

# A Characterization of Commutative Clean Rings

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**ABSTRACT.** A commutative ring  $A$  is said to be *clean* if every element of  $A$  can be written as a sum of a unit and an idempotent. This definition dates back to 1977 where it was introduced by W. K. Nicholson [7]. In 2002, V. P. Camillo and D. D. Anderson [1] investigated commutative clean rings and obtained several important results. In [4] Han and Nicholson show that if  $A$  is a semiperfect ring, then  $A[\mathbb{Z}_2]$  is a clean ring. In this paper we generalize this argument (for commutative rings) and show that  $A[\mathbb{Z}_2]$  is clean if and only if  $A$  is clean. We also show that if the group ring  $A[G]$  is a commutative clean ring, then  $G$  must be a torsion group. Our investigations lead us to introduce the class of 2-clean rings.

*Keywords:* commutative clean ring, commutative group ring

## 1. 2-Clean Rings

**Definition 1.1.** The element  $a \in A$  is said to be *clean* if there exists an idempotent  $e \in A$  such that  $a - e$  is invertible. If every element of  $A$  is clean, then  $A$  is said to be a *clean ring*.

The set of units and idempotents of  $A$  shall be denoted by  $\mathcal{U}(A)$  and  $Id(A)$ , respectively. As usual  $Max(A)$  denotes the set of maximal ideals of  $A$ , and it is equipped with the hull-kernel topology. This means that the collection of sets of the form  $V(a) = \{M \in Max(A) : a \in M\}$  for  $a \in A$  form a base for the closed sets. The complement of  $V(a)$  is denoted by  $U(a)$ .

It is well known that von Neumann regular rings, and more generally zero-dimensional rings are clean. Also, local rings are clean, and, in fact, are precisely the indecomposable clean rings. Thus, if  $A$  is an integral domain, then  $A$  is clean if and only if  $A$  is local. Some fundamental facts regarding clean rings are that every homomorphic image of a clean ring is a clean ring, and that a ring  $A$  is clean if and only if  $A/n(A)$  is clean where  $n(A)$  denotes the nilradical of  $A$ . Also, a direct product of rings is clean if and only if each factor is clean. Most of these facts can be found in [4] or [1]. We urge the interested reader to check there for their proofs.

**All rings are commutative and with identity.**

In [6] a list of several characterizations of commutative clean rings is given. Included in the list is one given by Johnstone [5] which we presently state.

**Theorem 1.2** [Johnstone].  *$A$  is a clean ring if and only if  $Max(A)$  is zero-dimensional and every prime ideal is contained in a unique maximal ideal.*

A topological space is said to be *zero-dimensional* if it has a base of clopen sets. It is known that if  $e \in Id(A)$ , then  $U(e)$  is a clopen subset of  $Max(A)$  but the reverse is not true in general. A clopen subset of  $Max(A)$  of the form  $U(e)$  for some  $e \in Id(A)$  will be called an *idempotent clopen*. In [6] it is shown that  $Max(A)$  is zero-dimensional and every clopen is an idempotent clopen precisely when  $A$  is a clean ring.

**Theorem 1.3.** *The following are equivalent for a ring  $A$*

- (i)  *$A$  is a clean ring.*
- (ii) *For each  $a \in A$  there exists an idempotent  $e$  such that  $V(a) \subseteq U(e)$  and  $V(a-1) \subseteq V(e)$ .*
- (iii) *The collection of idempotent clopen subsets of  $Max(A)$  is base for the topology on  $Max(A)$ .*
- (iv)  *$Max(A)$  is zero-dimensional and every clopen subset is of the form  $U(e)$  for some  $e \in Id(A)$ .*

Here is another characterization of a clean ring.

**Proposition 1.4.**  *$A$  is clean if and only if for every  $a \in A$ , there is an idempotent  $e \in A$  such that both  $a + e$  and  $a - e$  are invertible.*

**Proof.** The sufficiency is clear. As for the necessity, let  $a \in A$ . By Theorem 1.3, choose an idempotent  $e \in A$  such that  $V(a^2) \subseteq U(e)$  and  $V(a^2 - 1) \subseteq V(e)$ . Let  $M$  be any maximal ideal of  $A$ . If  $a \in M$ , then so is  $a^2$  and hence  $e \notin M$ . It follows that  $a + e, a - e \notin M$ . If  $a, e \notin M$ , then neither  $a + 1$  nor  $a - 1$  belong to  $M$ . Thus,  $a + e + M = a + 1 + M \neq M$  and  $a - e + M = a - 1 + M \neq M$ , whence  $a + e, a - e$  are invertible. ■

**Definition 1.5.** We call a commutative ring  $A$  *2-clean* if for every pair of elements  $a, b \in A$  there exists an idempotent  $e \in A$  such that  $a + b - e$  and  $a - b - e$  are both invertible. Observe that letting  $b = 0$  we obtain that a 2-clean ring is clean. Also, if  $\text{char} A = 2$ , then  $A$  is clean if and only if  $A$  is 2-clean.

The proof of our next proposition is straightforward.

**Proposition 1.6.** 1) *A homomorphic image of a 2-clean ring is 2-clean.*  
2) *A direct product  $A = \prod_{i \in I} A_i$  is 2-clean if and only if each factor  $A_i$  is 2-clean.*

**Proposition 1.7.** *The following are equivalent for a ring  $A$ .*

- (i)  *$A$  is 2-clean.*
- (ii)  *$A/n(A)$  is 2-clean.*
- (iii)  *$A$  is clean and  $A/\mathfrak{J}(A)$  is 2-clean.*
- (iv)  *$A/\mathfrak{J}(A)$  is 2-clean and idempotents lift modulo the Jacobson radical.*

**Proof.** The proof is exactly the same as the one used to show that a ring  $A$  is clean if and only if  $A/n(A)$  is clean. We leave the verification to the interested reader. ■

Our next result gives an easier way of checking whether a clean ring is 2-clean.

**Proposition 1.8.**  *$A$  is 2-clean if and only if  $A$  is clean and  $2 \in \mathfrak{J}(A)$ .*

**Proof.** Necessity. If  $A$  is 2-clean, then it is clean. Let  $a \in A$  be an arbitrary element. Using the 2-clean condition on the pair  $a, a$  there is an idempotent  $e \in A$  such that  $2a - e$  and  $-e$  (and hence  $e$ ) are invertible. The only idempotent that is invertible is  $e = 1$ . Therefore,  $2a - 1$  is invertible, whence  $2 \in \mathfrak{J}(A)$ .

Sufficiency. Let  $a, b \in A$ . Choose an idempotent  $e \in A$  such that  $a + b - e$  is invertible. We claim that  $a - b - e$  is also invertible. Let  $M$  be a maximal ideal of  $A$ . By hypothesis,  $2 \in M$  so that  $a - b - e + M = a + b - e + M \neq M$ . Therefore,  $a - b - e \notin M$  for any maximal ideal of  $A$ . It follows that both  $a - b - e$  and  $a + b - e$  are invertible. ■

**Theorem 1.9.** *Let  $A$  be a zero-dimensional ring. Then the following are equivalent:*

- (i)  *$A$  is 2-clean.*
- (ii)  *$\text{char} A = 2^k$  for some natural  $k$ .*
- (iii)  *$2$  is nilpotent.*
- (iv) *For each maximal ideal  $M$  of  $A$ ,  $\text{char} A/M = 2$ .*

**Proposition 1.10.** *Suppose  $A$  is a local ring. Then  $A$  is 2-clean if and only if  $\text{char} A/M = 2$ .*

**Proof.** The necessity is clear so suppose that  $\text{char} A/M = 2$ . This means that  $a + b + M = a - b + M$  for all  $a, b \in A$ . Therefore,  $a + b \in M$  if and only if  $a - b \in M$ . Thus, if both  $a + b, a - b \in M$ , then  $a + b - 1, a - b - 1 \notin M$  and hence both are invertible. Otherwise,  $a + b, a - b \notin M$  means that they are both invertible. ■

## 2. Commutative Clean Group Rings

In this section we address the question of when a group ring is clean. Some work on this question was done in [4] where they showed that if  $A$  is a boolean ring and  $G$  is a torsion group, then  $A[G]$  is a clean ring. We will only consider the case where the group in question is abelian. When a group ring is zero-dimensional or local has been characterized and therefore we have some sufficient conditions for a group ring to be a clean ring. We list [3] as our main reference on commutative group rings, though we warn the casual reader that most of the information in [3] is done for semi-group rings.

**All groups are abelian.**

**Definition 2.1.** Let  $A$  be a ring and  $(G, +)$  a group. The group ring is the set of formal sums of the form  $\sum_{i=1}^n a_i X^{g_i}$  under pointwise addition and multiplication defined by  $X^g X^h = X^{g+h}$  on monomials and extending this in the obvious way.

**Theorem 2.2.**  $A[G]$  is zero-dimensional if and only if  $A$  is zero-dimensional and  $G$  is a torsion group.

For commutative rings and abelian groups we obtain the following corollary.

**Corollary 2.3.** If  $A$  is a boolean ring and  $G$  is a torsion group, then  $A[G]$  is a clean ring.

**Theorem 2.4.**  $A[G]$  is local if and only if  $A$  is local,  $\text{char } A/M = p$ , and  $G$  is a  $p$ -group. In particular, if  $A[G]$  is local, then  $G$  is torsion.

Our aim is to show that for  $A[G]$  to be a clean ring it is necessary that  $A$  be clean and  $G$  be torsion. The first part is clear as  $A$  is a homomorphic image of  $A[G]$ . To obtain the second part we need a series of lemmas. The first is well known.

**Lemma 2.5.**  $A[G]$  is an integral domain if and only if  $A$  is an integral domain and  $G$  is torsion free.

**Lemma 2.6.** For any ring  $A$  and group  $G$ ,  $A[G]$  is clean if and only if  $A/n(A)[G]$  is clean.

**Proof.** The necessity is obvious since  $A/n(A)[G]$  is a homomorphic image of  $A[G]$ . As for the sufficiency it is known that  $n(A)[G] \leq n(A[G])$ , which means that  $A[G]/n(A[G])$  is a homomorphic image of  $A/n(A)[G]$  and therefore  $A[G]$  is clean. ■

**Proposition 2.7.** If  $A[G]$  is a clean ring, then  $A$  is clean and  $G$  is a torsion group.

**Proof.** Clearly,  $A$  is a clean ring. Let  $t(G)$  denote the torsion subgroup of  $G$ . Let  $P$  be a prime ideal of  $A$ . Then  $(A/P)[G/t(G)]$  is a homomorphic image of  $A[G]$  and hence is clean. Since  $G/t(G)$  is a torsion-free group and  $A/P$  is an integral domain it follows by the Lemma 2.5 that  $(A/P)[G/t(G)]$  is a clean integral domain and hence is local. By Theorem 2.4 it follows that  $G/t(G)$  is torsion, whence  $G = t(G)$ . ■

**Remark 2.8.** The converse of the previous proposition is certainly not true. In [4] it is shown that if  $A$  is the localization of  $\mathbb{Z}$  at the prime  $7\mathbb{Z}$  ( $A$  is clean because it is local) and  $G = \mathbb{Z}_3$  (clearly torsion), then  $A[G]$  is not clean. In general, discovering when  $A[G]$  is clean has eluded us except for in certain cases. In the next section we consider the specific case when  $G = \mathbb{Z}_2$ .

**Proposition 2.9.** *Suppose  $A$  is zero-dimensional. Then the following are equivalent:*

- (i)  $A[G]$  is clean.
- (ii)  $A[G]$  is zero-dimensional.
- (iii)  $G$  is a torsion group.

**Proof.** This is in fact a corollary to Proposition 2.7 and Theorem 2.2. ■

**Corollary 2.10.** *Let  $F$  be a field. Then  $F[G]$  is clean if and only if  $G$  is torsion.*

### 3. When $G = \mathbb{Z}_2$

In this section we investigate the cleanliness of the group ring  $A[\mathbb{Z}_2]$ .

**Definition 3.1.** Let  $e$  be an idempotent of  $A$ . The element  $eX^0 \in A[G]$  is an idempotent of  $A[G]$ . Any idempotent of  $A[G]$  of this form is said to be an *idempotent in  $A$* . If every idempotent of  $A[G]$  is in  $A$ , then we say that the *idempotents of  $A[G]$  are in  $A$* .

**Lemma 3.2.** *The element  $rX^0 + sX^1 \in A[\mathbb{Z}_2]$  is invertible if and only if  $r + s, r - s \in \mathcal{U}(A)$ .*

**Proof.** Suppose  $rX^0 + sX^1 \in \mathcal{U}(A[\mathbb{Z}_2])$ . Then there are  $f, g \in A$  such that  $(rX^0 + sX^1)(fX^0 + gX^1) = 1$ . This means that  $rf + sg = 1$  and  $rg + sf = 0$ . Adding these two equations gives us that  $(r + s)(f + g) = 1$ . Subtracting yields  $(r - s)(f - g) = 1$ . Therefore,  $r + s, r - s \in \mathcal{U}(A)$ .

Conversely, if  $r + s, r - s \in \mathcal{U}(A)$ , then so is  $r^2 - s^2$ . Let  $f = \frac{r}{r^2 - s^2}$  and  $g = \frac{-s}{r^2 - s^2}$ . A quick check shows that  $rf + sg = 1$  and  $rg + sf = 0$ . It follows that  $rX^0 + sX^1$  is an invertible element of  $A[\mathbb{Z}_2]$ . ■

**Lemma 3.3.** *The element  $cX^0 + dX^1 \in A[\mathbb{Z}_2]$  is an idempotent if and only if  $c^2 + d^2 = c$  and  $2cd = d$ . In this case,  $c + d, c - d \in Id(A)$ .*

**Proof.** The first statement follows from the equation  $cX^0 + dX^1 = (cX^0 + dX^1)^2 = (c^2 + d^2)X^0 + 2cdX^1$ . The second is obtained by adding (and subtracting) the equations. ■

**Proposition 3.4.** *The following are equivalent for the ring  $A$ :*

- (i)  $A$  is clean.
- (ii) For every  $a \in A$ , the element  $aX^0 \in A[\mathbb{Z}_2]$  can be written as the sum of a unit and an idempotent in  $A$ .
- (iii) For every  $a \in A$ , the element  $aX^1 \in A[\mathbb{Z}_2]$  can be written as the sum of a unit and an idempotent in  $A$ .

**Proof.** (i)  $\Rightarrow$  (ii) If  $A$  is clean, then for each  $a \in A$  let  $u \in \mathcal{U}(A)$  and  $e \in Id(A)$  such that  $a = u + e$ . Then

$$aX^0 = uX^0 + eX^0$$

is a representation of  $aX^0$  as a sum of a unit of and an idempotent in  $A$ .

(ii)  $\Rightarrow$  (i) Let  $a \in A$ . If

$$aX^0 = (rX^0 + sX^1) + eX^0$$

is a representation of  $aX^0$  as a sum of a unit and an idempotent in  $A$ , then  $s = 0$  and thus  $r = r + s \in \mathcal{U}(A)$ . Therefore,  $A$  is a clean ring.

(iii)  $\Rightarrow$  (i) Let  $a \in A$ . Suppose that  $aX^1$  is the sum of a unit and an idempotent in  $A$ , say

$$aX^1 = (rX^0 + aX^1) + eX^0.$$

for some  $e \in Id(A)$ . By Lemma 3.2  $r + a, a - r \in \mathcal{U}(A)$ . Since  $r = -e$  it follows that  $a - e$  and  $a + e$  are both invertible and so by Proposition 1.4  $A$  is clean.

(i)  $\Rightarrow$  (iii) Let  $a \in A$ . By Proposition 1.4 there exists an  $e \in Id(A)$  such that  $a - e, a + e$  are invertible. Set  $r = -e$  so that

$$aX^1 = (rX^0 + aX^1) + eX^0.$$

Since  $r + a = a - e$  and  $r - a = -(a + e)$  are invertible it follows that  $aX^1$  is the sum of a unit and an idempotent in  $A$ . ■

**Proposition 3.5.** *For every  $a \in A$ , the element  $aX^0 + aX^1 \in A[\mathbb{Z}_2]$  can be written as the sum of a unit and an idempotent in  $A$  if and only if  $2 \in \mathfrak{J}(A)$ . In this case every idempotent of  $A[\mathbb{Z}_2]$  belongs to  $A$ .*

**Proof.** Suppose  $aX^0 + aX^1 \in A[\mathbb{Z}_2]$  can be written as the sum of a unit and an idempotent in  $A$  for every  $a \in A$ , say

$$ax^0 + aX^1 = (rX^0 + sX^1) + eX^0.$$

Since  $s = a$  it follows that  $r + a, r - a \in \mathcal{U}(A)$  by Lemma 3.2. Notice that  $e = a - r$  which is invertible and hence  $e = 1$  and  $r = a - 1$ . Thus  $r + a = 2a - 1 \in \mathcal{U}(A)$  for every  $a \in A$ , whence  $2 \in \mathfrak{J}(A)$ .

Conversely, let  $a \in A$  and  $2 \in \mathfrak{J}(A)$ . Since

$$aX^0 + aX^1 = ((a - 1)X^0 + aX^1) + X^0$$

and  $(a - 1) + a = 2a - 1$  and  $(a - 1) - a = -1$  are both invertible, the result follows from Lemma 3.2.

Finally, suppose  $2 \in \mathfrak{J}(A)$  and let  $aX^0 + bX^1$  be idempotent. This means that  $a^2 + b^2 = a$  and  $2ab = b$ . The latter equality implies that  $(2a - 1)b = 0$ . By hypothesis,  $2 \in \mathfrak{J}(A)$  and therefore  $2a - 1$  is invertible. It follows that  $b = 0$  and so the idempotents of  $A[\mathbb{Z}_2]$  belong to  $A$ . ■

**Proposition 3.6.** *The following are equivalent.*

- (a) *For each  $a \in A$ , the element  $aX^0 + aX^1$  is clean.*
- (b) *For each  $a \in A$  there exists an  $e \in Id(A)$  and  $t, u \in A$  such that  $(2a - 1)t = (1 - e)$ , and  $2au = e$ .*
- (c) *For each  $a \in A$ , there exists an  $e \in Id(A)$  such that  $V(2a) \subseteq V(e)$  and  $V(2a - 1) \subseteq U(e)$ .*

**Proof.** (b) $\Rightarrow$ (a). Let  $a \in A$  and choose  $e \in Id(A)$  and  $t, u \in A$  satisfying (b). Let  $f = 1 - e$  so that  $ef = 0$  and  $e + f = 1$ . Observe that since both  $e$  and  $f$  are idempotent we may without loss of generality assume that  $ue = u$ , and  $tf = t$ . Let  $x = aue$  and observe that  $2x = e$  and  $ex = x$ . Let  $y = (a - 1)f$ ,  $z = (a - x)e$ , and  $r = y + z$ . Next let  $d = -x$  and  $c = d + 1 = 1 - x$ .

Now,

$$c^2 + d^2 = d^2 + 2d + 1 + d^2 = 2d^2 + 2d + 1 = 2x^2 - 2x + 1 = x + f.$$

Notice that  $2x = e = 1 - f$  so that  $c = 1 - x = x + f$  and therefore  $c^2 + d^2 = c$ . Next,

$$2cd = 2c(-x) = -ce = -(1 - x)e = ex - e = x - e = x - 2x = -x = d$$

and so  $cX^0 + dX^1$  is an idempotent. Now,

$$\begin{aligned} r + c &= y + z + c \\ &= (a - 1)f + (a - x)e + 1 - x \\ &= af - f + ae - ex + 1 - x \\ &= a - f - ex + 1 - x \\ &= a + e - 2x \\ &= a \end{aligned}$$

so that

$$aX^0 + aX^1 = (rX^0 + (r + 1)X^1) + (cX^0 + dX^1).$$

What is left to be shown is that  $(rX^0 + (r + 1)X^1)$  is invertible. To that end, first notice that

$$\begin{aligned} 2r + 1 &= 2y + 2z + 1 \\ &= 2(a - 1)f + 2(a - x)e + e + f \\ &= (2a - 1)f + 2ae. \end{aligned}$$

Next,

$$\begin{aligned} (2r + 1)(t + u) &= ((2a - 1)f + 2ae)(t + u) \\ &= (2a - 1)t + 2aue \\ &= f + e \\ &= 1. \end{aligned}$$

Therefore  $rX^0 + (r + 1)X^1$  is invertible by Lemma 3.2, whence  $aX^0 + aX^1$  is clean.

(a) $\Rightarrow$ (c). Suppose  $aX^0 + aX^1$  is clean and write it as a sum of a unit and idempotent, say

$$aX^0 + aX^1 = (rX^0 + sX^1) + (cX^0 + dX^1).$$

Note that  $a = r + c = s + d$ . Since  $(rX^0 + sX^1)$  is invertible,  $r + s \in \mathcal{U}(A)$ , and since  $(cX^0 + dX^1)$  is an idempotent, it follows that  $c + d \in Id(A)$ . Now  $2a = a + a = r + c + s + d = (r + s) + (c + d)$  and so  $2a$  is a clean element. It follows that there exists an idempotent  $e$  such that  $V(2a) \subseteq V(e)$  and  $V(2a - 1) \subseteq U(e)$ .

The equivalence of (b) and (c) is straightforward.

■

**Theorem 3.7.** *For a ring  $A$ , the following are equivalent:*

- (i)  $A$  is 2-clean.
- (ii)  $A[\mathbb{Z}_2]$  is clean and the idempotents of  $A[\mathbb{Z}_2]$  belong to  $A$ .
- (iii)  $A[\mathbb{Z}_2]$  is clean and  $2 \in \mathfrak{J}(A)$ .
- (iv) Every element of  $A[\mathbb{Z}_2]$  can be written as the sum of a unit and an idempotent in  $A$ .
- (v)  $A$  is clean and for every  $a, b \in A$ ,  $V(a + b) \cap V(a - b - 1) = \emptyset$ .

**Proof.** (i)  $\Rightarrow$  (ii) If  $A$  is 2-clean, then by Proposition 1.8 and Proposition 3.5 every idempotent of  $A[\mathbb{Z}_2]$  is in  $A$ . Let  $aX^0 + bX^1 \in A[G]$ . By the hypothesis there is an idempotent  $e$  such that  $a + b - e$  and  $a - b - e$  are both invertible. Set  $r = a - e$  so that  $r + b$  and  $r - b$  are invertible. It follows that

$$aX^0 + bX^1 = (rX^0 + bX^1) + eX^0.$$

is a clean representation, whence  $A[\mathbb{Z}_2]$  is a clean ring.

(ii)  $\Rightarrow$  (iii) Suppose that  $A[\mathbb{Z}_2]$  is clean and that idempotent elements are in  $A$ . Let  $a \in A$  and write

$$aX^0 + aX^1 = (rX^0 + aX^1) + eX^0$$

where  $e^2 = e$ ,  $r + e = a$ , and  $rX^0 + aX^1$  is invertible. As previously noted it follows that  $r + a, r - a$  are both invertible. Since  $r - a = -e$  is invertible it follows that  $e = 1$ . Therefore, the invertibility of  $2a - 1 = r + a$  and arbitrariness of  $a$  imply that  $2 \in \mathfrak{J}(A)$ .

(iii)  $\Rightarrow$  (ii) This is patent.

(ii)  $\Rightarrow$  (i) Let  $a, b \in A$ . Then the element  $aX^0 + bX^1$  is clean and thus we can write it as

$$aX^0 + bX^1 = (rX^0 + bX^1) + eX^0$$

where  $e \in Id(A)$  and  $(rX^0 + bX^1) \in \mathcal{U}(A[\mathbb{Z}_2])$ . Observe that  $r + e = a$  and  $r + b, r - b \in \mathcal{U}(A)$ . Since

$$a + b - e = r + b \text{ and } a - b - e = r - b$$

we are done.

(ii)  $\Rightarrow$  (iv). This is clear.

(iv)  $\Rightarrow$  (i). Let  $a, b \in A$ . By hypothesis there is a unit  $rX^0 + sX^1 \in A[\mathbb{Z}_2]$  and an idempotent  $e \in A$  such that

$$aX^0 + bX^1 = (rX^0 + sX^1) + eX^0.$$

It follows that  $s = b$  and that  $a - e = r$ . It is straightforward from here to show that  $A$  is 2-clean.

(i)  $\Rightarrow$  (v). If  $A$  is 2-clean, then  $V(a - b - 1) = V(a + b - 1)$  which is always disjoint from  $V(a + b)$ .

(v)  $\Rightarrow$  (1). For any  $a \in A$  we have  $\emptyset = V(a - a) \cap V(a + a - 1) = V(0) \cap V(2a - 1) = V(2a - 1)$  and therefore  $2a - 1$  is invertible. It follows that  $2 \in \mathfrak{J}(A)$ . ■

**Corollary 3.8.** *Let  $char A = 2$ . Then  $A[\mathbb{Z}_2]$  is clean if and only if  $A$  is clean.*

The next result was shown in [4]. We include a sketch of proof for completeness sake.

**Proposition 3.9.** *Let  $2 \in \mathcal{U}(A)$ . Then  $A[\mathbb{Z}_2] \cong A \times A$ . In this case  $A$  is clean if and only if  $A[G]$  is clean.*

**Proof.** Define  $\psi : A[G] \rightarrow A \times A$  by  $\psi(aX^0 + bX^1) = (a + b, a - b)$ . Then  $\psi$  is a ring homomorphism. When 2 is a regular element this homomorphism is injective, and when 2 is invertible the map is a surjection. ■

**Remark 3.10.** If  $V(2)$  is a clopen subset of  $Max(A)$ , then there is a decomposition of  $A = A_1 \oplus A_2$  where the rings  $A_1$  and  $A_2$  satisfy that  $2 \in \mathcal{U}(A_1)$  and  $2 \in \mathfrak{J}(A_2)$ . It follows that  $A[\mathbb{Z}_2]$  is a clean ring, but not every element of  $A[\mathbb{Z}_2]$  is the sum of a unit and a normal idempotent.

**Lemma 3.11.** *Suppose  $cX^0 + dX^1 \in Id(A[\mathbb{Z}_2])$ . Then both  $V(c), V(d) \setminus V(c)$  are idempotent clopen subsets of  $Max(A)$ . Moreover,  $V(d)$  is an idempotent clopen subset of  $Max(A)$ .*

**Proof.** The second statement clearly follows from the first. By hypothesis we know that  $c^2 + d^2 = c$ ,  $2cd = d$ , and  $c + d, c - d \in Id(A)$ . The second equation implies that  $V(2), V(c) \subseteq V(d)$ . In particular, we obtain that  $V(c) \subseteq V(c+d) \cap V(c-d)$ . Now, if  $M \in V(c+d) \cap V(c-d)$ , then both  $2c \in M$  and  $c^2 - d^2 \in M$ . If  $2 \in M$ , then the residue field has characteristic 2 and so

$$c + M = c^2 + d^2 + M = c^2 - d^2 + M = M$$

so that  $c \in M$ . It follows that  $V(c) = V(c+d) \cap V(c-d)$ , and so  $V(c)$  is an idempotent clopen subset of  $Max(A)$ .

Next we show that  $V(c+d) \cup V(c-d) = U(d) \cup V(c)$  from which it will follow that  $V(d) \setminus V(c)$  is an idempotent clopen subset of  $Max(A)$ . Suppose that  $d \notin M$ . Then  $2c + M = 1 + M$ . From this and  $c^2 + d^2 = c$  we conclude that  $2d + M = 1 + M$  or  $2d + M = -1 + M$ . (Observe that in any field of characteristic different from 2 we have that  $\frac{1}{2} - \frac{1}{2} = \frac{1}{2}$ ). Therefore,  $c + M = d + M$  or  $c + M = -d + M$  in which case  $M \in V(c-d)$  or  $M \in V(c+d)$ , whence  $U(d) \cup V(c) \subseteq V(c+d) \cap V(c-d)$ . The reverse is the easy case. ■

**Theorem 3.12.** *For a ring  $A$ ,  $A[\mathbb{Z}_2]$  is clean if and only if for each  $a, b \in A$  there exist idempotents  $e_1, e_2, e_3, e_4 \in Id(A)$  such that  $U(e_i) \cap U(e_j) = \emptyset, i \neq j$ ,*

$$1 = e_1 + e_2 + e_3 + e_4 \tag{1}$$

$$V(2) \subseteq U(e_3) \cup U(e_4) \tag{2}$$

$$V(a+b) \subseteq U(e_3) \cup U(e_1) \tag{3}$$

$$V(a+b-1) \subseteq U(e_4) \cup U(e_2) \tag{4}$$

$$V(a-b) \subseteq U(e_3) \cup U(e_2) \tag{5}$$

$$V(a-b-1) \subseteq U(e_4) \cup U(e_1) \tag{6}$$

**Proof.** Necessity. Suppose  $A[\mathbb{Z}_2]$  is clean and let  $a, b \in A$ . Then we can write

$$aX^0 + bX^1 = (rX^0 + sX^1) + (cX^0 + dX^1)$$

where  $rX^0 + sX^1 \in U(A[\mathbb{Z}_2])$  and  $(cX^0 + dX^1) \in Id(A[\mathbb{Z}_2])$ . This means that  $c^2 + d^2 = c$ ,  $2cd = d$ ,  $c + d, c - d \in Id(A)$  and both  $r + s$  and  $r - s$  are invertible. By Lemma 3.11 it follows

that there are idempotents  $e_1, e_2, e_3, e_4$  such that  $U(e_4) = V(c), U(e_3) = V(d) \setminus V(c), U(e_1) = V(c-d) \setminus V(c)$  and  $U(e_2) = V(c+d) \setminus V(c)$ . Observe that  $U(e_i) \cap U(e_j) = \emptyset$  for each  $i \neq j$  and that  $\bigcup_{1 \leq i \leq 4} U(e_i) = \text{Max}(A)$ . Without loss of generality we can assume that  $e_i e_j = 0$  so that  $e_1 + e_2 + e_3 + e_4$  is an invertible idempotent and hence (1) is satisfied. That (2) is satisfied follows from the fact that  $V(2) \subseteq V(d) = U(e_3) \cup U(e_4)$ .

Next, notice that  $a + b - (c + d) = r + s$  and so (3)  $V(a + b) \subseteq U(c + d) = U(e_1) \cup U(e_3)$ . Similarly,  $a - b - (c - d) = r - s$  and so (5)  $V(a - b) \subseteq U(c - d) = U(e_2) \cup U(e_3)$ . Let  $M \in V(a + b - 1)$ . If  $M \in U(e_1) = V(c - d) \setminus V(c)$ , then  $c + d \notin M$  (otherwise  $2c \in M$  and  $2 \notin M$  would imply  $c \in M$ .) Since  $c + d$  is idempotent,  $c + d + M = 1 + M$ . Hence

$$M = a + b - 1 + M = a + b - (c + d) + M$$

contradicting that  $r + s$  is invertible. Similarly, if  $M \in U(e_3) = V(d) \setminus V(c)$  then  $d + M = M$  and so  $c + M = 1 + M$ . Therefore

$$M = a + b - 1 + M = a + b - (c + d) + M.$$

Therefore, (4) is satisfied. A similar argument yields (6).

Sufficiency. Let  $aX^0 + bX^1 \in A[\mathbb{Z}_2]$  and suppose that  $e_1, e_2, e_3, e_4$  are idempotents satisfying the hypothesis. Observe that  $e_i e_j = 0$  for all  $i \neq j$ . We shall produce elements  $r, s, c, d \in A$  such that  $c^2 + d^2 = c$ ,  $2cd = d$ , and both  $r + s = a + b - (c + d)$  and  $r - s = a - b - (c - d)$  are invertible. Thus, it follows that

$$aX^0 + bX^1 = (rX^0 + sX^1) + (cX^0 + dX^1)$$

is a clean representation. To that end observe that (1) implies there is an  $x \in (e_1 + e_2)A$  such that  $2x = e_1 + e_2$ . We further suppose that  $x(e_1 + e_2) = x$ . Let  $y_1 = xe_1$  and  $y_2 = -xe_2$ . Set  $c = x + e_3$  and  $d = y_1 + y_2$ . Observe that

$$V(c) = U(e_4) \text{ and } V(d) = U(e_4) \cup U(e_3).$$

Since

$$\begin{aligned} c^2 + d^2 &= (x + e_3)^2 + (y_1 + y_2)^2 \\ &= x^2 + e_3 + x^2 e_1 + x^2 e_2 \\ &= 2x^2 + e_3 \\ &= x + e_3 \\ &= c \end{aligned}$$

and

$$\begin{aligned} 2cd &= 2(x + e_3)(xe_1 - xe_2) \\ &= 2(x^2 e_1 - x^2 e_2) \\ &= 2x^2(e_1 - e_2) \\ &= x(e_1 + e_2)(e_1 - e_2) \\ &= x(e_1 - e_2) \\ &= d \end{aligned}$$

it follows that  $cX^0 + dX^1 \in Id(A[\mathbb{Z}_2])$ . By construction  $c + d = 2xe_1 + e_3$  so that

$$V(c + d) = U(e_2) \cup U(e_4).$$

Let  $r = a - c$  and  $s = b - d$ . Then  $r + s = a + b - (c + d)$  and  $r - s = a + b - (c - d)$ .

If  $r + s$  is not invertible, then it lies in some maximal ideal  $M$ . If  $M \in V(a + b)$ , then  $M \in V(c + d)$  contradicting (3). Therefore,  $M \in U(a + b)$  and hence  $M \in U(c + d)$ . So  $a + b + M = c + d + M = 1 + M$ , i.e.  $M \in V(a + b - 1)$ . By (4)  $M \in U(e_2)$  (since  $M \notin V(c) = U(e_4)$ ). Our contradiction is

$$1 + M = c + d + M = xe_1 + xe_2 + e_3 + xe_1 - xe_2 + M = 2xe_1 + xe_3 + M = M$$

since  $M \in V(e_1) \cap V(e_3)$ . Therefore,  $r + s$  is invertible. A similar argument shows that  $r - s$  is invertible. ■

**Theorem 3.13.**  $A[\mathbb{Z}_2]$  is clean if and only if  $A$  is clean.

**Proof.** As we have pointed so many times already, if  $A[\mathbb{Z}_2]$  is clean, then so is  $A$ . So suppose that  $A$  is clean and let  $a, b \in A$ . Choose idempotents  $f_1, f_2 \in A$  such that  $V(a + b) \subseteq U(f_1)$ ,  $V(a + b - 1) \subseteq V(f_1)$  and  $V(a - b) \subseteq U(f_2)$ ,  $V(a - b - 1) \subseteq V(f_2)$ . Let  $e_1, e_2, e_3, e_4$  be idempotents such that

$$U(e_1) = U(f_1) \cap V(f_2),$$

$$U(e_2) = U(f_2) \cap V(f_1),$$

$$U(e_3) = U(f_1) \cap U(f_2),$$

$$U(e_4) = Max(A) \setminus (U(e_1) \cup U(e_2) \cup U(e_3)).$$

Observe that  $U(e_i) \cap U(e_j) = \emptyset$  whenever  $i \neq j$ . Without loss of generality  $e_i e_j = 0$  whenever  $i \neq j$ . Therefore, all the conditions of Theorem 3.12 are satisfied except for possibly (2).

We claim that  $V(2) \cap U(e_1)$  and  $V(a + b) \cup V(a - b - 1)$  are disjoint closed sets. To see this let  $M \in V(2) \cap U(e_1)$ . Suppose that  $M \in V(a + b)$ . Since  $M \in V(2)$ ,  $M \in V(a - b)$  and hence  $M \in U(f_2)$ , contradicting that  $M \in U(e_1)$ . If  $M \in V(a - b - 1)$ , then  $M \in V(a + b - 1)$  which implies that  $M \in V(f_1)$ , again contradicting that  $M \in U(e_1)$ . Similarly,  $V(2) \cap U(e_2)$  and  $V(a - b) \cap V(a + b - 1)$  are disjoint closed sets.

Since  $A$  is clean it follows that the collection of idempotent clopen sets form a base for the topology. It follows that there is an idempotent clopen subset  $K_1 \subseteq U(e_1)$  such that  $V(2) \cap U(e_1) \subseteq K_1$  and  $(V(a + b) \cup V(a - b - 1)) \cap K_1 = \emptyset$ . Similarly, there is an idempotent clopen subset  $K_2 \subseteq U(e_2)$  such that  $V(2) \cap U(e_2) \subseteq K_2$  and  $(V(a - b) \cup V(a + b - 1)) \cap K_2 = \emptyset$ . By combining  $U(e_4)$  together with  $K_1$  and  $K_2$  and shrinking down  $U(e_1)$  and  $U(e_2)$  in the appropriate manner, we obtain idempotent clopen subsets which satisfy conditions (1)-(6) of Theorem 3.12. Therefore,  $A[\mathbb{Z}_2]$  is clean. ■

**Corollary 3.14.** Suppose  $G$  is an elementary 2-group. Then  $A[G]$  is clean if and only if  $A$  is clean.

**Proof.** Since  $G$  is an elementary 2-group it follows that for each  $f \in A[G]$  there exists a finite subgroup  $H$  of  $G$  such that  $f \in A[H]$  and  $H$  is a finite direct product of copies of  $\mathbb{Z}_2$ . It follows that  $A[H]$  is a finite product of copies of  $A[\mathbb{Z}_2]$  which is clean. Therefore,  $f \in A[H]$  can be written as a sum of a unit and an idempotent in  $A[H]$ , and hence can be written as a sum of a unit and an idempotent in  $A[G]$ . ■

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