COMMUTATIVE SINGULAR f-RINGS

by

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ABSTRACT. This paper examines commutative f-rings A with identity 1, for which 1 is a singular element; (i.e., such that $0 \leq s \leq 1$ implies that $s \wedge (1-s) = 0$.) One of the main results is that for such f-rings the following are equivalent: (a) A is semihereditary; (b) A is a Prüfer ring; (c) the weak dimension of A does not exceed 1; (d) every subring B of the maximum ring of quotients QA which contains A is flat over A; (e) the lattice of all ideals of A is distributive. In the final section singular f-rings which are I-rings are discussed. It is shown that $C(X, \mathbb{Z})$ is an I-ring precisely when X is an extremally disconnected almost P-space.

1 INTRODUCTION

All rings will be commutative with identity. We shall need some facts from the theory of homological algebra. The central reference is [R], but we shall also have occasion to refer to [Va] and [Gl]. We assume that the reader is familiar with the terms *projective* and *injective* module; likewise with the elementary facts about localization in commutative algebra. By the term "domain" we mean "integral domain". Frequently our rings will be *semiprime*, which means that there are no non-zero nilpotent elements, or, equivalently, that the intersection of all the prime ideals is 0.

Let us now briefly review the notion of weak dimension.

DEFINITIONS & REMARKS 1.1 Suppose that A is a ring. (Some of the definitions which follow can be made without the assumptions of commutativity, but we shall have no use for more general settings.) An A-module F is *flat* if for each injective A-homomorphism of modules $f : G \longrightarrow H$, the induced A-homomorphism $F \otimes f : F \otimes G \longrightarrow F \otimes H$ of tensor products is also injective. It is well known that F is a flat A-module if and only if the localized A_M-module F_M is flat, for every maximal ideal M of A (see [R], Theorem 3.78, or Proposition 3.10 in [AM]).

We shall need the concept of weak dimension of a ring. A *flat resolution* of the A-module M is an exact sequence

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$$\ldots \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where each F_n is flat. If K_n stands for the kernel of the map $F_n \longrightarrow F_{n-1}$ (where F_{-1} = we refer to it as the *n-th yoke* of the resolution. If there is a flat resolution of M for \ the $m-th$ yoke is flat, but no yoke is flat for any $j < m$, we say that M has *flat dime* $m+1$. It is a fact that if one such resolution exists, then all flat resolutions of M hav feature. We write $fd(M) = m+1$. $fd(M) = 0$ means that M itself is flat. If there flat resolution of M with flat yokes, we say that $fd(M) = \infty$.

The *weak dimension* of the ring A, denoted $wd(A)$ is the supremum of the flat di sions, over all A-modules.

The following results, which we state, as illustrations, are well known. We shall re the second a number of times in this article. Recall that A is *von Neumann regular* each $a \in A$ there is a $b \in A$ such that $a^2b = a$

THEOREM 1.2 $wd(A) = 0$ *if and only if every A-module is flat if and only if A i Neumann regular.* (Theorem 4.16, [R])

THEOREM 1.3 $wd(A) \leq 1$ *if and only if every ideal of A is flat if and only if for prime ideal P of A, the localization A_P is a valuation domain.* ([Va], Proposition 1.2)

As to the topological notions that we shall need, all spaces will be Hausdorff, frequently also *Tychonoff*: for each point p and closed set K, not containing p, there continuous real-valued function g such that $g(p) = 0$ and $g(K) = \{1\}$. As is the cus βX denotes the Stone-Čech compactification of the Tychonoff space X.

Let us now turn to f-rings.

An *f-ring* is a lattice-ordered ring R in which $a \wedge b = 0$ implies that $a \wedge bc =$ (each $c \geq 0$ $(a, b, c \in R)$. In ZFC these are precisely the lattice-ordered rings which ca represented as subdirect products of totally ordered rings. For each Hausdorff topolo space X, we have the f-ring $C(X)$ of all continuous real-valued functions defined on

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The prototype for singular f-rings is the ring $C(X, \mathbf{Z})$ of all integer-valued contin functions defined on the space X. The reader is referred to [P] and [Al] for informatio $C(X, \mathbf{Z})$. Obviously, for this ring to be interesting, X had better be fairly disconne Indeed, without loss of generality, one may assume that X is *zero-dimensional*; that is, X possesses a base of clopen sets. $C(X, \mathbf{Z})$ is a lattice-subring of $C(X)$, and hence als f-ring.

The following theorem will follow from the main theorem of this section, Theorem We state it here for motivation.

THEOREM 2.1 *For any zero-dimensional Hausdorff space X, $wd(C(X,\mathbf{Z})) \leq 1$.*

Now, on to the greater generality.

DEFINITION & COMMENTS 2.2 First, all lattice-ordered groups (abbr. ℓ-groups) are assumed abelian. For standard material on the theory of ℓ-groups we refer the reader to [BKW]. We assume that the reader is familiar with such terms as *prime subgroup*, *value of an element*, etc. For example, if G is an ℓ-group and $g \in G$, then $Y(g)$ denotes the set of all the values of g. Under the hull-kernel topology, $Y(g)$ is a compact Hausdorff space, called the *Yosida space* of g.

Suppose now that G is an ℓ-group and $0 < s \in G$. We say that s is *singular* if for each $0 \leq g \leq s$, $g \wedge (s-g) = 0$. An ℓ-group G is *singular* if for each $g > 0$ there is a singular element $s > 0$ such that $s \leq g$. The group $\mathbf{Z}$, under addition, and each $C(X,\mathbf{Z})$ are singular ℓ-groups.

Let us now record some basic facts about singular elements. See [CMc], Lemmas 4.1 & 4.2.

PROPOSITION 2.3 *Suppose that G is an ℓ-group.*

(a) *If $0 < t \leq s$ and s is singular, then so is t.*

(b) *If ϕ is an ℓ-homomorphism of G onto the ℓ-group H, and s is singular, then, if $\phi s \neq 0$, it is singular.*

Next, we quote the following; see [CMc], Lemma 4.5 and Theorem 4.12.

PROPOSITION 2.4 *Suppose that G is an ℓ-group.*

(a) *Every value of a singular element is a minimal prime subgroup. Thus, if s is singular then $Y(s)$ is a zero-dimensional compact Haudorff space.*

(b) *If $s \in G$ is singular, and P is a value of s, then the coset $P+s$ is the smallest positive element of G/P.*

(c) *G is singular if and only if there is a maximal set of pairwise disjoint elements in G consisting of singular elements.*

DEFINITION & COMMENTS 2.5 A *singular* f-ring is an f-ring in which the identity is singular. Then A is a singular ℓ-group in its additive structure. It is this class of rings we wish to investigate, under a number of well known ring-theoretic conditions. In the next definition we bring most of those together.

First, let us make some simple observations about singular f-rings. Suppose that A is a singular f-ring. Clearly, every idempotent element of A is singular, and, conversely, if s is singular, then $s \wedge (1-s) = 0$, whence $s(1-s) = 0$, and it follows that s is idempotent.

For each $0 < a \in A$, $e = |a| \wedge 1$ is an idempotent. This makes it evident that a singular f-ring has no nonzero nilpotent elements.

DEFINITION & COMMENTS 2.6 (a) A ring A is *semihereditary* if every finitely generated ideal of A is projective. A domain is semihereditary if and only if it is a P rüfer domain.

We explain this term presently.

(b) We shall denote the classical (or total) ring of fractions of A by qA. Reca an A-submodule S of qA is a *fractional ideal* if there is a regular element $d \in A$ suc $dS \leq A$. Obviously, an ordinary ideal of A is a fractional ideal. A fractional ideal S to be *invertible* if there is a fractional ideal T of A such that $ST = A$. It is clear t $d \in A$ is regular, then Ad is invertible.

Now recall that a ring A is called a *Prüfer ring* if every finitely generated regula (that is, an ideal which contains a regular element) is invertible. For a fairly compreh treatment of Prüfer rings we refer the reader to [LMc], Chapter 10. The central the listing *thirteen* equivalent conditions, is Theorem 10.18 of [LMc].

It is useful to recall a well known result, usually only stated for domains: *Suppos I is a finitely generated regular ideal; then I is invertible precisely when it is project* is then obvious that every semihereditary ring is Prüfer.

A ring A is said to be *Bézout* if every finitely generated ideal of A is principal. It s be clear that every Bézout ring is Prüfer.

Conversely, and to illustrate with f-rings, it has been shown, independently, by E shear [Br] and De Marco [DM], that $C(X)$ is semihereditary if and only if X is ba disconnected (that is, the closure of every cozeroset is open). By constrast, it is s in [MW], that $C(X)$ is Prüfer if and only if X is a quasi F-space; (that is, every cozeroset of X is C^*-embedded.) From Theorem 14.25, [GJ], $C(X)$ is Bézout if and c X is an F-space.

For example, if D is an uncountable set with the discrete topology, and αD denot one-point compactification of D, then it is easy to see that αD is a quasi F-space (be it has *no* proper dense cozerosets) which is not an F-space (and therfore not bas disconnected). On the other hand, $\beta\omega \setminus \omega$ is an F-space which is not basically disconn (see [GJ], 6W and 14.27).

(c) A ring A is *coherent* if every finitely generated ideal I of A is finitely presented means that I can be put in a short exact sequence of A-homomorphisms

$$0 \longrightarrow K \longrightarrow F \longrightarrow I \longrightarrow 0,$$

where F is a free A-module, and both F and K are finitely generated. For an exc account of commutative coherent rings we refer the reader to [Gl].

(d) Recall that a ring A is *arithmetical* if its lattice of ideals is distributive. It is sta [LMc], p. 151, that A is arithmetical if for each maximal ideal M of A, the lattice of i of A_M forms a chain. From [LMc], Theorem 10.18(13), it follows that every arithm ring is Prüfer, and [LMc], Theorem 6.6(4) implies that a Prüfer domain is arithmetic

(e) In [M2] the following notion is introduced: A is said to be a *$P\#$-ring* if every su B of the maximum ring of quotients QA of A, which contains A, is flat over A. By virt Theorem 10.18(3), if A is a domain, or more generally if $qA = QA$, then A is Prüfer i only if it is a $P^{\#}$-ring. In general, every $P^{\#}$-ring is Prüfer. Again, by way of illustration, i is shown in [M2] that $C(X)$ is a $P^{\#}$-ring if and only if X is basically disconnected.

The main objective of this section is to show that the classes of rings introduced ii (a), (c), (d) and (e) coincide for singular f-rings. To that end we proceed. We need som additional generalities on commutative rings.

REMARKS 2.7 (a) We shall not review the construction of the maximum ring of quotients except to refer the reader to [L], where the construction is given, and the original paper c Utumi ([U]). In [Gl] it is discussed, in 4.2.14 and 4.2.15, but not referred to as a ring c quotients. In [M3] it is observed that, for each semiprime ring A, QA is the injective hul of A as a module over itself.

(b) Recall that A is *complemented* if for each $a \in A$ there is some $b \in A$ such that $ab =$ and $a+b$ is regular. It is observed in [M3], for semiprime f-rings, that A is complemented i and only if qA is von Neumann regular. This works as well for (abstract) semiprime rings

The space of minimal prime ideals of A (topologized via the hull-kernel topology) i denoted $Min(A)$. From Theorem 4.3 in [Hu], we have that QA is flat over A precisel when $Min(A)$ is a compact space. Now, if A is a semiprime f-ring then the following ar equivalent (see [M3]):

(i) A is complemented;

(ii) $Min(A)$ is compact;

(iii) qA is von Neumann regular.

Observe that, in general, (ii) above does not imply (iii). Quentel (see p. 118, [Gl]) first gav an example that illustrates this.

(c) We need Corollary 4.2.19 from [Gl]: for any ring A, the following are equivalent:

(i) A is semihereditary;

(ii) A is coherent, and $wd(A) \leq 1$;

(iii) qA is von Neumann regular, and for each maximal ideal M of A, A_M is a valuation domain.

Let us formally record an observation.

PROPOSITION 2.8 *Every semihereditary f-ring A is a semiprime $P^{\#}$-ring.*

PROOF: From (iii) in 2.7(c), if A is semihereditary, it is semiprime. Owing to the commen in 2.7(b), QA is flat over A, and since $wd(A) \leq 1$, it follows that every subring of Q containing A is flat over A. □

Next, some additional observations about singular f-rings are in order. Let us first re
the concept of projectability.

DEFINITION & COMMENTS 2.9 (a) For $a \in A$, let $a^{\perp}$ denote the annihilator o
Note the well known fact (see [BKW]) that, in a semiprime f-ring A, $ab = 0$ if and o
$|a| \wedge |b| = 0$ $(a, b \in A)$. The ring A is said to be *projectable* if, for each $a \in A$, $A = a^{\perp\perp} +$
It is easy to check that A is projectable if and only if, for each $a \in A$, $1 = e + f$, wh
$e \in a^{\perp\perp}$ and $f \in a^{\perp}$; if so, then e and f are idempotent, and $e^{\perp} = a^{\perp}$.

(b) Suppose that A is a semiprime ring. According to Lemma 3.3.4, [Gl], the pri
ideal P of A is minimal if and only if, for each $x \in P$, $x^{\perp} \not\subseteq P$. With this it is easy to sh
that if A is also projectable then, for any two distinct minimal prime ideals P and Q of
$A = P + Q$. This implies that every maximal ideal of A contains a unique minimal pri
ideal.

PROPOSITION 2.10 *Suppose that A is a singular f-ring. Then*
(a) *$Y(1)$, the Yosida space of 1, is $Min(A)$, which is compact.*
(b) *A is projectable.*

PROOF: (a) Since 1 is singular, its values are precisely the minimal prime subgroups o
(by Proposition 2.4(a)). As A is semiprime, the minimal prime subgroups are precisely
minimal prime ideals of A (see [BKW], 9.3.2).

(b) For each $a \in A$, let $e = |a| \wedge 1$. We leave it to the reader to verify that $1 = e + (1 -$
with $e \in a^{\perp\perp}$ and $1 - e \in a^{\perp}$. By 2.9(a), we have that A is projectable. □

COROLLARY 2.10.1 *If A is a singular f-ring, then qA is von Neumann regular, Q
flat over A, and every maximal ideal of A contains a unique minimal prime ideal.*

PROOF: Apply 2.7, (b) and (c), and 2.9(b). □

COROLLARY 2.10.2 *For a singular f-ring A the following are equivalent:*
(a) *A is arithmetical;*
(b) *A is semihereditary;*
(c) $wd(A) \leq 1$.

PROOF: That (b) implies the others has already been observed. If $wd(A) \leq 1$, then, si
qA is von Neumann regular, we apply 2.7(c) to obtain that A is semihereditary. Likew
if A is arithmetical, then by 2.7(c) and 2.6(d), A is semihereditary. □

Now to the equivalence of the trio in Corollary 2.10.2 with the Prüfer condition.
following two simple lemmas do the trick.

LEMMA 2.11.1 *If A is a singular f-ring, and I is an ideal of A, then $a \in I$ if and
if a^{+} and a^{-} belong to I. In particular every ideal of A is generated by positive eleme*

and every finitely generated ideal is generated by finitely many positive elements.

PROOF: Suppose that I is an ideal of A, and that $a \in I$. Let $e = a^+ \wedge 1$. Then it is ea[illegible] to verify that $ea = a^+$, whence a^+ and $a^- = a^+ - a$ both belong to I. The converse obvious, as is the fact that the rest of the lemma follows from the first claim. □

LEMMA 2.11.2 *If A is a singular f-ring and I is a finitely generated ideal, then there is finitely generated regular ideal J of which I is a direct summand.*

PROOF: By Lemma 2.11.1, I has a finite number of positive generators $a_1, \cdots, a_k$. L $e = 1 \wedge (a_1 + \cdots + a_k)$, and observe that $e^\perp = (a_1 + \cdots + a_k)^\perp = \cap_{i=1}^k a_i^\perp$ is the annihilat[illegible] of I. Letting $J = I + A(1-e)$, one sees that $J = I \oplus A(1-e)$, and that the regular eleme[illegible] $(a_1 + \cdots + a_k) + (1-e) \in J$. □

THEOREM 2.12 *For a singular f-ring A, the following are equivalent:*

(a) *A is arithmetical.*
(b) *A is a $P^\#$-ring.*
(c) *A is a Prüfer ring.*
(d) *A is semihereditary.*
(e) $wd(A) \leq 1$.

PROOF: We have already observed (in Corollary 2.10.2) that (a), (d) and (e) are equivale[illegible] From comments in 2.6 and Proposition 2.8, (d) implies (b), which, in turn implies (c). Th[illegible] it suffices to show that (c) implies (d).

Let I be a finitely generated ideal. According to Lemma 2.11.2, I is the direct summa[illegible] of J, a regular, finitely generated ideal of A, which is projective if (c) is assumed. But th[illegible] I too is projective, proving that A is semihereditary. □

COROLLARY 2.12.1 *Every Bézout singular f-ring is semihereditary.* (Note that Theore[illegible] 2.1 is a corollary of this assertion.)

One more corollary, which will be of interest in light of the discussion that follows:

COROLLARY 2.12.2 *Suppose that A is a singular f-ring. Then A is a Prüfer ring if a[illegible] only if, for each minimal prime ideal P of A, A/P is a Prüfer domain.*

PROOF: First observe that $wd(A) \leq 1$ if and only if $wd(A_M) \leq 1$, for each maximal id[illegible] M of A. Next, for each minimal prime ideal P, and each maximal ideal M containing $A_M \cong (A/P)_{M/P}$. Since each A/P is a totally ordered singular domain, the corollary n[illegible] follows from Theorem 2.12. □

The preceding discussion and its culmination in Theorem 2.12 and its corollaries prom[illegible] two obvious questions, neither of which we have been able to answer:

QUESTIONS 2.13 (a) *Is every singular f-ring coherent?*

From a theorem of Chase (see Theorem 2.3.2 in [Gl]), a ring A is coherent if and (if every $a^{\perp}$ is finitely generated and the intersection of any two finitely generated ideal A is finitely generated. Since it is obvious that, in a singular f-ring A, for each $a \in$ $a^{\perp} = e^{\perp} = A(1-e)$, where $e = |a| \wedge 1$, to show that A is coherent, it suffices to prove the intersection of any two finitely generated ideals of A is finitely generated.

(b) *Is a Prüfer singular f-ring necessarily Bézout?*

As we shall presently prove, and in view of Corollary 2.12.2, this question reduce the following one for totally ordered domains: *Suppose that A is a totally ordered Pr domain, in which 1 is the least positive element; is A necessarily a Bézout domain?*

A restricted version of this question is of particular interest: *Is every archimed singular f-ring which is Prüfer also Bézout?* Observe that the answer to this one migh in the affirmative, even if the larger question has a negative answer.

The proof of the following proposition only glancingly involves the ordering on the r and is probably valid in much greater generality.

PROPOSITION 2.14 *Suppose that A is a singular f-ring. Then A is Bézout if and on A/P is a Bézout domain, for each minimal prime ideal P of A.*

PROOF: The necessity is obvious. Before proceeding to prove the sufficiency, observe following: any ring A is Bézout if and only if, for each $a, b \in A$ there exist d, r, s, u, v such that $d = ra + sb$, while $a = ud$ and $b = vd$.

Assume now that A/P is a Bézout domain, for each minimal prime ideal P of A, suppose that $a, b \in A$. Let $I = Aa + Ab$, and denote the set of all minimal prime id which do not contain I by $\mathbf{K}$. Since I is finitely generated, $\mathbf{K}$ is a clopen set in the comp Hausdorff space $Min(A)$. For each $Q \in \mathbf{K}$ there exist $d_Q, r_Q, s_Q, u_Q, v_Q \in A$ such t $a \equiv u_Q d_Q \, modQ$, $b \equiv v_Q d_Q \, modQ$, and $d_Q \equiv r_Q a + s_Q b \, modQ$. Since $Min(A)$ is zero-dimensional, there is an idempotent $e_Q \notin Q$ so that $d_Q e_Q = r_Q e_Q a + a_Q e_Q b$, $a e_Q = u_Q d_Q e_Q$ and $b e_A = v_Q d_Q e_Q$. The open sets $\mathbf{u}(e_Q) = \{ P \in Min(A) : e_Q \notin P \}$ (c all $Q \in \mathbf{K}$) define an open cover of $\mathbf{K}$.

This means that there exist $Q_1, \ldots, Q_m$ so that the $\mathbf{u}(e_{Q_k})$ $(k = 1, \ldots, m)$ cover Without loss of generality (and relabeling) we may assume that the $e_k \equiv e_{Q_k}$ are pairv disjoint. (Indeed relabel everything in sight so as to simplify Q_k to plain k.) Now de $u = \Sigma_{k=1}^m u_k e_k$, $v = \Sigma_{k=1}^m v_k e_k$, $d = \Sigma_{k=1}^m d_k e_k$, $r = \Sigma_{k=1}^m r_k e_k$, $s = \Sigma_{k=1}^m s_k e_k$, and fin $e = \Sigma_{k=1}^m e_k$. Obviously $a = ae$ and $b = be$; since the e_k are disjoint it follows that $a =$ $b = vd$ and $d = ra + sb$, which proves that A is Bezout. □

Here are two examples, which may offer some insight into the questions posed in 2.

EXAMPLE 2.15 *An example of an archimedean singular f-ring which is coherent, but Prüfer.*

Suppose A denotes the ring of all integer-valued sequences, which are eventually polynomial. More specifically, an integer-valued sequence f belongs to A if there is a polynomial $F(T) \in \mathbb{Z}[T]$ and a positive number n_0, so that, for all $n \geq n_0$, $f(n) = F(n)$. The lattice-ordering on A is pointwise; thus, A is an ℓ-subring of $C(\omega, \mathbb{Z})$. It is clearly singular.

Now let S denote the ideal of all sequences in A which are eventually zero. It is easy to see that S is a minimal prime ideal. (Indeed, $Min(A)$ is homeomorphic to $\alpha\omega$, the one-point compactification of the natural numbers, and S is the point at infinity.) Moreover $A/S \cong \mathbb{Z}[T]$, where the latter is lexicographically ordered, with $1 << T \cdots << T^n << \cdots$ Since $\mathbb{Z}[T]$ is not Prüfer (since otherwise it would be Dedekind, which it is not), it follows from Corollary 2.12.2 that neither is A Prüfer.

To see that A is coherent, use the characterization mentioned in 2.13(a). Note first that every $a^{\perp}$ is clearly finitely generated, and, indeed, principal. Next, suppose that I is a finitely generated ideal. By Lemma 2.11.1, we may assume that I has a finite set of positive generators. Then, the generators of I may be divided into two types: those which belong to S and those that are polynomials with integer coefficients on their cozerosets. It is easy to see that one can choose a set of positive generators $a_1, \cdots, a_n$ for I, so that $a_1, \cdots, a_m \in S$ and $|coz(a_i)| = 1$, for each $i = 1, \cdots, m$; and $a_{m+1}, \cdots, a_n$ *having the same zeroset, so that each is a polynomial with integer coefficients on the common cozeroset.* For convenience, let us call such a generating set a *split set of generators.*

Now if I and J are finitely generated ideals of A, we can, with a little more care, find split sets of generators for them, so that the ones outside S all have the same zeroset. How then to choose a finite generating set for $I \cap J$? Suppose that $n \in \mathbf{N}$; if n supports generators of I or J from S, let $c^{(n)}$ be defined by: $c^{(n)}(k)$ is the least common multiple of those generators, when $n = k$, and 0, otherwise. We obtain finitely many $c^{(n_1)}, \cdots, c^{(n_t)}$ (in $I \cap J$) in this manner, running through all n which support a generator of I or J from S. Finally, choose $d \in A$ as follows: $d(n) = 0$, for each n in the common zeroset of the generators of I or J which do not belong to S, and d is the least common multiple of the polynomials representing these generators, otherwise.

We leave it to the reader to verify that $\{c^{(n_1)}, \cdots, c^{(n_t)}, d\}$ is a generating set for $I \cap J$. This shows that the intersection of any two finitely generated ideals of A is finitely generated and, hence, that A is coherent.

Next, an example which illustrates, perhaps, that finding totally ordered domains in which 1 is the least positive element, which are Prüfer, but not Bézout, may not be so easy.

EXAMPLE 2.16 Consider $\mathbb{Z}[\sqrt{10}]$, with the ordering it inherits as a subring of the real numbers. This is a Dedekind domain, as is well known, and therefore Prüfer, but, as it is also Noetherian, to be Bézout it would have to be a principal ideal domain, which it is not. The reader will, however, note that $0 < 4 - \sqrt{10} < 1$.

To conclude this section, we have some remarks intended as an application of the preceding to archimedean singular ℓ-groups. We shall appeal to some facts from [HM2].

REMARKS 2.17 Suppose that G is an archimedean ℓ-group with singular weak order $e > 0$. Note that G is then a singular ℓ-group, in the sense of Definition 2.2. Let . denote the subgroup generated by the singular elements of G. This is a Specker grou defined by Conrad in [C3]. $S(G)$ is an ℓ-subgroup of G, and, since G is singular, $S($ convex as well. As it is a Specker group, $S(G)$ admits a unique ring structure, in whicl product of singular elements is their infimum. It is in this sense that we shall view $S($ a ring; note that it is a singular f-ring.

$S(G)$ will be referred to as the *Specker subring* of G. Indeed, $S(G) = C(Y(e), \mathbf{Z})$ Corollary 2.12.1, $S(G)$ is semihereditary, and so its weak dimension is 1.

DEFINITION & COMMENTS 2.18 (a) Recall that, for any (not-necessarily archimed ℓ-group H, we say that H is *laterally complete* if every set of pairwise disjoint elements a supremum. H' is said to be the *lateral completion* of H if H is a dense ℓ-subgroup o H' is laterally complete, and no proper ℓ-subgroup of H' containing H is laterally comp The existence and uniqueness of lateral completions was demonstrated for abelian ℓ-gr by Conrad [C1], and in general by Bernau [Be1]. The lateral completion of H is den by H^L. Recall that every archimedean laterally complete ℓ-group is projectable [Be2].

It is shown in [HM2] (Proposition 2.8) that if G is an archimedean ℓ-group with sing weak unit $e > 0$, then $G^L = S(G)^L$.

(b) Recall that in any ℓ-group G, and for any subset S,

$$S^{\perp} = \{g \in G : |g| \wedge |s| = 0, \text{ for each } s \in S\}.$$

If $S = \{x\}$, then we write $x^{\perp}$. Each $S^{\perp}$ is an ℓ-ideal of G; an ℓ-ideal C for which $C = ($ is called a *polar* of G. It is well known that the polars of G form a boolean algebra $\mathcal{I}$ under inclusion. As we did for rings, we say that an ℓ-group G is *projectable* if, for . $g \in G$, $G = g^{\perp\perp} + g^{\perp}$. From [HM2], we have that if G is a projectable archimedea group with singular weak unit e, then G is an $S(G)$-module, by putting, for each $g \in$ $(m_1e_1 + \cdots + m_ke_k) * g = m_1g_1 + \cdots + m_kg_k$, where g_i denotes the projection of g in . Note that, for any singular element f, $f * g = g$ if and only if $f \geq |g| \wedge e$; in particular, e as an identity under scalar multiplication, which establishes one of the standard defi conditions for a unital module.

In particular, as already observed, if G is an archimedean ℓ-group with singular v unit e, then G^L is projectable. Let us recall, from [HM2], what G^L actually is. For Hausdorff space X, let $D(X, \mathbf{Z})$ stand for all the functions defined on X with integer val or else $\pm\infty$, which are continuous (relative to the two-point compactification of the integ and integer-valued on a dense subset of X. $D(X, \mathbf{Z})$ is a lattice under pointwise operati though, in general, not a group or ring under these.

If, however, X is *extremally disconnected* (the closure of every open set is open) t $D(X, \mathbf{Z})$ is an archimedean ℓ-group, and, indeed, laterally complete. In fact, from Theo 4.7 in [HM2], if G is an archimedean ℓ-group with singular weak unit e, then $G^L = D(X$ for a suitable compact, extremally disconnected space X. (More on this in Remark 3.6

(c) Concerning maximum rings of quotients, recall this: if A is a semiprime ring, then QA is von Neumann regular ([L], p. 42, Proposition 1). Earlier we observed that QA is the A-injective hull of A. In [M3] it is shown that if A is an archimedean f-ring, then QA contains a copy of A^L as an ℓ-subring.

All the preliminaries are now in place for the following theorem.

THEOREM 2.19 *Suppose that G is a projectable archimedean ℓ-group with singular unit e. Then G is a flat $S(G)$-module.*

PROOF: We embed G in its lateral completion $S(G)^L$. This in turn embeds in $Q(S(G))$. As observed in 2.17, $wd(S(G)) \leq 1$. Thus, if we can show that $Q(S(G))$ is flat over $S(G)$ we are done, since G is a submodule of $Q(S(G))$. However, as $Q(S(G))$ is the $S(G)$-injective hull of $S(G)$, we may invoke 2.7(b), as $Min(S(G)) = Y(e)$, which is compact. □

We conclude the section with a remark about lateral completion of a singular f-ring.

PROPOSITION 2.20 *The lateral completion of a singular f-ring is singular.*

PROOF: First, if A is a singular f-ring, then it is well known that A^L is an f-ring; see [C2]. On the other hand, it follows from Corollary 54.9 in [D] that the supremum of any number of singular elements is singular. Thus, to verify that A^L is singular, it suffices to show that 1 remains singular in the lateral completion. If $0 \leq s \in A^L$, then, since A is projectable, each positive element of A^L is the disjoint supremum of elements of A (which is implicit in Corollary 48.4 of[D]). Hence, s is a supremum of elements of A, which are necessarily singular. It follows that s too is singular. □

3 I-Rings

In this section we investigate when an archimedean singular f-ring is an I-ring.

DEFINITION & REMARKS 3.1 We say the ring A is an *I-ring* if every extension of A in QA is integrally closed. This notion was first introduced in [Eg]. Note that every I-ring is a $P^{\#}$-ring (Proposition 1.4, [M2]), and thus, by 2.6(e), every I-ring is Prüfer. It is shown in [M2] that a Tychonoff space is extremally disconnected if and only if $C(X)$ is an I-ring.

Observe the following, from [Eg]: (Theorem 7) *Let A be a Prüfer ring and qA an I-ring. Then A is also an I-ring.* Since $Q(qA) = QA$, it is immediate that, if A is a Prüfer ring then A is an I-ring if and only if qA is an I-ring.

For singular f-rings things are neater.

PROPOSITION 3.2 *Suppose A is a Prüfer singular f-ring. Then A is an I-ring if and only if $qA = QA$.*

PROOF: Assume that $qA = QA$; then the notions of "I-ring", "$P\#$-ring" and "Prü ring coincide. Conversely, since A is singular, it is semiprime, and then Theorem 9 of [takes care of the rest. □

EXAMPLE 3.3 The notions of Prüfer rings and I-rings are different for singular f-ri consider the following example. Let $A = C(\alpha\omega, \mathbf{Z})$, the ring of eventually constant inte valued sequences. Then it is easily seen that $qA \neq QA$. On the other hand, A is Béz (proved in [Al]), hence a Prüfer ring as well.

This does bring up the question whether all singular f-rings which are also I-rings in fact Bézout.

REMARKS 3.4 It was shown in [M3] (Theorem 2.5) that, for any semiprime f-ring $QA = q(oA)$, where $o(\cdot)$ denotes the orthocompletion. (Without getting into a discuss of the orthocompletion, let us remind the reader that an f-ring is *orthocomplete* if i both laterally complete and projectable.) We have already noted that an archimedean ring which is laterally complete is automatically projectable, and hence it is orthocomple Moreover, in [M3] it is also shown (Theorem 2.4) that if A is projectable then the applicat of $o(\cdot)$ and $q(\cdot)$ may be reversed. Furthermore, since qA is projectable when A is, QA $(qA)^L$, for any projectable f-ring.

We now apply this to singular f-rings.

LEMMA 3.5 *Suppose that A is a Prüfer singular f-ring. Then the following are equivale*

(a) *A is an I-ring.*

(b) *qA is laterally complete.*

(c) $A^L \leq qA$.

PROOF: Assume (a); then $QA = qA$ is laterally complete ([M3], so (b) follows. (b) clea implies (c). Finally, assuming (c), we have that $QA = q(A^L) \leq qA$, by the remarks in 3 proving that $QA = qA$, and so, since A is assumed to be Prüfer, we may conclude that is an I-ring. □

The rest of this section is concerned with the question of when $C(X, \mathbf{Z})$ is an I-ring.

REMARKS 3.6 We assume that X is a zero-dimensional Hausdorff space. Then $C(X, \mathbf{Z})^L$ $D(X^*, \mathbf{Z})$, where X^* stands for the minimal prime space of $CX, \mathbf{Z})$. As we indicated 2.18(b), if X is "sufficiently" disconnected, then $D(X, \mathbf{Z})$ is an archimedean singular f-ri under pointwise operations. For details we refer the reader to [HM2].

The reader should also recall that $Min(C(X, \mathbf{Z}))$ is homeomorphic to $\beta_0 X$, the so-call *Banaschewski compactification* of X (see [M1]). Without further ado, it is (projectivel the largest among all zero-dimensional compactifications. Recall that a space X is *stron zero-dimensional* if every pair of disjoint zerosets can be separated by a clopen partitio Then (see [PW], 4.7(h)) X is strongly zero-dimensional precisely when $\beta_0 X = \beta X$.

In any event, the X^* mentioned in the description of the lateral completion of $C(X,$

is the Gleason absolute of $\beta_0 X$. For a discussion of the Gleason absolutes the reader i referred to Chapter 10 of [Wa] or else Chapter 6 of [PW].

To facilitate the discussion let us indulge in the following definition.

DEFINITION 3.7 X is an *I-space* if $C(X, \mathbf{Z})$ is an I-ring.

REMARKS 3.7a It is now time to invoke some facts from [HM1]. If A is a semiprime f-rin and $QA = qA$ then $Min(A)$ is extremally disconnected (that is, the closure of every ope set is open); see Theorem 1.1. This gives us the following preliminary result.

LEMMA 3.8 *If X is an I-space, then X is extremally disconnected.*

PROOF: First, the classical and maximum quotient rings of $C(X, \mathbf{Z})$ coincide, whenc $Min(C(X, \mathbf{Z})) = \beta_0 X$ is extremally disconnected, by 3.7a. On the other hand, any dens subspace of an extremally disconnected space is C^*-embedded ([GJ], 6M.2), from which w conclude that X itself is extremally disconnected. □

Summing up the situation so far, we have the following corollary.

COROLLARY 3.8.1 *X is an I-space if and only if X is extremally disconnected ar $D(\beta X, \mathbf{Z}) \leq qC(X, \mathbf{Z})$.*

To make further headway, we consider an alternative view of the lateral completion ‹ $C(X, \mathbf{Z})$.

NOTATION 3.9 Suppose $\mathcal{F}$ is a filter of dense subsets of a space X. $C[\mathcal{F}, \mathbf{Z}]$ stands for tl direct limit of the $C(U, \mathbf{Z})$, over all the members $U \in \mathcal{F}$, where the bonding maps are tl restrictions.

We then have the following observation. $\mathcal{G}$ denotes the filter of dense open subsets.

PROPOSITION 3.10 *For an extremally disconnected space X, $D(\beta X, \mathbf{Z}) = C[\mathcal{G}, \mathbf{Z}]$.*

PROOF: View $f \in C(U, \mathbf{Z})$ (with $U \in \mathcal{G}$) as a continuous function into the two-poi compactification of $\mathbf{Z}$. Then f extends uniquely to f' on βU. Since U dense in X, ar X is extremally disconnected, it follows that $\beta U = \beta X$. Next, since $U \subseteq (f')^{-1}(\mathbf{R}$ $f' \in D(\beta X, \mathbf{Z})$. This implies that $C(U, \mathbf{Z}) \leq D(\beta X, \mathbf{Z})$, whence $C[\mathcal{G}, \mathbf{Z}] \leq D(\beta X, \mathbf{Z})$.

As for the reverse containment, if $f \in D(\beta X, \mathbf{Z})$ then by restriction $f \in C(f^{-1}(\mathbf{R})$ $X, \mathbf{Z})$. □

Obviously, one ought to settle when $C(X, \mathbf{Z})$ is laterally complete, which brings us the next theorem. First we recall two constructions in topology.

DEFINITION & NOTATION 3.11 For any Tychonoff space X, υX denotes the subspa of βX consisting of the "real" points of the compactification. (The reader should rec that each point $p \in \beta X$ corresponds to a maximal ideal M^p of $C(X)$. The points vX are those for which $C(X)/M^p = \mathbf{R}$. Recall as well (see Chapter 8 of [GJ]) that *real-compact* if $X = vX$. Furthermore, vX is characterized (up to homeomorphism X) as the realcompact space Y, in which X is dense, and such that each $f \in C(X)$ ha extension to some $f' \in C(Y)$.

For zero-dimensional spaces X, there is a parallel "N-compactification". $v_{\mathbf{N}}X$ cons of the points of $\beta_0 X$, which correspond to minimal prime ideals P of $C(X, \mathbf{Z})$ for wl $C(X, \mathbf{Z})/P = \mathbf{Z}$. A zero-dimensional space X is **N**-*compact* if $v_{\mathbf{N}}X = X$. As with r compactness, $v_{\mathbf{N}}X$ is characterized as the (zero-dimensional) N-compact space Y, wl contains X densely, and so that each continuous, integer-valued function on X can extended continuously to Y. (For discussion of N-compactness, see [PW].)

THEOREM 3.12 *The following are equivalent for an extremally disconnected space X.*

(1) $C(X, \mathbf{Z})$ *is laterally complete.*

(2) $C(X, \mathbf{Z}) = D(\beta X, \mathbf{Z})$.

(3) $C(X, \mathbf{Z}) \cong C(U, \mathbf{Z})$, *via the restriction map, for each dense open subset* $U \subseteq X$.

(4) *For each dense open subset* $U \subseteq X$, $v_{\mathbf{N}}U = v_{\mathbf{N}}X$.

(5) *For each dense open subset* $U \subseteq X$, $vU = vX$.

PROOF: For any extremally disconnected space X, $vX = v_{\mathbf{N}}X$ ([PW], 5G(3)), since X strongly zero-dimensional, and so it is clear that (4) and (5) are equivalent. The equivale of (1) and (2) is evident from the comments in 3.6. As for (2) $\rightarrow$ (3), we have, by Proposit 3.10, that $C(X, \mathbf{Z}) = C[\mathcal{G}, \mathbf{Z}]$. Since $C(X, \mathbf{Z}) \leq C(U, \mathbf{Z})$ via the restriction map, (3) follo In the same manner, again using Proposition 3.10, we get that, in fact, the first th statements are equivalent.

Next, suppose that (3) holds. First, (see [PW]) to prove (4) we note that it suffices show that for each $f \in C(U, \mathbf{N})$ there is a continuous extension f' from $v_{\mathbf{N}}X$ into $\mathbf{N}$. I by assumption f may be extended to f^* on X and then all the way to $v_{\mathbf{N}}X$. Restrict back to U we get the desired extension f'.

Finally, suppose that (4) holds and let $f \in C(U, \mathbf{Z})$. Without loss of generality (si the inclusion of $C(X, \mathbf{Z})$ inside $C(U, \mathbf{Z})$ is an ℓ-homomorphism,) we may assume that $f \geq$ i.e., that $f \in C(U, \mathbf{N})$. Then extend f to $v_{\mathbf{N}}X$, and restrict to X; denote this restriction f_o, and observe that f_o restricts to f, whence the restriction map is surjective. This shc that (4) implies (3), and the theorem is proved. □

We are almost to the characterization of those $C(X, \mathbf{Z})$ which are I-rings. But firs small, but very useful lemma.

LEMMA 3.13 *For any zero-dimensional space X, $qC(X, \mathbf{Z}) \cap D(\beta_0 X, \mathbf{Z}) = C(X, \mathbf{Z})$.*

PROOF: Let $g \in qC(X, \mathbf{Z}) \cap D(\beta_0 X, \mathbf{Z})$. Then $g \colon \beta_0 X \longrightarrow \mathbf{Z} \cup \{\pm\infty\}$, and without l of generality $g \geq 0$. Now we may write $g = \frac{a}{b}$, where both $a, b \geq 0$ and $a, b \in C(X,$ Note that $\frac{a}{b}(x) = \frac{a(x)}{b(x)} \in \mathbf{Q}$, for all $x \in \beta_0 X$, for which $|b(x)| < \infty$. But then $g(x)$ is inte valued, whence $\frac{a}{b}(x) \in \mathbf{Z}$. Thus, for each $x \in X$, $b(x)$ divides $a(x)$, so that the function h defined by $h(x)b(x) = a(x)$, for each $x \in X$, is continuous on X. This shows that, in fact $g = \frac{a}{b} = h \in C(X, \mathbf{Z})$, which is what we wanted. □

Here is the main theorem of this section. It is immediate from Theorem 3.12, Lemm 3.13 and Corollary 3.8.1.

THEOREM 3.14 *For a zero-dimensional space X, the following are equivalent.*

(1) *X is an I-space.*

(2) *$C(X, \mathbf{Z})$ is laterally complete.*

(3) *X is extremally disconnected and $C(X, \mathbf{Z}) = D(\beta X, \mathbf{Z})$.*

(4) *X is extremally disconnected, and $C(X, \mathbf{Z}) \cong C(U, \mathbf{Z})$, via the restriction map, fc each dense open subset $U \subseteq X$.*

(5) *X is extremally disconnected, and for each dense open subset $U \subseteq X$, $\upsilon_{\mathbf{N}}U = \upsilon_{\mathbf{N}}X$*

(6) *X is extremally disconnected, and for each dense open subset $U \subseteq X$, $\upsilon U = \upsilon X$.*

REMARKS 3.15 With an eye toward generalization of some of the above to archimedea Prüfer singular f-rings, let us recall some material from [HM2], namely, the "integer-valued version of Yosida embedding. If A is any archimedean singular f-ring, then A may t identified with an f-ring in $D(Min(A), \mathbf{Z})$. If A is also an I-ring, then $Min(A)$ is extremall disconnected (Theorem 1.1 of [HM1]), which means that, for each $a \in A$, the set on whic a is integer-valued is C^*-embedded in $Min(A)$. This means that Lemma 3.13 (and i proof) are valid, *as long as one assumes that A is Bézout*: in that development the greate common divisor of a and b is b, and so h (as defined there) belongs to A. Together wit Lemma 3.5, this proves the necessity in the following result.

THEOREM 3.16 *Suppose that A is a singular archimedean f-ring, which is Bézout. Th A is an I-ring if and only if A is laterally complete.*

PROOF: The reader easily verifies that, for each compact extremally disconnected spa X, $D(X, \mathbf{Z})$ is a Bézout ring, and therefore also Prüfer. This comment and Lemma 3 prove the sufficiency. □

REMARK 3.17 Lemma 3.13 is not valid for archimedean singular f-rings, in generε Consider the following example. Let A be the ring of all integer valued sequences whi are eventually a polynomial in the ring $\mathbf{Z}[T^2, T^3]$. A is an ℓ-subring of the ring in examp 2.15. Both have $\alpha\omega$, the one point compactification of the discrete natural numbers, as the space of minimal prime ideals. Therefore, both embed $D(\alpha\omega, \mathbf{Z})$. In addition, these rin have the same classical ring of quotients, as the identity sequence $j(n) = n$ is eventual the rational function T^3/T^2. Thus, $A \neq qA \cap D(Min(A), \mathbf{Z})$.

This is an issue which has to do with integral closure of a singular archimedean f-rii in its classical ring of quotients, which we take up elsewhere.

DEFINITION & REMARKS 3.18 Recall that a point p in the Tychonoff space X is call an *almost P-point* if every zeroset containing p has nonempty interior. A space X is c an *almost P-space* if every point of X is an almost P-point. The following equival are well known (see [Lv]): (a) X is an almost P-space; (b) every nonempty zerose nonempty interior; (c) the only dense cozeroset is X; (d) every non-empty G_δ-se non-empty interior. In addition, it is known that in an extremally disconnected s of nonmeasurable cardinality every almost P-point is isolated (see 6O in [PW] and [Lv]). Thus, every extremally disconnected almost P-space, of nonmeasurable cardinal discrete. For a definition of measurable cardinals we refer the reader to [GJ], Chapter

What we are after is the final theorem of this article. We leave out any discussi the existence of nondiscrete almost P-spaces which are extremally disconnected, once a referring the reader to [GJ], 12H, and [Lv].

THEOREM 3.19 *Let X be a zero-dimensional space. Then X is an I-space if and o it is an extremally disconnected almost P-space.*

PROOF: Suppose first that X is an I-space. We already know it is extremally disconne Now, let V be a dense cozeroset of X. Express $V = coz(f)$, for some $f \in C(X)^+$. B or (6) in Theorem 3.14, we may extend the function f^{-1} to all of X. A routine check s that this impossible unless $V = X$.

Conversely, if X is an extremally disconnected almost P-space, then each $f \in D(\beta X$ must be real-valued, as $f^{-1}(\mathbf{R})$ is a dense cozeroset. Thus, $D(\beta X, \mathbf{Z}) = C(X, \mathbf{Z})$, and claim follows from Theorem 3.14. □

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