

Chapter 1

Algebras

1. Lattices

The first part of this course will deal with the branch of mathematics known as Universal Algebra. At this point the reader should be familiar with three classes of algebras: groups, rings, modules. We would like to add a few more examples which shall be brought up several times throughout the course. We begin with relations.

Definition 1.1. Let A be a nonempty set. A subset $R \subseteq A \times A$ is called a *relation*. The relation R is called

1. *reflexive* if $(x, x) \in R$ for all $x \in A$.
2. *symmetric* if $(x, y) \in R$ implies $(y, x) \in R$.
3. *anti-symmetric* if (x, y) and $(y, x) \in R$ imply that $x = y$.
4. *transitive* if $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$.

A relation that is reflexive, symmetric, and transitive is called an *equivalence relation*. A relation that is reflexive, anti-symmetric, and transitive is called a *partial order*. A set equipped with a partial order is called a *partially-ordered set*. We shall usually use \equiv or \sim to denote equivalence relations. An equivalence relation induces a partition of the set A and conversely.

Usually, we shall use \leq to denote a partial order. Note that $x \leq y$ denotes that (x, y) is in the partial order. If (A, \leq) is a partially ordered set and the order is such that for any $a, b \in A$ with $a \leq b$ or $b \leq a$ then the order is called a *total-order*. Finally, a poset is called *well-ordered* if every nonempty subset has a least element. Observe that a well-ordered set is in fact totally-ordered.

Definition 1.2. Suppose (A, \leq) is a partially-ordered set and let $B \subseteq A$. If there exists an $x \in A$ such that $b \leq x$ for every $b \in B$, then x is called an *upper bound* for B . We dually define a *lower bound* for B . If x is an upper bound B which also has the added bonus that if y is also an upper bound for B then $x \leq y$ we say that x is a *least upper bound* for a and b . Dually for *greatest lower bound*. We employ the symbols $\sup B$ or $\vee B$, or say the *join* of B to mean least upper bound for B (if it exists). Dually, $\inf B$, $\wedge B$ or the *meet* of B to mean the greatest lower bound. If the set B is indexed, say $B = \{x_i : i \in I\}$ we may write $\inf_i x_i$ or $\wedge_i x_i$.

If A has an upper bound, then it is unique and is called the *top* element. Dually, we have the notion of a bottom element.

A *lattice* is a partially-ordered set (A, \leq) satisfying every (non-empty) finite subset of A has a least upper bound and greatest lower bound. The order is often-times called a *lattice-order*. If every subset has a sup and an inf then it is called a *complete lattice*.

Example 1.3. (i) Let A be any nonempty subset and define a partial-order on A by $a \leq b$ if and only if $a = b$. This is called the trivial order on A . Observe that this partial order is a lattice precisely when A is a singleton set.

(ii) Let $A \subset \mathbb{R}$ with the usual order. Then A is a lattice. Observe that any total order is a lattice-order.

(iii) Let A be any set and $\mathcal{P}(A)$ its power set. Order $\mathcal{P}(A)$ by inclusion. Then for any $B_1, B_2 \in \mathcal{P}(A)$ we have $B_1 \vee B_2 = B_1 \cup B_2$ and $B_1 \wedge B_2 = B_1 \cap B_2$. Thus, $\mathcal{P}(A)$ under this ordering is a lattice. Note that $\mathcal{P}(A)$ is totally-ordered by inclusion if and only if $|A| \leq 1$.

Definition 1.4. Suppose L is a lattice. An element $x \in L$ is called *compact* if $x \leq \bigvee_I x_i$ implies that $x \leq \bigvee_F x_i$ for some finite subset $F \subseteq I$. A lattice is *algebraic* if it is complete and every element is the join of compact elements.

An understanding of example (iii) of 1.3 leads us to following definitions.

Definition 1.5. Suppose (L, \leq) is a lattice. If for every $x, y, z \in L$ we have

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

and

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$$

then L is said to be a *distributive lattice*.

Note: The class of distributive lattices form a subclass of the modular lattices. A lattice is called *modular* if it satisfies: $(c \vee a) \wedge b = (c \wedge b) \vee a$ for all $a, b, c \in L$ with $a \leq b$. We suggest the interested reader consult [C81]. As an example, the set of all normal subgroups of a given group forms a modular lattice.

Example 1.6. Let (G, \cdot) be a group and $\mathcal{N}(G)$ its set of normal subgroups. $\mathcal{N}(G)$ is a lattice when partially-ordered under inclusion. Recall that for any $A, B \in \mathcal{N}(G)$, $A \wedge B = A \cap B$ and $A \vee B = AB$. We shall now prove that it is a modular lattice.

Suppose $A \leq B$ and C are all normal subgroups. To verify that $\mathcal{N}(G)$ is modular is to show that $CA \cap B \subseteq (C \cap B)A$. So let $b \in CA \cap B$, say $b = ca$ (where $c \in C, a \in A$). Since $a^{-1} \in A \leq B$ we have that $c = ba^{-1} \in B$, whence $b = ca \in (C \cap B)A$.

(Where is normality used in this example?)

Definition 1.7. Let (L, \leq) be a lattice with top element 1 and bottom element 0. Two elements $x, y \in L$ are called *complementary* if $x \vee y = 1$ and $x \wedge y = 0$. Two elements which have a common complement $x \in L$ are said to be *x-related* in L (or simply related.)

Suppose now that every element x of L has a complement x' satisfying

- (i) $x = x''$ for every $x \in L$;
- (ii) $(x \wedge y)' = x' \vee y'$ for every $x, y \in L$;
- (iii) $(x \vee y)' = x' \wedge y'$ for every $x, y \in L$;

then L is called a *complemented* lattice.

Lemma 1.8. *A lattice L is distributive if and only if it satisfies*

$$(z \vee x) \wedge y \leq (z \wedge y) \vee x \quad \text{for all } x, y, z \in L.$$

Proof: If L is distributive then

$$(z \vee x) \wedge y = (z \wedge y) \vee (x \wedge y) \leq (z \wedge y) \vee x.$$

Conversely, then

$$(x \vee y) \wedge z \leq (y \wedge z) \vee x;$$

meeting both sides with z we obtain

$$(x \vee y) \wedge z \leq [(y \wedge z) \vee x] \wedge z \leq (y \wedge z) \vee (x \wedge z).$$

Since $x \vee y \geq (x \wedge z) \vee (y \wedge z)$ and $z \geq (x \wedge z) \vee (y \wedge z)$, we have $(x \vee y) \wedge z \geq (x \wedge z) \vee (y \wedge z)$, which together with the hypothesis establishes the distributivity of L . ■

Theorem 1.9. *A lattice L is distributive if and only if for each interval I , any two elements of I which are related in I are equal.*

Recall that an interval is a set of the form $[a, b] = \{x : a \leq x \leq b\}$. Finally, a complemented distributive lattice is called a *boolean algebra*. We shall discuss these later on.

Exercise 1.10. 1. Show that \vee and \wedge are commutative operations.

2. Prove that if $\vee B$ exists, then it is unique. Dually for \wedge .
3. Prove by induction that the po-set (L, \leq) is a lattice if and only every pair of elements has a sup and an inf.
4. Prove that a total-order is a lattice order.
5. Prove that $\mathcal{P}(A)$ is totally-ordered by inclusion if and only if $|A| \leq 1$.
6. Suppose (A, \leq) is totally-ordered and let $x \in A$. Prove that x is compact if and only if it a least element or a successor (that is, there is an immediate previous element).
7. Prove that a lattice L is distributive if and only if, for all $a, b, c \in L$ $a \wedge b \leq c$ and $a \leq b \vee c$ imply $a \leq c$.
8. (\mathbb{N}, \leq) where $a \leq b$ if $a|b$. Prove that this ordering is a partial-order. Furthermore, show that it is a lattice-order by determining $a \vee b$ and $a \wedge b$ for arbitrary naturals. What is the bottom element?
9. Prove that for any set A , $\mathcal{P}(A)$ is boolean algebra when ordered by inclusion.
10. Let X be a topological space. Prove that the collection of all clopen subsets of X forms a boolean algebra.

11. Let $(R, +, \cdot)$ be a boolean ring (it satisfies $x^2 = x$ for all $x \in R$). Define an ordering on R by $x \leq y$ if $x = xy$. Prove that this is a bona-fide partial order. In fact prove it is a boolean algebra where $x \wedge y = xy$, $x \vee y = y + x + xy$, and $x' = 1 - x$.
12. Let R be a Bézout domain, i.e., every finitely generated ideal is principally generated. Show that the set of principal ideals forms a sublattice of the lattice of all ideals. Prove that this sublattice is distributive. (Hint: Use Theorem 1.9.)

2. Boolean Algebras

In this section we assume the type $(0, 1, ()', \vee, \wedge)$ of boolean algebras. The objective is to prove that every boolean algebra can be embedded as a subalgebra of a power set. We begin with some preliminaries and notation.

Definition & Remarks 2.1. As before let $\mathcal{P}(X)$ stand for the power set of the set X ; this is, for the record, the collection of all the subsets of X . On the other hand, we denote by $\mathbf{2}$ the least boolean algebra, $\mathbf{2} = \{0, 1\}$, in which $0 < 1$. Now, if X is any set, then $\mathbf{2}^X$ stands for the set of all functions of X into $\mathbf{2}$. One can also look at $\mathbf{2}^X$ as the cartesian product of X copies of $\mathbf{2}$.

Let us assume that $\mathbf{2}^X$ has the direct product structure of boolean algebras. Thus, we have for each $x \in X$,

$$(f \vee g)(x) = \max\{f(x), g(x)\} \quad \text{and} \quad f'(x) = f(x)'$$

On $\mathcal{P}(X)$ the boolean operations are the familiar ones of set-theoretic union, intersection and complementation.

Finally, if $E \subseteq X$ then $\chi_E \in \mathbf{2}^X$ denotes the *characteristic function* of E , defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{otherwise.} \end{cases}$$

We are now set up for the following proposition.

Definition 2.2. Suppose $(A, \vee, \wedge, ', 0, 1)$ and $(B, \vee, \wedge, ', 0, 1)$ are boolean algebras. A function $f : A \rightarrow B$ is called a *boolean algebra homomorphism* if f preserves all the operations, that is $f(a \wedge b) = f(a) \wedge f(b)$, $f(a \vee b) = f(a) \vee f(b)$, $f(0) = 0$, $f(1) = 1$, and $f(a') = f(a)'$. If f happens to be a bijection then we call f a *boolean algebra isomorphism* and say A and B are isomorphic as boolean algebras.

We shall discuss homomorphisms in more detail later in §7.

Proposition 2.3. *The function $E \mapsto \chi_E$, from $\mathcal{P}(X) \rightarrow \mathbf{2}^X$, is an isomorphism of boolean algebras.*

Proof. Let $\chi(E) \equiv \chi_E$; we verify that χ preserves suprema and complements, and leave the rest for the reader to check. Suppose that $E, F \subseteq X$. Then for each $x \in X$ we have,

$$\chi(E \cup F)(x) = \begin{cases} 0, & \text{if } x \notin E \cup F \\ 1, & \text{otherwise} \end{cases} = \max\{\chi_E(x), \chi_F(x)\} = (\chi_E \vee \chi_F)(x),$$

which shows that $\chi(E \cup F) = \chi(E) \vee \chi(F)$. As for complements, note that

$$(\chi(E))'(x) = \chi(E)(x)' = \begin{cases} 0, & \text{if } x \in E \\ 1, & \text{otherwise} \end{cases} = \chi(X \setminus E)(x). \blacksquare$$

Based upon the isomorphism of Proposition 2.3, we shall freely identify subsets with their characteristic functions, as indicated.

Definition 2.4. A *maximal* ideal of the boolean algebra B is a proper ideal of B , which is not properly contained in any proper ideal of B . (For a definition of ideal of a boolean algebra, see Exercise 7.6, and note also Exercise 7.7; it tells us that a maximal ideal is one which is a maximal ideal in the ring-theoretic sense.)

Proposition 2.5. Suppose that I is a proper ideal of the boolean algebra B . Then I is maximal if and only if, for each $x \in B$, either $x \in I$ or $x' \in I$, (but not both).

Proof. If I is maximal and $x \notin I$, then the ideal generated by I and x , denoted (I, x) , properly contains I . Now, verify that

$$(I, x) = \{b \in B : b \leq x \vee a, \text{ for some } a \in I\}.$$

Thus, $1 = x \vee a$, for some $a \in I$, and taking the infimum with x' , one obtains that $x' = 0 \vee (a \wedge x') = a \wedge x' \in I$.

The other half of the proof is left to the reader. ■

The next proposition says, effectively, that the maximal ideals of a boolean algebra are precisely the kernels of homomorphisms into $\mathbf{2}$.

Proposition 2.6. Suppose that B is a boolean algebra. If f is a homomorphism $B \rightarrow \mathbf{2}$, then $M_f \equiv \{b \in B : f(b) = 0\}$ is a maximal ideal of B . Conversely, if M is a maximal ideal of B , then $g_M : B \rightarrow \mathbf{2}$ defined by

$$g_M(b) = \begin{cases} 0, & \text{if } b \in M, \\ 1, & \text{otherwise,} \end{cases}$$

is a homomorphism.

Proof. Use the preceding proposition. ■

Now the crucial, Zornian step, of verifying that there are “enough” maximal ideals in a boolean algebra. The next result is often referred to as the “Boolean Prime Ideal Theorem”.

Proposition 2.7. Suppose that B is a boolean algebra, and $x \neq y$ in B . Then there exists a maximal ideal M , so that either $x \in M$ and $y \notin M$, or the reverse.

Proof. Apply Zorn’s Lemma. ■

Putting the preceding propositions together, we have the main theorem. We say under the circumstances of Theorem 2.8 that the algebra B is represented as a *field of sets*.

Theorem 2.8. (Stone’s Representation Theorem) Suppose that B is a boolean algebra. Then there is a set X , and a one-to-one boolean homomorphism $g : B \rightarrow \mathbf{2}^X$.

Proof. Let $X = \text{Max}(B)$, the set of all maximal ideals of B . By Proposition 2.7, $\bigcap \text{Max}(B) = \{0\}$. Now define $g : B \rightarrow \mathbf{2}^X$ as follows: for each maximal ideal M , let g_M be the homomorphism of Proposition 2.6; now define

$$g(b)(M) \equiv g_M(b).$$

This way $g(b)$ is well-defined as an element of $\mathbf{2}^X$, because of Proposition 2.6. We leave it to the reader to verify that g is a homomorphism of boolean algebras. Since $\bigcap \text{Max}(B) = \{0\}$, g is one-to-one. ■

Exercise 2.9. Suppose that $X = \{0\} \cup \{1/n : n = 1, 2, \dots\}$ is the topological space obtained by restricting the ordinary metric topology on the real line to X . Compute the boolean algebra of clopen (closed-and-open) sets.

Exercise 2.10. An element $0 < s \in B$, in the boolean algebra B is said to be an *atom* provided the interval $[0, s]$ consists of 0 and s alone. Prove the following “refinement” of Stone’s Theorem.

There is a representation $g : B \rightarrow \mathbf{2}^X$ of B as a field of subsets of some set X such that the range $g(B)$ contains all finite sets if and only if every $b > 0$ in B exceeds an atom.

In fact, the following is true: in any representation of B as a field of sets the image of an element b is a finite set if and only if b is a finite supremum of atoms.

Boolean algebras with this property are said to be *atomic*.

Here is an example of a boolean algebra without any atoms.

Example 2.11. Consider $\mathbf{2}^{\mathbb{N}}$, and let B be the subalgebra of all periodic sequences. (A sequence s is *periodic* if there is an integer k such that $s(m+k) = s(m)$, for each m .) The reader will easily verify that B is a subalgebra of A , and that it has no atoms.

Remark 2.12. Stone’s Representation Theorem guarantees that every boolean algebra can be viewed as a subalgebra of a power set, where the infimum and supremum are set-theoretic intersection and union, respectively. Theorem 2.8 does not say that we can assume that arbitrary suprema and infima agree with unions and intersections. We shall return to this point later in the course when we discuss Stone Duality.

As to atoms, there is an important family of boolean algebras without atoms, namely the free boolean algebras. The reader may look at Halmos’ very readable introduction to boolean algebra [Ha63]. After a discussion of Stone Duality, then those with some topological background may understand that the fact that free boolean algebras have no atoms comes from the well known theorem in topology which characterizes the Cantor Set as the only compact metric space which is totally disconnected and has no isolated points.

3. Brouwerian Lattices

We will soon be discussing universal algebras. One topic is the lattice of subalgebras. It turns out that although they are not boolean algebras they are fairly close. How close you ask? Well that’s what this section is about. As for the history of the subject... this stuff goes back to the 20s and 30s. The main players in lattice theory have been (and not necessarily in any order): Robert P. Dilworth, Arend Heyting, V. Glivenko, Garrett Birkhoff, Luitzen Egbertus J. Brouwer, Marshall H. Stone, etc.

Lemma 3.1. *Suppose that (L, \leq) is a lattice and $x, y, z \in L$. If $y \leq z$, then $x \vee y \leq x \vee z$ and dually $x \wedge y \leq x \wedge z$.*

Proof: All that is needed is to show that $x \vee z$ is an upper bound for both x and y from which it follows by the definition of join that $x \vee y \leq (x \vee z)$.

To that end clearly, $x \leq (x \vee z)$. As simple is that $y \leq z \leq (x \vee z)$ by transitivity of the partial order. ■

Lemma 3.2 [Distributive Inequality]. *Let L be a lattice and $x, y, z \in L$. We have the following inequalities:*

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

$$x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z).$$

Proof: Observe that $y \wedge z \leq y$ and hence by the previous lemma $x \vee (y \wedge z) \leq x \vee y$. Similarly, $y \wedge z \leq z$ so that $x \vee (y \wedge z) \leq x \vee z$. Thus, $x \vee (y \wedge z)$ is a lower bound for $x \vee y$ and $x \vee z$, whence the top inequality holds. The second inequality holds by duality. ■

Definition 3.3. A lattice B is called a *brouwerian lattice* if for any pair of elements $a, b \in B$, the set of all elements $x \in B$ for which $a \wedge x \leq b$ contains a greatest element which we shall call the *relative pseudo-complement of a in b* . We shall denote this largest element by $b : a$. We should remark to the reader that this class has gone by several different names. (a) a Heyting algebra, due to A. Heyting (1930). (b) over the last 15-20 years researchers in algebra have started to use the term “frame” for a complete brouwerian lattice. (Analysts have used frame to mean something completely different.) (c) A relatively pseudo-complemented distributive lattice. We have made our choice as it is often used by lattice-ordered groupers.

This definition is fairly technical but our aim is to give a characterization of brouwerian lattices using meets and joins and then apply what we find to our usual class of “algebras”. First, let’s put the class of brouwerian lattices in its proper place.

Theorem 3.4. *A brouwerian lattice is distributive.*

Proof: Given $a, b, c \in B$ form $d = (a \wedge b) \vee (a \wedge c)$ and consider $d : a$. Since $a \wedge b \leq d$ and $a \wedge c \leq d$, we have $b \leq d : a$ and $c \leq d : a$. Hence $b \vee c \leq d : a$. This leads to the string of inequalities

$$a \wedge (b \vee c) \leq a \wedge d : a \leq d.$$

The distributive inequality shows that $d \leq a \wedge (b \vee c)$ and so anti-symmetry forces

$$(a \wedge b) \vee (a \wedge c) d = a \wedge (b \vee c).$$

Duality is the final nail in the coffin. ■

Theorem 3.5. *A boolean algebra is a brouwerian lattice.*

Proof: Let A be a boolean algebra and $a, b \in A$. Set $x = a' \vee b$. Obviously,

$$a \wedge x = a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b) = a \wedge b \leq b$$

Furthermore, suppose y has the property that $a \wedge y \leq b$. Then taking the join of both sides with a' we obtain

$$y \leq (a' \vee y) = a' \vee (a \wedge y) \leq a' \vee b = x$$

which shows that x is the largest element satisfying the desired property, i.e., x is the relative pseudo-complement of a in b . Whence A is a brouwerian lattice. ■

Combining the previous two theorems together we get that a boolean algebra is a brouwerian lattice is a distributive lattice. We adduce that the implications may not be reversed.

Example 3.6. 1) Let A be a chain with a top element 1. Then A is a brouwerian lattice. To see this let $a, b \in A$. If $a \leq b$, then 1 is the largest element satisfying $a \wedge 1 \leq b$. If $b \leq a$, then $b : a = b$.

2) Let X be a non-discrete topological space and τ its topology of open sets. We may partially order τ under inclusion in which case τ is a sublattice of $\mathcal{P}(X)$, i.e., \vee and \wedge denote union and intersection. Clearly, τ is a distributive lattice which is not a boolean algebra. Observe that it is also brouwerian. We leave it to the reader to check that if O, V are arbitrary open subsets of X , then $V : O = (X - (U - V))^\circ \cap U$.

3) Let X be a non-discrete topological space and κ its lattice of closed subsets. Then as above κ is a distributive lattice. In this case it is not brouwerian. For example if $X = \mathbb{R}$, $U = [0, 2]$, $V = [1, 3]$ then one can show that a relative pseudo complement would have to contain $[1, \infty)$ as well as $(-\infty, 0)$. But this set is not closed. Check this.

It is now time to characterize the class of brouwerian lattices within the setting of complete lattices. In some manuscripts, authors have used this characterization as the basis for their definition. For example, our description below is used in [D]. But as we shall see later on his only applications are to complete lattices and so there shouldn't be any confusion. The definitions given in 3.3 above (due to [B]) hopefully leads the reader to understand the class as properly between the class of distributive lattices and the class of boolean algebras. Looking ahead, when we define polar subgroups our way shall be the right way.

Theorem 3.7. *Let L be a complete lattice. Then L is brouwerian if and only if the join operation is completely distributive on meets. That is,*

$$a \wedge \left(\bigvee_i x_i \right) = \bigvee_i (a \wedge x_i)$$

for any set $\{x_i\}$.

Proof: Suppose L is brouwerian and $\{x_i\} \subset L$. Set $b = \bigvee_i (a \wedge x_i)$ and $d = b : a$. By definition, $a \wedge x_i \leq b$ and so $x_i \leq d$. Thus, $\bigvee_i x_i \leq d$. Since also by definition $a \wedge d \leq b$ we obtain that $a \wedge (\bigvee_i x_i) \leq a \wedge d \leq b$. On the other hand, the distributive inequality implies that for each i $a \wedge x_i \leq a \wedge (\bigvee_i x_i)$, whence $\bigvee_i (a \wedge x_i) \leq a \wedge (\bigvee_i x_i)$.

For the reverse direction suppose that L is completely distributive on meets and let $a, b \in L$. Let P be the set of all $x \in L$ such that $a \wedge x \leq b$. Denote the sup of P by d . We now demonstrate with a string of inequalities that $d \in P$ and hence is the largest element of P , whence $d = b : a$.

The hypothesis together with the fact that b is an upper bound for the set of $\{a \wedge x\}_{x \in P}$ gives us

$$a \wedge d = a \wedge \left(\bigvee_{x \in P} x \right) = \bigvee_{x \in P} (a \wedge x) \leq b.$$

This proves that L is brouwerian. ■

Exercise 3.8. Let L be a lattice with least element. Suppose that for any $a \in L$ $0 : a$ exists. We call L a *pseudo-complemented distributive lattice* (an obvious generalization of a brouwerian lattice). Set $a^* = 0 : a$. Prove the following:

- (i) $a \leq b$ implies $b^* \leq a^*$;
- (ii) $a \leq a^{**}$;

- (iii) $a^* = a^{***}$;
- (iv) $(a \vee b)^* = a^* \wedge b^*$.

Exercise 3.9. Let L be a brouwerian lattice with 0 and 1. Define an element $a \in L$ to be *regular* (ugh!!) if $a = a^{**}$ and denote the set of all regular elements of L by L^{**} . (It is fact that L^{**} is a boolean algebra.) Prove that the following are equivalent. Such lattices are called *Stone lattices* (after their first appearance by M.H. Stone (1937)).

- (i) for all $a, b \in L$, $(a \wedge b)^* = a^* \vee b^*$;
- (ii) for all $a \in L$, $a^* \vee a^{**} = 1$;
- (iii) every regular element has a complement in A ;
- (iv) L^{**} is a sublattice of L .

4. Universal Algebra

Definition 4.1. An n -ary operation on a set A is a function from the n -fold cartesian product $A^n = A \times \cdots \times A$ (n times) into A . “Operation” in these notes means “ n -ary operation” for a suitable non-negative integer n . An operation will typically be designated by a small greek letter, say ϕ , and its action on an n -tuple (a_1, a_2, \dots, a_n) by $\phi(a_1, a_2, \dots, a_n)$.

Here are some examples which should be familiar:

Remark 4.2. (a) Consider the multiplication in any group, multiplication in any ring, or the operation in a lattice which computes the supremum or else the infimum of any two elements. These are examples of 2-ary or *binary* operations.

(b) Consider a group, and the map $a \mapsto a^{-1}$ in the group. This is an example of a 1-ary or *unary* operation. So is the complementation in a boolean algebra. Another example of a unary operation: suppose that M is a left R -module (where R is any ring with identity); pick $r \in R$, and look at the function $x \mapsto rx$ on M .

(c) The case $n = 0$; what is a *zero-ary* or *nullary* operation? Reflect that $|A^0| = 1$, and thus the nullary operations, being maps from a singleton into A , are simply designated elements of the set A . For example, the identity of a multiplicative group G , say e , can be so designated as a nullary operation, when one talks about a group. Likewise, in a boolean algebra one may designate two special elements as nullary operations, the least element 0 and the largest element 1.

Definition & Remarks 4.3. (a) An *algebra* is a set, together with a certain number of operations (possibly infinitely many). The operations that are agreed upon are referred to as the *type* of the algebra. This is imprecise, for the moment. It is one of the rare times when it is best to proceed intuitively towards a rigorous formulation.

Thus a group $(G; 1, ()^{-1}, \cdot)$, with one binary operation, one unary operation and one nullary operation can be said to have the *type* $(1, ()^{-1}, \cdot)$. Often the operations themselves are “identified” with the cardinalities of the sets of n -ary operations under consideration. Thus, the type of group theory can also be viewed as the triple $(1,1,1)$, meaning that a group has 1 nullary, 1 unary and 1 binary operation.

A ring with multiplicative identity affords a choice. One may view it as $(R; 0, 1, -(), +, \cdot)$, by designating the multiplicative identity, in which case it has the type $(0, 1, -(), +, \cdot)$, with 2 nullary, 1 unary and 2 binary operations. Depending upon the theory one is interested in, the identity might be ignored, and the ring R viewed as an algebra of type $(0, -(), +, \cdot)$, or $(1, 1, 2)$. In commutative algebra, for example, the choice is almost always to treat a ring as an algebra of type $(2, 1, 2)$. As we will see, this choice forces one to deal only with homomorphisms which preserve the multiplicative identity.

Similarly, a boolean algebra most commonly is regarded as an algebra of type $(0, 1, ()', \vee, \wedge)$, or $(2, 1, 2)$, where 0 and 1 denote the least and largest element of the algebra, $()'$ the boolean complement, while \vee and \wedge stand for the supremum and infimum, respectively. However, one might wish to “forget” that a boolean algebra is boolean, and just “remember” that it is a distributive lattice, and view it as an algebra of type (\vee, \wedge) , or $(0, 0, 2)$.

Let us now be more precise.

(b) For each nonnegative integer n , an n -ary symbol is a fixed function θ , the domain of which is the class of all sets, so that, for each set A , θ_A is an n -ary operation on A . A *type* is a set T of n -ary symbols (for various n 's), partitioned by

$$T = T_0 \cup T_1 \cup T_2 \cup \cdots,$$

where T_n is a set of n -ary symbols. (T_n is frequently empty.)

An algebra of type T is a pair, denoted $(A; T)$, where A is a set. T is referred to as the *operator domain of A* . Thus, one should think of the designation $(A; T)$ as associating the operations θ_A (for each $\theta \in T$) with the set A . We are choosing some operations for A , encoding them via the type T .

In practice we shall frequently identify the operator symbols with the actual operations. Indeed, until we introduce homomorphisms, no confusion can ensue from identifying the type with the corresponding sequence of cardinalities of the T_i . (If $T_k = \emptyset$, for all $k > m$, for a suitable m , then instead of listing zeroes in all positions k , for $k > m$, one simply truncates the sequence after the m -ary position.)

The reader should keep in mind that “type” implies a set of operator symbols, each having (usually) a suggestive connotation. Strictly speaking, however, when one views a ring with identity as an algebra of type $(0, 1, -(), +, \cdot)$, and a boolean algebra as one of type $(0, 1, ()', \vee, \wedge)$, even though the symbols have particular interpretations in each case, it could be argued that the algebras are of the same type.

Example 4.4. If M is a (left) module over the ring R , then the module may be viewed as an algebra of type $(1, 1 + |R|, 1)$, written as

$$(M; 0, -(), \{ \phi_r : r \in R \}, +).$$

ϕ_r designates the left scalar multiplication by r , $\phi_r a = ra$, which is a unary operation. The reader should take note that here we have an algebra with $1 + |R|$ -many unary operations.

Definition 4.5. Let $\{ A_i : i \in I \}$ be a family of algebras of type T . Put $A = \prod_{i \in I} A_i$ (the cartesian product over I of the A_i). We make A into an algebra of type T , by coordinatewise operations. Recall that the members of this cartesian product are functions from I into the disjoint union of the A_i , for which $g(i) \in A_i$.

Now, suppose that $\phi \in T_n$. If $n = 0$ we define $\phi_A \in A$ by $\phi_A(i) = \phi_{A_i}$, the nullary symbol corresponding to ϕ , in A_i . If $n \geq 1$, and a_1, a_2, \dots, a_n are in A , define

$$\phi_A(a_1, \dots, a_n)(i) = \phi_{A_i}(a_1(i), \dots, a_n(i)).$$

This defines an algebra structure on A of type T . This is the *direct product* of the algebras A_i . When no structure is mentioned explicitly on a cartesian product, one assumes that it is the direct product. We also note that, henceforth, in the context of direct products, we shall suppress subscripts on the operations.

5. Examples of “Algebras”

We define the working notion of a “subobject”, and begin to consider how a subobject is generated.

Definition 5.1. Suppose that $(A; T)$ is an algebra (in $\mathbf{Al}(T)$). A subset B of A is called a *subalgebra* of A , if for each $n = 0, 1, 2, \dots$ and each $\phi \in T_n$, and all $b_1, b_2, \dots, b_n \in B$, we have $\phi(b_1, \dots, b_n) \in B$. The set of all subalgebras of A will be denoted by $\mathbb{S}(A)$. If B is a subalgebra of A we write $B \leq A$.

Remarks 5.2. (a) If G is a group (viewed as an algebra of type $(1,1,1)$) then a subalgebra H *must* be nonempty, as it must contain the designated identity e . Moreover, H is closed under multiplication and inversion, and since it inherits the laws that make G a group, then H too is a group. Thus, a subalgebra (as defined here) is a subgroup (in the familiar sense). The reader ought to reflect and become convinced that the reverse is true.

(b) If a ring R with identity 1 is regarded as an algebra of type $(2,1,2)$ (that is, if 1 is designated a nullary operation), then “subalgebra” means “subring which inherits the identity”. If R is regarded as an algebra of type $(1,1,2)$, then “subalgebra” means “subring”, period!

Exercise 5.3. (a) Let $(A; T)$ be an algebra. The empty subset of A is a subalgebra if and only if T contains no nullary operations.

(b) The intersection of any number of subalgebras is a subalgebra.

Definition & Remarks 5.4. Suppose that $(A; T)$ is an algebra, and consider $\mathbb{S}(A)$. A is a subalgebra of itself, which means that $\mathbb{S}(A)$ is not void. As has already been remarked, \emptyset is a subalgebra, precisely when there are no nullary operations.

Now let $\{B_i : i \in I\}$ be any family of subalgebras containing $X \subseteq A$. By Exercise 5.3(b), $B = \bigcap_i B_i$ is a subalgebra of A , which clearly contains X . Thus, we may speak of the *least* subalgebra containing the subset X , and we call it the subalgebra *generated* by X . It will be denoted by $\langle X \rangle_T$.

In particular, let $X = T_0$, the set of all nullary operations. Since every subalgebra must contain T_0 , it is clear that $\langle T_0 \rangle_T$ is the smallest subalgebra of A .

We then have the following:

Proposition 5.5. *The set $\mathbb{S}(A)$ is a complete lattice with largest element A and least element $\langle T_0 \rangle_T$, and in which*

$$\bigwedge_i B_i = \bigcap_i B_i \quad \text{and} \quad \bigvee_i B_i = \langle \bigcup_i B_i \rangle_T,$$

where $\{B_i : i \in I\}$ is a set of subalgebras of A .

Exercise 5.6. (a) Suppose that $(A; T)$ is an algebra for which $T_k = \emptyset$, for all $k \geq 2$; (that is, so that T consists of nullary and unary operations only). Then the union of any number of subalgebras of A is a subalgebra of A .

Give an example (say, from group theory) to show that if n -ary operations are allowed, with $n \geq 2$, then the supremum in Proposition 5.5 need not be the union.

(b) Suppose that $\{B_i : i \in I\}$ is a set of subalgebras of A , so that I is partially ordered so that $i \leq j$ implies that $B_i \leq B_j$, and for each pair $i, j \in I$, there is a $k \geq i, j$, such that both $B_i, B_j \leq B_k$. Prove that $\cup_{i \in I} B_i$ is a subalgebra of A .

(c) If B is a subalgebra of A , and C is a subalgebra of B , then C is a subalgebra of A .

Exercise 5.7. Let G be a group with more than one element. Define a binary operation θ as follows:

$$\theta(a, b) = \begin{cases} ab^{-1}, & \text{if one of these is a power of the other;} \\ e, & \text{otherwise.} \end{cases}$$

Prove that, as an algebra of type $\{\theta\}$, in $\mathbb{S}(G)$ the union of subalgebras is a subalgebra.

6. The Algebraic Nature of Lattices

Suppose that $(A; T)$ is an algebra and that B is a subalgebra of A . We say that B is *finitely generated* if there is a finite subset F of A , for which $B = \langle F \rangle_T$.

Proposition 6.1. Suppose that $(A; T)$ is an algebra and that $B \leq A$. Then B is finitely generated if and only if the following holds:

for any family of subalgebras of A , $\{C_i : i \in I\}$, so that $B \leq \vee_i C_i$, there is a finite subset $I' \subseteq I$, such that $B \leq \vee \{C_i : i \in I'\}$.

(Recall that this condition means that the finitely generated elements of $\mathbb{S}(A)$ are precisely the compact elements. We shall prove the proposition with the help of two lemmas.)

Lemma 6.2. Suppose that $B \leq A$, and that $\langle X \rangle_T = B$. Define $X_0 = X \cup T_0$; assume now that X_0, X_1, \dots, X_{k-1} have been defined so that $X_i \subseteq X_{i+1}$, for all $i = 0, 1, \dots, k-1$. Define

$$X_k = X_{k-1} \cup \{a \in A : a = \phi(a_1, \dots, a_n), \text{ for some } \phi \in T_n, \text{ and } a_i \in X_{k-1}\}.$$

$$\cup_{k=0}^{\infty} X_k = B.$$

Proof of Lemma: To do this let U be the union of the X_k . One should first show by induction that B contains each X_k and thus U . Then, one must show that U is in fact an algebra, which contains X and hence B . ■

Lemma 6.3. Suppose that $B \leq A$, and that $\langle X \rangle_T = B$. For each $a \in B$ there exists a finite subset $X' \subset X$ such that $a \in \langle X' \rangle$.

Proof of Lemma: Suppose by means of contradiction that there is an $a \in B$ which does not have the desired property. Let k be the least integer such that X_k contains such an element, call it a . Observe that $k > 0$. Thus, $a \in X_k \notin X_{k-1}$. By definition there exists an n -ary ϕ and $b_1, \dots, b_n \in X_{k-1}$ such that $\phi(b_1, \dots, b_n) = a$. For each $1 \leq j \leq n$, (since $b_j \in X_{k-1}$) there exists a finite subset, say Y_j of X such that $b_j \in \langle Y_j \rangle_T$. Letting $X' = \cup_{1 \leq j \leq n} Y_j$, a finite subset of X , we obtain that $a \in \langle X' \rangle_T$, the desired contradiction. ■

Proof of Proposition: Sufficiency: suppose B is a compact element in the lattice of subalgebras. In general for any algebra we have

$$B = \bigvee_{b \in B} \langle b \rangle .$$

Compactness says there is a finite subset of B , say $\{b_1, \dots, b_n\}$ such that

$$B = \langle b_1 \rangle \vee \dots \vee \langle b_n \rangle = \langle \{b_1, \dots, b_n\} \rangle$$

hence B is finitely generated.

Necessity: we assume B is finitely generated, say by $X = \{b_1, \dots, b_m\}$ and that

$$B \leq \bigvee_i C_i .$$

Let $C = \bigcup_i C_i$ and observe that $B \leq \langle C \rangle$. By lemma 6.3, it follows that for each $i \in I$ we have a finite subset $D_i \subset C$ such that $b_i \in \langle D_i \rangle$. Letting D be the (finite) union of the D_i we are left with

$$B = \langle X \rangle = \langle \{b_1, \dots, b_m\} \rangle = \langle b_1 \rangle \vee \dots \vee \langle b_m \rangle \leq \langle D_1 \rangle \vee \dots \vee \langle D_m \rangle = \langle D \rangle .$$

Enumerating $D = \{d_1, \dots, d_k\}$ and noticing that each $d_j \in C_{i_j}$, we combine this with the above string of inequalities to get

$$B \leq \langle D \rangle = \langle d_1 \rangle \vee \dots \vee \langle d_k \rangle \leq C_{i_1} \vee \dots \vee C_{i_k}$$

whence B is compact. ■

Examples 6.4. One of the complicated things to describe, with some grace, is exactly what is contained in a subalgebra generated by a finite set F . Let us look at a few examples. We fix an algebra A and a subset $F = \{x_1, x_2, \dots, x_k\}$ of A . The objective is to give a description of $\langle F \rangle_T$. Remember that every subalgebra must contain every nullary operation.

(a) Suppose that A is an (additive) abelian group, viewed as an algebra of type (1,1,1), written as $(A; 0, -(), +)$. Then

$$\langle F \rangle_T = \{m_1 x_1 + m_2 x_2 + \dots + m_k x_k : m_i \in \mathbb{Z}\} .$$

(\mathbb{Z} denotes the set of integers.)

(b) Let's complicate matters; now A is a (not-necessarily abelian) group, written multiplicatively. Verify that $\langle F \rangle_T$ consists of all the expressions of the form $g_1 g_2 \dots g_s$, where each g_i is one of the x_j or the inverse of one.

(c) If A is a commutative ring with identity, regarded as an algebra of type (2,1,2), then $\langle F \rangle_T$ is the set of all polynomials in the x_i ; that is, all expressions of the form

$$f(x_1, x_2, \dots, x_k) = m_0 + m_1 f_1 + m_2 f_2 + \dots + m_s f_s,$$

where each m_i is an integer, and each f_i is a product of the form $x_1^{d_1} x_2^{d_2} \dots x_k^{d_k}$, where each d_j is a non-negative integer.

Exercise 6.5. Work out expressions for the members of $\langle F \rangle_T$ when A is

- (a) a commutative ring of type (1,1,2);
- (b) a not-necessarily commutative ring with identity, regarded as having type (2,1,2);
- (c) a boolean algebra, with type (2,1,2).

One often expresses the content of the following proposition by saying that every member of $\mathbb{S}(A)$ is the supremum of its compact elements.

Proposition 6.6. *Let $(A; T)$ be an algebra. In $\mathbb{S}(A)$ each element is the supremum of its finitely generated subalgebras.*

Exercise 6.7. Let $(A; T)$ be an algebra. The element $a \in A$ is said to be a *non-generator* if, for any $X \subseteq A$, $\langle X \cup \{a\} \rangle_T = A$ implies that $\langle X \rangle_T = A$. (Think of a non-generator this way: it is an element which can be omitted from any generating set.)

- (a) Let $\text{Fr}(A)$ denote the set of all non-generators of A . Prove that $\text{Fr}(A)$ is a subalgebra of A .
- (b) Show that $\text{Fr}(A)$ is the intersection of all maximal (proper) subalgebras of A . (If A has no maximal subalgebras, then $\text{Fr}(A) = A$; this can happen! Read on.)
- (c) Let p be a prime number, and \mathbb{Z}_{p^∞} be the multiplicative group of all p^n -th complex roots of 1, for every natural number n . (View \mathbb{Z}_{p^∞} as an abelian group, of type $(1, (\)^{-1}, \cdot)$.) Show that \mathbb{Z}_{p^∞} has no proper maximal subgroups.

(d) Likewise: The additive group \mathbb{Q} of rational numbers has no proper maximal subgroups. Prove that. (Hint: if H is a proper subgroup of \mathbb{Q} , then, for each $n \in \mathbb{N}$, let

$$H_n = \{x \in \mathbb{Q} : nx \in H\}.$$

Prove that H_n is a subgroup of \mathbb{Q} , and that $H < H_n$, for some $n \geq 2$.)

Exercise 6.8. Suppose that $(A; T)$ is a finitely generated algebra. Prove that every proper subalgebra B is contained in a maximal proper subalgebra of A . (Hint: Zorn's Lemma! Any proper subalgebra fails to contain one of the generators.)

These “Noetherian” conditions might look familiar from the theory of commutative rings. As one can see, rings have nothing to do with what makes these conditions equivalent.

Exercise 6.9. Let $(A; T)$ be an algebra. Prove that the following are equivalent.

- (a) Every subalgebra of A is finitely generated.
- (b) $\mathbb{S}(A)$ satisfies the *ascending chain condition*; that is, $\mathbb{S}(A)$ has no infinite chains of the form $A_1 < A_2 < \dots$, consisting of subalgebras of A .
- (c) Every nonempty family of proper subalgebras of A has a maximal element.

7. Homomorphism, Kernels, and Congruences

Starting now it will become important to think of the type of an algebra as the set of operator symbols itself, rather than the sequence of their cardinalities! This presents a problem, at least in principle: for example, as we may consider a boolean algebra and a ring with identity to be of the same type (2,1,2), we must now “label” the operation in each set T_n of n -aries. In the case at hand we must therefore decide whether we regard \vee and $+$ as being the same operation, or else, say \wedge and $+$. This is determined by context, and as said before, involves a choice or a labelling.

Definition 7.1. Suppose that A and B are algebras of type T , and $f : A \longrightarrow B$ is a function. We say that f is a T -homomorphism (or, if the type is understood, a *homomorphism*) if, for each $\phi \in T_n$ ($n \geq 1$) and $a_1, a_2, \dots, a_n \in A$,

$$f(\phi_A(a_1, \dots, a_n)) = \phi_B(f(a_1), \dots, f(a_n));$$

if $\phi \in T_0$, then $f(\phi) = \phi$. A T -isomorphism is a T -homomorphism which is one-to-one and surjective. (Yes, and henceforth we suppress all subscripts on operations.)

It bears emphasizing that homomorphisms are always considered between algebras of the same type!

If $f : A \longrightarrow B$ is any function, then the relation

$$\ker(f) = \{ (x, y) \in A^2 : f(x) = f(y) \}$$

is an equivalence relation. If f is a T -homomorphism, then $\ker(f)$ has the following additional feature: if $\phi \in T_n$ and $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n) \in \ker(f)$, then, if $x = \phi(x_1, \dots, x_n)$ and $y = \phi(y_1, \dots, y_n)$, it follows that $(x, y) \in \ker(f)$. Any equivalence relation which satisfies this condition is called a T -congruence. If f is a T -homomorphism, $\ker(f)$ is referred to as the *kernel congruence of f* . We shall get to the precise relationship between kernels, congruences and homomorphisms shortly. When speaking of a congruence, we shall refer to the equivalence classes as *congruence classes*.

First, let us make an observation, and then discuss some examples.

Proposition 7.2. For an equivalence relation σ on the algebra $(A; T)$ the following are equivalent:

- (a) σ is a T -congruence;
- (b) σ (as a subset of A^2) is a subalgebra of A^2 (taken with the direct product structure).

Examples 7.3. (a) Suppose that we consider the type $(0, 1, -(\), +, \cdot)$ of rings with identity. A homomorphism $f : A \longrightarrow B$ as defined here is one which preserves addition, multiplication, the additive and multiplicative identities, and additive inverses. As the reader knows well enough, if a function between rings with identity preserves addition and multiplication, then the 0 and additive inverses are automatically preserved as well. This is a consequence of the laws of rings. In general, the multiplicative identity is not preserved, and this feature must be stipulated by spelling out what the type is. Similar comments apply to other algebraic structures.

(b) For the type $(0, 1, (\)', \vee, \wedge)$ of boolean algebras, notice that a map $f : A \longrightarrow B$ which preserves \vee and \wedge need not preserve either the least or the largest element. For suppose that $A = B = \mathcal{P}(X)$, the power set of $X = \{a, b\}$, and consider the map $f : A \longrightarrow B$ defined by

$f(\emptyset) = f(\{a\}) = \{a\}$, and $f(\{b\}) = f(X) = X$. Then f preserves union and intersection, but is not a homomorphism of the type under consideration.

On the other hand, if $f : A \rightarrow B$ is a function between two boolean algebras which preserves \vee and complements, then one can show that (because of DeMorgan's Laws), 0 , 1 and \wedge are also preserved, so that f is a (boolean) homomorphism.

To summarize, the point of these examples is to caution: identities which govern an algebraic structure of a certain type T , may actually make the algebra one of a larger type T' . On the other hand, one should be precise in defining the domain of discourse; the type determines (in terms of the homomorphisms allowed) the structure under consideration.

Definition & Remarks 7.4. Suppose that $(A; T)$ is an algebra, and τ is a congruence on it. Denote by A/τ the set of all the congruence classes. We now define an algebra structure of type T on A/τ . Denote the congruence class of a by $\tau[a]$.

Suppose that $\phi \in T_n$ ($n \geq 1$), and $a_1, a_2, \dots, a_n \in A$. Define

$$\phi(\tau[a_1], \tau[a_2], \dots, \tau[a_n]) = \tau[\phi(a_1, a_2, \dots, a_n)].$$

Let us verify that this n -ary operation is well-defined. Suppose that $\tau[a_i] = \tau[b_i]$, for each $i = 1, 2, \dots, n$. Since each a_i is τ -related to b_i , it follows that $\phi(a_1, a_2, \dots, a_n)$ is τ -related to $\phi(b_1, b_2, \dots, b_n)$. Thus,

$$\tau[\phi(a_1, a_2, \dots, a_n)] = \tau[\phi(b_1, b_2, \dots, b_n)].$$

For the nullary operation $\phi \in T_0$, set

$$\phi_{A/\tau} = \tau[\phi_A].$$

This defines an algebra structure on A/τ of type T . It is called the *factor algebra of of A type T , modulo τ* . Observe that under these definitions, the canonical map $\tau^* : A \rightarrow A/\tau$ by $\tau^*(a) = \tau[a]$ is a T -homomorphism.

Remark 7.5. The reader is familiar with the concept of "kernel" from the group and ring theories. It is the subset of the domain which a homomorphism sends to the additive identity of the target structure. More precisely, and to the point here, the kernel is one of the equivalence classes in the congruence $\ker(f)$, when $f : A \rightarrow B$ is a homomorphism.

Specifically, suppose that we are dealing with rings, of type $T = (0, -(), +, \cdot)$, and $f : A \rightarrow B$ is a homomorphism. Then the set $K_f \equiv \{a \in A : (a, 0) \in \ker(f)\}$, is the kernel of f . Note that K_f is an ideal of the ring A . Conversely, suppose that I is an ideal of A ; by defining

$$\sigma_I = \{(x, y) \in A^2 : x - y \in I\},$$

the result is a congruence, and the factor algebra A/σ_I , as defined in 7.4 is none other than the factor ring A/I of lore. Observe as well that the assignment $I \mapsto \sigma_I$ is a one-to-one correspondence between the set of ideals and the set of congruences on A ; it preserves the lattice operations of these two sets.

Similar comments apply to groups, modules and some other structures, where the underlying additive group structure plays a prominent role. We shall revisit this kind of situation later, after a discussion of functors and some category theory. For now, simply observe the connections with familiar settings, but be advised of two important points: first, kernel congruences are not necessarily associated with subsets of the domain (and even when they are the association is less than one would like; see the next exercise); on the other hand, the set of all congruences on an

algebra of type T is, by Proposition 7.2 identical with the lattice $\mathbb{S}(A^2)$, where A^2 has the direct product structure.

If $(A; T)$ is an algebra, let $\mathcal{C}(A)$ denote the lattice of all congruences on A . It should be said, for the sake of emphasis, that the infimum in $\mathcal{C}(A)$ is set-theoretic intersection, whereas the supremum of a family of congruences $(\sigma_i)_{i \in I}$, $\sigma = \vee_i \sigma_i$ is simply the least congruence on A which contains all the σ_i .

Exercise 7.6. This concerns the type $(0, \vee, \wedge)$, with cardinalities $(1, 0, 2)$, and distributive lattices with least element 0. Suppose that A is a distributive lattice (with least element 0). An *ideal* of A is a nonempty subset J of A , which is closed under \vee , and so that if $b \leq a \in J$, then $b \in J$.

Prove the following:

- (a) If σ is a congruence on A , then $\sigma[0]$ is an ideal of A .
- (b) Find a distributive lattice with least element, and a pair of distinct congruences σ_1 and σ_2 , such that $\sigma_1[0] = \sigma_2[0]$. (Which means that one can form distinct factor algebras with the same ideal!)

Recall that a *boolean ring* R is a ring in which every element is idempotent ($x^2 = x$, for each $x \in R$). Recall that if R is a boolean ring, then it is necessarily commutative and of characteristic 2.

Exercise 7.7. Suppose that $(A; 0, 1, ()', \vee, \wedge)$ is a boolean algebra. Define

$$a + b = (a \wedge b') \vee (a' \wedge b), \quad \text{and} \quad a \cdot b = a \wedge b, \quad \text{and} \quad -a = a.$$

- (a) Prove that $(A; 0, 1, -(), +, \cdot)$ is a boolean ring with 1.
- (b) Conversely, suppose that $(A; 0, 1, -(), +, \cdot)$ is a boolean ring with identity. Define

$$a \vee b = a + b + ab \quad \text{and} \quad a \wedge b = ab, \quad \text{and} \quad a' = 1 + a.$$

Show that $(A; 0, 1, ()', \vee, \wedge)$ is a boolean algebra.

- (c) Suppose that A is a boolean ring with 1. Show that a subset J of A is an ideal in the ring sense if and only if it is an ideal in the sense of Exercise 7.6, relative to the associated boolean algebra structure.

Exercise 7.8. If $f : A \longrightarrow B$ is any T -homomorphism between algebras of the same type T , then show that the image $f(A)$ of f is a subalgebra of B .

Exercise 7.9. Use De Morgan's Laws in boolean algebra to show that a boolean algebra B , with type $(0, 1, ()', \vee, \wedge)$ is isomorphic to itself but with type $(1, 0, ()', \wedge, \vee)$. (The point of this exercise being, to reinforce the notion that the choice of a type is a premeditated act. The concept of isomorphism is defined in §8.)

Exercise 7.10. If $\{A_i : i \in I\}$ is a set of algebras of type T , show that the direct product A is characterized by being the only algebra structure of type T on the cartesian product, which makes all projection maps T -homomorphisms.

8. The Homomorphism and Isomorphism Theorems

Definition 8.1. A T -homomorphism f from one algebra of type T to another, such that its inverse exists is a T -isomorphism. It is a routine matter to settle that, if f is a T -isomorphism then its inverse f^{-1} is a T -homomorphism as well, and therefore also a T -isomorphism.

Thus the statement " A is isomorphic to B ", meaning that there is a T -isomorphism from the algebra A of type T to B of the same type, is an equivalence relation on $\mathbf{Al}(T)$. We shall frequently use the symbol \cong to denote the isomorphism between two algebras.

We shall use the terms *injective* and *surjective* for one-to-one and onto maps, respectively.

Here is the most basic result about homomorphisms:

Theorem 8.2. (Induced Homomorphism Theorem) *Suppose that $f : A \rightarrow B$ and $g : A \rightarrow C$ are T -homomorphisms between algebras of the same type T . If $\ker(f) \subseteq \ker(g)$ and f is onto B , then there is a unique T -homomorphism $g^* : B \rightarrow C$ so that $g^* \cdot f = g$.*

Proof. (Existence) Define $g^*(b) = g(a)$, when $f(a) = b$. Since f is surjective, such an a exists for each b . Now, if $f(a) = f(a') = b$, then $(a, a') \in \ker(f) \subseteq \ker(g)$, whence $g(a) = g(a')$, so that g^* is well-defined. We now verify that g^* is a T -homomorphism, checking for n -ary operations, with $n \geq 1$. We leave the nullary case to the reader. Observe, though, before going further, that $g^* \cdot f = g$ holds, by definition.

So let $\phi \in T_n$ ($n \geq 1$), and $b_1, b_2, \dots, b_n \in B$. Find (for each $i = 1, \dots, n$), an element $a_i \in A$, such that $f(a_i) = b_i$. Now, since $f(\phi(a_1, \dots, a_n)) = \phi(f(a_1), \dots, f(a_n))$, and the corresponding identity holds with g , we get:

$$\begin{aligned} g^*(\phi(b_1, \dots, b_n)) &= g^*(\phi(f(a_1), \dots, f(a_n))) \\ &= g^*(f(\phi(a_1, \dots, a_n))) \\ &= g(\phi(a_1, \dots, a_n)) \\ &= \phi(g(a_1), \dots, g(a_n)) \\ &= \phi(g^*(b_1), \dots, g^*(b_n)). \end{aligned}$$

(Uniqueness) If $h : B \rightarrow C$ is a T -homomorphism, such that $hf = g$, then, for all $b \in B$, and $a \in A$, so that $f(a) = b$, $h(b) = h(f(a)) = g(a)$. Thus, $h = g^*$. ■

Corollary 8.3. (First Isomorphism Theorem) *Suppose that $g : A \rightarrow C$ is a surjective homomorphism. Then there is a unique isomorphism $g^* : A/\ker(g) \rightarrow C$, for which $g^* \cdot \tau_g = g$, where τ_g is the canonical homomorphism $A \rightarrow A/\ker(g)$.*

Proof. In Theorem 8.2 let $f = \tau_g$. Verify that g^* is one-to-one because the two kernel congruences agree. g^* is surjective because g is. ■

For the next result, let's develop some notation. Suppose that $(A; T)$ is an algebra, and that σ is a congruence on A . Let $\mathcal{C}(A \geq \sigma)$ denote the set of all congruences on A , which contain σ .

For each congruence $\tau \in \mathcal{C}(A \geq \sigma)$, define $\tau/\sigma \in \mathcal{C}(A/\sigma)$ as follows: $(\sigma[a], \sigma[a']) \in \tau/\sigma$ if and only if $(a, a') \in \tau$. Conversely, if $\mu \in \mathcal{C}(A/\sigma)$, then define μ^c by:

$$(x, y) \in \mu^c \Leftrightarrow (\sigma[x], \sigma[y]) \in \mu.$$

Theorem 8.4. (Congruence Correspondence Theorem) *With the notation just developed, the maps $\tau \rightarrow \tau/\sigma$ and $\mu \rightarrow \mu^c$ are mutually inverse bijections between $\mathcal{C}(A \geq \sigma)$ and $\mathcal{C}(A/\sigma)$, which preserve inclusion. Thus, $\mathcal{C}(A \geq \sigma)$ and $\mathcal{C}(A/\sigma)$ are lattice-isomorphic.*

Sketch of the Proof. The trick is to show that τ/σ is well-defined; this is where the assumption that $\tau \geq \sigma$ comes into play. So suppose that $(a, a') \in \tau$. What must be shown is that, if $(a, b), (a', b') \in \sigma$, then $(b, b') \in \tau$. But this is obvious, since $\sigma \leq \tau$, and τ is both symmetric and transitive.

Now, suppose that $\tau \in \mathcal{C}(A \geq \sigma)$; then $(x, y) \in (\tau/\sigma)^c$ precisely when $(\sigma[x], \sigma[y]) \in \tau/\sigma$, which holds exactly when $(x, y) \in \tau$, proving that $\tau = (\tau/\sigma)^c$.

It remains to verify that $\mu^c/\sigma = \mu$, and that τ/σ and μ^c are congruences, as claimed. We leave this to the reader. That these maps preserve inclusion is obvious. Then, it is also evident that if a bijection between two lattices is order-preserving, it is a lattice-isomorphism; that is to say, it preserves all suprema and infima. ■

Theorem 8.5. (Second Isomorphism Theorem) *Let $(A; T)$ be an algebra, and σ and τ be congruences on A , with $\sigma \leq \tau$. Then the map $g : A/\sigma \rightarrow A/\tau$ by $g(\sigma[x]) = \tau[x]$, is a surjective T -homomorphism, and $\ker(g) = \tau/\sigma$. It follows that*

$$(A/\sigma)/(\tau/\sigma) \cong A/\tau.$$

Proof. Exercise! ■

Theorem 8.6. (Third Isomorphism Theorem) *Suppose that $(A; T)$ is an algebra, B a subalgebra of A , and $\sigma \in \mathcal{C}(A)$. Let*

$$B' = \{ \sigma[b] : b \in B \}.$$

Then $B/(\sigma \cap B^2) \cong B'$.

Proof. Note first that $\sigma \cap B^2$ simply denotes the restriction of the relation σ to pairs from B . Observe as well that B' , being the image of a subalgebra of A , is a subalgebra of A/σ .

Now, we sketch the argument: define $g : B \rightarrow B'$ by $g(b) = \sigma[b]$. Verify that this is a T -homomorphism, which is onto B' (by definition of B'). Its kernel congruence is $\sigma \cap B^2$. Then apply the First Isomorphism Theorem. □

Remark 8.7. A nonidiotic exercise: to reconcile these isomorphism theorems with their special counterparts in group, ring and module theory!!! (Note that in the first two cases "congruence" corresponds with "normal subgroup" and "twosided ideal", respectively.)

Chapter 2

Some Set Theory

1. Axioms and Choice

We shall give a basic treatment of set theory, namely the framework in which most of our mathematics will take place. Our undefined terms are *class* and the relation \in . For any two classes A and B , $A \in B$ is either true or false. An interested reader might want to check [Ku80] or [Du66].

Definition 1.1. We write $A \subset B$ if $x \in A$ implies $x \in B$. We say A and B are equal and write $A = B$ if $A \subset B$ and $B \subset A$.

For a given class A if there exists a class B such that $A \in B$ then we call A a *set*. Otherwise it is a *proper class*.

Axiom 1 **The Axiom of Existence** There exists a set which has no elements.

Axiom 2 **The Axiom of Extensionality** If every element of X is an element of Y and every element of Y is an element of X then $X = Y$.

Axiom 3 **The Axiom Schema of Comprehension** For each formula p in which only set variables are quantified and in which the class variable A does not appear, there is a class A whose members are just those sets having property p .

Axiom 4 **The Axiom of Pair** For any sets A and B , there is a set C such that $x \in C$ if and only if $x = A$ or $x = B$. We write $\{A, B\}$.

Axiom 5 **The Axiom of Union** For any set S , there is a set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

Axiom 6 **The Axiom of Power Set** For any set A , there is a set $\mathcal{P}(A)$ called the *power set of A* such that $X \in \mathcal{P}(A)$ if and only if $X \subset A$.

Axiom 7 **The Axiom of Replacement** Let ϕ be a formula such that for every set x there is a unique set y for which $\phi(x, y)$ is true. For every set A there is a set B such that for all $x \in A$ there is a $y \in B$ for which $\phi(x, y)$ holds.

Axiom 7' **The Axiom of Replacement** If A is a set and $f : A \rightarrow B$ is a function, then the range of f is a set.

Axiom 8 **The Axiom of Infinity** There exists a set A with the properties: (i) $\emptyset \in A$ and (ii) if $a \in A$, then $a \cup \{a\} \in A$.

Axiom 9 The Axiom of Foundation For every nonempty set A there is a $x \in A$ such that $x \cap A = \emptyset$.

Axiom 10 The Axiom of Choice Given any nonempty family $\{A_i : i \in I\}$ where each A_i and I are sets, then there exists a set S consisting of exactly one element from each A_i .

It can be shown that there is a unique set satisfying A.1. It is called the *empty set* and written as \emptyset . Axiom 4 and 7 give us that we may take Cartesian products and hence define functions. Thus, the reason why there are two Axiom of Replacement listed.

Axioms 1-9 are called Zermelo-Frankel set theory or often times **ZF**. When one includes Axiom 10, we write it as **ZFC**. Choice is indeed independent of the other axioms. We obtain some interesting facts and little lemmas:

Lemma 1.2 [ZF]. *There does not exist a set x satisfying $x \in x$.*

Proof: Otherwise, by the Axiom of Power Set we could form the set $\{x\}$. Now, this set must satisfy foundation but it only has one element, namely a and so $A \cap A = \emptyset$. ■

We end this section with a well-known theorem which is the impetus for most of set theory and mathematics, including algebra.

Theorem 1.3. *The following three statements are equivalent:*

1. *The axiom of choice.*
2. *Zorn's lemma: Let X be a partially ordered set. If each chain in X has an upper bound, then X has a maximal element.*
3. *Zermelo's Theorem: Every set can be well-ordered.*

2. Ordinals

Definition 2.1. A set A is called *transitive* if every element of A is a subset of A . The set A is called an *ordinal* if it is transitive and well-ordered by \in .

Example 2.2. \emptyset , $\{\emptyset\}$, and $\{\emptyset, \{\emptyset\}\}$ are examples of ordinals (ordered by inclusion). In fact, these are the first three ordinals. What are the next 3? In fact, in set theory the process which obtains these examples is the only way to construct the natural numbers, i.e., $0 = \emptyset$, $1 = \{\emptyset\}$, and $2 = \{\emptyset, \{\emptyset\}\}$. How would you obtain \mathbb{N} ?

Proposition 2.3. *If x is an ordinal and $y \in x$, then y is an ordinal.*

Proof: Clearly the restriction of the ordering to y maintains the well-ordering. This holds true if y were simply as subset of x . So we want to show that if $z \in y$ then $z \subset y$. Let $w \in z$ then in y (or x) we have $w \in z \in y$ and transitivity means that $w \in y$, whence $z \subset y$. ■

Lemma 2.4. *Suppose α is an ordinal. In each nonempty set $A \subset \alpha$, there is a unique $a \in A$ such that for each $x \in A$ we have either $a \in x$ or $a = x$. This element is called the first element of A .*

Proof: By Foundation there is an $a \in A$ with $a \cap A = \emptyset$. For any $x \in A$ we must have either $x \in a$ or $a \in x$ (by total ordering.) But our choice of a implies that we can't have $x \in a$.

As for uniqueness suppose that b also had the desired property. Then both $a \in b$ and $b \in a$ so that $a = b$. ■

Proposition 2.5. (1) If a and b are distinct ordinals, then $a \in b$ if and only if $a \subset b$.

(2) If a and b are any two ordinals then either $a \subset b$ or $b \subset a$.

Proof: (1) If $a \in b$ then by definition $a \subset b$. Conversely, if $a \subset b$ and let $x \in b$ be the first element of $b - a$. For any $y \in x$ we get that y must be in a whence $x \subset a$. Now assume that $y \in a$. Since $x \in y$ or $x = y$ contradicts that $x \notin a$, we obtain that $y \in x$. But this shows that $a \subset x$. Therefore $a = x \in b$.

(2) It should be immediate that $a \cap b$ is an ordinal. Suppose that $a \cap b$ were a proper subset of both a and b . By (1) we would get that $a \cap b \in a$ and $a \cap b \in b$. But then this means that $a \cap b \in a \cap b$, a contradiction. ■

Corollary 2.6. The collection **ON** of all ordinals is well-ordered by inclusion (or equivalently by \in).

3. Cardinals

Although we could continue discussing set-theoretic concepts we shall not do so due to time constraints and our enthusiasm to do some more algebra. For now we shall end this chapter with a section on cardinalities and their relation to ordinals. There are several ways to do so but hopefully this treatment shines out as being natural and a subtle way of showing off the power of the axiom of choice.

Theorem 3.1. Any well-ordered set is order isomorphic to a unique ordinal.

Definition 3.2. If the set A can be well-ordered, then $|A|$ is the least ordinal which is order isomorphic to A .

Clearly, if together with **ZF** we assume the negation of the Axiom of Choice then $|A|$ is not defined for all sets. Regardless of Choice $|x|$ is always defined for ordinals.

Definition 3.3. The ordinal x is called a *cardinal* if $x = |x|$.

The positives of this definition are that cardinals are clearly tied in to ordinals, which in doing a treatment of axiomatic set-theory makes perfect sense as ordinals are usually the only objects employed to define the naturals, induction, and transfinite induction. The negative is that in this treatment there may exist sets which do not have a cardinality. But for those of us who strongly believe in Choice this point doesn't even concern us.

Now, for those of you who prefer a streamlined version of cardinality we shall do so now:

Definition 3.4. Let A and B be sets. We write $A \leq B$ if there exists a 1-1 function from A into B . We say A and B are equipotent if there exists a bijection between them and write $A \cong B$.

It is easily seen that \leq is transitive and that \cong is an equivalence relation on the (proper) class of all sets.

Theorem 3.5 [Schröder-Bernstein]. *If $A \leq B$ and $B \leq A$, then $A \cong B$.*

The proof of the Schröder-Bernstein theorem is long and technical. As it does not illustrate anything new we shall leave it to the interested reader to find a suitable reference.

At this point there are one of two ways to define cardinal numbers. One is to simply define the cardinality of a set A to be the equivalence class under \leq . But through this way you obtain that cardinalities are not sets but proper classes. The other way is to innocuously define cardinals as a collection of sets with the property that each arbitrary set A is assigned a unique cardinal $|A|$ and that $A \cong B$ if and only if $|A| = |B|$. You then can induce the order onto the class of cardinals.

Now, again the last two have the feature that if you simply want to discuss cardinalities and obtain known results such as

Theorem 3.6 [Cantor's Theorem]. *For any set A , we have $|A| \leq |\mathcal{P}(A)|$.*

But what we lose is the fact that class of all cardinals are well-ordered since this holds true for all ordinals. This then takes the area down the road of ordinal and hence cardinal arithmetic and then leading to ideas such as the Continuum Hypothesis, Martin's Axiom, measurable cardinals, strongly inaccessible cardinals, \diamond , Suslin's Hypothesis etc. (Don't forget forcing which even I don't understand.)

Remark 3.7. Just for S& G's. It is known that the consistency of **ZFC** would imply that there is a consistent system of set theory, namely **ZFC** without the Axiom of Power Set where all sets are countable.

Chapter 3

Category Theory

1. Examples of Categories

The main objects in mathematics are usually sets with some extra structure and the functions between these objects. The idea of category theory is to somehow be able to discuss relationships between different types of structures. This discussion is enveloped under the umbrella called category theory. Roughly speaking a category is two classes; objects and the functions between them. The formal definitions shall be given in this section. We take these ideas from [HS79], [PW??],

Definition 1.1. A *category* \mathbf{C} consists of two classes: a class $obj(\mathbf{C})$ of objects and a class $mor(\mathbf{C})$ of *morphisms* satisfying the following axioms.

- (i) With each morphism f there are associated two objects called the *domain* and *codomain* of f . If $A, B \in obj(\mathbf{C})$, then $\text{Hom}(A, B)$ denotes the *set* of morphisms whose domain is A and whose codomain is B . If $f \in \text{Hom}(A, B)$ We shall represent this by $f : A \rightarrow B$. If we need to distinguish between categories we shall write $\text{Hom}_{\mathbf{C}}(A, B)$.
- (ii) Let $A, B, C, D \in obj(\mathbf{C})$. If $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(B, C)$, then there is a unique morphism written $g \circ f \in \text{Hom}(A, C)$ called the *composition* of g with f .
- (iii) Composition is associative, i.e., if $h \in \text{Hom}(C, D)$ then $h \circ (g \circ f) = (h \circ g) \circ f$.
- (iv) For each object A there is an *identity morphism* written $1_A \in \text{Hom}(A, A)$ with the following properties: if B is an object and $f \in \text{Hom}(A, B)$ (resp. $g \in \text{Hom}(B, A)$) then $f \circ 1_A = f$ (resp. $1_A \circ g = g$).

Example 1.2. 1. **Set** stands for the category whose objects are sets and whose morphisms are simply functions.

2. **Group** is the category of groups and the group homomorphisms between them.
3. **Abel** is teh category of all abelian groups and group homomorphisms.
4. **Rng** is the category of all rings and ring homomorphisms.
5. **Rng1** is the category of all rings with identity and those homomorphisms which preserve the identity.
6. **BA** is the category of boolean algebras whose morphisms are boolean algebra homomorphisms.

7. **Lat** is the category of lattices whose morphisms are lattice homomorphisms.
8. For a fixed ring R , ${}_R\text{Mod}$ is the category of all left unital R -modules with all left R -homomorphisms. Mod_R is analogously defined.
9. **Top** is the category of topological spaces whose morphisms are continuous functions between topological spaces.
10. **Top₂** is the category of all Hausdorff spaces and all continuous maps.
11. **KTop₂** is the category of all compact Hausdorff spaces and all continuous maps.
12. Let I be a *quasi-ordered class*; that is to say, a class with a relation \leq , which is reflexive and transitive. We regard I as a category, whose object class is I itself, and so that, for $i, j \in I$, the morphism set $I(i, j)$ is empty, unless $i \leq j$, and if $i \leq j$, then $I(i, j)$ has exactly one element, the “arrow” $i \longrightarrow j$.

Definition 1.3. Suppose \mathbf{B}, \mathbf{C} are categories. We say \mathbf{B} is a *subcategory* of \mathbf{C} if every object and morphism of \mathbf{B} is also an object or morphism of \mathbf{C} . If \mathbf{B} is a subcategory of \mathbf{C} we say \mathbf{B} is *full* if for any $A, B \in \mathbf{b}$ and $f \in \text{Hom}_{\mathbf{C}}(A, B)$ it is true that $f \in \text{Hom}_{\mathbf{B}}(A, B)$.

- Example 1.4.**
1. Every category is a subcategory of itself.
 2. The category of finite sets and functions is a full subcategory of **Set**.
 3. **Abel** is a full subcategory of **Group**.
 4. **KTop₂** is a full subcategory of **Top₂** and **Top₂** is a full subcategory of **Top**.
 5. **BA** is not a full subcategory of **Lat**.
 6. **Rng1** is not a full subcategory of **Rng**.

2. Examples of Functors

Definition 2.1. Suppose \mathbf{C} and \mathbf{D} are categories. A *covariant functor* $F : \mathbf{C} \rightarrow \mathbf{D}$ is a mapping between categories which assigns to each object C of \mathbf{C} a unique object $F(C)$ of \mathbf{D} and to each morphism f in \mathbf{C} a unique morphism $F(f)$ of \mathbf{D} which satisfies

If $f : A \rightarrow B$ is a morphism of \mathbf{C} then $F(f) : F(A) \rightarrow F(B)$.

For each $B \in \mathbf{C}$, $F(1_B) = 1_{F(B)}$.

For each $f \in \text{Hom}_{\mathbf{C}}(A, B)$ and $g \in \text{Hom}_{\mathbf{C}}(B, C)$, $F(g \circ f) = F(g) \circ F(f)$.

If you were to replace all the arrows in the target of the functor you have what is called a contravariant functor.

Examples 2.2. There follows a list of examples of functors, which, once again, is but a tiny part from of a wealth of examples to be found in [HS79].

- (A) Let **Group** be the category of groups, **Abel** the full subcategory of all abelian groups (both with all homomorphisms). If G is a group, let G' stand for the commutator subgroup, and $A(G) = G/G'$. Note that $A(G)$ is abelian. Now if $f : G \rightarrow H$ is any homomorphism between two groups, then, since the image under f of any commutator is again a commutator, it follows that $f(G') \leq H'$; hence, by the Induced Homomorphism Theorem, there is a unique homomorphism $A(f) : A(G) \rightarrow A(H)$, so that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\mu_G} & A(G) \\ \downarrow f & & \downarrow A(f) \\ H & \xrightarrow{\mu_H} & A(H) \end{array}$$

Note: $\mu_G : G \rightarrow A(G)$ denotes the canonical map.

The assignment $A : \mathbf{Group} \rightarrow \mathbf{Abel}$ is a covariant functor.

- (B) For a fixed natural number n , let \mathbf{Ab}_n be the full subcategory of **Abel** consisting of all abelian groups G in which $ng = 0$, for all $g \in G$. For each abelian group G , let nG denote the subgroup $\{ng : g \in G\}$. Letting $G^{(n)} = G/nG$, and μ_G once again denote the canonical map $G \rightarrow G^{(n)}$, we have for each homomorphism $f : G \rightarrow H$, an induced homomorphism $f^{(n)} : G^{(n)} \rightarrow H^{(n)}$, so that the diagram below commutes:

$$\begin{array}{ccc} G & \xrightarrow{\mu_G} & G^{(n)} \\ \downarrow f & & \downarrow f^{(n)} \\ H & \xrightarrow{\mu_H} & H^{(n)} \end{array}$$

The assignment $()^{(n)} : \mathbf{Abel} \rightarrow \mathbf{Ab}_n$ is a covariant functor.

- (C) Let **TfAb** denote the full subcategory of **Abel** consisting of all torsion free abelian groups; (recall $G \in \text{obj}(\mathbf{Abel})$ is *torsion free* if $ng = 0$ implies that $g = 0$, in G .) For each abelian group G , let $T(G)$ stand for the set of all elements of G of finite order. This is a subgroup of G . Let $t_G : G \rightarrow G/T(G)$ denote the canonical homomorphism. Since, under any homomorphism of groups, the image of an element of finite order also has finite order, it follows that, if $f : G \rightarrow H$ is a homomorphism of abelian groups, then $f(T(G)) \leq T(H)$. By the Induced Homomorphism Theorem there is a unique homomorphism $\tau(f) : G/T(G) \rightarrow H/T(H)$, such that $\tau(f) \cdot t_G = t_H \cdot f$. Letting $\tau(G) = G/T(G)$, we obtain a covariant functor $\tau : \mathbf{Abel} \rightarrow \mathbf{TfAb}$.

We feature the Hom functors, which will be of special importance in the theory of rings and modules. There are two variables, and this furnishes us with a covariant functor in one, and a contravariant one in the other.

Definition & Remarks 2.3. Suppose that \mathbf{C} is any category, and $A \in \text{obj}(\mathbf{C})$. The assignment $\text{Hom}(A, _) : \mathbf{C} \rightarrow \mathbf{Set}$ defined by setting, for each morphism $f : B \rightarrow B'$, $\text{Hom}(A, f) : \text{Hom}(A, B) \rightarrow \text{Hom}(A, B')$, defined by

$$\text{Hom}(A, f)(g) = f \cdot g,$$

is a covariant functor.

Check: $\text{Hom}(A, 1_B)(g) = 1_B \cdot g = g$, so that $\text{Hom}(A, 1_B) = 1_{\text{Hom}(A, B)}$. Also: if $f : B \rightarrow B'$ and $g : B' \rightarrow B''$ are \mathbf{C} -morphisms, then

$$\text{Hom}(A, g \cdot f)(h) = (g \cdot f) \cdot h = g \cdot (f \cdot h) = \text{Hom}(A, g)(\text{Hom}(A, f)(h)),$$

which means that $\text{Hom}(A, g \cdot f) = \text{Hom}(A, g) \cdot \text{Hom}(A, f)$.

Dually, for each \mathbf{C} -object A , the function $\text{Hom}(_, A) : \mathbf{C} \rightarrow \mathbf{Set}$ which is defined, for each \mathbf{C} -morphism $f : B \rightarrow B'$, as $\text{Hom}(f, A) : \text{Hom}(B', A) \rightarrow \text{Hom}(B, A)$, as $\text{Hom}(f, A)(h) = h \cdot f$, is a contravariant functor.

This example will be revisited many times and in a number of variations, throughout the course.

If it's pure algebra you're after then the following example is a bit far afield. However, it is an important one, and is typical of other like it, in algebraic geometry, which do interest algebraists a great deal.

Example 2.4. *The Ring of Continuous Functions on a Compact Hausdorff Space.* We define a contravariant functor from \mathbf{KTop}_2 , the category of all compact Hausdorff spaces and all continuous maps between them, to $\mathbf{CRng1}$, the category of all commutative rings with identity, with all homomorphisms that preserve the identity.

Let X be a compact Hausdorff space, and $C(X)$ stand for the ring of all continuous real valued functions defined on X . This is a commutative ring with identity, under pointwise addition and multiplication, in which the constant 1 is the multiplicative identity. For each continuous map $f : X \rightarrow Y$ between compact Hausdorff spaces, define $C(f) : C(Y) \rightarrow C(X)$, by $C(f)(g) = g \cdot f$. Since the composite of two continuous functions is continuous, this makes sense. We leave it to the reader to verify that

- (i) $C(f)$ is a ring homomorphism, which preserves the identity, and
- (ii) that $C(_)$ is a contravariant functor.

Notice that this is one of the variations promised in 2.3. The next example is yet another one.

Example 2.5. Recall that a Hausdorff space is said to be *zero-dimensional* if there is a base for the open sets consisting of clopen (= closed-and-open) sets. Now we define a contravariant functor from the category of compact zero-dimensional spaces and all continuous maps, denoted \mathbf{KZero} , to \mathbf{Bool} . For each $X \in \text{obj}(\mathbf{KZero})$, let $\mathfrak{B}(X)$ stand for the set of all clopen subsets. $\mathfrak{B}(X)$ is a boolean algebra, by defining supremum and infimum to be set-theoretic union and intersection, respectively, and complements to be set-theoretic complements. The least and largest elements are, respectively, \emptyset and X itself, both of which are clopen.

Since, under a continuous map, the inverse image of any open (resp. closed) set is open (resp. closed), it follows that the following is well defined: for each continuous map $f : X \rightarrow Y$, set $\mathfrak{B}(f) : \mathfrak{B}(Y) \rightarrow \mathfrak{B}(X)$ by $\mathfrak{B}(f)(W) = f^{-1}(W)$.

The reader ought to verify that \mathfrak{B} defines a contravariant functor between the categories indicated here.

Example 2.6. Forgetful Functors. Let \mathbf{C} be any category; any covariant functor $G : \mathbf{C} \rightarrow \mathbf{Set}$ such that for all $f, g \in \text{Hom}(A, B)$, $G(f) = G(g)$ implies that $f = g$, will be called an *underlying-set functor*. Examples abound: for any category \mathbf{C} of algebras of a given type T , an algebra $(A; T)$ has a “natural” underlying set $G(A)$, and any T -homomorphism f is, first and foremost, a function $G(f)$. Since composition of homomorphisms is just functional composition, the underlying set assignment is a covariant functor, which clearly has the attribute of injectivity.

Likewise, out of \mathbf{Top} , one has the underlying set $G(X)$ of a topological space X , and for each continuous function f , the underlying function $G(f)$.

The label “forgetful” is another label frequently used in this context, although that encompasses a larger class of functors. “Grounding” functor is yet another. An underlying-set functor is forgetful in the sense that whatever additional structure was present is ignored. A category which admits an underlying-set functor will be called a *concretizable* category. If \mathbf{C} is already furnished with an underlying-set functor $G : \mathbf{C} \rightarrow \mathbf{Set}$ we say that the pair (\mathbf{C}, G) is a *concrete category*.

As is pointed out in [HS79], the relationship between concrete categories and concretizable categories is roughly like the one between metric spaces (i.e., topological spaces already endowed with a metric) and *metrizable spaces* (topological spaces, on which a metric can be defined).

There are examples of nonconcretizable categories, but they are surprisingly difficult to define. For more on this subject, see Exercise 12L in [HS79].

We highlight the injectivity feature of underlying-set functors.

Definition 2.7. A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ with the property that, for each pair $f, g \in \text{Hom}_{\mathbf{C}}(A, B)$, $F(f) = F(g)$ implies that $f = g$, will be called a *faithful functor*.

The free constructions of the previous chapter are special examples of free functors.

Example 2.8. Free Functors. Suppose that (\mathbf{C}, G) is a fixed concrete category. Suppose that for each set X there is an $F(X) \in \text{obj}(\mathbf{C})$ so that the following properties are satisfied:

(fr1) For each set X , there is a function $f_X : X \rightarrow G(F(X))$, which is one-to-one, such that

(fr2) for each $A \in \text{obj}(\mathbf{C})$ and each function $g : X \rightarrow G(A)$, there exists a *unique* \mathbf{C} -morphism $g^* : F(X) \rightarrow A$ such that $g = G(g^*) \cdot f_X$; i.e., such that the diagram below commutes:

$$\begin{array}{ccc}
 X & \xrightarrow{f_X} & G(F(X)) & & F(X) \\
 & \searrow g & \downarrow G(g^*) & & \downarrow g^* \\
 & & G(A) & & A
 \end{array}$$

These conditions actually define a covariant functor $F : \mathbf{Set} \rightarrow \mathbf{C}$. To see this, let $g : X \rightarrow Y$ be a function, and apply (fr2) to the composite $f_Y \cdot g$. Let $F(g)$ (by definition) be the unique morphism predicted by (fr2). It makes the following diagram commute

$$\begin{array}{ccc}
 X & \xrightarrow{f_X} & G(F(X)) \\
 \downarrow g & & \downarrow G(F(g)) \\
 Y & \xrightarrow{f_Y} & G(F(Y))
 \end{array}
 \quad (*)$$

We leave to the reader the verification that F , thus defined on morphisms, is actually a covariant functor. Such an F is said to be the *free functor for G* . (We note, with emphasis, that a free functor is “paired” with a specific underlying-set functor. Notice as well, that in (fr1) it is required that f_X be injective; this is not standard in the literature. We shall return to this point later.)

Definition 2.9. In most of the above examples of covariant functors there was a little more to what was going on. Suppose \mathcal{C} is a category and \mathcal{B} is a subcategory. A covariant functor $r : \mathcal{C} \rightarrow \mathcal{B}$ is called a *reflection* if to each object $A \in \text{obj}(\mathcal{C})$ there is a \mathcal{C} -morphism $r_A : A \rightarrow r(A)$ with the property that for every morphism $f : A \rightarrow B$ where $B \in \text{obj}(\mathcal{B})$ there is a unique \mathcal{B} -morphism $f' : r(A) \rightarrow B$ making the following diagram commute:

$$\begin{array}{ccc}
 A & \xrightarrow{r_A} & r(A) \\
 & \searrow f & \downarrow f' \\
 & & B
 \end{array}$$

A subcategory together with a reflection is called a *reflective subcategory*.

Definition 2.10. Let \mathcal{C} be a category and $f : A \rightarrow B$ a \mathcal{C} -morphism. We define f to be a *monomorphism* (or simply *monic*) if provided that for all \mathcal{C} -morphisms h and k such that $f \circ h = f \circ k$, it follows that $h = k$.

Analogously, f is called an *epimorphism* provided for all \mathcal{C} -morphisms h and k such that $h \circ f = k \circ f$, it follows that $h = k$.

Exercise 2.11. Prove that in the category \mathbf{Set} monomorphisms and epimorphisms are precisely injective and surjective functions, respectively. This is precisely the motivation for the definition.

Generally speaking show that if \mathcal{C} is a concrete category then injectives are always monic and surjectives are always epic.

Exercise 2.12. (1) Consider **DivAbel** the category of divisible abelian groups (a full subcategory of **Abel**). Prove that the quotient map $\pi : \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ is a monomorphism which is not injective.

(2) Let **Field** be the category of all fields and field homomorphisms. Prove that every morphism is a monomorphism. (Remember that a field there is the assumption that $0 \neq 1$.)

Exercise 2.13. (1) Prove that in **Top₂** the epimorphisms are precisely those continuous functions $f : X \rightarrow Y$ with dense images, i.e., $f(X)$ is a dense subspace of Y .

(2) Prove that in **TfAbel** (see (C) of 2.2) a function $f : A \rightarrow B$ is an epimorphism if and only if $B/f(A)$ is a torsion group.

Chapter 4

Lattice-Ordered Groups

1. Exercises

Exercise 1.1. 1. Determine the type of a lattice-ordered group.

2. Prove that $D(\mathbb{R})$ the set of differentiable real-valued functions on \mathbb{R} partially-ordered component-wise is not an ℓ -group.
3. For an $f \in C(X)$, the *support of f* (denoted $\text{supp}(f)$) is the closure of the cozero set of f .

In $C(\mathbb{R})$ show that

$$J = \{f \in C(X) : \text{supp}(f) \text{ is compact.}\}$$

is an ℓ -ideal. Find $f, g \in C(\mathbb{R})^+$ for which $g + J \ll f + J$ in $C(\mathbb{R})/J$. Conclude that $C(\mathbb{R})$ is not hyper-archimedean.

4. Show that $\mathbf{C}(G)$ is an algebraic lattice.
5. Assume that L is an algebraic, Brouwerian lattice; $k(L)$ denotes the subset of compact elements. Pick $0 < b \in L$. Prove that the following are equivalent:
 - (i) $[0, b] = \{x \in L : 0 \leq x \leq b\}$ has no non-trivial disjoint pairs.
 - (ii) For each compact element $0 < a \leq b$, $a^\perp = b^\perp$.
 - (iii) b^\perp is a prime element of L .
 - (iv) b^\perp is a minimal prime element of L .
 - (v) $b^{\perp\perp}$ is maximal among the y 's for which $[0, y]$ satisfies (i).
 - (vi) $b^{\perp\perp}$ is a minimal (non-zero) polar.
 - (vii) b^\perp is a maximal proper polar.

An element which satisfies any of the above equivalent conditions is called *basic*. We call a positive element b of an ℓ -group G *basic* if $G(b)$ is a basic element of $\mathbf{C}(G)$. We say that G has a basis if there is a maximal pairwise disjoint set consisting of basic elements.

6. Prove that in an archimedean ℓ -group G , $0 < a$ is special if and only if it is basic and that $G = G(a) + a^\perp$.
7. Suppose that X is a compact Hausdorff space. Prove that $0 < f \in C(X)$ is basic if and only if $f = re$, where e is the characteristic function of an isolated point.

8. Suppose that G is an ℓ -group. Show the following are equivalent.
- (a) G has a finite basis consisting of n elements.
 - (b) G has n minimal prime subgroups.
 - (c) $\mathfrak{P}(G)$ is a finite boolean algebra with n atoms.
 - (d) Each $A \in \mathbf{C}(G)$ can be written as $A = A_1 + \cdots + A_k$, with $k \leq n$, and where each A_i is indecomposable in $\mathbf{C}(G)$.
 - (e) each pairwise disjoint subset of G has at most n -elements.
9. Let $G = C(\mathbb{N})$ and let H be the ℓ -subgroup of those sequences with finite support. G is an archimedean ℓ -group with a basis. Show that G/H is neither archimedean nor has a basis.

Chapter 5

Appendix

1. Topological Spaces

Definition 1.1. A *topology* on a set X is a collection τ of subsets of X , called the *open sets*, satisfying:

- 1) Any union of elements of τ belongs to τ .
- 2) Any finite intersection of elements of τ belong to τ .
- 3) \emptyset and X belong to τ .

A set equipped with a topology is called a topological space.

Example 1.2. a) If (X, d) is a metric space, then the open sets relative to the metric form a topology called the metric topology.

b) Let X be a set and $\tau = \mathcal{P}(X)$. τ is clearly a topology which we call the *discrete topology*.

c) Let X be a set and $\tau = \{\emptyset, X\}$. This is called the *indiscrete* or *trivial topology*.

d) Let $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then τ is a topology on X .

e) Let X be an infinite space and τ consist of all *co-finite* sets (that, is a complement of a finite set.) Then τ is a topology called the *co-finite topology*.

f) [**ZF**] Let X be an infinite space with cardinality greater than \aleph_0 . Let τ be the set of all *co-countable* subsets of X . This is called the co-countable topology on X .

Definition 1.3. Let (X, τ) be a topological space. A subset $E \subset X$ is called *closed* if $X - E$ is open. If a set is both open and closed then we say it is *clopen*. Observe that there always at least two clopen subsets of a nonempty topological space, namely \emptyset and X . If these are the only two then the space is said to be *connected*. Otherwise, it is said to be *disconnected*.

The next theorem is an application of DeMorgan's Laws.

Theorem 1.4. The set of closed subsets, \mathcal{F} , of a topological space (X, τ) satisfy:

- a) Any intersection of elements in \mathcal{F} belong to \mathcal{F} .
- b) Any finite union of elements in \mathcal{F} belong to \mathcal{F} .

c) \emptyset and X belong to \mathcal{F} .

Sometimes it possible to describe the topology on X without describing each and every open set. This used through a *base* for the topology.

Definition 1.5. Let (X, τ) be a topological space. A *base* for τ is a collection $\mathcal{B} \subset \tau$ such that

$$\tau = \{\cup_{B \in \mathcal{C}} B \mid \mathcal{C} \subset \mathcal{B}\}.$$

That is, τ can be recovered from \mathcal{B} by taking all possible unions of subcollections from \mathcal{B} .

Observe that each element in a base is inherently open. If there is a base consisting of clopen sets, then we say the topology is zero-dimensional. The rationals, the irrationals, the one-point compactification of the naturals, and the Cantor set are all examples of zero-dimensional spaces.

Proposition 1.6. Let (X, τ) be a topological space and $\mathcal{B} \subset \tau$. \mathcal{B} is a base for τ if and only if whenever O is an open set in X and $p \in O$, there is some $B \in \mathcal{B}$ such that $p \in B \subset O$.

Proof: Suppose \mathcal{B} is a base, O is open and $p \in O$. Then $O = \cup_{B \in \mathcal{C}} B$, where $\mathcal{C} \subset \mathcal{B}$. Then $p \in O$ means that for some $B \in \mathcal{C}$ and hence $B \in \mathcal{B}$ we have $p \in B$. Clearly, $B \subset O$.

Conversely, if O is open, then for each $p \in O$ we have $B_p \in \mathcal{B}$ such that $p \in B_p \subset O$, then $O = \cup_{p \in O} B_p$. Let $\mathcal{C} = \{B_p : p \in O\}$ which clearly satisfies $\mathcal{C} \subset \mathcal{B}$ and

$$O = \cup_{B \in \mathcal{C}} B.$$

■

Example 1.7. (1) If (X, d) is a metric space, then the set of all open balls centered at points forms a base for the metric topology.

(2) The set of singletons forms a base for the discrete topology on any set.

(3) In \mathbb{R}^2 , the set of open rectangles forms a base for the metric topology.

Definition 1.8. Let (X, τ) be a topological space and $A \subset X$. The *closure* of A is defined to be

$$\bar{A} = \cap \{K : K \text{ is closed and } A \subset K\}.$$

A subset A for which $\bar{A} = X$ is called *dense*. The rationals are a dense subset of \mathbb{R} in the usual topology.

The interior of a set is defined to be

$$A^\circ = \cup \{O : O \text{ is open and } O \subset A\}.$$

Exercise 1.9. (1) Show that $\bar{A} = X - (X - A)^\circ$.

(2) Show that for any A , A° is open. Similarly, \bar{A} is closed.

(3) Show that if $A \subset B$, then a) $\bar{A} \subset \bar{B}$ and b) $A^\circ \subset B^\circ$.

(4) Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$. Give an example that shows this is not true for union.

Exercise 1.10. A space X is called *second countable* if it has a countable base. A space is called *separable* if it has a countable dense subset. Prove that a separable metric space is second countable.

Exercise 1.11. Given a subset $A \subset X$, prove that the closure of A is the set of $x \in X$ having the property that for any open set $x \in O$ we have that $O \cap A \neq \emptyset$.

Definition 1.12. Let (X, τ) be a topological space and $Y \subset X$. Define

$$\sigma = \{O \cap Y \mid O \in \tau\}.$$

Then (Y, σ) is a topological space (Check this!!). This topology on Y is called the *subspace topology* on Y .

2. Hausdorff Spaces, Continuous Maps and Compactness

Definition 2.1. A topological space (X, τ) is said to be a *Hausdorff space* (or T_2) if for every pair of distinct points $a, b \in X$ there exists disjoint open sets O_1 and O_2 such that $a \in O_1$ and $b \in O_2$.

Metric spaces are Hausdorff spaces. Example 1.2 d), e), and f) are not Hausdorff spaces.

Exercise 2.2. 1) Show that a subspace of a Hausdorff space is again Hausdorff.

2) Show that in a Hausdorff space, singleton sets are closed.

Definition 2.3. Let (X, τ) and (Y, σ) be topological spaces and $f : X \rightarrow Y$ a function. We say f is *continuous* if $f^{-1}(S) \in \tau$ for every $S \in \sigma$. In other words, the inverse image of an open set is open. Equivalently, the inverse image of a closed set is closed. (We may change the above definition to σ being a base. Also, one should check that if X and Y are metric space, then the above definition coincides with the usual δ, ϵ definition.

The collection of all continuous maps from X into Y is denoted by $C(X, Y)$. If $Y = \mathbb{R}$, then as stated before we simply write $C(X)$.

Definition 2.4. If X and Y are topological spaces, a function $f : X \rightarrow Y$ is called a *homeomorphism* (and X are said to be *homeomorphic*) if f is a bijection, and both f and its inverse map f^{-1} are continuous. If f is everything but onto then we call f an embedding. in this case X and $f(X)$ are homeomorphic.

Theorem 2.5. Suppose, $f, g : X \rightarrow Y$ are continuous and Y is Hausdorff. Then $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X .

Proof: Let $A = \{x \in X \mid f(x) = g(x)\}$ and suppose y has the property defined in Exercise 1.11 yet $f(y) \neq g(y)$. Since Y is Hausdorff we may find disjoint open sets V_1, V_2 of Y such that $f(y) \in V_1$ and $g(y) \in V_2$. Since f and g are continuous, taking inverse images and setting $O_1 = f^{-1}(V_1)$ and $O_2 = g^{-1}(V_2)$, we have that $y \in O_1 \cap O_2$ where the meet happens to be an open set. By our choice of y we get that $A \cap (O_1 \cap O_2) \neq \emptyset$, say $a \in A \cap (O_1 \cap O_2)$. Finally, $f(a) \in V_1$, $g(a) \in V_2$ together with $f(a) = g(a)$ contradicts the fact that V_1 and V_2 are disjoint. Thus, $f(y) = g(y)$, whence A is closed. ■

Corollary 2.6. *Suppose $f, g : X \rightarrow Y$ are continuous and Y is Hausdorff. If they agree on a dense set then $f = g$.*

Definition 2.7. Let (X, τ) be a topological space. A collection of subsets \mathcal{O} is called a *cover of X* if for every $x \in X$ there is some $O \in \mathcal{O}$ such that $x \in O$; equivalently, if $\cup \mathcal{O} = X$. A subcollection $\mathcal{V} \subset \mathcal{O}$ which also happens to be a cover of X is called a *subcover*. If \mathcal{V} is finite (resp. countable) then it is called a finite (resp. countable) subcover.

If each element in \mathcal{O} is open then we say \mathcal{O} is an *open cover* and similarly an open subcover.

Definition 2.8. Let (X, τ) be a topological space. We call X *compact* if every open cover of X has a finite subcover.

Exercise 2.9. Show that a space X is compact if and only if every collection of closed sets with the finite intersection property has nonempty intersection. (A collection \mathcal{V} is said to have the finite intersection property if for any $V_1, \dots, V_n \in \mathcal{V}$ we have $V_1 \cap \dots \cap V_n \neq \emptyset$.)

Theorem 2.10 [Heine-Borel]. *Let X be a Euclidean space, i.e., $X = \mathbb{R}^n$ equipped with the metric topology. Then a subset of X is compact if and only if it is closed and bounded.*

Proposition 2.11. (a) *Any closed subset of a compact subset is again compact.*

(b) *Any compact subset of a Hausdorff space is closed.*

Definition 2.12. A space is called *locally compact* if every point has a compact neighbourhood. (A neighbourhood of a point x is a set N with $x \in N^\circ$.)

Example 2.13. Let (X, τ) be a locally compact, Hausdorff space which is not compact. We shall construct a compact, Hausdorff space from X .

Let $\alpha X = X \cup \{\infty\}$ where ∞ is any point not in X . (Use the Axiom of Pairing if you have to.) We shall define a topology on αX . Let $U \subset \alpha X$. We consider two cases in defining the open sets.

Case 1: If $\infty \notin U$, then we say U is open if $U \in \tau$.

Case 2: If $\infty \in U$, the U is open iff $X - U$ is a compact subset of X .

We shall let the reader verify that this collection forms a topology on αX . As for compactness, let \mathcal{O} be an open cover of αX . Then $p \in O$ for some $O \in \mathcal{O}$. Now by definition, $L = X - O$ is a compact subset. Since \mathcal{O} also covers L it follows that there exist $O_1, \dots, O_n \in \mathcal{O}$ covering L . It should be obvious that $\{O, O_1, \dots, O_n\}$ is open subcover of αX .

That αX is Hausdorff is patent. Finally, X is an open, dense subspace of αX . We call αX the *one-point compactification of X* . It is also often times referred to as the Alexandroff compactification of X .

Note: By a compactification of a space X we mean a compact space K together with an embedding $h : X \rightarrow K$ where $h(X)$ is a dense subspace of K .

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