Algebra First Year Examination Exercises

1. Let G be a finite group acting on the set S. Suppose that H is a normal subgroup of G so that for any $s_1, s_2 \in S$ there is a unique $h \in H$ so that $hs_1 = s_2$. For each $s \in S$, let $G_s = \{\, g \in G \,:\, gs = s \,\}$. Prove

   (a) $G = G_s H$, and $G_s \cap H = \{e\}$;

   (b) if H is contained in the center of G, then G_s is normal and G is (isomorphic to) a direct product of G_s and H.

2. Suppose that $\phi : G \longrightarrow H$ and $\theta : G \longrightarrow K$ are homomorphisms between groups. Assume that ϕ is surjective. Show that if $Ker(\phi)$ is contained in $Ker(\theta)$ then there is a unique homomorphism $\theta^* : H \longrightarrow K$ such that $\theta^* \cdot \phi = \theta$.

3. Prove that if G is a finite group then any subgroup of index 2 is normal.

4. Prove that any subgroup of a cyclic group is cyclic.

5. Find all the automorphisms of order 3 of $\mathbb{Z}_{91}$. (Hint: How can $\mathbb{Z}_3$ act nontrivially on $\mathbb{Z}_{91}$?) Does $\mathbb{Z}_{91}$ have any automorphisms of order 5? Explain.

6. Suppose that $\phi : G \longrightarrow H$) is a surjective homomorphism of groups. Prove the following about) the assignment $\phi^* : N \mapsto \phi^{-1}(N)$. Assume that) it maps subgroups to subgroups.

   (a) ϕ^* is a bijection between the lattice of subgroups of H and the set of subgroups of G that contain $Ker(\phi)$.

   (b) $N_1 \subseteq N_2$ if and only if $\phi^*(N_1) \subseteq \phi^*(N_2)$.

   (c) $\phi^*(N)$ is normal in G if and only if N is normal in H.

7. Prove that $(\mathbb{Q}, +)$, the additive group of rational numbers is not cyclic.

8. Prove that $Aut(\mathbb{Z}_k)$ is isomorphic to the group $U(k)$ of integers i, with $1 \leq i < k$, which are relatively prime to k, under multiplication modulo k.

9. Show that if G is a simple abelian group then it is cyclic of prime order.

10. For the additive group of rational numbers $(\mathbb{Q}, +)$, show that the intersection of any two nontrivial subgroups is nontrivial.

11. Show that the group $(\mathbb{Q}, +)$ of additive rational numbers has no maximal subgroups. (Hint: Use the lattice isomorphism theorem (Exercise 6) and Exercise 9.)

12. The *commutator subgroup* G' of a group G is defined as the subgroup generated by the set

$$\{\, x^{-1}y^{-1}xy \,:\, x, y \in G \,\}.$$

   Prove that:

   (a) Show that G' is a normal subgroup of G.

   (b) Show that G/G' is abelian.

(c) Show that if $\phi : G \longrightarrow H$ is a homomorphism into the abelian group H, then there exists a unique homomorphism $\hat{\phi} : G/G' \longrightarrow H$ such that $\hat{\phi}(G'x) = \phi(x)$, for each $x \in G$.

13. State and prove Cayley's Theorem about finite groups.

14. (a) Prove that two elements of the symmetric group S_n are conjugate if and only if their cycle types are the same.

    (b) Is this true for the alternating groups? Justify your answer.

15. Prove that if $|G| = 12$ and G has 4 Sylow 3-subgroups, then $G \cong A_4$. (Hint: let G act by conjugation on the 4 Sylow 3-subgroups.)

16. Let p and q be prime numbers, and suppose that $p < q$. If G is a group of order pq and p does not divide $q - 1$, show that G must be cyclic.

17. Suppose that G is a nonabelian group of order 21. Prove:

    (a) $Z(G) = \{e\}$;

    (b) G has an automorphism which is not inner.

18. Suppose that G is a group and H is a proper subgroup of index k. Show that

    (a) $g * (xH) = gxH$ defines a group action of G on the set $\Omega = (G/H)_l$ of left cosets of H;

    (b) the kernel of the induced homomorphism into the permutation group on Ω is the intersection of all the conjugates of H.

    (c) Now suppose that G is simple and that $k > 1$ is the index of H. Then show that G is isomorphic to a subgroup of S_k.

19. Prove that a group of order 30 must have a normal subgroup of order 15.

20. Classify the groups of order 70.

21. Show that if G is a subgroup of S_n (n a natural number) containing an odd permutation, then half the elements of G are odd and half are even.

22. Use 21 to prove that if G is a group of order $2m$, with m odd, then G cannot be simple, and, indeed, contains a subgroup of index 2.

23. Prove that S_4 contains no non–abelian simple groups.

24. Use the result of 23 to prove that if G is a nonabelian simple group, then every proper subgroup of G has index at least 5.

25. Let P be a Sylow p-subgroup of H and $H \leq K$. If P is normal in H and H is normal in K, prove that P is normal in K. Deduce that if $P \in Syl_p(G)$ then $N_G(P)$ is selfnormalizing.

26. Prove that if G is a finite group, and each Sylow p-subgroup is normal in G, then G is a direct product of its Sylow subgroups.

27. Classify the abelian groups of order $2^5 \cdot 5^2 \cdot 17^3$.

28. Give examples of each of the following, with a brief explanation in each case:

(a) A solvable group with trivial center.

(b) An abelian p-group which is isomorphic to one of its proper subgroups and also one of its proper homomorphic images.

(c) An abelian group having no maximal subgroups.

(d) A direct product of nilpotent groups which is not nilpotent.

(e) A semidirect product of abelian groups which is not nilpotent.

(f) A finite nonabelian group in which every proper subgroup is cyclic.

29. Each three-cycle in S_n has $\frac{1}{3}n(n-1)(n-2)$ conjugates. Prove this and conclude from it that A_4 is the only subgroup of S_4 of order 12.

30. Prove that A_5 is a simple group.

31. For $n \geq 5$, prove that A_n is the only proper, nontrivial normal subgroup of S_n.

32. Suppose that G is a group and H is a normal subgroup. Prove that $G/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$. (Hint: let G act on H by conjugation.)

33. Let G be a group of 385 elements. Prove that the Sylow 11-subgroups are normal, and that any Sylow 7-subgroup lies in the center.

34. Suppose that $|G| = 105$. If G has a normal Sylow 3-subgroup, prove that it must lie in the center of G.

35. Let G and H be the cyclic groups of orders n and k, respectively. Prove that the number of homomorphisms from G to H is the sum of all $\phi(d)$, where d runs over all common divisors of n and k, and ϕ denotes the Euler ϕ-function.