Extensions of functions which preserve the continuity on the original domain

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Abstract

We say that a pair of topological spaces \((X, Y)\) is good if for every \(A \subseteq X\) and every continuous \(f : A \to Y\) there exists \(\tilde{f} : X \to Y\) which extends \(f\) and is continuous at every point of \(A\). We use this notion to characterize several classes of topological spaces, as hereditarily normal spaces, hereditarily collectionwise normal spaces, \(Q\)-spaces, and completely metrizable spaces. We also show that if \(X\) is metrizable and \(Y\) is locally compact then \((X, Y)\) is good and we answer a question of Arhangel’skiĭ’s about weakly \(C\)-embedded subspaces. For separable metrizable spaces our classification of good pairs is almost complete, e.g., if \(X\) is uncountable Polish then \((X, Y)\) is good if and only if \(Y\) is Polish as well. We also show that if \(Y\) is Polish and \(X\) metrizable then \(\tilde{f}\) can be chosen to be of Baire class 1.

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1. Introduction

Given two topological spaces \(X\) and \(Y\), a subset \(A\) of \(X\), and a continuous function \(f : A \to Y\), the problem of extending \(f\) to a continuous \(F : X \to Y\) has been studied extensively. The prototype of the results in this area is the Tietze–Urysohn extension theorem [8, Theorem 2.1.8]: if \(X\) is normal, \(Y\) a closed interval in \(\mathbb{R}\), \(A\) closed in \(X\) then the continuous extension \(F\) does exist. A similar result is the theorem of Stone [8, Exercise 3.2.J] that asserts the existence of \(F\) under the same hypothesis on \(Y\) whenever \(X\) is Tychonoff and \(A\) compact. In general the existence of \(F\) is rather exceptional, since in most cases \(f\) is not extendible to a continuous function on the whole of \(X\).

* The problem studied in this paper arose from a question asked by Alessandro Andretta and the second author. Andretta gave a significant contribution to the early part of this research.

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The classical way of overcoming this difficulty is to weaken the requirement that $F$ is defined on the whole of $X$. Along these lines we have Bourbaki’s extension theorem [3, Théorème I.8.1] which states that if $A$ is dense in the topological space $X$ and $Y$ is regular then $f$ can be extended to a continuous $F : B \to Y$, where $B = \{ x \in X \mid f$ has a continuous extension to $A \cup \{x\}\}$. A similar result (which is attributed in [8] to Lavrentieff, in [15] to Kuratowski, and in [19] to Hahn) will be useful in the sequel of the paper.

**Theorem 1.1.** Let $X$ be a topological space, $Y$ a completely metrizable space, $A$ dense in $X$ and $f : A \to Y$ continuous. Then there exist a $G_δ$ subset $B$ of $X$ such that $A \subseteq B$ and a continuous extension $F : B \to Y$ of $f$. If $X$ is metrizable (actually if every closed subset of $X$ is $G_δ$) the hypothesis that $A$ is dense in $X$ is not necessary.

**Proof.** See, e.g., [8, Theorem 4.3.20] or, for the case of $X$ metrizable [15, Theorem 3.8]. □

The research in this area is still very active (see, e.g., [19]). Results that are particularly interesting are those providing characterizations of topological properties in terms of existence of extensions. For example, Bourbaki’s extension theorem characterizes the $Y$’s which are regular amongst the Hausdorff spaces [3, Exercise I.8.19], the first statement in Theorem 1.1 characterizes the $Y$’s which are Čech-complete and have a $G_δ$ diagonal amongst the completely regular spaces [4], Corollary 1.7 of [19] characterizes normal spaces amongst the regular spaces, and Exercise 5.5.1(c) of [8] characterizes collectionwise normal spaces amongst the $T_1$ spaces.

A different approach to the problem of the existence of extensions is to insist that the extension should be defined on the whole $X$, but weaken the requirement that it should be continuous everywhere. A more modest requirement is to demand the extension only to preserve the continuity at every point of the original domain $A$. Our main question is therefore the following:

Let $X$ and $Y$ be topological spaces, $A \subseteq X$ and $f : A \to Y$ a continuous function; can we extend $f$ to a function $\tilde{f} : X \to Y$ which is continuous at every point of $A$?

Some instances of this question were already considered by Arhangel’skii in [1, pp. 91–92], where he introduced the notion of a subset $A$ being *weakly C-embedded into* a topological space $X$: this means that every $f : A \to \mathbb{R}$ can be extended to a $\tilde{f} : X \to \mathbb{R}$ which is continuous at every point of $A$. Arhangel’skii proved several results and asked a few questions about this notion. Our approach here is broader (because we do not confine ourselves to real-valued functions) and we are able to answer positively (actually for a much wider class of spaces than the one for which the question was originally asked) to one of Arhangel’skii’s questions (see Theorem 3.2).

We show that under appropriate hypotheses the answer to our question is positive: e.g., if $Y$ is locally compact it suffices either that $X$ is metrizable (and $A$ arbitrary) or that $A$ is dense in $X$ (and $X$ arbitrary).
We are mainly interested (and in this respect our viewpoint is rather different from [1]) in pairs \((X, Y)\) of topological spaces such that the answer to our question is positive for every \(A \subseteq X\) and every continuous \(f: A \rightarrow Y\). We call such a pair good; if a pair is not good we say it is bad. A similar notion when the extension is required to be continuous everywhere has been studied (for bounded real-valued functions), e.g., in [24]. In this case we will use the following terminology: the pair \((X, Y)\) is strongly good if for every \(A \subseteq X\) and every continuous \(f: A \rightarrow Y\) there exists a continuous \(F: X \rightarrow Y\) extending \(f\).

Using the notion of good pair we provide characterizations of several classes of topological spaces: Theorem 3.3 characterizes hereditarily normal spaces amongst the \(T_1\) ones (they are exactly the \(X\)'s such that the pair \((X, \{0, 1\})\) is good or such that the pair \((X, \mathbb{R})\) is good), Theorem 3.4 characterizes hereditarily collectionwise normal spaces amongst the \(T_1\) ones (they are exactly the \(X\)'s such that the pair \((X, Y)\), with \(Y\) the discrete space of cardinality the weight of \(X\), is good), Theorem 5.1 characterizes \(\mathbb{Q}\)-spaces amongst the metrizable ones (they are exactly the \(X\)'s such that for every \(Y\) the pair \((X, Y)\) is good), and Theorem 5.2 characterizes completely metrizable spaces amongst the metrizable ones (they are exactly the \(Y\)'s such that for every metrizable \(X\) the pair \((X, Y)\) is good).

Restricting ourselves to metrizable spaces, which are the main focus of our investigation, we prove that if \(X\) is a \(\sigma\)-space and \(Y\) is countable then \((X, Y)\) is good (and thus it is consistent with ZFC that there exist good pairs consisting of a non-\(\mathbb{Q}\)-space and a non-completely metrizable space), while if \(X\) is not a \(\lambda\)-space and \(Y\) is not hereditarily Baire then \((X, Y)\) is bad.

We now explain the organization of the paper. In Section 2 we list some elementary facts about our main question and review the definitions and some basic properties of the classes of topological spaces that turn out to be relevant to it. The results mentioned above appear in Sections 3 and 5: in Section 3 we deal with general topological spaces, while in Section 5—we focus on metrizable spaces. In the final part of Section 5 we obtain results and use techniques that have a distinct descriptive set-theoretic flavor: we show that if \(X\) is Polish uncountable and \(Y\) separable metrizable then \((X, Y)\) is good if and only if \(Y\) is Polish. We also obtain, using both the nonseparable descriptive set theory developed by Hansell and the nonseparable determinacy results of Martin, generalizations of this result to the nonseparable case. The descriptive set-theoretic flavor is even more intense in Section 6, which deals with the complexity of \(\tilde{f}\) and where we show that if \(Y\) is Polish then \(\tilde{f}\) can be chosen to be of Baire class 1. In Section 7 we generalize a result about strongly good pairs which is well known in the real-valued case.

2. Basic facts and definitions

The following are simple but useful observations about our question.

**Proposition 2.1.** Let \(X\) and \(Y\) be topological spaces. If \(A \subseteq B \subseteq X\), with \(B\) open, \(f: A \rightarrow Y\) is continuous and there exists \(f^*: B \rightarrow Y\) which extends \(f\) and is continuous at every point of \(A\) then any \(\tilde{f}: X \rightarrow Y\) which extends \(f^*\) has the same properties.
Definition 2.2. If $X$ is a topological space and $A \subseteq X$ we say that $A$ is a weak retract of $X$ if there exists $r : X \to A$ which is the identity on $A$ and is continuous at every point of $A$.

Notice that $A$ being a weak retract of $X$ means that the identity on $A$ has an extension to $Y$ which is continuous at every point of $A$, i.e., that a particular instance of our main question has a positive answer.

Proposition 2.3. Let $X$ and $Y$ be topological spaces. Suppose $X'$ is a subspace of $X$ and $Z \subseteq Y$ is a weak retract of $Y$. If $A \subseteq X'$, $f : A \to Z$ is continuous and there exists $f^* : X \to Y$ which extends $f$ and is continuous at every point of $A$ then there exists $\tilde{f} : X' \to Z$ with the same properties.

Therefore if $(X, Y)$ is good then $(X', Z)$ is also good.

Proof. If $r$ witnesses that $Z$ is a weak retract of $Y$ let $\tilde{f} = (r \circ f^*) | X'$. □

Proposition 2.4. Let $X$ and $Y$ be topological spaces. Suppose $X'$ is a subspace of $X$ and $Z \subseteq Y$ is open nonempty. If $A \subseteq X'$, $f : A \to Z$ is continuous and there exists $f^* : X \to Y$ which extends $f$ and is continuous at every point of $A$ then there exists $\tilde{f} : X' \to Z$ with the same properties.

Therefore if $(X, Y)$ is good then $(X', Z)$ is also good.

Proof. Any $r : Y \to Z$ which is the identity on $Z$ shows that $Z$ is a weak retract of $Y$. Thus the result follows from Proposition 2.3. □

Let us recall the following definitions (see, e.g., [8]).

Definition 2.5. Let $X$ be a topological space. $X$ is hereditarily normal (or $T_5$) if every subset of $X$ is normal. $X$ is hereditarily collectionwise normal if every subset of $X$ is collectionwise normal (a space is collectionwise normal if it is $T_1$ and for every discrete family $\{F_s\}$ of closed sets there exists a family $\{U_s\}$ of pairwise disjoint open sets such that $F_s \subseteq U_s$ for every $s$).

We will need the following characterization of hereditarily normal spaces (see [8, Theorem 2.1.7]).

Definition 2.6. Two subsets $A_0, A_1$ of a topological space $X$ are separated if $\overline{A_0} \cap A_1 = A_0 \cap \overline{A_1} = \emptyset$ (or, equivalently, if they are disjoint and both clopen in $A_0 \cup A_1$).

Lemma 2.7. Let $X$ be a $T_1$ space. $X$ is hereditarily normal if and only if for every $A_0, A_1 \subseteq X$ which are separated there exist disjoint open sets $U_0, U_1$ such that $A_i \subseteq U_i$ for $i = 0, 1$.

An analogous characterization holds for hereditarily collectionwise normal spaces (see [8, Exercise 5.5.1(a)]).
Lemma 2.8. Let \( X \) be a \( T_1 \) space. \( X \) is hereditarily collectionwise normal if and only if for every family \( \{ A_s \} \) which is discrete in \( A = \bigcup_s A_s \) (and hence consists of sets closed in \( A \)), there exists a family \( \{ U_s \} \) of pairwise disjoint open sets such that \( A_s \subseteq U_s \) for every \( s \).

We will often deal with completely metrizable spaces: the basic fact we will use is that a metrizable space is completely metrizable if and only if it is \( G_\delta \) in any completely metrizable space in which it is embedded (see, e.g., [8, Theorems 4.3.23–24] or [15, Theorem 3.11]). Let us also recall (see, e.g., [8, Exercise 4.4.K]) that for every cardinal \( \kappa \geq \aleph_0 \) the Hilbert space \( \ell^2(\kappa) \) is universal for all metrizable spaces of weight \( \leq \kappa \), i.e., any such space is homeomorphic to a subspace of \( \ell^2(\kappa) \). As usual, by a Polish space we mean a separable completely metrizable space.

Definition 2.9. A topological space \( X \) is hereditarily Baire if every closed subspace of \( X \) is a Baire space, i.e., satisfies the Baire category theorem.

If \( X \) is metrizable then \( X \) is hereditarily Baire if and only if no closed subset of \( X \) is homeomorphic to \( \mathbb{Q} \) if and only if no \( G_\delta \) subset of \( X \) is homeomorphic to \( \mathbb{Q} \) (see [5], the second equivalence follows from Proposition 1.2). Every completely metrizable space is hereditarily Baire and ZFC proves the existence of separable metrizable spaces which are hereditarily Baire but not completely metrizable. However these sets are defined using the axiom of choice and need to be highly undefinable: in ZFC they cannot be coanalytic (see, e.g., [15, Corollary 21.21]), assuming large cardinal axioms they cannot be projective and in \( \text{ZF} + \text{DC} + \text{AD} \) they do not exist (this follows, e.g., from [16, Theorem 4]).

Other classes of topological spaces which are relevant to our problem are the following.

Definition 2.10. A \( Q \)-space is a topological space \( X \) such that every subset of \( X \) is \( G_\delta \) (and hence also \( F_\sigma \)) in \( X \). A \( \sigma \)-space is a topological space \( X \) such that every Borel subset of \( X \) is \( G_\delta \) (and hence also \( F_\sigma \)) in \( X \). A \( \lambda \)-space is a topological space \( X \) such that every countable subset of \( X \) is \( G_\delta \) in \( X \).

Clearly every \( Q \)-space is a \( \sigma \)-space and every \( T_1 \) \( \sigma \)-space is a \( \lambda \)-space. The properties of being \( Q \), \( \sigma \) and \( \lambda \) are hereditary: if a space contains a non-\( \lambda \) (respectively \( Q \), \( \sigma \))-space it is not a \( \lambda \) (respectively \( Q \), \( \sigma \))-space. Every \( \lambda \)-space is \( T_1 \) and every \( \sigma \)-discrete (in particular, countable) space is a \( Q \)-space. Spaces which are not \( \lambda \)-spaces include those containing separable \( T_1 \) Baire spaces which have no isolated points: in particular every uncountable Polish space is not a \( \lambda \)-space.

A \( Q \)-set is an uncountable \( Q \)-space which is a subset of the real line (and hence is metrizable and separable). \( \sigma \)- and \( \lambda \)-sets are defined analogously. These sets have been studied extensively (see [17, §40], [22] and its references). The existence of \( \lambda \)-sets can be established in ZFC (see, e.g., [17, Theorem 40.III.2]), which proves the existence of \( \lambda \)-sets of cardinality \( \mathfrak{b} \) (see [6] for both the definition of \( \mathfrak{b} \) and a proof [6, Theorem 9.1]). On the other hand the existence of \( Q \) and \( \sigma \)-sets is independent of ZFC (however Balogh in [2] has shown that ZFC proves the existence of a regular \( Q \)-space which is not \( \sigma \)-discrete).
The existence of a $Q$-set of cardinality $\kappa$ implies $2^\kappa \leq 2^{\aleph_0}$, and therefore $2^{\aleph_1} > 2^{\aleph_0}$ (in particular, CH) implies that $Q$-sets do not exist. The existence of $\sigma$-sets follows from CH or, more generally, from MA (because every Sierpiński set is a $\sigma$-set, see [22, Theorem 4.1] or [21, Theorem 17]); however, as pointed out in [17, Theorem 40.VI.3], CH implies that there exist $\lambda$-sets which are not $\sigma$-sets. Therefore the existence of $\lambda$-sets which are not $\sigma$-sets and the existence of $\sigma$-sets which are not $Q$-sets are both (even simultaneously) consistent with ZFC. MA + $\neg$CH implies that every uncountable subset of the real line of cardinality less than the continuum is a $Q$-set (for a proof see [23, §5]). Miller has shown [21, Theorem 22] that it is consistent that $\sigma$-sets do not exist, and hence that $\sigma$-sets and $Q$-sets coincide. As far as we know it is unknown whether it is consistent that $\lambda$-sets and $\sigma$-sets coincide.

Although the existence of $\lambda$-sets which are not $\sigma$-sets and the existence of $\sigma$-sets which are not $Q$-sets are consistent with ZFC, such sets need to be highly undefinable, since the perfect set property fails for any such set: in ZFC they cannot be analytic (see, e.g., [15, Theorem 29.1]), assuming large cardinal axioms they cannot be projective and in ZF + DC + AD they do not exist.

We will need the following simple fact about metrizable spaces which are not $\lambda$-spaces.

**Proposition 2.11.** If $X$ is a metrizable space which is not a $\lambda$-space there exists $Q \subseteq X$ which is homeomorphic to $\mathbb{Q}$ and not a $G_\delta$ in $X$.

**Proof.** Since $X$ is not a $\lambda$-space there exists $A \subseteq X$ which is countable and not $G_\delta$ in $X$. By applying the Cantor–Bendixson procedure we can write $A = Q \cup S$ where $Q$ is perfect and $S$ is scattered. As is well known every scattered space is completely metrizable and hence $S$ is $G_\delta$ in $X$. Since $A$ cannot be the union of two $G_\delta$'s, $Q$ is not $G_\delta$. Since $A$ is countable so is $Q$ and hence, being nonempty, countable and perfect, it is homeomorphic to $\mathbb{Q}$. 

If $d$ is a metric on the space $X$, $x \in X$, $\varepsilon > 0$ and $A \subseteq X$, we denote by $B(x; \varepsilon)$ the open ball of center $x$ and radius $\varepsilon$, i.e., the set $\{ y \in X \mid d(x, y) < \varepsilon \}$, and by $d(x, A)$ the distance of $x$ from $A$, with the convention that $d(x, \emptyset) = \infty$.

3. **Good pairs: topological spaces**

We start by proving a couple of results dealing with the case in which $Y$ is locally compact. In the first theorem we assume that $X$ is metrizable.

**Theorem 3.1.** Let $X$ be a metrizable space and $Y$ a locally compact space. Then $(X, Y)$ is good.

**Proof.** Let $A \subseteq X$ and a continuous $f : A \to Y$ be given. Fix a compatible metric $d$ for $X$. For every $x \in X$ and $n \in \mathbb{N}$ let 

$$C_n(x) = f \left( B(x; d(x, A) + 2^{-n}) \cap A \right).$$
so that $C_n(x) \neq \emptyset$ and $C_{n+1}(x) \subseteq C_n(x)$. Let

$$B = \{ x \in X | \exists n \text{ such that } C_n(x) \text{ is compact} \}.$$ 

so that if $x \in B$ then $\bigcap_1^n C_n(x) \neq \emptyset$. It is immediate to check that $A \subseteq B$ and $B$ is open (because if $d(y, x) < 2^{-n-2}$ then $C_{n+1}(y) \subseteq C_n(x)$). Thus by Proposition 2.1 it suffices to find $f^*: B \to Y$ which extends $f$ and is continuous at every point of $A$.

Let $f^*: B \to Y$ be any extension of $f$ which satisfies

$$\forall x \in B \quad f^*(x) \in \bigcap_n C_n(x).$$

By our observation above it is clear that there exists such an $f^*$. We claim that $f^*$ is continuous at every $a \in A$.

Let $a \in A$ and an open neighborhood $V$ of $f(a)$ in $Y$ be given. Since $Y$ is locally compact there exists an open neighborhood $W$ of $f(a)$ such that $W$ is compact and $W \subseteq V$. Since $f$ is continuous there exists $\delta > 0$ such that $f(B(a; \delta) \cap A) \subseteq W$. We will show that $f^*(B(a; \delta/2) \cap B) \subseteq V$, thereby completing the proof. Let $x \in B(a; \delta/2) \cap B$. Notice that $d(x, A) < \delta/2$ and hence for some $n$ we have $d(x, A) + 2^{-n} < \delta/2$ which implies $B(x; d(x, A) + 2^{-n}) \subseteq B(a; \delta)$. Therefore $C_n(x) = f(B(x; d(x, A) + 2^{-n}) \cap A) \subseteq f(B(a; \delta) \cap A) \subseteq W$ and $f^*(x) \in C_n(x) \subseteq W \subseteq V$. 

The next result is obtained by generalizing the proof of Theorem 3.1 to the nonmetrizable case, but needs an additional assumption on $A$ and cannot be stated using our good/bad terminology. It answers a question of Arhangel’skii’s [1, Problem 14]. Actually Arhangel’skii asked the question only for the case in which $Y = \mathbb{R}$ and $X$ is a regular $T_1$ space (in a sub-question he was willing even to put further restrictions on $A$), while here we show that to obtain a positive answer no hypotheses on $X$ are necessary and the only relevant property of $\mathbb{R}$ is the local compactness.

**Theorem 3.2.** Let $X$ be a topological space and $Y$ a locally compact space. If $A \subseteq X$ is dense and $f: A \to Y$ is continuous there exists $f: X \to Y$ which extends $f$ and is continuous at every $a \in A$.

**Proof.** For every $x \in X$ let $\{U^x_i \}_{i \in I_x}$ be a basis of open neighborhoods of $x$ and, for every $i \in I_x$, let

$$C_i(x) = f(U^x_i \cap A),$$

so that $C_i(x) \neq \emptyset$ (because $A$ is dense). We now essentially repeat the proof of Theorem 3.1, defining

$$B = \{ x \in X | \exists i \in I_x \text{ such that } C_i(x) \text{ is compact} \},$$

so that $B \supseteq A$ is open, and if $x \in B$ then $\bigcap_{i \in I_x} C_i(x) \neq \emptyset$. Let $f^*: B \to Y$ be an extension of $f$ which satisfies

$$\forall x \in B \quad f^*(x) \in \bigcap_{i \in I_x} C_i(x).$$
The proof of the continuity of \( f^* \) at every \( a \in A \) follows the pattern of Theorem 3.1: in place of \( B(a; \delta) \) we find an open neighborhood \( U \) of \( a \) such that \( f(U \cap A) \subseteq W \), and for every \( x \in U \cap B \) we prove \( f^*(x) \in W \subseteq V \). If \( x \in U \) there exists \( i \in I_x \) such that \( U_i \subseteq U \); therefore \( f^*(x) \in C_i(x) \subseteq f(U \cap A) \subseteq W . \)

The following results characterize hereditarily normal and hereditarily collectionwise normal spaces in terms of good pairs. They should be compared with the characterizations of regular, normal and collectionwise normal spaces mentioned in the introduction.

**Theorem 3.3.** Let \( X \) be a \( T_1 \) space. The following conditions are equivalent:

(i) \( X \) is hereditarily normal;

(ii) \( (X, [0,1]) \) is good (throughout the paper we give \([0,1]\) the discrete topology);

(iii) \( (X, \mathbb{R}) \) is good;

(iv) there exists a topological space \( Y \) which contains two nonempty disjoint open sets (such spaces are called non-irreducible) such that \( (X,Y) \) is good.

**Proof.** To prove the implication from (i) to (ii) let \( A \subseteq X \) and a continuous \( f : A \to [0,1] \) be given. For \( i = 0,1 \) let \( A_i = f^{-1}(i) \): the continuity of \( f \) implies that \( A_0 \) and \( A_1 \) are open in \( A \), and therefore separated in \( X \). Since \( X \) is hereditarily normal, by Lemma 2.7 there exist disjoint open sets \( U_0, U_1 \) such that \( A_i \subseteq U_i \). Any \( f : X \to [0,1] \) such that \( f(x) = i \) whenever \( x \in U_i \) extends \( f \) and is continuous at every point of \( A = A_0 \cup A_1 \).

The implication from (i) to (iii) is contained in [1, Theorem 20].

The implication from either (ii) or (iii) to (iv) is trivial.

To prove that (iv) implies (i) we will again use Lemma 2.7. Let \( Y \) be such that \((X,Y)\) is good and let \( V_0, V_1 \subseteq Y \) be nonempty disjoint open sets. For \( i = 0,1 \) pick \( y_i \in V_i \). Let \( A_0, A_1 \subseteq X \) be separated (and, in particular, disjoint) and let \( A = A_0 \cup A_1 \). Define \( f : A \to Y \) by letting \( f(a) = y_i \) if and only if \( a \in A_i \). Since \( A_0 \) and \( A_1 \) are separated in \( X \) they are open in \( A \); therefore \( f \) is continuous. Since \( (X,Y) \) is good there exists \( \bar{f} : X \to Y \) which extends \( f \) and is continuous at every point of \( A \). Let \( U_i = \text{Int}(\bar{f}^{-1}(V_i)) \). Clearly \( U_0 \) and \( U_1 \) are open and disjoint. Moreover \( A_i \subseteq U_i \) because \( \bar{f} \) is continuous at every point of \( A_i \), and this completes the proof. \( \square \)

The following theorem is proved essentially in the same way of Theorem 3.3, using Lemma 2.8 in place of Lemma 2.7.

**Theorem 3.4.** Let \( X \) be a \( T_1 \) space. Let \( \kappa \) be the weight of \( X \). The following conditions are equivalent:

(i) \( X \) is hereditarily collectionwise normal;

(ii) \( (X, \kappa) \) is good (here \( \kappa \) is given the discrete topology);

(iii) there exists a topological space \( Y \) which contains \( \kappa \) nonempty disjoint open sets such that \( (X,Y) \) is good.

**Proof.** To prove the implication from (i) to (ii) let \( A \subseteq X \) and a continuous \( f : A \to \kappa \) be given. For \( \alpha \in \kappa \) let \( A_\alpha = f^{-1}(\alpha) \): the continuity of \( f \) implies that every \( A_\alpha \) is clopen in
A and hence that the family $\{A_\alpha\}$ is discrete in $A$. Since $X$ is hereditarily collectionwise normal, by Lemma 2.8 there exists a family $\{U_\alpha\}$ of pairwise disjoint open sets in $X$ such that $A_\alpha \subseteq U_\alpha$ for every $\alpha$. Any $\tilde{f} : X \to \kappa$ such that $\tilde{f}(x) = \alpha$ whenever $x \in U_\alpha$ extends $f$ and is continuous at every point of $A$.

The implication from (ii) to (iii) is trivial.

To prove that (iii) implies (i) we use Lemma 2.8. Let $Y$ be such that $(X, Y)$ is good and let $\{V_\alpha \mid \alpha \in \kappa\}$ be a family of $\kappa$ nonempty disjoint open sets in $Y$. For every $\alpha$ pick $y_\alpha \in V_\alpha$. Since the weight of $X$ is $\kappa$, any family of subsets of $X$ which is discrete in its union has at most $\kappa$ nonempty elements. Therefore without loss of generality we may consider a family $\{A_\alpha \mid \alpha \in \kappa\}$ which is discrete in $A = \bigcup_\alpha A_\alpha$ and consists of sets closed in $A$. Define $f : A \to Y$ by letting $f(a) = y_\alpha$ if and only if $a \in A_\alpha$. Since $\{A_\alpha\}$ is discrete in $A$, $f$ is continuous; since $(X, Y)$ is good there exists $\tilde{f} : X \to Y$ which extends $f$ and is continuous at every point of $A$. Let $U_\alpha = \text{Int}(\tilde{f}^{-1}(V_\alpha))$. Clearly the $U_\alpha$’s are open and pairwise disjoint. Moreover $A_\alpha \subseteq U_\alpha$ because $\tilde{f}$ is continuous at every point of $A_\alpha$, and the proof is complete. $\square$

4. Basic constructions for metrizable spaces

We now focus on metrizable spaces and start by proving some lemmas that will be used to establish the main results of next section. These lemmas are useful in proving that a pair is good (Lemma 4.1) or bad (Lemma 4.3).

**Lemma 4.1.** Let $X$ be a metrizable space, $Y$ a topological space, $A$ a $G_\delta$ subset of $X$ and $f : A \to Y$ a continuous function. Then there exists $\tilde{f} : X \to Y$ which extends $f$ and is continuous at every point of $A$.

**Proof.** Fix a compatible metric $d$ for $X$. Since $A$ is $G_\delta$ its complement $X \setminus A$ is $F_\sigma$ and we can write $X \setminus A = \bigcup_n F_n$, where each $F_n$ is closed in $X$. We may assume that $F_n \subseteq F_{n+1}$ for every $n$. For every $x \in X \setminus A$ let $n_x$ be the least $n \in \mathbb{N}$ such that $x \in F_n$ and let $\rho_x = 2^{-n_x} + d(x, A)$. Notice that $B(x; \rho_x) \cap A \neq \emptyset$.

Let $\tilde{f} : X \to Y$ be any extension of $f$ such that for every $x \in X \setminus A$

$$\tilde{f}(x) \in f\left(B(x; \rho_x) \cap A\right).$$

We will show that $\tilde{f}$ is continuous at every $a \in A$.

Let $a \in A$ and an open neighborhood $V$ of $\tilde{f}(a) = f(a)$ in $Y$ be given. Since $f$ is continuous at $a$ there exists $\eta > 0$ such that $f(B(a; \eta) \cap A) \subseteq V$. Let $n \in \mathbb{N}$ be such that $2^{-n} \leq \eta$ and let $\delta = \min\{\frac{\eta}{4}, d(a, F_n)\}$. We have $\delta > 0$ because $a \notin F_n$. We claim that $\tilde{f}(B(a; \delta)) \subseteq V$.

To prove the claim let $x \in B(a; \delta)$. If $x \in A$ then, since $\delta < \eta$, we have $x \in B(a; \eta) \cap A$ and hence $\tilde{f}(x) = f(x) \in V$. If $x \notin A$ we have $d(x, A) \leq d(x, a) < \delta$. Moreover, by our choice of $\delta$, we have $n_x > n$ and therefore $\rho_x < 2^{-n_x-1} + \delta \leq \frac{1}{4}\eta$. By (a) there exists $b \in B(x; \rho_x) \cap A$ such that $\tilde{f}(x) = f(b)$. Since $d(a, b) \leq d(a, x) + d(x, b) < \frac{1}{2}\eta + \frac{3}{4}\eta = \eta$ we have $b \in B(a; \eta) \cap A$ and $\tilde{f}(x) = f(b) \in V$. $\square$
Lemma 4.1 implies that if \( Y \) is metrizable Proposition 2.4 can be extended to \( G_\delta \) subsets of \( Y \).

**Proposition 4.2.** Let \( X \) be a topological space and \( Y \) a metrizable space. Suppose \( X' \) is a subspace of \( X \) and \( Z \subseteq Y \) is \( G_\delta \) nonempty. If \( A \subseteq X' \), \( f : A \to Z \) is continuous and there exists \( f^* : X \to Y \) which extends \( f \) and is continuous at every point of \( A \) then there exists \( \tilde{f} : X' \to Z \) with the same properties.

Therefore if \((X, Y)\) is good then \((X', Z)\) is also good.

**Proof.** The identity function on \( Z \) is a continuous function from a \( G_\delta \) subset of \( Y \) to a topological space; by Lemma 4.1 it has an extension \( r : Y \to Z \) which is continuous at every point of \( Z \). \( r \) shows that \( Z \) is a weak retract of \( Y \) and by Proposition 2.3 the proof is complete. \( \Box \)

**Lemma 4.3.** Let \( X \) and \( Y \) be metrizable spaces, \( A \) a subset of \( X \) which is not \( G_\delta \) in \( X \) and \( f : A \to Y \) a continuous function which is open on its image. If \( f(A) \) is \( G_\delta \) in \( Y \) and for every \( y \in Y \) the set \( f^{-1}(y) \) is closed in \( X \), there does not exist \( \tilde{f} : X \to Y \) which extends \( f \) and is continuous at every point of \( A \).

**Proof.** Since \( f(A) \) is \( G_\delta \) in \( Y \), by Proposition 4.2 it suffices to prove that \( f \) cannot be extended to \( \tilde{f} : X \to f(A) \). Hence we may assume that \( f \) is onto \( Y \).

Let \( d \) be a compatible metric for \( X \) and \( \tilde{f} : X \to Y \) be any extension of \( f \). For every \( x \in X \), \( f^{-1}(\tilde{f}(x)) \subseteq A \) is closed in \( X \) and hence, if \( x \notin A \), we have \( d(x, f^{-1}(\tilde{f}(x))) > 0 \). For every \( n \in \mathbb{N} \) let

\[
M_n = \{ x \in X \mid d(x, f^{-1}(\tilde{f}(x))) > 2^{-n} \},
\]

so that \( X \setminus A = \bigcup_n M_n \) and hence \( X \setminus A \subseteq \bigcup_n \overline{M_n} \). Since \( A \) is not \( G_\delta \) in \( X \), its complement \( X \setminus A \) is not \( F_\sigma \), and the inclusion is proper. Therefore there exist \( a \in A \) and \( i \in \mathbb{N} \) such that \( a \in \overline{M_i} \). We claim that \( \tilde{f} \) is not continuous at \( a \).

To prove the claim we argue by contradiction and assume that \( \tilde{f} \) is continuous at \( a \). Since \( f \) is open, \( V = f(B(a; 2^{-i-1}) \cap A) \) is an open neighborhood of \( f(a) = \tilde{f}(a) \). Therefore there exists an open neighborhood \( U \) of \( a \) in \( X \) such that \( f(U) \subseteq V \). Since \( U \cap B(a; 2^{-i-1}) \) is an open neighborhood of \( a \) in \( X \) and \( a \in \overline{M_i} \), there exists \( x \in M_i \cap U \cap B(a; 2^{-i-1}) \). We have \( \tilde{f}(x) \in V \), i.e., there exists \( b \in B(a; 2^{-i-1}) \cap A \) such that \( f(b) = \tilde{f}(x) \), and hence \( b \in f^{-1}(\tilde{f}(x)) \). Then, from \( x \in M_i \) we get \( d(x, b) \geq d(x, f^{-1}(\tilde{f}(x))) > 2^{-i} \), while from \( x, b \in B(a; 2^{-i-1}) \) we get \( d(x, b) < 2^{-i} \): a contradiction. \( \Box \)

A particular case of Lemma 4.3 that will be very useful is the following corollary.

**Corollary 4.4.** Let \( X \) and \( Y \) be metrizable spaces, \( A \) a subset of \( X \) which is not \( G_\delta \) in \( X \) and \( f : A \to Y \) a topological embedding. If \( f(A) \) is \( G_\delta \) in \( Y \) there does not exist \( \tilde{f} : X \to Y \) which extends \( f \) and is continuous at every point of \( A \).
5. Good pairs: Metrizable spaces

We start by characterizing the metrizable spaces which are always members of good pairs.

**Theorem 5.1.** Let $X$ be a metrizable space. The following conditions are equivalent:

(i) $X$ is a $Q$-space;

(ii) for every topological space $Y$ the pair $(X, Y)$ is good;

(iii) for every metrizable space $Y$ the pair $(X, Y)$ is good.

**Proof.** The implication from (i) to (ii) is an immediate consequence of Lemma 4.1 and of the definition of $Q$-space. The implication from (ii) to (iii) is trivial. It remains to prove that if $X$ is not a $Q$-space then there exists a metrizable space $Y$ such that $(X, Y)$ is bad. Let $A$ be a subset of $X$ which is not $G_δ$, and $F : A → Y$ the identity map. We are in the hypotheses of Corollary 4.4 and any $f : X → Y$ which extends $F$ is not continuous at some $a ∈ A$. This shows that $(X, Y)$ is bad. □

**Theorem 5.2.** Let $Y$ be a metrizable space. Let $κ$ be the weight of $Y$. The following conditions are equivalent:

(i) $Y$ is completely metrizable;

(ii) for every metrizable space $X$ the pair $(X, Y)$ is good;

(iii) for every completely metrizable space $X$ the pair $(X, Y)$ is good;

(iv) $(ℓ^2(κ), Y)$ is good.

**Proof.** To show that (i) implies (ii) let $X$ be metrizable, $Y$ completely metrizable, $A ⊆ X$ and $f : A → Y$ continuous. By Theorem 1.1 there exist $B ⊆ X$ which is $G_δ$, and $F : B → Y$ continuous such that $A ⊆ B$ and $F$ extends $f$. By Lemma 4.1 $F$ can be extended to $f : X → Y$ which is continuous at every point of $B$: a fortiori $f$ extends $F$ and is continuous at every point of $A$.

The implications from (ii) to (iii) and from (iii) to (iv) are trivial.

To prove that (iv) implies (i) let $A ⊆ ℓ^2(κ)$ be homeomorphic to $Y$ and denote by $f : A → Y$ the homeomorphism. If $Y$ is not completely metrizable $A$ is not $G_δ$ in $ℓ^2(κ)$, thus we are in the hypotheses of Corollary 4.4 and we can conclude that $(ℓ^2(κ), Y)$ is bad. □

Notice that in (iv) of Theorem 5.2 any universal space for all metrizable spaces of weight $κ$ can substitute $ℓ^2(κ)$; e.g., when $Y$ is separable the Hilbert cube $[0, 1]^κ$ will do.

Theorems 5.1 and 5.2 may lead to conjecture that if $X$ and $Y$ are metrizable, $X$ is not a $Q$-space and $Y$ is not completely metrizable then the pair $(X, Y)$ is bad. We will now show that in this generality the conjecture is not provable in ZFC.

**Theorem 5.3.** Let $X$ be a metrizable $σ$-space and $Y$ be a countable topological space. Then $(X, Y)$ is good.
Proof. Let $A \subseteq X$ and a continuous $f : A \to Y$ be given. For every $y \in Y$, let $C_y = f^{-1}(y) \subseteq X$: then $\bigcup_{y \in Y} C_y$ is $F_\sigma$, and hence $G_\delta$, in $X$; therefore $B = X \setminus \bigcup_{y \in Y} C_y$ is $F_\sigma$ in $X$.

Let $X_1 = A \cup B = (\bigcup_{y \in Y} f^{-1}(y)) \cup B$: of course, $B$ is $F_\sigma$ in $X_1$, too, and hence by Lemma 4.1 there exists an extension $\tilde{f}_1$ of $f$ to $X_1$ which is continuous at every point of $A$.

We now define an extension $\tilde{f}_2$ of $f$ to $X_2 = \bigcup_{y \in Y} C_y$: for every $x \in X_2$, if $x \in A$ let $\tilde{f}_2(x) = f(x)$, while if $x \notin A$ let $\tilde{f}_2(x) = y$, where $y$ is any point of $Y$ such that $x \in C_y$. We will show that $\tilde{f}_2 : X_2 \to Y$ is continuous at every point of $A$. Indeed, if $a \in A$ and $V$ is a neighborhood of $\tilde{f}_2(a) = f(a)$ in $Y$, fix an open neighborhood $U$ of $a$ in $X$ such that $f(U \cap A) \subseteq V$: we claim that $\tilde{f}_2(U \cap X_2) \subseteq V$. To prove the claim notice that if $x \in U \cap X_2$ then $\tilde{f}_2(x) = y$, where $x \in C_y = f^{-1}(y)$. Hence $U \cap f^{-1}(y) \neq \emptyset$, i.e., there exists $b \in U \cap A$ such that $f(b) = y$. Since $f(U \cap A) \subseteq V$, we have that $\tilde{f}_2(x) = y = f(b) \in V$.

Consider now the extension $\tilde{f}$ of $f$ to $X$, defined by:

$$\tilde{f}(x) = \begin{cases} f(x) = \tilde{f}_1(x) = \tilde{f}_2(x) & \text{if } x \in A, \\ \tilde{f}_1(x) & \text{if } x \in X_1 \setminus A, \\ \tilde{f}_2(x) & \text{if } x \in X_2 \setminus A. \end{cases}$$

Since $\tilde{f}_1$ and $\tilde{f}_2$ are both continuous at every point of $A$, $\tilde{f}$ is continuous at every point of $A$. \hfill $\Box$

Corollary 5.4. It is consistent with ZFC that there exists a good pair $(X, Y)$ where $X$ and $Y$ are separable metrizable, $X$ is not a $Q$-space and $Y$ is not completely metrizable.

Proof. As mentioned in Section 2, it is consistent with ZFC (e.g., it follows from CH) that there exists a $\sigma$-set $X$ which is not a $Q$-set. Then the pair $(X, \emptyset)$ is good by Theorem 5.3. \hfill $\Box$

Even if ZFC does not prove that all pairs consisting of a non-$Q$-space and of a non-completely metrizable space are bad, it suffices to strengthen slightly the hypotheses to obtain the conclusion. The relationships between the stronger hypotheses and the original ones have been discussed after the relevant definitions in Section 2.

Theorem 5.5. Let $X$ and $Y$ be metrizable spaces. If $X$ is not a $\lambda$-space and $Y$ is not hereditarily Baire then $(X, Y)$ is bad.

Proof. By Proposition 2.11, let $A \subseteq X$ be homeomorphic to $\mathbb{Q}$ and not $G_\delta$ in $X$. Since $Y$ is not hereditarily Baire there exists $Q \subseteq Y$ which is closed and homeomorphic to $\mathbb{Q}$. Let $f : A \to Y$ be a homeomorphism between $A$ and $Q$. We are in the hypotheses of Corollary 4.4 and $f$ cannot be extended to $\tilde{f} : X \to Y$ which is continuous at every $a \in A$. Therefore $(X, Y)$ is bad. \hfill $\Box$

If $\kappa$ is an infinite cardinal we define the Baire space of weight $\kappa$ to be $\kappa^N$ (sometimes denoted $B(\kappa)$ in the literature) where $\kappa$ is given the discrete topology and $\kappa^N$ the product
topology; when \( \kappa = \aleph_0 \) we obtain the usual Baire space which is homeomorphic to the irrationals and denoted by \( \mathbb{N}^\mathbb{N} \). Since \( \mathbb{N}^\mathbb{N} \) is not universal for all metrizable spaces of weight \( \kappa \), Theorem 5.2 leaves open the possibility that there exist metrizable spaces \( Y \) of weight \( \kappa \) which are not completely metrizable but are such that the pair \((\kappa, Y)\) is good. We now show that in many cases this is not the case. We start with the separable case (i.e., \( \kappa = \aleph_0 \)) where the results of descriptive set theory are available.

**Theorem 5.6.** If \( X \) contains a subset homeomorphic to the Baire space \( \mathbb{N}^\mathbb{N} \) and \( Y \) is separable metrizable but not completely metrizable, then \((X, Y)\) is bad.

**Proof.** By Proposition 2.4 it suffices to prove the theorem when \( X = \mathbb{N}^\mathbb{N} \). Fix a compatible metric \( d \) for \( Y \) and let \( \hat{Y} \) be the completion of \( Y \) with respect to \( d \). We will denote by \( d \) also the metric on \( \hat{Y} \). \( \hat{Y} \) is a Polish space and \( Y \), being not completely metrizable, is not \( G_\delta \) in \( \hat{Y} \). If \( Y \) is Borel (or even coanalytic) in \( \hat{Y} \) then \( Y \) is not hereditarily Baire (see, e.g., \cite[Corollary 21.21]{15}) and the theorem follows from Theorem 5.5 (obviously \( \mathbb{N}^\mathbb{N} \) is not a \( \lambda \)-space).

Therefore we assume that \( Y \) is not Borel in \( \hat{Y} \). Since \( \hat{Y} \) is Polish there exist a closed \( C \subseteq \mathbb{N}^\mathbb{N} \) and a continuous bijection \( g : C \to \hat{Y} \) (see, e.g., \cite[Theorem 7.9]{15}). We will show that \((C, Y)\) is bad: by Proposition 2.4 this suffices to show that \((\mathbb{N}^\mathbb{N}, Y)\) is bad. Let \( A = g^{-1}(Y) \subseteq C \). If \( A \) were Borel (in \( C \) or, equivalently, in \( \mathbb{N}^\mathbb{N} \)) \( Y \) would be the one-to-one continuous image of a Borel set, and hence Borel (see, e.g., \cite[Theorem 15.1]{15}). Therefore \( A \) is not Borel, and in particular not \( G_\delta \), in \( C \). Let \( f = g \upharpoonright A : A \to Y \). \( f \) is a continuous bijection, but in general it is not open, so Lemma 4.3 does not apply here. However we can mimic its proof, using the fact that \( g \) is defined also on \( C \setminus A \), and obtain the same conclusion, i.e., that \( f \) cannot be extended to \( C \) preserving the continuity at every point of \( A \).

To this end let \( \tilde{f} : C \to Y \) be any extension of \( f \). For every \( n \in \mathbb{N} \) let

\[
M_n = \{ x \in C \mid d(f(x), \tilde{f}(x)) \geq 2^{-n} \}.
\]

Since \( A = g^{-1}(Y) \) we have \( C \setminus A = \bigcup_n M_n \) and hence \( C \setminus A \subseteq \bigcup_n \overline{M_n} \). Since \( A \) is not \( G_\delta \) in \( C \) the inclusion is proper. Therefore there exist \( a \in A \) and \( i \in \mathbb{N} \) such that \( a \in \overline{M_i} \). We claim that \( \tilde{f} \) is not continuous at \( a \).

Notice that for \( x \in M_i \) we have

\[
d(\tilde{f}(x), \tilde{f}(a)) \geq d(f(x), \tilde{f}(x)) - d(g(x), \tilde{f}(a)) \geq 2^{-i} - d(g(x), \tilde{f}(a)).
\]

Since \( g \) is continuous there exists an open neighborhood \( W \) of \( a \) such that \( g(W) \subseteq B(\tilde{f}(a); 2^{-i-1}) \) (recall that \( g(a) = f(a) = \tilde{f}(a) \)). For every open neighborhood \( U \) of \( a \) there exists \( \tilde{x} \in M_i \cap U \cap W \); we have \( d(\tilde{f}(\tilde{x}), \tilde{f}(a)) \geq 2^{-i} - 2^{-i-1} = 2^{-i-1} \). Therefore \( \tilde{f} \) is not continuous at \( a \). \( \square \)

**Corollary 5.7.** If \( X \) is an uncountable Polish space and \( Y \) is separable metrizable then \((X, Y)\) is good if and only if \( Y \) is Polish.
Proof. Every uncountable Polish space contains a subset homeomorphic to \( \mathbb{N}^\mathbb{N} \). Hence the corollary is a consequence of Theorems 5.2 and 5.6.

We do not know whether Corollary 5.7 holds for all metrizable non-\( \sigma \)-spaces \( X \) (or at least for some class wider than that of uncountable Polish spaces). However the picture of good and bad pairs of separable metrizable spaces is almost complete: in Fig. 1 we summarize our results for this case.

We will now prove analogues of Theorem 5.6 for \( Y \) not separable. To this end we need to extend the results of classical descriptive set theory (which deals with separable metrizable spaces) to nonseparable metrizable spaces: [25] is an overview of some results in this area (but more has been done since that paper). For nonseparable metrizable spaces the usual class of Borel sets (i.e., the \( \sigma \)-algebra generated by the open sets) is too narrow and in the early 70’s Hansell defined the class of the extended Borel sets as the smallest \( \sigma \)-algebra containing the open sets and closed under arbitrary unions of discrete families of sets. Hansell proved many results about these sets, including the generalized Souslin theorem (which follows from either [10, Theorem 12] or [11, Theorem 4.15], and from the main result of [12]).

More than 15 years after the first papers by Hansell, Martin (unaware of Hansell’s work) in [20] also extended the class of Borel sets, in the case of strongly zero-dimensional
spaces (this terminology from general topology does not occur in [20], but, e.g., [8, Theorem 7.3.15] shows that these are indeed the spaces considered by Martin), to nonseparable spaces by introducing the quasi-Borel sets. In [20] the generalized Souslin theorem is proved as well, and this is a way of seeing that, at least for strongly zero-dimensional spaces, the notions of extended Borel and quasi-Borel coincide (for separable spaces they both coincide with the classical notion of Borel). In his paper Martin proved the determinacy of the quasi-Borel sets and this has some nice consequences: one can prove Wadge lemma for extended Borel subsets of strongly zero-dimensional spaces (e.g., by repeating verbatim the proof of Theorem 21.14 of [15]), and then use it to prove the generalization of (a special case of) Theorem 22.10 of [15]:

**Theorem 5.8.** If $X$ is a strongly zero-dimensional completely metrizable space and $A \subseteq X$ is extended Borel but not $G_3$ then $A$ is $F_\sigma$-hard, i.e., whenever $Y$ is a strongly zero-dimensional space and $B \subseteq Y$ is $F_\sigma$ there exists a continuous $f : Y \to X$ such that $y \in B$ if and only if $f(y) \in A$.

Another difficulty in extending the separable theory to nonseparable spaces is that arbitrary continuous functions appear to be too “wild”, and one needs to put further restrictions on the continuous functions that are used. Hansell (see again [25] for an overview) gave the following definition.

**Definition 5.9.** A function $f : X \to Y$ is co-$\sigma$-discrete if whenever $\{U_i \mid i \in I\}$ is a discrete family of subsets of $X$ the family $\{f(U_i) \mid i \in I\}$ of subsets of $Y$ has a $\sigma$-discrete base, i.e., there exists a $\sigma$-discrete family $\{V_j \mid j \in J\}$ of subsets of $Y$ such that every $f(U_i)$ is the union of some of the $V_j$’s.

We gave the definition of co-$\sigma$-discrete function for the sake of completeness, but we will use this notion as a black box, quoting some of Hansell’s results in the proof of the next theorem. Hansell used the co-$\sigma$-discrete functions (in combination with the extended Borel sets) to extend several classical results to the nonseparable case.

We say that a metrizable space $Y$ is **absolutely extended Borel** if it is extended Borel in one (and hence in all) of its completions.

**Theorem 5.10.** Let $Y$ be a metrizable space which is not completely metrizable. Let $\kappa$ be the weight of $Y$. Then:

1. if $X$ contains a subset homeomorphic to the generalized Baire space $\kappa^{\aleph_0}$ and $Y$ is not absolutely extended Borel then $(X, Y)$ is bad;
2. if $X$ contains a subset homeomorphic to $\aleph_0^{\aleph_0}$ and $Y$ is absolutely extended Borel and strongly zero-dimensional then $(X, Y)$ is bad.

**Proof.** The proof follows the same ideas of the proof of Theorem 5.6, so we start by considering $\widehat{Y}$, the completion of $Y$ with respect to a compatible metric $d$. $\widehat{Y}$ is completely metrizable and $Y$, being not completely metrizable, is not $G_3$ in $\widehat{Y}$. The result of Hansell
[13, Theorem 5.6] we use is that there exist \( C \subseteq \kappa^N \) closed and a continuous co-\( \sigma \)-discrete bijection \( g : C \rightarrow \hat{Y} \).

If we are in case (1), i.e., \( Y \) is not extended Borel in \( \hat{Y} \), we follow closely the proof of Theorem 5.6: let \( A = g^{-1}(Y) \). \( A \) is not extended Borel in \( C \) by Corollary 5.7 of [13] (which states that a one-to-one continuous co-\( \sigma \)-discrete image of an extended Borel set is extended Borel), and in particular not \( G_\delta \). The proof of Theorem 5.6 now can be repeated verbatim and shows that \((C, Y)\) is bad. By Proposition 2.4 \((X, Y)\) is bad.

If we are in case (2), i.e., \( Y \) is strongly zero-dimensional and extended Borel in \( \hat{Y} \), we use Theorem 5.8: since \( Y \) is not \( G_\delta \) in \( \hat{Y} \) it is \( F_\sigma \)-hard. Let \( Q \subseteq 2^N \) be homeomorphic to \( Q \). Since \( Q \) is \( F_\sigma \) in \( 2^N \) there exists \( f : 2^N \rightarrow \hat{Y} \) continuous such that \( x \in Q \) iff \( f(x) \in Y \) for every \( x \in 2^N \). Let \( Z \) be the image of \( f \): \( Z \) is compact metrizable and hence Polish. \( Y \cap Z \) is not \( G_\delta \) in \( Z \) (otherwise \( f^{-1}(Y \cap Z) = Q \) would be \( G_\delta \) in \( 2^N \)) and hence not completely metrizable. On the other hand \( Z \), and a fortiori \( Y \cap Z \), is separable: hence, by Theorem 5.6 \((X, Y \cap Z)\) is bad. \( Y \cap Z \) is closed, and hence \( G_\delta \), in \( Y \) and Proposition 4.2 implies that \((X, Y)\) is also bad.

We can summarize both results of Theorem 5.10 in a single statement, which however follows easily also from Corollary 4.4, because every strongly zero-dimensional space of weight \( \kappa \) is homeomorphic to a subset of \( \kappa^N \) [8, Theorem 7.3.15].

**Corollary 5.11.** Let \( Y \) be a metrizable space which is strongly zero-dimensional and not completely metrizable. If \( \kappa \) is the weight of \( Y \) and \( X \) contains a subset homeomorphic to \( \kappa^N \) then \((X, Y)\) is bad.

**Proof.** Immediate from Theorem 5.10, since \( \kappa \geq \aleph_0 \) (because \( Y \) is not completely metrizable) and therefore \( N^N \subseteq \kappa^N \).

We had to assume strong zero-dimensionality in (2) of Theorem 5.10 (and hence in Corollary 5.11) because in the nonseparable case we do not know whether the analogue of [15, Corollary 21.21] or the extension of Theorem 5.8 to arbitrary completely metrizable \( X \)’s hold. (The latter holds in the separable case: see [15, Exercises 22.11 and 24.20].)

6. The complexity of the extensions

When the answer to our original question is positive, i.e., when \( \tilde{f} : X \rightarrow Y \) extending \( f \) and preserving the continuity at every point of \( A \) does exist, another natural question comes up: how complicated is \( \tilde{f} \)? As pointed out in the introduction, \( \tilde{f} \) can be continuous (on the whole of \( X \)) only in exceptional circumstances, but in some cases (including those interesting from the viewpoint of descriptive set theory) we will prove that \( \tilde{f} \) can be the “next best thing”, i.e., of Baire class 1. The proof is based on an effectivization of the choice principle needed in the proof of Lemma 4.1 to find \( \tilde{f} \) satisfying condition (⋆).

Recall the following definitions.
Definition 6.1. Let \( Y \) be a topological space and \( \Gamma \) a class of subsets of \( X \). We say that \( g : X \to Y \) is \( \Gamma \)-measurable if \( f^{-1}(V) \in \Gamma \) for every open \( V \subseteq Y \). Let \( F(Y) \) be the hyperspace of the closed nonempty subsets of \( Y \). A function \( G : X \to F(Y) \) is lower \( \Gamma \)-measurable if \( \{ x \in X \mid G(x) \cap V \neq \emptyset \} \in \Gamma \) for every open \( V \subseteq Y \).

The \( F_\sigma \)-measurable functions are known as functions of Baire class 1.

Theorem 6.2. Let \( X \) be a metrizable space, \( Y \) a Polish space, \( A \subseteq X \) and \( f : A \to Y \) a continuous function. Then we can extend \( f \) to a Baire class 1 function \( \tilde{f} : X \to Y \) which is continuous at every point of \( A \).

Proof. Let \( f : A \to Y \) be continuous at every point of \( A \). Theorem 5.3 we may assume that \( A \) is \( G_\delta \). Let \( X \setminus A = \bigcup_n F_n \), where each \( F_n \) is closed in \( X \). Fix a compatible metric \( d \) for \( X \) and, as in the proof of Lemma 4.1, for every \( x \in X \setminus A \) let \( n_x \) be the least \( n \in \mathbb{N} \) such that \( x \in F_n \) and define \( \rho(x) = 2^{-n_x} + d(x, A) \). The map \( \rho : X \setminus A \to \mathbb{R} \) is continuous on every \( F_n \setminus \bigcup_{m<n} F_m \) and hence \( F_\sigma \)-measurable on \( X \setminus A \).

Now consider the map \( x \mapsto f(B(x; \rho(x)) \cap A) \), \( X \setminus A \to F(Y) \). We claim that this map is lower \( F_\sigma \)-measurable.

Since we have
\[
\bar{f}(B(X; \rho(x)) \cap A) \cap V \neq \emptyset \iff f(B(x; \rho(x)) \cap A) \cap V \neq \emptyset \\
\iff B(x; \rho(x)) \cap f^{-1}(V) \neq \emptyset \\
\iff d(x, f^{-1}(V)) < \rho(x)
\]
this follows from the fact that \( \rho \) is \( F_\sigma \)-measurable.

By the Kuratowski–Ryll–Nardzewski selection theorem [18] there exists a Baire class 1 function \( \tilde{f} : X \setminus A \to Y \) such that \( \tilde{f}(x) \in \bar{f}(B(x; \rho(x)) \cap A) \) for every \( x \in X \setminus A \). By letting \( \tilde{f}(x) = f(x) \) when \( x \in A \) we have a function defined on \( X \) which extends \( f \).

\( \tilde{f} \) does not precisely fit the pattern of the proof of Lemma 4.1, since when \( x \in X \setminus A \) we have
\[
\tilde{f}(x) \in \bar{f}(B(X; \rho(x)) \cap A)
\]
which is weaker than (\*). However it is routine to check that (\**) suffices to carry out the proof of the continuity of \( \tilde{f} \) at every point of \( A \).

We now claim that \( \tilde{f} \) is of Baire class 1. To prove the claim let \( V \) be open in \( Y \); let \( B = \tilde{f}^{-1}(V) \cap A = f^{-1}(V) \). For every \( a \in B \) let \( U_a \) be an open neighborhood of \( a \) such that \( \tilde{f}(U_a) \subseteq V \). Then
\[
\tilde{f}^{-1}(V) = (\tilde{f}^{-1}(V) \cap (X \setminus A)) \cup \bigcup_{a \in B} U_a
\]
is the union of an \( F_\sigma \) set (in \( X \setminus A \) and hence in \( X \), because \( X \setminus A \) is \( F_\sigma \)) and an open set, and hence is \( F_\sigma \).

Let us notice that the preceding proof does not generalize to the nonseparable case: indeed Hansell [14] constructed a lower (weakly, in the terminology of that paper) \( F_\sigma \)-measurable function from \( N_1^\omega \) to \( N_1 \) which has no extended Borel-measurable selector.
We can however easily prove a result similar to Theorem 6.2 if we shift the separability assumption from $Y$ to $X$.

**Corollary 6.3.** Let $X$ be a separable metrizable space, $Y$ a completely metrizable space, $A \subseteq X$ and $f : A \to Y$ a continuous function. Then we can extend $f$ to a Baire class 1 function $\tilde{f} : X \to Y$ which is continuous at every point of $A$.

**Proof.** The range of $f$ is a separable subset of $Y$ and hence its closure in $Y$ is a Polish space. By Theorem 6.2 there exists $\tilde{f} : X \to \overline{f(A)}$ of Baire class 1 which extends $f$ and is continuous at every point of $A$. Clearly $\tilde{f}$ is of Baire class 1 also as a function into $Y$. □

7. Strongly good pairs

We will need the following standard definition:

**Definition 7.1.** A topological space $X$ is *extremally disconnected* if it is $T_2$ and for every open $U \subseteq X$ the set $\overline{U}$ is open.

The following characterization of extremally disconnected spaces will be useful (see [8, Theorem 6.2.26]):

**Lemma 7.2.** Let $X$ be a $T_2$ space. $X$ is extremally disconnected if and only if $U \cap \overline{V} = \emptyset$ for every disjoint open sets $U, V \subseteq X$.

A similar characterization holds for hereditarily extremally disconnected spaces:

**Lemma 7.3.** Let $X$ be a $T_2$ space. $X$ is hereditarily extremally disconnected if and only if $A \cap \overline{B} = \emptyset$ for every $A, B \subseteq X$ which are separated.

**Proof.** Assume $X$ is hereditarily extremally disconnected. If $A$ and $B$ are separated then they are open in $M = A \cup B \cup (\overline{A} \cap \overline{B})$. Since $M$ is extremally disconnected Lemma 7.2 implies that the closures of $A$ and $B$ in $M$ are disjoint, which in turn implies that $\overline{A} \cap \overline{B} = \emptyset$.

For the other direction let $M \subseteq X$ and observe that if $A, B \subseteq M$ are disjoint and open in $M$ then $A$ and $B$ are separated in $X$. Hence $\overline{A} \cap \overline{B} = \emptyset$ and a fortiori the closures of $A$ and $B$ in $M$ are disjoint. By Lemma 7.2 $M$ is extremally disconnected. □

The following equivalence is folklore:

**Proposition 7.4.** Let $X$ be a $T_1$ space. The following conditions are equivalent:

(i) $X$ is hereditarily normal and hereditarily extremally disconnected;

(ii) $X$ is normal and hereditarily extremally disconnected;

(iii) $X$ is hereditarily normal and extremally disconnected;

(iv) $(X, [0, 1])$ is strongly good.
Proof. (i) implies (ii) and (i) implies (iii) are both obvious and (iii) implies (i) is straightforward (see, e.g., [8, Exercise 6.2.G.d]) using Lemma 2.7. The equivalence between (ii) and (iv) is contained, e.g., in Exercise 6R2 of [9]. Since (iv) is a hereditary property this shows also that (iv) implies (i).

\( \beta \omega \) is normal and extremally disconnected but not hereditarily extremally disconnected: hence in (i) of Proposition 7.4 both “hereditarily” cannot be removed at the same time.

Our result about strongly good pairs generalizes Proposition 7.4 by showing that (iv) is equivalent to a much more general property.

**Theorem 7.5.** Let \( X \) be a \( T_1 \) space. The following conditions are equivalent:

(i) \( X \) is (hereditarily) normal and hereditarily extremally disconnected;

(ii) \( (X, Y) \) is strongly good for every compact metrizable space \( Y \);

(iii) \( (X, \{0, 1\}) \) is strongly good.

Proof. We first dispose of the easiest implications: (ii) implies (iii) is obvious and to prove that (iii) implies (i) assume \( (X, \{0, 1\}) \) is strongly good. Using Lemmas 2.7 and 7.2 the proofs of hereditary normality and hereditary extremal disconnectedness are essentially the same. Let \( A_0 \) and \( A_1 \) be separated, \( A = A_0 \cup A_1 \) and define \( f : A \to \{0, 1\} \) by \( f(a) = i \) if \( a \in A_i \). Each \( A_i \) is open in \( A \) and hence \( f \) is continuous. Since \( (X, \{0, 1\}) \) is strongly good there exists \( F : X \to \{0, 1\} \) which is a continuous extension of \( f \). Then \( F^{-1}(0) \) and \( F^{-1}(1) \) are disjoint clopen sets containing \( A_0 \) and \( A_1 \): for hereditary normality it matters that they are open, for hereditary extremal disconnectedness it matters that they are closed.

To show that (i) implies (ii) assume \( X \) is hereditarily normal and hereditarily extremally disconnected and \( Y \) is compact metrizable with metric \( d \). Let \( A \subseteq X \) and \( f : A \to Y \) continuous. The proof uses the following lemma (to be proved later):

**Lemma 7.6.** Assume the above hypotheses on \( X, Y, A \) and \( f \). Let \( \Omega \subseteq X \) be clopen and \( W \subseteq Y \) be open such that \( f(\Omega \cap A) \subseteq W \). Then for every \( \epsilon > 0 \) there exist a finite clopen partition \( \{\Omega_1, \ldots, \Omega_m\} \) of \( \Omega \) and corresponding nonempty open subsets \( W_1, \ldots, W_m \) of \( W \), such that for every \( \epsilon = 1, \ldots, m \):

1. \( \Omega_i \subseteq W_i \);
2. \( \text{diam}(W_i) \leq \epsilon \);
3. \( \text{diam}(f(\Omega_i \cap A)) \leq W_i \).

Using the lemma we construct a sequence \( \mathcal{P}^n \) of clopen partitions of \( X \) with \( \mathcal{P}^n = \{\Omega^n_1, \ldots, \Omega^n_m\} \) and a corresponding sequence \( \mathcal{W}^n \) of collections of nonempty open subsets of \( Y \) with \( \mathcal{W}^n = \{W^n_1, \ldots, W^n_m\} \) such that for every \( n \):

(a) \( \mathcal{P}^{n+1} \) is a refinement of \( \mathcal{P}^n \);
(b) \( f(\Omega^n_i \cap A) \subseteq W^n_i \) for every \( i = 1, \ldots, m_n \);
(c) for every \( i = 1, \ldots, m_n \) and \( j = 1, \ldots, m_{n+1} \) if \( \Omega^{n+1}_j \subseteq \Omega^n_i \) then \( W^{n+1}_j \subseteq W^n_i \);
(d) \( \text{diam}(W^n_i) \leq 2^{-n} \) for every \( i = 1, \ldots, m_n \).
We start by applying Lemma 7.6 to \( \Omega = X \), \( W = Y \) and \( \varepsilon = 1/2 \): we obtain a clopen partition \( P^1 = \{G_1, \ldots, G_m\} \) of \( X \) and a collection \( W^1 = \{W_1, \ldots, W_k\} \) of nonempty open subsets of \( X \) that satisfy (b) and (d) (which are the properties not mentioning \( P^1 \)).

Suppose now that \( P^1, \ldots, P^n \) and \( W^1, \ldots, W^n \) have been defined, so that (a)–(d) hold for all appropriate \( n' \leq n \). For every \( i = 1, \ldots, m_n \) apply Lemma 7.6 with \( \Omega = \Omega_n \), \( W = W_n \) (since (b) holds this is possible) and \( \varepsilon = 2^{-(n+1)} \). We obtain a clopen partition \( P_{i+n} = \{G_{i+1}, \ldots, G_{i+m_n}\} \) of \( \Omega_n \) and a collection \( W_{i+n} = \{W_{i+1}, \ldots, W_{i+m_n}\} \) of nonempty open subsets of \( W_i \) satisfying the conclusion of the lemma. Let

\[
m_{i+n} = \sum_{i=1}^{m_n} \ell_i, \quad P_{i+n} = \bigcup_{i=1}^{m_n} P_{i+n}, \quad W_{i+n} = \bigcup_{i=1}^{m_n} W_{i+n}.
\]

Clearly (a) and (d) hold. If we reindex \( P_{i+n} = \{G_{i+1}, \ldots, G_{i+m_n}\} \) and \( W_{i+n} = \{W_{i+1}, \ldots, W_{i+m_n}\} \) by working in parallel, we will have that (b) and (c) hold. This completes our construction.

We now use the \( P^n \)'s and the \( W^n \)'s to define a continuous extension \( f \) of \( f \). Fix \( x \in X \). For every \( n \) let \( i(x, n) \) be the unique \( i \in \{1, \ldots, m_n\} \) such that \( x \in \Omega_i^n \). Every \( P^n \) is a partition so that, by (a), we have \( P^n_{i(x, n)+1} \subseteq \Omega_i^n \). Hence, by (c), \( \bigcap_{i=n}^{m_n} W^n_{i(x, n)+1} \subseteq W^n_{i(x, n)} \).

Since, by (d), \( \text{diam}(W^n_{i(x, n)}) \to 0 \) and the metric is complete, \( \bigcap_{i=n}^{m_n} W^n_{i(x, n)} \) is a singleton: let \( F(x) \) be its unique element. To check that \( F \) extends \( f \) fix \( x \in A \); by (b), \( f(x) \in W^n_{i(x, n)} \) for every \( n \) and hence \( F(x) = f(x) \).

To check that \( F \) is continuous fix \( x \in X \) and \( \varepsilon > 0 \). Let \( n \) be such that \( 2^{-n} < \varepsilon \). Since \( \text{diam}(W^n_{i(x, n)}) \leq 2^{-n} \) and \( F(x) \in W^n_{i(x, n)} \) we have that \( W^n_{i(x, n)} \subseteq B(F(x); \varepsilon) \). \( \Omega^n_{i(x, n)} \) is an open neighborhood of \( x \) and it suffices to show \( F(\Omega^n_{i(x, n)}) \subseteq W^n_{i(x, n)} \). This is immediate because for any \( x' \in \Omega^n_{i(x, n)} \) we have \( i(x', n) = i(x, n) \) and hence \( F(x') \in W^n_{i(x, n)} \). \( \square \)

**Proof of Lemma 7.6.** Since \( Y \) is normal there exists an open set \( V \subseteq Y \) such that \( f(\Omega \cap A) \subseteq V \subseteq \overline{V} \subseteq W \). Since \( V \) is totally bounded there exists a finite collection \( G = \{G_1, \ldots, G_m\} \) of open subsets of \( V \) such that \( \bigcup G = V \) and \( \text{diam}(G_i) \leq \frac{1}{2^n} \) for every \( i = 1, \ldots, m-1 \).

Fix \( G_i \in G \). Let \( G_i^* = \bigcup \{G_j \mid G_j \cap G_i = \emptyset\} \), so that \( G_i \cap G_i^* = \emptyset \). Therefore \( f^{-1}(G_i) \cap \Omega \) and \( f^{-1}(G_i^*) \cap \Omega \) are disjoint open subsets of \( A \), and hence are separated in \( X \). \( X \) is hereditarily extremally disconnected and Lemma 7.3 yields \( f^{-1}(G_i) \cap \Omega \cap f^{-1}(G_i^*) \cap \Omega = \emptyset \). Since \( \Omega \) is closed we have \( f^{-1}(G_i) \cap \Omega \cap f^{-1}(G_i^*) \cap \Omega \subseteq \emptyset \). By normality of \( X \) there exist disjoint open sets \( L_i, L_i^* \) such that \( f^{-1}(G_i) \cap \Omega \subseteq L_i \) and \( f^{-1}(G_i^*) \cap \Omega \subseteq L_i^* \). Since \( \Omega \) is open we may assume \( L_i, L_i^* \subseteq \Omega \). Let \( \Omega_i^* = \overline{L_i^*} \) so that (by Lemma 7.2) \( \Omega_i^* \) and \( \Omega_i^* \) are disjoint clopen subsets of \( \Omega \) such that \( f^{-1}(G_i) \cap \Omega \subseteq \Omega_i^* \) and \( f^{-1}(G_i^*) \cap \Omega \subseteq \Omega_i^* \). Denote as usual by \( \text{St}(G_i, \mathfrak{g}) \) the star of \( G_i \) with respect to \( \mathfrak{g} \) (i.e., \( \text{St}(G_i, \mathfrak{g}) = \bigcup \{G_j \mid G_j \cap G_i \neq \emptyset\} \)) and observe (using \( \Omega_i^* \)) that \( f(\Omega_i^* \cap A) \subseteq \text{St}(G_i, \mathfrak{g}) \). Since \( \text{St}(G_i, \mathfrak{g}) \subseteq V \subseteq W \) every \( y \) in \( \text{St}(G_i, \mathfrak{g}) \) has an open neighborhood \( U_y \) such that \( U_y \subseteq W \) and \( \text{diam}(U_y) \leq \frac{1}{2^n} \). By compactness of \( \text{St}(G_i, \mathfrak{g}) \), we can find \( y_1, \ldots, y_k \in \text{St}(G_i, \mathfrak{g}) \) such that \( \text{St}(G_i, \mathfrak{g}) \subseteq \bigcup_{y=1}^{k} U_{y_k} \). Let \( W_i = \bigcup_{y=1}^{k} U_{y_k} \). Recall that \( \text{diam}(G_j) \leq \frac{1}{2^n} \) for every \( j = 1, \ldots, m-1 \); thus.
diam(St(G_i, G_j)) = diam(St(G_i, G_j)) ≤ \frac{1}{2}ε and diam(W_i) ≤ ε. Moreover W_i ⊆ W, and \( f(Ω^k \cap A) ⊆ St(G_i, G_j) \subseteq W_i \), (i.e., (1)–(3) hold for \( i = 1, \ldots, m - 1 \)). The only problem is that \( \{Ω^k_1, \ldots, Ω^k_{m-1}\} \) is not in general a partition of \( Ω \); in fact its elements may be not pairwise disjoint and they may not cover \( Ω \).

For every \( i = 1, \ldots, m - 1 \) let \( Ω_i = Ω^k_i \setminus (Ω^k_1 \cup \cdots \cup Ω^k_{i-1}) \). Then the \( Ω_i \)'s are clopen and pairwise disjoint and \( f(Ω_i \cap A) ⊆ W_i \) (so that (3) still holds). Let also \( Ω_m = Ω \setminus (Ω_1 \cup \cdots \cup Ω_{m-1}) \), so that \( \{Ω_1, \ldots, Ω_m\} \) is a clopen partition of \( Ω \). It is straightforward to check that \( Ω_m \cap A = \emptyset \); thus any open nonempty \( W_m \) satisfying (1) and (2) will complete the construction. \( \square \)

We do not know whether the metrizability hypothesis can be removed from (ii) of Theorem 7.5. In contrast, the following result shows how much the compactness hypothesis is essential.

**Proposition 7.7.** Let \( Y \) be a \( T_2 \) topological space. If \( Y \) is not compact there exists a normal and hereditarily extremally disconnected space \( X \) such that \((X,Y)\) is not strongly good.

**Proof.** Let \( V \) be an open covering of \( Y \) which has no finite subcover. Let \( F = \{Y \setminus \bigcup V' \mid V' \subseteq V \text{ is finite}\} \). \( F \) generates a filter on \( Y \) which is proper because \( V \) has no finite subcover. Hence there exists an ultrafilter \( U \supseteq F \). \( U \) is nonprincipal because \( V \) is a covering.

Let \( z \notin Y \) and \( X = Y \cup \{z\} \). Endow \( X \) with the topology which makes \( Y \) discrete and such that the neighborhoods of \( z \) are of the form \( M \cup \{z\} \) with \( M \in U \). \( X \) is paracompact and hereditarily extremally disconnected (use Lemma 7.3).

Let \( A = Y \subseteq X \) and \( f : A \to Y \) be the identity function: notice that \( f \) is continuous (on \( A \) the topology is finer than the original topology of \( Y \)). We claim that \( f \) cannot be extended to a continuous \( F : X \to Y \), thus completing the proof.

To prove the claim let \( F : X \to Y \) be an extension of \( f \) and let \( y = F(z) \). Since \( y \in Y \) there exists \( V \in \mathcal{V} \) such that \( y \in V \). Let \( W = Y \setminus V \in \mathcal{F} \subseteq U \). Let \( M \cup \{z\} \) with \( M \in U \) be an arbitrary neighborhood of \( z \) in \( X \). Since \( M \cap W \in \mathcal{U} \) we have \( M \cap W \neq \emptyset \), and let \( a \in M \cap W \); then \( F(a) = f(a) = a \in W \) and hence \( F(a) \notin V \). Therefore for every neighborhood \( M \cup \{z\} \) of \( z \) we have \( F(M \cup \{z\}) \nsubseteq V \) and \( F \) is not continuous at \( z \). \( \square \)

The reader may wonder whether there are interesting examples of spaces which are both normal and hereditarily extremally disconnected, e.g., spaces of this kind which have no isolated points. Using Lemma 7.3 it is fairly easy to see that any maximal space (which obviously has no isolated points) is hereditarily extremally disconnected and El’kin [7] constructed maximal spaces of any infinite cardinality which are hereditarily collectionwise normal.

Using a different kind of construction it is possible to show the existence of a (normal and) hereditarily extremally disconnected space which is hereditarily paracompact (but not maximal). Actually, this space turns out to fulfill also a separation property which strengthens normality in a way different from paracompactness.
**Definition 7.8.** A topological space $X$ is **structurally normal** if it is $T_1$ and every closed $C \subseteq X$ has a basis $\mathcal{V}_C$ of open neighborhoods such that for all closed $C_1, C_2 \subseteq X$ and for all $V_1 \in \mathcal{V}_{C_1}$ and $V_2 \in \mathcal{V}_{C_2}$ if $C_1 \cap V_2 = C_2 \cap V_1 = \emptyset$ then $V_1 \cap V_2 = \emptyset$.

It is straightforward to show that a $T_1$ space $X$ is structurally normal if and only if every $x \in X$ has a basis $\mathcal{V}_x$ of open neighborhoods such that for all $x_1, x_2 \in X$ and for all $V_1 \in \mathcal{V}_{x_1}$ and $V_2 \in \mathcal{V}_{x_2}$ if $x_1 \notin V_2$ and $x_2 \notin V_1$ then $V_1 \cap V_2 = \emptyset$.

It is also immediate that structural normality is a hereditary property and that every structurally normal space is (hereditarily) collectionwise normal and (hereditarily) strongly zero-dimensional. In general structural normality does not imply paracompactness: e.g., the first uncountable ordinal with the order topology is easily seen to be structurally normal.

However if the structurally normal space $X$ is also equipped with a tree structure which respects structural normality then $X$ is paracompact. Indeed suppose $< \! < \!$ is a partial ordering on $X$ which is a tree (i.e., for every $x \in X$ the set $\{y \in X \mid y < x\}$ is well-ordered by $< \!$) and for every $x \in X$ let $ht(x)$ be the order type of $\{y \in X \mid y < x\}$. Suppose moreover that for every $x \in X$ and $V \in \mathcal{V}_x$ we have $ht(y) > ht(x)$ for every $y \in V \setminus \{x\}$. Then it can be shown that $X$ is paracompact. If for every $x \in X$ and $V \in \mathcal{V}_x$ we have $x < y$ for every $y \in V \setminus \{x\}$ then $X$ is hereditarily paracompact.

We proceed now to sketch the construction of the promised space, without carrying out the proofs in all details. Let $X$ be the set of all finite sequences of natural numbers, fix a nonprincipal ultrafilter $U$ on $\mathbb{N}$ and declare $U \subseteq X$ to be open if and only if $\forall s \in U \{n \mid s \not\subseteq \{n\} \in U\} \in U$. This is indeed a topology on $X$, it has no isolated points and is both hereditarily normal and (hereditarily) extremally disconnected, yet it is not maximal (the proof of extremal disconnectedness uses an inductive argument). Moreover $X$ can be shown to be structurally normal in a way which is compatible with the obvious tree structure of $X$ (i.e., the one given by the relation of being an initial segment), so that by the considerations made above $X$ is hereditarily paracompact.

Notice that it is hard to get a similar example with “nicer” properties: e.g., an infinite compact space cannot be hereditarily extremally disconnected (see [9, Exercise 6R4]).

**References**