = 1, \cdots, n$. By Theorem 2 there is a chained ideal above K, call it V. Notice that V does not lie inside any of the maximal ideals M_i since K does not. But since $\mathcal{S}$ is a summit it follows that V is not co-maximal to one of the J_i. Therefore, $V + J_i$ (and therefore V) lies below a maximal ideal. But the only possible such maximal ideal is M_i, a contradiction. Therefore, the only maximal ideals of R are $M_1, \cdots, M_n$.■

Theorem 11. *For a Prufer domain the following statements are equivalent.*

i) *R has finite character.*
ii) *R has a Clifford semigroup*
iii) *R satisfies $(*)$.*
iv) *R satisfies Property (F).*

Proof. Bazzoni showed the equivalency of i) and ii), as well as the implication ii) $\Rightarrow$ iii). Theorem 9 is the implication iii) $\Rightarrow$ iv).

(iv) $\Rightarrow$ (i): Suppose R satisfies Property (F) and let I be a (proper) finitely generated ideal of R. By Proposition 8, R_{S_I} also satisfies Property (F). By the proof of Proposition 7 IR_M is a non-zero finitely generated ideal lying below the Jacobson radical of R_{S_I}. Consequently, by Proposition 10, R_{S_I} has a finite number of maximal ideals. Finally, by Proposition 7, there are only a finite number of maximal ideals of R containing I. We conclude that R has finite character.■

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