h-embeddings

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Abstract. Given a topological space \( X \), a subspace \( Y \) of \( X \) is said to be \( C \)-embedded in \( X \) if every continuous real valued function on \( Y \) has a continuous extension to all of \( X \). The notion of \( C \)-embedding is well understood and documented, as well as that of the more general notions of \( C^* \)-embedding and \( z \)-embedding. In this article we define a generalization of \( C \)-embedding and investigate some of its interesting properties. In Sections we show prove some fundamental properties of \( h \)-embeddings, e.g. every closed subset of Tychonoff space is \( h \)-embedded. In Section we characterize those spaces for which every open subset is \( h \)-embedded.

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1. Introduction

This article aims to generalize the concept of an \( h \)-point which is discussed in [Mc]. To begin with \( C(X) \) denotes the ring of all real-valued continuous functions defined on the topological space \( X \). We assume that all spaces are Tychonoff, that is, completely regular and Hausdorff. For a given point \( p \in X \), we denote the set \( X \setminus \{p\} \) by \( X_p \). If \( p \) is not isolated then there are some natural questions regarding the natural ring embedding \( \pi_{X_p} : C(X) \to C(X_p) \) given by restriction. For instance, when \( \pi_{X_p} \) is surjective then we say that \( X_p \) is a \( C \)-embedded subspace of \( X \), i.e., every continuous function on \( X_p \) may be extended to a continuous function defined on all of \( X \). It is also common to say that \( p \) is a \( C \)-point of \( X \). A \( C^* \)-point is similarly defined. In [Mc] an intermediate ring is discussed and similar questions are asked.

The set \( M_p \) denotes the fixed maximal ideal of \( C(X) \) consisting of those continuous functions which vanish at \( p \).

\[
\text{Hom}(M_p) = \{ f \in C(X_p) : \forall g \in M_p, \exists h \in C(X), h(x) = f(x)g(x) \forall x \in X_p \}
\]

In other words, \( \text{Hom}(M_p) \) is the set of those continuous functions on \( X_p \) whose product with each member of \( M_p \) may be continuously extended to all of \( X \). Clearly, \( C(X) \leq \text{Hom}(M_p) \leq C(X_p) \). The point \( p \in X \) is called an \( h \)-point if \( \text{Hom}(M_p) = C(X_p) \). In [Mc] several results are given regarding this interesting class of spaces. As previously mentioned it is the aim of this article to generalize this concept and prove some results in the context of Tychonoff spaces.

2. \( h \)-embeddings

Definition 2.1. Let \( Y \) be a subspace of \( X \). We let \( \phi_Y \) be the ring homomorphism

\[
\phi_Y : C(X) \longrightarrow C(Y)
\]

defined by restriction. If \( Y \) is clear from the context we shall omit the subscript. It is known that the space \( Y \) is a \( C \)-embedded subspace of \( X \) precisely when \( \phi \) is a surjective map. If we resign ourselves to only consider bounded functions then we are dealing with
the notion of $C^*$-embedded. A space $X$ is called a $C$-space ($C^*$-space) if $X_p$ is $C$-embedded ($C^*$-embedded) in $X$ for every $p \in X$.

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Let $X$ of $g$ and $f$ each $y$ mentioned in the first section when

Let write Lemma 2.3.

A simpler example is given by the following. (Tychonoff) space produces a closed $h$ shall soon show. Thus, as a consequence of Urysohn’s Extension Theorem, any non-normal $C$-embedded subspace which is not $C^*$-spaces are $h$-spaces. The reverse is easily seen to be false and urge the reader to check [Mc] for examples. In fact, all closed sets are $h$-embdedded as we shall soon show. Thus, as a consequence of Urysohn’s Extension Theorem, any non-normal (Tychonoff) space produces a closed $h$-embedded subspace which is not $C^*$-embedded. A simpler example is given by the following.

**Definition 2.2.** Let $Y \subseteq X$. We say $Y$ is $h$-embedded in $X$ if $H(Y|X) = H(Y)$. As mentioned in the first section when $Y = X_p$ then we say $p$ is an $h$-point of $X$. In case every point of $X$ is an $h$-point then $X$ is called $h$-space. Clearly, $C$-embedded subsets are $h$-embedded and so $C$-spaces are $h$-spaces. The reverse is easily seen to be false and urge the reader to check [Mc] for examples. In fact, all closed sets are $h$-embdedded as we shall soon show. Thus, as a consequence of Urysohn’s Extension Theorem, any non-normal (Tychonoff) space produces a closed $h$-embedded subspace which is not $C^*$-embedded. A simpler example is given by the following.

**Lemma 2.3.** Suppose $Y \subseteq X$ is a co-dense. Then $Y$ is $h$-embedded.

**Proof.** If $Y$ is a co-dense subset of $X$, then $M_{X \setminus Y} = 0$.

**Example 2.4.** The set of rationals $\mathbb{Q}$ is an $h$-embedded subset of $\mathbb{R}$ which is not $C^*$-embedded.

The following theorem was proved in [FGL] for open subsets.

**Theorem 2.5.** Let $Y \subseteq X$. Then $C^*(Y) \leq H(Y|X)$.

**Proof.** Let $f \in C^*(Y)$ and $g \in C(X)$ for which $g$ vanishes on the complement of $Y$. Define $fg$ on $X$ to be the usual product on $Y$ and 0 elsewhere. We now show that $fg \in C(X)$. Let $x \in X$ and let $M \in \mathbb{R}$ such that $|f| \leq M$. If $x \in cl_X(X \setminus Y)$ then $g(x) = 0$ and hence $fg(x) = 0$. Let $\epsilon > 0$. Choosing an open set $O$ of $X$ for which $|g(y)| \leq \frac{\epsilon}{M}$ we have that for each $y \in O$, $|fg(y)| < \epsilon$ and so the product is continuously defined at $x$.

Otherwise, suppose $x \notin cl_X(X \setminus Y)$. Then this means that $x \in int_X Y$ and so it follows that since $f$ is continuous on a neighbourhood of $x$ then so is the product $fg$.

Observe that what we may gather from the previous proof is that for any $f \in C(Y)$ and $g \in M_{X \setminus Y}$ the product is always continuously extendable to all points in the interior of $X \setminus Y$. This easily gives us:

**Proposition 2.6.** Let $Y$ be a closed subset of $X$. Then $Y$ is $h$-embedded in $X$.

**Proposition 2.7.** Let $O$ be a cozeroset of $X$. Then $O$ is an $h$-embedded subset of $X$ if and only if it is clopen.
Proof. Since clopen sets are $C$-embedded they are also $h$-embedded. To see the reverse let $O = \text{coz}(g)$ for some $g \in C(X)$. Set $f = \frac{1}{g^2} \in C(O)$. It is straightforward to show that $fg \notin C(X)$ unless $O$ is clopen. □

Corollary 2.8. Suppose $p \in X$ is a nonisolated $G_\delta$-point. Then $p$ is not an $h$-point and hence not a $C$-point.

Example 2.9. Let $\Sigma = \mathbb{N} \cup \{\sigma\}$, for some $\sigma \in \beta\mathbb{N} - \mathbb{N}$, and $\Sigma$ is equipped with the subspace topology inherited from $\beta\mathbb{N}$. $\Sigma$ is an example of a $C^*$-space that is not an $h$-space nor a $C$-space. Totally-ordered $P$-spaces are examples of $h$-spaces that are not $C$-spaces (and hence not $C^*$-spaces.)

Proposition 2.10. Let $Y \subseteq X$. If $\phi_Y(C(X)) = H(Y|X)$ holds then $Y$ is a $C^*$-embedded subset of $X$. In this case $Y$ is $C$-embedded in $X$ if and only it is $h$-embedded.

Proof. This is clear because of the following string of inequalities

$$C^*(Y) \leq H(Y|X) = \phi_Y(C(X)) \leq C(Y).$$

□

In [Mc] it is shown that when $Y = X_p$, then the converse of Proposition 2.10 is true. In general, this is not the case. Any non pseudocompact space $X$ which is $h$-embedded in $\beta X$ witnesses this. (Examples will be provided in a later section.) We do have a positive result which generalizes the co-finite case. First another definition.

Definition 2.11. In [BH] the authors say a subspace $Y$ of $X$ is said to satisfy property ($\gamma$) if it is completely separated from every zeroset (of $X$) disjoint to it. Later in [A] this property is given the name well-embedding, and is referred to Moran ([M]). There is an algebraic characterization of well-embeddings given in [BH] stating that a subspace $Y$ of $X$ is well-embedded if and only if every continuous function on $Y$ that extends continuously to a cozeroset of $X$ containing $Y$ may be further extended continuously to $X$. Also, observe that a cozero set is well-embedded if and only if it is clopen. This leads us to believe there are some similarities between well-embeddings and $h$-embeddings. In general these notions do not coincide as we shall eventually show.

Now, Aull (Theorem 1, [A]) proved that every dense subspace of $X$ is well-embedded if and only if $X$ is an almost $P$-space. It is apparent that if every dense subspace is $h$-embedded, then $X$ is necessarily an almost $P$-space. Shortly, we will characterize when every dense subspace is $h$-embedded. This class of spaces will be strictly smaller than the class of all almost $P$-spaces. Therefore, there are (dense) subspaces which are well-embedded that are not $h$-embedded. For example, the one-point compactification of an uncountable discrete space, $\alpha D$, is known to be an almost $P$-space. In [Mc] it is shown that $\alpha D$ is not an $h$-space and hence $D$ is not $h$-embedded in $\alpha D$. Finally, since not all closed subspaces are well-embedded it follows that there are subspaces which are $h$-embedded but not well-embedded. Though these ideas are different we do obtain the following nice result.

Theorem 2.12. Suppose $Y \subseteq X$ is a dense $C^*$-embedded subspace. The following are equivalent:

(i) $Y$ is $C$-embedded in $X$.

(ii) For every subspace $Y \subseteq S \subseteq X$, $S$ is $C$-embedded in $X$.

(iii) For every subspace $Y \subseteq S \subseteq X$, $S$ is $h$-embedded in $X$.

(iv) There are no zerosets of $X$ disjoint from $Y$.

(v) The only cozero set of $X$ containing $Y$ is $X$.

**Proof.** If $Y$ is a dense $C$-embedded subset of $X$, then so is every subspace between $Y$ and $X$ and hence they are all $h$-embedded. Conversely, if every subspace between $Y$ and $X$ is $h$-embedded in $X$ then it follows that no subspace between $Y$ and $X$ is a cozero set of $X$ and hence there is no zeroset of $X$ disjoint from $Y$. This means that $Y$ is vacuously well-embedded in $X$. According to Theorem [BH] it follows that $Y$ is $C$-embedded in $X$. □

**Corollary 2.13.** Let $p$ be a nonisolated point of $X$ and suppose that $X_p$ is $C^*$-embedded in $X$. Then $X_p$ is $C$-embedded in $X$ if and only if $X_p$ is $h$-embedded in $X$.

Density is required for Theorem 2.12. Observe that $[0, 1]$ is $C$-embedded in $\mathbb{R}$ but according to Proposition 2.7, $(-1, 2)$ is not an $h$-embedded subset of $\mathbb{R}$.

For the following theorem recall that a space $X$ is said to be a $P$-space if the topology is closed under countable intersections. Equivalently, for a Tychonoff space, $X$ is a $P$-space if and only if every cozero set is clopen.

**Theorem 2.14.** The following are equivalent:

(a) Every open set of $X$ is $h$-embedded.

(b) Every dense open subspace of $X$ is $h$-embedded.

(c) Every cozero set is $h$-embedded.

(d) $X$ is a $P$-space.

(e) Every subspace of $X$ is $h$-embedded.

(f) Every dense subspace is $h$-embedded.

(g) Every subspace is well-embedded.

(h) Every cozeroset is well-embedded.

**Proof.** The implications (a) $\rightarrow$ (b) and (e) $\rightarrow$ (a) are obvious. (b) $\rightarrow$ (a) follows from Proposition 2.7 and Proposition 3.1, which we shall prove shortly. That (a) implies (d) follows by applying Proposition 2.7. Similarly, (c) and (d) are equivalent. That (d), (g), and (h) are all equivalent can be found in [A].

(d) $\rightarrow$ (e): If $X$ is a $P$-space and $Y \subseteq X$ is any subspace. Then for any $f \in C(Y)$ and $g \in M_{X \setminus Y}$ define $fg(x) = 0$ for all $x \in X \setminus Y$. Since for each $x \notin Y$, we have $x \in Z(g) \subseteq Z(fg)$ and $Z(g)$ is clopen it follows this is a continuous extension of $fg$.

Finally, (e) implies (f) and (f) implies (b). □
Remark 2.15. Observe that by (b) of the previous theorem, in a $P$-space every regular open set is $h$-embedded. (A set is called regular open if it is the interior of a closed set.) The converse is not true. In an extremally disconnected space $X$ every regular open set is clopen and hence $h$-embedded, yet most examples of extremally disconnected spaces are not $P$-spaces.

3. Transitive Properties

In the section we explore the different kinds of ”transitive” embedding relations found in the literature.

Proposition 3.1. Suppose $S \subseteq T \subseteq X$. If $S$ is $h$-embedded in $T$ and $T$ is $h$-embedded in $X$, then $S$ is $h$-embedded in $X$. In other words, being $h$-embedded is transitive.

Proof. Let $f \in C(S)$. Since every element of $M_{X \setminus S}$ may be decomposed into the difference of positive elements we assume without loss of generality that $g \in M_{X \setminus S}$ and $g \geq 0$. Let $h = \sqrt{g}$ and $\overline{h}$ be the restriction of $h$ to $T$. Since $g$, and hence $h$ vanishes on $X \setminus S$, it follows that $\overline{h}$ vanishes on $T \setminus S$. By the hypothesis that $S$ is $h$-embedded in $T$ it follows that $f\overline{h} \in C(T)$. Since $T$ is $h$-embedded in $X$ and $h$ vanishes on $X \setminus T$ we then obtain that $fg = (f\overline{h})h \in C(X)$. □

Definition 3.2. Let $\mathcal{P}$ be a property of embeddings and let $S \subseteq T \subseteq X$. The property $\mathcal{P}$ is said to be metatransitive if whenever $S$ is $\mathcal{P}$-embedded in $X$ and $T$, and $S$ is a dense subspace of $X$, then $T$ is $\mathcal{P}$-embedded in $X$.

There is a stronger condition called paratransitive. One need only assume that $S$ is a dense subspace of $T$ in the above condition. It is known that $C$-embeddings and $C^*$-embeddings are paratransitive. To prove this fact the density of $S$ in $T$ is required.

Example 3.3. We produce an example of sets $S \subseteq T \subseteq X$ where $S$ is a dense $h$-embedded subspace of $X$, but $T$ is not $h$-embedded in $X$. Let $S = \mathbb{Q}$, $T = \mathbb{R} \setminus \{\sqrt{2}\}$, and $X = \mathbb{R}$. Since $S$ is co-dense in $T$ and $X$ it follows that $M_{T \setminus S} = 0 = M_{X \setminus S}$ and hence $H(S|X) = C(S)$ and $H(S|T) = C(S)$. Now, as $T$ is a (non clopen) cozeroset of $X$ it follows that $T$ is not $h$-embedded in $X$. Thus, $h$-embedding is not metatransitive.

The same proof would work for any dense, co-dense subspace contained in a proper cozeroset.

Definition 3.4. Let $S \subseteq T \subseteq X$. Observe that if $S$ is $C$-embedded ($C^*$-embedded) in $X$, then $S$ is $C^*$-embedded in $T$. An embedding with this property is called hereditary.

Example 3.5. Let $S = \mathbb{Q}$, $T = S \cup \{\sqrt{2}\}$, and $X = \mathbb{R}$. As in the previous example, $S$ is co-dense in $X$ and so $S \subseteq X$ is an $h$-embedding. Alas, since $S$ is a cozeroset of $T$ it follows that $S$ is not $h$-embedded in $T$. From this it follows that $h$-embeddings are not hereditary.

Proposition 3.6. Let $S \subseteq T \subseteq X$ and suppose that $S$ is $h$-embedded in $X$ and that $T$ is a closed subspace of $X$. Then $S$ is $h$-embedded in $T$. Therefore, $S$ is $h$-embedded in $X$ if and only if $S$ is $h$-embedded in its closure in $X$.  

4. $X$ as an $h$-embedded subspace of $\beta X$.

As a consequence of Theorem 2.14 and the fact that compact $P$-spaces are finite it follows that for any infinite space $X$ there is a dense open subspace of $\beta X$ which is not $h$-embedded. In this section we ask a different type of question regarding $\beta X$. As the title of the section suggests we are concerned with answering when $X$ is an $h$-embedded subspace of $\beta X$. As it plays an important role we remind the reader that a space is locally compact if every point has a compact neighbourhood. The class of non-compact locally compact spaces are precisely those that have a one-point compatification. Another characterization is that a space is locally compact precisely when it is an open subset of its Stone-Čech compactification. Now, if $X$ is locally compact then it is easy to determine when $X$ is $h$-embedded in $\beta X$. The answer is that $X$ must be pseudocompact. If $X$ is not pseudocompact, then there is a countable subset of $X$, say $\{x_i\}$, on which we can define a continuous $f : X \to \mathbb{R}$ where $f(x_n) = n^2$. There would also exist a $g \in C^*(X) = C(\beta X)$ whose restriction to $\beta X \setminus X$ equals the constant 0 and so that $g(x_n) = \frac{1}{n}$. As we have seen previously, the function $fg$ cannot be extended to all of $X$. Now, if $X$ happens to be nowhere compact (that is $X$ does not have any nonempty compact neighbourhoods) then it is known that $X$ is co-dense in $\beta X$ and therefore it is also $h$-embedded. We now proceed to characterize those spaces for which $X$ is $h$-embedded in $\beta X$.

**Definition 4.1.** The corona (or remainder) of $X$ is simply $\beta X \setminus X$. We shall denote this set by $X^\ast$. It then follows that we may write $M_{X^\ast}$ instead of $M_{\beta X \setminus X}$. Furthermore, throughout this section we shall let $L_X = X \setminus (cl_{\beta X} X^\ast)$.

**Lemma 4.2.** For any space $X$, $L_X = (\beta X \setminus cl_{\beta X} X^\ast)$ and is therefore an open subspace of $X$. For each $x \in L_X$ there is a compact subset of $K$ of $X$ such that $x \in \overline{\beta X \setminus K}$. Furthermore, $L_X$ is locally compact. $X \setminus L_X$ is a closed nowhere compact subset.

**Proof.** The first statement is clear. As to the second, if $x \in L_X = X \setminus cl_{\beta X} X^\ast$, then since $\beta X$ is compact we may choose an $f \in C^*(X) = C(\beta X)$ such that $0 \leq f \leq f(x) = 1$ and $f(y) = 0$ for all $y \in X^\ast$. Observe that $K = f^{-1}(\frac{2}{3}, 1]$ is a closed neighbourhood of $x$ in $\beta X$ and hence is a compact neighbourhood of $x$. Since $K$ is disjoint from the zero set of $f$ which contains the corona of $X$ it follows that $K$ is a compact neighbourhood of $x$ in $X$.

To see that $L_X$ is locally compact notice that for a given $x \in L_X$ let $K$ be as in the above proof. We claim that $K \subseteq L_X$. But this is clear as $K \cap cl_{\beta X} X^\ast = \emptyset$. Therefore, each $x \in L_X$ lies in a compact neighbourhood in $L_X$. The rest of the proof is patent.□

**Theorem 4.3.** $X$ is $h$-embedded in $\beta X$ if and only if $L_X$ is a bounded subset of $X$.

**Proof.** Sufficiency: suppose $L_X$ is a bounded subset of $X$. Let $f \in C(X)$ and $g \in C(\beta X)$ such that $g(y) = 0$ for all $x \in X^\ast$. By design cos$(g) \subseteq L_X$. Also, there is a positive real number $M$ for which $|f(x)| < M$ for each $x \in L_X$. Define $h : \beta X : \mathbb{R}$ by

$$
h(x) = \begin{cases} f(x)g(x), & x \in X \\ 0, & \text{otherwise.} \end{cases}$$
Since $L_X \subseteq X$ is open subset of $\beta X$ it follows that $h$ is continuous on all of $L_X$. Also, if $x \notin L_X$, then $x$ belongs to $cl_{\beta X}X^*$ and therefore $h(x) = 0$. Let $\epsilon > 0$. Find an open neighbourhood $O$ of $x$ such that $|g(y)| < \frac{\epsilon}{M}$ for all $y \in O$. Then for every $y \in O$ we have $h(y) = 0$ or otherwise $h(x) = f(x)g(x) < M \cdot \frac{\epsilon}{M}$. Therefore, $h$ is a continuous extension of $fg$ to all of $\beta X$, whence $X$ is $h$-embedded in $\beta X$.

Necessity: suppose $L_X$ is not a bounded subset of $X$. Let $h \in C(X)$ witness this. It follows that there is $C$-embedded subset, say $J = \{x_n\}$, of $X$ with $J \subseteq L_X$. Since being $C^*$-embedded is a transitive property we obtain that $J \cup cl_{\beta X}X^*$ are $C^*$-embedded subsets of $\beta X$. Choose $g \in C(\beta X)$ such that $g(x_n) = \frac{1}{n}$ and $g(y) = 0$ for each $y \in cl_{\beta X}X^*$. Next, select $f \in C(X)$ for which $f(x_n) = n^2$. It follows that $fg$ does not have any continuous extension to all of $\beta X$. Thus, $X$ is not $h$-embedded in $\beta X$.

**Corollary 4.4.** Suppose $X$ is locally compact. Then the following are equivalent:

(i) $X$ is pseudocompact.

(ii) $X$ is $h$-embedded in $\beta X$.

(iii) $X$ is $C$-embedded in $\beta X$.

**Proof.** If $X$ is locally compact then $X = L_X$. Therefore, by Theorem 4.3 we have that $X$ is $h$-embedded in $\beta X$ if and only if $X$ is pseudocompact. That (ii) and (iii) are equivalent is well-known. □

**Corollary 4.5.** Suppose $X$ is nowhere compact. Then $L_X = \emptyset$ and hence is vacuously bounded in $X$. Therefore, $X$ is $h$-embedded in $\beta X$. □

We conclude this section by determing which spaces are absolutely $h$-embedded.

**Definition 4.6.** We call a space $X$ absolutely $h$-embedded if it is $h$-embedded in every larger Tychonoff space containing it. Observe that by previous comments this is equivalent to saying it is $h$-embedded in any Tychonoff space containing it as a dense subspace.

**Theorem 4.7.** For a Tychonoff space $X$ the following are equivalent:

(a) $X$ is absolutely $h$-embedded.

(b) $X$ is $h$-embedded in every larger space containing it as a dense subspace.

(c) $X$ is pseudocompact.

**Proof.** Our proof is simply to show that $\upsilon X = \beta X$. If this is not the case then let $p \in \beta X \setminus \upsilon X$ and set $T = X \cup \{p\}$. It is known that $p$ is a $G_\delta$-point of $T$ and therefore it follows that $X = X_p$ is not $h$-embedded in $T$. □
REFERENCES


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