A Natural Equivalence for the Category of Coherent Frames

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ABSTRACT. The functor on the category of bounded lattices induced by reversing their order, gives rise to a natural equivalence of coherent frames. We investigate the spectra as well as some well-known frame properties like zero-dimensionality and normality.

1. Introduction

During the 9$^{\text{th}}$ annual International Conference on Ordered Algebraic Structures in Gainesville, Florida, 2006 (a celebration of the life and work of Bernhard Banaschewski [7]) a presentation was given that showed there are several pairs of classes of algebraic frames which are dual, in some sense, to each other. Then, later, the authors realized that the well-known "Lemma on Ultrafilters" could be placed in a more algebraic context. We set set out to uncover the deeper meaning of these phenomena. We demonstrate that the correspondence between minimal prime elements and ultrafilters gives rise to a systematic way of constructing naturally equivalent classes of complete algebraic lattices.

We use the remainder of this section to define and introduce the fundamental concepts that will be used throughout. In section 2, we recall the several interesting classes from the theory of lattices and frames and determine the relationship between them as pertaining to the construction mentioned above. The remaining sections concentrate on applications.

Remark 1.1. Recall that an element a of L is called *compact* if, whenever $a \leq \bigvee S$ for some subset $S \subseteq L$, then $a \leq \bigvee T$ for a finite subset $T \subseteq S$. Denote the set of all compact elements of L by $\mathfrak{K}(L)$ and notice that $\mathfrak{K}(L)$ is a join-semilattice with bottom element. The lattice L is called *algebraic* if every element of L is the supremum of a collection of compact elements. An algebraic lattice is called *coherent* if it is complete and finite infima of compact elements are again compact. A lattice homomorphism between two bounded algebraic lattices is called a *coherent lattice homomorphism*, if it maps the top element (resp. bottom element) to top element (resp. bottom element) and maps compact elements to compact elements.

2000 *Mathematics Subject Classification*: Primary: 06D22; Secondary: 06D50, 06B10.
Key words and phrases: coherent frame, algebraic frame, lattice, natural equivalence.

For a given join-semilattice with bottom element $(L, \leq, \vee, 0_L)$, we let $\mathrm{Idl}(L)$ denote the lattice of all ideals of L. Recall that an ideal I is a nonempty subset of L that satisfies the following properties:

(1) $0_L \in I$;
(2) if $a \leq b \in I$, then $a \in I$;
(3) if $a, b \in I$, then $a \vee b \in I$.

$\mathrm{Idl}(L)$ is a complete algebraic lattice whose compact elements are precisely the principal ideals $\downarrow a = \{x \in L : x \leq a\}$, where $a \in L$.

Recall the well-known theorem that states that a complete lattice is algebraic if and only only if it is isomorphic to the ideal lattice of its compact elements (see Theorem 8, pg 186 [6]). This theorem extends to an equivalence between the category of join-semilattices with bottom element and the category of complete algebraic lattices with coherent lattice homomorphisms. In particular, we may view $\mathrm{Idl}(\cdot)$ as a covariant functor from the category of join-semilattices to the category of complete algebraic lattices; it induces a natural equivalence between the two categories. Its inverse is the functor that assigns to a complete algebraic lattice L, its join-semilattice of compact elements. Then for a given complete algebraic lattice, say L, $\mathrm{Idl}(\mathfrak{K}(L))$ is naturally isomorphic to L. Moreover, the restriction of this equivalence to the category of bounded distributive lattices is a natural equivalence between this category and that of coherent frames.

For the sake of completeness we recall that a *frame* is a complete distributive lattice L which satisfies the strengthened distributive law

$$a \wedge \bigvee_{s \in S} s = \bigvee \{a \wedge s : s \in S\}$$

for all $a \in L$ and all $S \subseteq L$. This identity will be referred to as the *frame law*. For an element $a \in L$ the *polar of a* (or *pseudo-complement of a*) is given by

$$a^{\perp} = \bigvee \{x \in L : x \wedge a = 0\}$$

and by the frame law $a^{\perp}$ is the largest element of L disjoint to a. An element of the form $a^{\perp}$ is called a *polar of L*. When $a \vee a^{\perp} = 1$, we say that a is a *complemented element of L*. A frame for which every element is complemented is called a *boolean frame*. A frame is said to satisfy the *finite intersection property on compact elements* (or *the FIP for short*) if whenever $a, b \in \mathfrak{k}(L)$, then $a \wedge b \in \mathfrak{k}(L)$. We call an algebraic frames satisfying the FIP an *M-frame*; this is not standard terminology but in our opinion it is more efficient to say M-frame then algebraic frame satisfying the FIP. An M-frame which is itself compact is a *coherent frame*. A *frame homomorphism* is a lattice homomorphism $f \colon L \to M$ between two frames L and M such that $f(\bigvee S) = \bigvee \{f(s) : s \in S\}$ for all subsets S of L. It is noteworthy that a complete algebraic distributive lattice is a frame.

To get to our construction we recall the definition of a filter. For a meet-semilattice with top element $(L, \leq, \wedge, 1_L)$ recall that a filter (a.k.a. a dual ideal) on L is a nonempty subset, say $F \subseteq L$ such that

(1) $1_L \in F$;
(2) if $a \geq b \in F$ and $b \in F$, then $a \in F$;
(3) if $a, b \in F$, then $a \wedge b \in F$.

The set of all filters of L is denoted by $\mathfrak{F}(L)$ and when ordered by inclusion it is known that $\mathfrak{F}(L)$ is a complete lattice since the intersection of filters is again a filter. For any $a \in L$ the set $\mathcal{F}_a = \{x \in L : a \leq x\}$ is called the *principal filter generated by* a. (We use the notation $\mathcal{F}_a$ instead of $\uparrow a$ when we want to emphasize the filter property of the set instead of it being a quotient meet-semilattice.) Notice that for any $x, y \in L$ $x \leq y$ exactly when $\mathcal{F}_y \leq \mathcal{F}_x$. Thus, $\mathcal{F}_x \wedge \mathcal{F}_y = \mathcal{F}_{x \vee y}$ and $\mathcal{F}_x \vee \mathcal{F}_y = \mathcal{F}_{x \wedge y}$

We now state two important theorems. For a proof of the first theorem the reader is urged to consult [10], Theorem 2.5. Recall (see [8]) that for a poset $(P, \leq)$, we let $\leq^*$ denote the reverse order on P, that is, $x \leq^* y$ precisely when $y \leq x$. Then $(P^*, \leq^*)$ is also a poset called the *dual of* $(P, \leq)$. Observe that $(P, \leq)$ is a join semilattice precisely when $(P^*, \leq^*)$ is a meet-semilattice.

Theorem 1.2. *Let L be a meet-semilattice with top element.*

(1) *$\mathfrak{F}(L)$ is a complete algebraic lattice whose compact elements are*
$$\mathfrak{K}(\mathfrak{F}(L)) = \{\mathcal{F}_a : a \in L\}$$
(2) *$\mathfrak{F}(L)$ is a compact lattice if and only if L has a bottom element.*
(3) *$\mathfrak{K}(\mathfrak{F}(L))^* \cong L$ as meet-semilattices and $\mathfrak{K}(\mathfrak{F}(L)) \cong L^*$ as join-semilattices.*
(4) *$\mathfrak{F}(L)$ is an M-frame if and only if L is a distributive lattice.*
(5) *$\mathfrak{F}(L)$ is a coherent frame if and only if L is a bounded distributive lattice.*

Theorem 1.3. *Suppose L is a lattice. The following statements are equivalent.*

(1) *L is a complete algebraic lattice.*
(2) *L is isomorphic to the lattice of all ideals of a join-semilattice with bottom element.*
(3) *L is isomorphic to the lattice of all filters of a meet-semilattice with top element.*

Proof. Suppose L is a complete algebraic lattice. Consider the meet-semilattice with top element $\mathfrak{K}(L)^*$, the dual of $\mathfrak{K}(L)$. It is straightforward to check that $\mathfrak{F}(\mathfrak{K}(L)^*)$ and L are isomorphic as lattices. □

Remark 1.4. Observe that for a given pair of meet-semilattices with top element, say L and M, if $g \colon L \to M$ is a meet-semilattice homomorphism (mapping 1_L to 1_M), then the map $\mathfrak{F}(g) \colon \mathfrak{F}(L) \to \mathfrak{F}(M)$ defined by
$$\mathfrak{F}(g)(F) = \{y \in M : \text{ there is an } x \in F \text{ such that } g(x) \leq y\}$$

is well-defined and is a lattice homomorphism. Moreover, this map produces a covariant functor between the category of meet-semilattices with top element and the category of complete algebraic lattices. As mentioned before for a complete algebraic lattice L, $\mathfrak{F}(\mathfrak{K}(L)^*)$ is naturally isomorphic to L. It follows that $\mathfrak{F}$ induces an natural equivalence between the two categories just mentioned.

Now what if for a given complete compact algebraic lattice L, $\mathfrak{K}(L)$ is a sublattice to begin with? We are then in a position where we need not pass to the dual of $\mathfrak{K}(L)$ to construct a filter lattice. Thus the question of whether L and $\mathfrak{F}(\mathfrak{K}(L))$ are isomorphic arises. In general, the answer to the question is negative, but it does leave us in a nice position to start from. From this point out we will turn our attention to coherent frames and specifically try to relate properties of L to $\mathfrak{F}(\mathfrak{K}(L))$. For a given coherent frame L we denote by L' the coherent frame $\mathfrak{F}(\mathfrak{K}(L))$. We will also have occasion to call L' the *prime of* L. Note that L is naturally isomorphic to its double prime L''. We say two coherent frames L and M are *prime* to each other if $L' \cong M$ (or equivalently, $M' \cong L$).

The proof of the next theorem is left to the interested reader.

Proposition 1.5. *Two coherent frames L and M are prime to each other if and only if $\mathfrak{K}(L)^* \cong \mathfrak{K}(M)$ if and only if $\mathfrak{K}(L) \cong \mathfrak{K}(M)^*$.*

2. Priming Properties

Now that we are equipped with our main construction we turn to several interesting classes of frames and try to determine the relationships between them. Throughout this section we assume that L and M are coherent frames. We begin by recalling the regularity conditions of [14]; this is an excellent source on algebraic frames and M-frames.

Definition 2.1. Define $a \in L$ to be *well-below* $b \in L$ if $a^\perp \vee b = 1$; if so we write $a \preceq b$. $a \in L$ is *regular* if

$$a = \bigvee \{x \in L : x \preceq a\}.$$

The frame L is said to be *regular* if each $a \in L$ is regular.

An element $a \in L$ which satisfies

$$a = \bigvee \{c^{\perp\perp} : c \leq a, c \in \mathfrak{K}(L)\}$$

is called a *d-element* of L. For an extensive article on regular frames we encourage the reader to consult [14]. In particular, the authors defined the following four regularity conditions on algebraic frames:

Reg(1): L is regular.
Reg(2): Each d-element of L is regular.

Reg(3): Each polar of L is regular.
Reg(4): Each $c^{\perp}$ with $c \in \mathfrak{K}(L)$, is regular.

In [14] it is shown that a coherent frame L satisfies Reg(1) if and only if it is a *zero-dimensional* frame, that is, every element of L is the supremum of complemented elements. Furthermore, an algebraic frame L satisfies Reg(2) if and only if it satisfies Reg(3), and both of these conditions are equivalent to the frame being a *projectable frame*, that is, for every $a \in \mathfrak{K}(L)$, $a^{\perp}$ is a complemented element. Finally, Reg(4) is equivalent to the condition that for any $a, b \in \mathfrak{K}(L)$ if $a \wedge b = 0$, then $a^{\perp} \vee b^{\perp} = 1$.

Definition 2.2. In the articles [2] and [12] the authors introduced new classes of frames with the goal of characterizing well-known classes of commutative rings. The former article studied the frame of radical ideals of a ring (see next section), while the latter article investigated the frame of multiplicative filters of ideals of the ring. We recall these classes of frames now.

A coherent frame L is called *weakly zero-dimensional* if for every $a, b \in \mathfrak{K}(L)$ with $a \vee b = 1$ there is a complemented $c \in \mathfrak{K}(L)$ such that $c \leq a$ and $c^{\perp} \leq b$. A coherent zero-dimensional frame is weakly zero-dimensional.

A coherent frame L is called *feebly projectable* if for every disjoint $a, b \in \mathfrak{K}(L)$ there is a complemented element $d \in \mathfrak{K}(L)$ such that $a \leq d$ and $b \leq d^{\perp}$. In [12] it is shown that every projectable frame is feebly projectable, and that every feebly projectable frame satisfies Reg(4). (The implications are proper.)

For completeness sake we remind the reader that a coherent frame L is said to be *normal* if whenever $a, b \in L$ and $1 = a \vee b$, then there are $c, d \in L$ such that $c \wedge d = 0$ while $a \vee d = 1 = b \vee c$.

Recall that the Duality Principle states: "If a statement Φ is true in all posets, then its dual is also true in all posets." The construction of L' from L leads to what we call the Priming Principle; this principle is in the same spirit as the Duality Principle.

Theorem 2.3 (**The Priming Principle**). *Let L be an coherent lattice. If a statement Φ is true for $\mathfrak{K}(L)$, then its dual is true for $\mathfrak{K}(L')$.*

We can apply this theorem to coherent frames and

Theorem 2.4. *Let L be a coherent frame. The following statements are equivalent.*

(1) *L is a regular frame.*
(2) *L is a zero-dimensional frame.*
(3) *L' is a zero-dimensional frame.*
(4) *L' is a regular frame.*

Proof. As mentioned before, the proof of the equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) can be found in [14]. To show that an algebraic frame is zero-dimensional it is necessary and sufficient to demonstrate that every compact element is complemented. So if L is a zero-dimensional frame, then since an arbitrary compact element of L' is of the form $\mathcal{F}_a$ for some $a \in \mathfrak{K}(L)$ it follows that a is complemented in L and therefore $\mathcal{F}_a$ is complemented in L'. Since L is isomorphic to L'' we conclude that (2) and (3) are equivalent. □

Theorem 2.5. *Let L be a coherent frame. The following statements are equivalent.*

(1) *L satisfies* Reg(2).
(2) *L satisfies* Reg(3).
(3) *L is a projectable frame.*
(4) *For every $a \in \mathfrak{K}(L')$, $\bigwedge\{x \in \mathfrak{K}(L') : x \vee a = 1_{L'}\}$ is a complemented element in L'.*

Proof. Once again we cite [14] as a source for the proof of the equivalence of (1), (2), and (3). So suppose L is a projectable frame and let $\mathcal{F}_a \in \mathfrak{K}(L')$. Since $a \in \mathfrak{K}(L)$, $a^{\perp}$ is a complemented element. Since L is compact it follows that $a^{\perp}$ and $a^{\perp\perp}$ are also compact elements. Thus,

$$\mathcal{F}_{a^{\perp}} = \bigwedge\{\mathcal{F}_x \in \mathfrak{K}(L') : \mathcal{F}_x \vee \mathcal{F}_a = \mathcal{F}_0\}$$

is a complemented element of L'.

Conversely, we observe that for $a \in \mathfrak{K}(L)$,

$$\bigwedge\{\mathcal{F}_x \in \mathfrak{K}(L') : \mathcal{F}_x \vee \mathcal{F}_a = \mathcal{F}_0\} = \bigwedge\{\mathcal{F}_x \in \mathfrak{K}(L') : x \wedge a = 0\}$$

and so if this element is complemented, and hence compact, it is of the form $\mathcal{F}_t$ for some $t \in \mathfrak{K}(L)$, which also happens to be complemented. By the equality above $t = a^{\perp}$. □

Theorem 2.6. *Let L be a coherent frame. L is a feebly projectable frame if and only if L' is a weakly zero-dimensional frame.*

Proof. This is a direct consequence of the Priming Principle since the dual of the statement defining a feebly projectable frame is that statement which defines a weakly zero-dimensional frame. (Again, notice that a complemented element in a coherent frame is compact.) □

Theorem 2.7. *Let L be a coherent frame. The following statements are equivalent.*

(1) *L satisfies* Reg(4).
(2) *L for any $a, b \in \mathfrak{K}(L)$ if $a \wedge b = 0$, then $a^{\perp} \vee b^{\perp} = 1$.*
(3) *L' is a normal frame.*

Proof. First of all we observe that a coherent frame is normal if and only if whenever $a, b \in \mathfrak{K}(L)$ and $a \vee b = 1$, then there are $c, d \in \mathfrak{K}(L)$ such that $c \wedge d = 0$ and $1 = a \vee d = b \vee c$. We leave it to the interested reader to check that the dual of this is equivalent to statement (2) for coherent frames. □

3. $\mathrm{Rad}(A)$ and $\mathfrak{C}_A$

Throughout this section A will denote a commutative ring with identity and $\mathrm{Lat}(A)$ denotes its coherent lattice of ideals. For any ideal I of A the radical of I is the ideal $\sqrt{I} = \{x \in A : x^n \in I \text{ for some } n \in \mathbb{N}\}$. $\sqrt{I}$ is also the intersection of all prime ideals containing I. An ideal is called a *radical ideal* if $I = \sqrt{I}$, i.e. it is an intersection of prime ideals. $\mathrm{Rad}(A)$ denotes the collection of all radical ideals of A. When ordered by inclusion $\mathrm{Rad}(A)$ is a coherent frame whose compact elements are the ideals of the form $\sqrt{\{u_1, \cdots, u_n\}}$ for some finite subset $\{u_1, \cdots, u_n\} \subseteq A$. For more information on the frame of radical ideals of a commutative ring with identity we encourage the reader to have a look at [1], [2], and [3]. In [3] it is proved that every coherent frame is isomorphic to $\mathrm{Rad}(A)$ for an appropriate commutative ring A with identity. Therefore, we find it important to consider the prime of $\mathrm{Rad}(A)$. We shall need to recall a different frame associated to a commutative ring with identity.

In [12] the authors investigated the frame $\mathfrak{M}_A$ of multiplicative filters of ideals of a ring A. A filter F of ideals is said to be *multiplicative* if $IJ \in F$ whenever $I, J \in F$. The arbitrary intersection of multiplicative filters is again a multiplicative filter. This gives that the supremum of two multiplicative filters is the filter generated by the finite ideal products of elements in each of the filter. In [9] the authors investigated a subcollection of $\mathfrak{M}_A$. A multiplicative filter F is said to be a *noetherian filter* if whenever $I \in F$ there is a finitely generated ideal $J \in F$ such that $J \leq I$. In other words, F has a filter base consisting of finitely generated ideals. The collection of all noetherian filters of ideals of A is denoted by $\mathfrak{C}_A$. Now, the arbitrary intersection of noetherian filters need not be noetherian, but the suprema (in $\mathfrak{M}_A$) of noetherian filters are again noetherian filters. In [9] it is shown that $\mathfrak{C}_A$ is a coherent frame where finite infima are given by intersection and suprema can be expressed as follows. If $\{F_i\}_{i \in I}$ is a collection of noetherian filters, then

$$\bigvee_{i \in I} F_i = \{I \in \mathrm{Lat}(A) : I_{i_1} \cdots I_{i_n} \leq I \text{ for some finitely generated } I_{i_k} \in F_{i_k}\}$$

The compact elements of $\mathfrak{C}_A$ are precisely the filters of the form

$$\mathfrak{F}_I = \{J \in \mathrm{Lat}(A) : I^n \leq J, n \in \mathbb{N}\}$$

where I is a finitely generated ideal of A.

Theorem 3.1. *The coherent frames* $\mathrm{Rad}(A)$ *and* $\mathfrak{C}_A$ *are prime to each other.*

Proof. Since $\mathrm{Rad}(A)$ and $\mathfrak{C}_A$ are coherent frames, it suffices to show that there is an order reversing isomorphism between their respective compact elements. Define $\Psi \colon \mathfrak{K}(\mathfrak{C}_A) \to \mathfrak{K}(\mathrm{Rad}(A))$ by

$$\Psi(\mathfrak{F}_I) = \sqrt{I}.$$

Notice that I is assumed to be finitely generated. Also, if $\mathfrak{F}_I = \mathfrak{F}_J$ then for appropriate $n, m \in \mathbb{N}$, $J^n \leq I$ and $I^m \leq J$. By one of the standard definitions of prime ideals it follows that a prime ideal P contains I precisely when it contains J. Therefore, $\sqrt{I} = \sqrt{J}$, whence the map Ψ is well-defined. Next, suppose that $\mathfrak{F}_I \leq \mathfrak{F}_J$ for finitely generated ideals $I, J \in \mathrm{Lat}(A)$. Then for some $n \in \mathbb{N}$, $J^n \leq I$ and so any prime ideal containing I contains J. Therefore, $\sqrt{J} \leq \sqrt{I}$ which shows that Ψ is order reversing. The map Ψ is clearly surjective and we leave it to the interested reader to check that it is injective. Finally,

$$\Psi(\mathfrak{F}_I \vee \mathfrak{F}_J) = \Psi(\mathfrak{F}_{IJ}) = \sqrt{IJ} = \sqrt{I}\sqrt{J} = \Psi(\mathfrak{F}_I) \wedge \Psi(\mathfrak{F}_J)$$

and

$$\Psi(\mathfrak{F}_I \wedge \mathfrak{F}_J) = \Psi(\mathfrak{F}_{I+J}) = \sqrt{I+J} = \sqrt{\sqrt{I}+\sqrt{J}} = \Psi(\mathfrak{F}_I) \vee \Psi(\mathfrak{F}_J)$$

Consequently, by Proposition 1.5, $\mathrm{Rad}(A)$ and $\mathfrak{C}_A$ are prime coherent frames. □

Corollary 3.2. *Regarding a lattice L the following statements are equivalent.*

(1) *L is a coherent frame.*
(2) *There is a commutative ring with identity, say A, such that $L \cong \mathrm{Rad}(A)$ as frames.*
(3) *There is a commutative ring with identity, say A, such that $L \cong \mathfrak{C}_A$ as frames.*

Proof. That the first two items are equivalent is proved in [3]. That the last two statements are equivalent follows from Theorem 3.1. □

Definition 3.3. Recall that a commutative ring with identity, say A, is a von Neumann regular if for every $a \in A$ there is an $x \in A$ such that $a^2x = a$. If we denote the Jacobson radical of A by $\mathfrak{J}(A)$, then a ring is called *semiregular* if $A/\mathfrak{J}(A)$ is von Neumann regular and idempotents lift module $\mathfrak{J}(A)$. A von Neumann regular ring has Krull dimension 0 (i.e., it a zero-dimensional ring) and are precisely the semiprime rings of Krull dimension 0. Von Neumann regular rings are examples of clean ring.

A clean ring is a ring for which every element is the sum of a unit and an idempotent. Clean rings have played a very important role in ring theory in the recent past; for an extensive reference in the commutative case we urge the readers to check [15]. It should be pointed out that clean rings are also known as exchange rings, although these are two different concepts in the non-commutative context.

We denote the Jacobson radical of A by $\mathfrak{J}(A)$. The ring A is called *semiregular* if $A/\mathfrak{J}(A)$ is a von Neumann regular ring and idempotents lift modulo $\mathfrak{J}(A)$. Every semiregular ring is clean.

Clean rings are examples of Gelfand rings. Recall that the ring A is called a Gelfand ring if whenever $a + b = 1$ then for some $r, s \in A$, $(1 + ar)(1 + bs) = 0$.

Corollary 3.4. *Let A be a commutative ring with identity. The following statements are equivalent.*

(1) *A is a zero-dimensional ring.*
(2) Rad(A) *is a zero-dimensional frame.*
(3) *$\mathfrak{C}_A$ is a zero-dimensional frame.*

Proof. That (1) and (2) are equivalent is Theorem 2.3 of [9]. The rest follows from Theorem 3.1. □

Remark 3.5. Proposition 4.7 of [12] characterizes those rings for which $\mathfrak{M}_A$ is a zero-dimensional frame as a finite product of fields. Since there are zero-dimensional rings which are not a finite product of fields it follows that there is a ring A for which $\mathfrak{C}_A$ is a zero-dimensional frame, while $\mathfrak{M}_A$ is not.

Corollary 3.6. *Let A be a commutative ring with identity. The following statements are equivalent.*

(1) *A is a clean ring.*
(2) Rad(A) *is weakly zero-dimensional.*
(3) *$\mathfrak{C}_A$ is a feebly projectable frame.*
(4) *$\mathfrak{M}_A$ is a feebly projectable frame.*

Proof. Proposition 2 of [2] proves that (1) and (2) are equivalent, while Proposition 4.11 of [12] proves that (1) and (4) are equivalent. The rest may be obtained from Theorem 3.1. □

Corollary 3.7. *Let A be a commutative ring with identity. The following statements are equivalent.*

(1) *A is a Gelfand ring.*
(2) Rad(A) *is a normal frame.*
(3) *$\mathfrak{C}_A$ satisfies* Reg(4).
(4) *$\mathfrak{M}_A$ satisfies* Reg(4).

Proof. Proposition 1 of [2] demonstrates that (1) and (2) are equivalent. Next, (1) and (4) are equivalent by Proposition 4.12 of [12]. □

Proposition 3.8. *Let A be a commutative ring with identity. The following statements are equivalent.*

(1) *A is a semiregular ring.*
(2) Rad(A) *satisfies the property that for every finitely generated ideal I of A there is an idempotent $e \in A$ such that $I + eA = A$ and whenever $I + J = A$, then $e \in J$.*
(3) *$\mathfrak{C}_A$ is a projectable frame.*

Proof. The proof follows from Proposition 2.4 of [9] and Theorem 3.1. □

Remark 3.9. Proposition 3.8 2. gives a new characterization of semiregular rings. Theorem 4.6 of [12] states that $\mathfrak{M}_A$ is a projectable frame precisely when A is a finite product of local rings. Since there are semiregular rings which are not finite products of local rings it follows that there are rings for which $\mathfrak{C}_A$ is a projectable frame, yet $\mathfrak{M}_A$ is not a projectable frame.

4. Prime Spectra and Priming

For a coherent frame L we recall that an element $p \in L$ is called prime if $p < 1$ and whenever $a \wedge b \leq p$, then $a \leq p$ or $b \leq p$. A compactness argument yields that to check that p is prime it is enough to check that whenever $a, b \in \mathfrak{K}(L)$ and $a \wedge b \leq p$, then either $a \leq p$ or $b \leq p$. It is known that the element $p \in L$ is prime precisely when the set $\{c \in \mathfrak{K}(L) : c \leq p\}$ is a prime ideal of $\mathfrak{K}(L)$. We prefer to view things in their frame theoretic context.

The collection of prime elements of L is denoted by $\mathrm{Spec}(L)$. Since L is assumed to be coherent we can discuss maximal elements of L and state that $\mathrm{Max}(L) \subseteq \mathrm{Spec}(L)$. Furthermore, a Zorn's Lemma argument produces minimal prime elements of L which we denote the collection of these by $\mathrm{Min}(L)$. We next recall the Zariski and inverse topologies on $\mathrm{Spec}(L)$ and the inherited subspace topologies on $\mathrm{Max}(L)$ and $\mathrm{Min}(L)$. For $x \in \mathfrak{K}(L)$, let $U(x) = \{p \in \mathrm{Spec}(L) : x \nleq p\}$ and $V(x) = \mathrm{Spec}(L) \smallsetminus U(x)$. The collection $\{U(x) : x \in \mathfrak{K}(L)\}$ forms a base for a topology on $\mathrm{Spec}(L)$ known as the Zariski topology (a.k.a. the hull-kernel topology). On the other hand, since a coherent frame satisfies the FIP the collection $\{V(x) : x \in \mathfrak{K}(L)\}$ also forms a base for the topology known as the inverse topology. For more information on the Zariski topology and the inverse topology the reader is encouraged to consult [4], [5], and [15].

We now give a corresponding prime theorem for the spectra of coherent frames.

Theorem 4.1. *Let L be a coherent frame. There is an order-reversing bijection between* $\mathrm{Spec}(L)$ *and* $\mathrm{Spec}(L')$*. Moreover, this bijection maps a basic open subset of* $\mathrm{Spec}(L)$ *onto a basic closed subset of* $\mathrm{Spec}(L')$*. The bijection restricts to a bijection between* $\mathrm{Min}(L)$ *and* $\mathrm{Max}(L')$ *and a bijection between* $\mathrm{Max}(L)$ *and* $\mathrm{Min}(L')$*.*

Proof. This statement is simply a realization of the fact that the complement of a prime ideal is a prime filter. For completeness sake we include the short proof.

We begin by constructing the bijection. Let $p \in \mathrm{Spec}(L)$. We claim that the set $F_p = \{x \in \mathfrak{K}(L) : x \nleq p\}$ belongs to $\mathrm{Spec}(L')$. First notice that $1 \in F_p$ and that if $x, y \in F_p$ then so is $x \wedge y$ since p is a prime element of p. Also, it is clear that if $x \in F_p$ and $x \leq z \in \mathfrak{K}(L)$, then $z \in F_p$. It follows that $F_p \in L'$. To show that $F_p \in \mathrm{Spec}(L')$ suppose that $\mathcal{F}_x, \mathcal{F}_y \in \mathfrak{K}(L')$ (for $x, y \in \mathfrak{K}(L)$ and that $\mathcal{F}_x \wedge \mathcal{F}_y \leq F_p$. But this means that $\mathcal{F}_{x \vee y} \leq F_p$ and so $x \vee y \nleq p$.

Therefore, either $x \nleq p$ or $y \nleq p$. Thus, either $\mathcal{F}_x \leq F_p$ or $\mathcal{F}_y \leq F_p$ ensuring that $F_p \in \mathrm{Spec}(L')$.

This yields the map $\Psi\colon \mathrm{Spec}(L) \to \mathrm{Spec}(L')$ defined by $\Psi(p) = F_p$. By construction the map is clearly injective. As for the reverse if $F \in \mathrm{Spec}(L')$ then the set $\{x \in \mathfrak{K}(L) : x \notin F\}$ is a prime ideal which yields a prime element $p \in \mathrm{Spec}(L)$. Following the construction again produces the equality $\Psi(p) = F$.

We leave it to the interested reader to check that for each $x \in \mathfrak{K}(L)$, $\Psi(U(x)) = V(\mathcal{F}_x)$ and $\Psi(V(x)) = U(\mathcal{F}_x)$. □

Corollary 4.2. *Let L be a coherent frame. The Zariski topology on* $\mathrm{Max}(L)$ *is homeomorphic to the inverse topology on* $\mathrm{Min}(L')$*, while the inverse topology on* $\mathrm{Max}(L)$ *is homeomorphic to the Zariski topology on* $\mathrm{Max}(L')$*.*

Remark 4.3. If one begins with an M-frame that is not necessarily coherent then the above arguments still hold but by first taking $\mathfrak{N}(L) = \mathfrak{K}(L) \cup \{1_L\}$ and then constructing $L' = \mathfrak{F}(\mathfrak{N}(L)$. In this case we lose the correspondence that $L \cong L''$ but we do have some nice relationships between L and L'. Notice that $1_L \in L$ is compact if and only if there are $x, y \in \mathfrak{K}(L)$ such that $x, y < 1$ yet $1 = x \vee y$. Consequently, this translates into the following statement that every nonzero element of L' is dense precisely when 1_L is not compact. In this case $0_{L'} \in \mathrm{Spec}(L')$.

This means that the map $\Psi\colon \mathrm{Spec}(L) \to \mathrm{Spec}(L')$ maps onto the nonzero prime elements of $\mathrm{Spec}(L')$. Therefore, it can be said that the Zariski topology on $\mathrm{Min}(L)$ is homeomorphic to the inverse topology on $\mathrm{Max}(L')$. If L is not coherent, then it is possible that there are no maximal elements of L while L' has a unique minimal prime element $0_{L'}$.

Turning to choice principles, Banaschewski (Lemma 7, [2]) proved that The Prime Ideal Theorem holds if and only if every coherent pm-frame is normal. (Recall that a pm-frame whenever each prime element is below a unique maximal element.) In this manuscript the only time we invoked any sort of choice principle was to ensure that maximal elements exist and it is well-known that the Prime Ideal Theorem is sufficient for this. Observe that the existence of maximal elements in a coherent frame yields the existence of minimal prime elements in a coherent frame. Therefore, we may call a coherent frame a pmin-*frame* if every prime exceeds a unique minimal prime.

Lemma 4.4. *Suppose L is a coherent frame. L is a* pm-*frame if and only if L' is* pmin-*frame, and vice-versa.*

Theorem 4.5. *The Prime Ideal Theorem holds if and only if every coherent* pmin-*frame satisfies Reg(4).*

Proof. By Theorem 2.7 and Lemma 4.4 a coherent frame L is a pmin-frame that satisfies Reg(4) precisely when L' is a coherent normal pm-frame. □

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