

March 2002

K. Magaard¹, T. Shaska, S. Shpectorov², and H. Völklein³Wayne State University, University of California at Irvine,
Bowling Green State University, and University of Florida

Abstract: Let G be a finite group, and $g \geq 2$. We study the locus of genus g curves that admit a G -action of given type, and inclusions between such loci. We use this to study the locus of genus g curves with prescribed automorphism group G . We completely classify these loci for $g = 3$, and for $g \leq 10$ we classify those loci corresponding to large G .

1 Introduction

There is a vast literature on automorphism groups of compact Riemann surfaces, beginning in the 19th century with Schwartz, Klein, Hurwitz, Wiman and others. However, most of the literature is quite recent. In the first part of the paper, we survey the main results.

By the covering space theory, a finite group G acts faithfully on a genus g curve if and only if it has a genus g generating system (see section 5). Using this purely group-theoretic condition, Breuer [Br] classified all groups that act on curves of genus ≤ 48 . This was a major computational effort using the computer algebra system GAP [GAP]. It greatly improved on several papers dealing with small genus, by various authors (see the references in part I).

Of course, for each group on Breuer's list, all subgroups are also on the list. This raises the question how to select those groups that occur as the *full automorphism group* of a genus g curve. This question is answered in Part II of the paper. Let \mathcal{M}_g be the moduli space of genus g curves. We study the locus \mathcal{L} in \mathcal{M}_g of curves admitting a G -action of given ramification type (resp., signature). All components of \mathcal{L} have the same dimension which depends only on the signature of the G -action. Restricting the action to a subgroup H of G yields an inclusion of \mathcal{L} into the corresponding locus \mathcal{L}' for the action of H . If $\dim \mathcal{L} < \dim \mathcal{L}'$ then for "most" curves in \mathcal{L}' , the H -action does not extend to a G -action of the type defining \mathcal{L} . Thus, one is led to classify the pairs $(\mathcal{L}, \mathcal{L}')$ with $\dim \mathcal{L} = \dim \mathcal{L}'$. This is done in Lemma 4.1 below. It turns out that such pairs exist only in very restricted cases, in particular only if $\dim \mathcal{L} \leq 3$. From that we derive a necessary and sufficient condition for a group to occur as the full automorphism group of a genus g -curve (Theorem 5.3).

After Part II was written, we found references to papers of Singerman [Si2] and Ries [Ri] which contain similar results. Their method is different, of analytic nature, using Teichmüller theory and Fuchsian groups. Our approach is based on algebraic geometry, using the algebraic structure of Hurwitz spaces and the moduli spaces \mathcal{M}_g . Therefore, it can be used to obtain information on fields of definition. This aspect may be studied in later work.

In Part III we apply the above criterion to the data compiled by Breuer (available from his website). Our main application is in the case of genus 3, which is already quite rich and shows the power of our group-theoretic method. We obtain the full list of automorphism groups plus equations for the corresponding curves. This has been done in [KuKo1] by other methods, but that paper contains several errors which we correct here.

The locus \mathcal{L} of curves in \mathcal{M}_g with a fixed automorphism group G consists of finitely many components. In Part III we also present a complete list of components of the loci \mathcal{L} for the so-called large G up to genus 10. The list also shows the embeddings between the components, thus giving a rather detailed description of the loci \mathcal{L} . Going beyond the large groups requires computational tools that will be developed in later work (mapping class group action on generating systems). In the special case of curves of orbit genus 0 (which is always the case for large G), the mapping class group action is just the braid group action studied in [FV], [V].

¹Partially supported by the NSA

²Partially supported by the NSA grant MDA904-01-1-0023

³Partially supported by the NSF grant DMS-9970357

Then the components of \mathcal{L} correspond to the braid orbits on the generating systems in G , and the latter can be computed with our GAP package BRAID [MSV].

Notation: We will use the term "curve" to mean "compact Riemann surface", and $X = X_g$ will denote a curve of genus $g \geq 2$. Further, $\text{Aut}(X)$ denotes the group of analytic (equivalently, algebraic) automorphisms of X .

Part I

Survey of known results

1.1 The beginnings in the 19th century

The group $\text{Aut}(X)$ acts on the finite set of Weierstrass points of X . This action is faithful unless X is hyperelliptic, in which case its kernel is the group of order 2 containing the hyperelliptic involution of X . Thus, in any case, $\text{Aut}(X)$ is a finite group. This was first proved by Schwartz.

F. Klein [Kl] (1879) studied the genus 3 projective curve

$$x^3y + y^3z + z^3x = 0 \tag{1}$$

which is now known as Klein's quartic. He found all its automorphisms. They form a simple group of order 168. Further, he showed that there is a degree 168 covering from this curve to \mathbb{P}^1 branched over three points with ramification indices 2, 3 and 7. See [Le] for a detailed history of Klein's quartic (including an English translation of Klein's original article) as well as an account of further developments influenced by it.

Poincaré introduced Fuchsian groups in 1882. They are basic for the analytic theory of Riemann surfaces and have been used heavily in the study of $\text{Aut}(X)$ (see [Br], [Ri] for example). In this paper we take a different (algebraic) point of view.

The next milestone was Hurwitz's seminal paper [Hu] published in 1893, where he discovered what is now called the Riemann-Hurwitz formula. From this he derived that

$$|\text{Aut}(X_g)| \leq 84(g - 1),$$

now known as the Hurwitz bound. This bound is attained, for example, by Klein's quartic. Curves which attain this bound are called *Hurwitz curves* and their automorphism groups *Hurwitz groups*. Hurwitz proved that a finite group is a Hurwitz group if and only if it has generators a , b and c of orders 2, 3 and 7, respectively, with $abc = 1$.

1.2 Hurwitz groups

Klein's quartic is the only Hurwitz curve of genus $g \leq 3$. Fricke showed that the next Hurwitz group occurs for $g = 7$ and has order 504. It is isomorphic to $SL(2, 8)$. An equation for the corresponding Hurwitz curve was computed by Macbeath [Mb] in 1965. Klein's quartic and Macbeath's curve are the only Hurwitz curves whose equations are known. Further Hurwitz curves occur for $g = 13$ and $g = 17$ (and for no other values of $g \leq 19$).

There is a lot of papers by group-theorists on Hurwitz groups, surveyed by Conder [Co]. It follows from Hurwitz's presentation that a Hurwitz group is *perfect*. Every its quotient is again a Hurwitz group, and if such a quotient is minimal then it is a non-abelian simple group. Several infinite series of simple Hurwitz groups have been found by Conder, Malle, Kuribayashi, Zalessky, Zimmermann and others. In 2001, Wilson [Wl] showed that the sporadic Monster simple group is a Hurwitz group.

1.3 Large automorphism groups.

For a fixed $g \geq 2$ denote by $N(g)$ the maximum of $|\text{Aut}(X_g)|$. Accola [Ac1] and Maclachlan [Mc1] independently show that $N(g) \geq 8(g + 1)$ and that this bound is sharp for infinitely many g 's. If g is divisible by 3 then $N(g) \geq 8(g + 3)$.

The following terminology is standard. We say $G \leq \text{Aut}(X_g)$ is a *large automorphism group* in genus g if

$$|G| > 4(g-1).$$

In this case the quotient of X_g by G is a curve of genus 0, and the number of points of this quotient ramified in X_g is 3 or 4 (see [Br], [FK]). Singerman [Si3] (1974) shows that Riemann surfaces with large cyclic, Abelian, or Hurwitz groups are symmetric (admit an involution). Kulkarni [Ku] (1997) classifies Riemann surfaces admitting large cyclic automorphism groups and works out full automorphism groups of these surfaces. Matsuno [Mt] (1999) investigates the Galois covering of the projective line by compact Riemann surfaces with large automorphism groups.

1.4 Cyclic groups as automorphism groups.

Let t be the order of an automorphism of X_g . Hurwitz [Hu] showed that $t \leq 10(g-1)$. In 1895, Wiman improved this bound to $t \leq 2(2g+1)$ and showed this to be best possible. If the order of t is a prime then $t \leq 2g+1$. Homma [Ho] (1980) shows that this bound is achieved if and only if the curve is birationally equivalent to

$$y^{m-s}(y-1)^s = x^q, \text{ for } 1 \leq s < m \leq g_x + 1.$$

1.5 The canonical representation of $\text{Aut}(X)$

Each group $G \leq \text{Aut}(X_g)$ acts faithfully on the g -dimensional vector space of holomorphic differential forms on X . The resulting finite subgroups of $GL(g, \mathbb{C})$ satisfy a number of special conditions, studied by I. Kuribayashi, A. Kuribayashi, Kimura, Ohmori, Kobayashi, Hayakawa and others. Kuribayashi and others (see [KuKi] and the references in that paper) have completely listed such finite linear groups up to genus 5. Using computational group theory (more precisely, the computer algebra system GAP [GAP]), Breuer [Br] extended this up to genus 48. He computed all groups $G \leq \text{Aut}(X_g)$ together with their character on the space of holomorphic differential forms. He avoids *ad-hoc* methods by using the GAP library of small groups by H. Besche and B. Eick [BE] available from the website <http://www-public.tu-bs.de:8080/~beick/so.html>. Breuer's own results are collected in a database (a GAP library). It can be downloaded from his website located at <http://www.math.rwth-aachen.de/~Thomas.Breuer/>

1.6 Full automorphism groups.

For each group on Breuer's list, all subgroups are also on the list. This raises the question how to select the groups that occur as the *full automorphism group* of a genus g curve. Answering this question requires a moduli space argument: either analytically, using Fuchsian groups and Teichmüller space, as done by Singerman [Si2] and Ries [Ri], or algebraically, as done in Part II of this paper. Using these general results, one can compute explicitly all groups $\text{Aut}(X_g)$ for small g . We give the complete list for $g = 3$ in Part III, and the list of large groups $\text{Aut}(X_g)$ up to $g = 10$. Further results will be given in later work, together with additional information.

For $g > 3$, there are no lists of full automorphism groups in the literature. The case $g = 2$ is easy, but already for $g = 3$ there is no satisfactory account. Results by A. Kuribayashi and various co-authors (see Part III for details) for $g = 3$ are scattered over several long papers, not available in most libraries. They use *ad-hoc* methods.

Part II

Subgroups of $\text{Aut}(X_g)$ not occurring as full automorphism groups in genus g

2 Ramification type and signature of a G -curve

Fix an integer $g \geq 2$ and a finite group G . Let C_1, \dots, C_r be conjugacy classes $\neq \{1\}$ of G . Let $\mathbf{C} = (C_1, \dots, C_r)$, viewed as unordered tuple, repetitions are allowed. We allow r to be zero, in which case \mathbf{C} is empty. Let c_i be the order of the elements in C_i .

Consider pairs (X, μ) , where X is a curve and $\mu : G \rightarrow \text{Aut}(X)$ is an injective homomorphism. Mostly, we will suppress μ and just say X is a curve with G -action, or a G -curve, for short. Two G -curves X and X' are called equivalent if there is a G -equivariant isomorphism $X \rightarrow X'$.

We say a G -curve X is of *ramification type* (g, G, \mathbf{C}) if the following holds: First of all, g is the genus of X . Second, the points of the quotient X/G that are ramified in the cover $X \rightarrow X/G$ can be labelled as p_1, \dots, p_r so that C_i is the conjugacy class in G of the distinguished inertia group generator over p_i (for $i = 1, \dots, r$). (Distinguished inertia group generator is the generator that acts in the tangent space as multiplication by $\exp(2\pi\sqrt{-1}/\epsilon)$, where ϵ is the ramification index). For short, we will just say X is of type (g, G, \mathbf{C}) .

If X is a G -curve of type (g, G, \mathbf{C}) then the genus g_0 of X/G is given by the Riemann-Hurwitz formula

$$(1) \quad \frac{2(g-1)}{|G|} = 2(g_0-1) + \sum_{i=1}^r \left(1 - \frac{1}{c_i}\right).$$

We call g_0 the *orbit genus* associated with the action of G on X . Note that g_0 depends only on the *signature* (g, G, \mathbf{c}) of the G -curve X , where $\mathbf{c} = (c_1, \dots, c_r)$. (This definition of signature is a slight modification of the usual one; Breuer [Br] calls this a signature-group pair).

3 Hurwitz spaces and moduli of curves

Define $\mathcal{H} = \mathcal{H}(g, G, \mathbf{C})$ to be the set of equivalence classes of G -curves of type (g, G, \mathbf{C}) . By the covering space theory (or the theory of Fuchsian groups), \mathcal{H} is non-empty if and only if G can be generated by elements $\alpha_1, \beta_1, \dots, \alpha_{g_0}, \beta_{g_0}, \gamma_1, \dots, \gamma_r$ with $\gamma_i \in C_i$ and

$$(2) \quad \prod_j [\alpha_j, \beta_j] \prod_i \gamma_i = 1.$$

Here $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$.

Let \mathcal{M}_g be the moduli space of genus g curves, and $\mathcal{M}_{g_0, r}$ the moduli space of genus g_0 curves with r distinct marked points, where we view the marked points as unordered (contrary to the usual procedure). Consider the map

$$\Phi : \mathcal{H} \rightarrow \mathcal{M}_g$$

forgetting the G -action, and the map

$$\Psi : \mathcal{H} \rightarrow \mathcal{M}_{g_0, r}$$

mapping the class of a G -curve X to the class of the quotient curve X/G together with the (unordered) set of branch points p_1, \dots, p_r . If $\mathcal{H} \neq \emptyset$ then Ψ is surjective and has finite fibers, by the covering space theory. Also Φ has finite fibers, since the automorphism group of a curve of genus ≥ 2 is finite.

By [Be], the set \mathcal{H} carries the structure of quasi-projective variety (over \mathbb{C}) such that the maps Φ and Ψ are finite morphisms. If $\mathcal{H} \neq \emptyset$ then all components of \mathcal{H} map surjectively onto $\mathcal{M}_{g_0, r}$ (through a finite map), hence they all have the same dimension

$$\delta(g, G, \mathbf{C}) := \dim \mathcal{M}_{g_0, r} = 3g_0 - 3 + r.$$

(This dimension is always positive because $r \geq 1$ if $g_0 = 1$ and $r \geq 3$ if $g_0 = 0$; the latter holds because $g \geq 2$). Let $\mathcal{M}(g, G, \mathbf{C})$ denote the image of Φ , *i.e.*, the locus of genus g curves admitting a G -action of type (g, G, \mathbf{C}) . Since Φ is also a finite map, we get

Lemma 3.1 *If $\mathcal{M}(g, G, \mathbf{C})$ is non-empty then each of its components has dimension $\delta(g, G, \mathbf{C})$.*

Note that $\delta(g, G, \mathbf{C})$ depends only on the signature, so we write $\delta(g, G, \mathbf{c}) := \delta(g, G, \mathbf{C})$. The lemma continues to hold if we replace \mathbf{C} by \mathbf{c} .

4 Restriction to a subgroup

Let H be a subgroup of G . Then each G -curve can be viewed as an H -curve by restriction of action. Let X be a G -curve of type (g, G, \mathbf{C}) . Then the resulting H -curve is of type (g, H, \mathbf{D}) , where \mathbf{D} is obtained as follows. Choose $\gamma_i \in C_i$ and let $\sigma_{i1}, \sigma_{i2}, \dots$ be a set of representatives for the double cosets $\langle \gamma_i \rangle \sigma H$ in G . Let m_{ij} be

the smallest integer ≥ 1 such that the element $\sigma_{ij}^{-1} \gamma_i^{m_{ij}} \sigma_{ij}$ lies in H , and let D_{ij} be the conjugacy class of this element in H . Then \mathbf{D} is the tuple consisting of all D_{ij} with $D_{ij} \neq \{1\}$. (More precisely, the tuple \mathbf{D} is indexed by the set of possible pairs (i, j) , and its (i, j) -entry is D_{ij}). The definition of \mathbf{D} does not depend on the choice of the γ_i and σ_{ij} . Note that the signature of the H -curve depends on the type of the G -curve and not only on its signature.

We have

$$\mathcal{M}(g, G, \mathbf{C}) \subset \mathcal{M}(g, H, \mathbf{D}).$$

Hence their dimensions satisfy

$$\delta(g, G, \mathbf{C}) \leq \delta(g, H, \mathbf{D}).$$

If this is a strict inequality then the complement of the closure of $\mathcal{M}(g, G, \mathbf{C})$ in $\mathcal{M}(g, H, \mathbf{D})$ is open and dense; then, in particular, it is not true that every H -curve of type (g, H, \mathbf{D}) is the restriction of a G -curve of type (g, G, \mathbf{C}) .

Lemma 4.1 *Suppose H is a subgroup of G of index $n > 1$. We let \bar{G} denote the permutation group induced by G on the coset space G/H . Let X be a G -curve of type (g, G, \mathbf{C}) , where we label the C_i so that $c_1 \leq \dots \leq c_r$. Let $\mathbf{c} = (c_1, \dots, c_r)$. The H -curve obtained by restriction of action is of type (g, H, \mathbf{D}) , where \mathbf{D} is defined above; let (g, H, \mathbf{d}) be its signature. Let s (resp., r) be the length of \mathbf{D} (resp., \mathbf{C}). Let h_0 (resp., g_0) be the genus of X/H (resp., X/G). If $\delta := \delta(g, G, \mathbf{C}) = \delta(g, H, \mathbf{D})$ then $g_0 = 0$, and one of the following holds:*

- (I) $\delta = 3$ and, furthermore, $n = 2$, $h_0 = 2$, $s = 0$, $r = 6$, all $c_i = 2$ and $C_i \not\subset H$;
- (II) $\delta = 2$ and, furthermore, $n = 2$, $h_0 = 1$, $s = 2$, $r = 5$, $C_i \not\subset H$, $c_i = 2$, for $i = 1, \dots, 4$, and $\mathbf{d} = (c_5, c_5)$;
- (III) $\delta = 1$ and one of the following holds:
 - (a) $n = 2$, $h_0 = 1$, $s = 1$, $r = 4$, $C_i \not\subset H$, for all i , $c_1 = c_2 = c_3 = 2$ and $\mathbf{d} = (c_4/2)$;
 - (b) $n = 2$, $h_0 = 0$, $s = r = 4$, $c_1 = c_2 = 2$, $C_3, C_4 \subset H$ and $\mathbf{d} = (c_3, c_3, c_4, c_4)$;
 - (c) $h_0 = 0$, $s = r = 4$, $c_1 = c_2 = c_3 = 2$, $C_4 \subset H$, H is normal in G with G/H isomorphic to the Klein 4-group, and $\mathbf{d} = (c_4, c_4, c_4, c_4)$;
- (IV) $\delta = 0$ and, furthermore, $h_0 = 0$, $s = r = 3$ and one of the following holds:
 - (a) $n = 2$, $h_0 = 0$, $s = r = 3$, $c_1 = 2$, $c_i > 2$ is even and $C_j \subset H$, where $\{i, j\} = \{2, 3\}$, and $\mathbf{d} = (c_i/2, c_j, c_j)$;
 - (b) $n = 3$, $\bar{G} \cong \mathbb{Z}/3$, one $C_i \subset H$ and the other two have $c_j = 3$; here $\mathbf{d} = (c_i, c_i, c_i)$;
 - (c) $n = 3$, $\bar{G} \cong S_3$, $c_1 = 2$, $c_2 = 3$, $c_3 > 2$ is even, $\mathbf{d} = (2, c_3/2, c_3)$;
 - (d) $n = 4$, $\bar{G} \cong A_4$, $c_1 = 2$, $c_2 = 3$, $c_3 > 3$ is divisible by 3, $\mathbf{d} = (3, c_3/3, c_3)$;
 - (e) $n = 6$, $\bar{G} \cong PGL_2(5)$, $\mathbf{c} = (2, 4, 5)$, $\mathbf{d} = (4, 4, 5)$, $g \equiv 1 \pmod{3}$;
 - (f) $n = 8$, $\bar{G} \cong PSL_2(7)$, $\mathbf{c} = (2, 3, 7)$, $\mathbf{d} = (3, 3, 7)$, $g \equiv 1 \pmod{2}$;
 - (g) $n = 9$, $\bar{G} \cong PSL_2(8)$, $\mathbf{c} = (2, 3, 7)$, $\mathbf{d} = (2, 7, 7)$, $g \equiv 1 \pmod{6}$;
 - (h) $n = 10$, $\bar{G} \cong PGL_2(9)$, $\mathbf{c} = (2, 3, 8)$, $\mathbf{d} = (3, 8, 8)$, $g \equiv 1 \pmod{15}$;
 - (i) There are 6 further cases when H is not maximal in G , and $n = 4, 6, 6, 12, 12, 24$, respectively.

The cases in (IV) are taken from [Ma1], Table 2 (c.f. 5b below).

Proof : We denote the hypothesis $\delta(g, G, \mathbf{C}) = \delta(g, H, \mathbf{D})$ by (Hyp). Let r_1 be the number of points of X/G that are ramified in X/H . Applying Riemann-Hurwitz to the covering $X/H \rightarrow X/G$ of degree n yields

$$(3) \quad 2(h_0 - 1) \geq 2n(g_0 - 1) + r_1.$$

Let s be the number of points of X/H that are ramified in X , and $r_2 := r - r_1$. The r_2 points of X/G that are ramified in X , but not in X/H , have a total of nr_2 distinct preimages in X/H . The latter points are ramified in X , hence $s \geq nr_2$.

Case 1: $g_0 > 1$

(Hyp) gives

$$3g_0 + r_1 + r_2 = 3h_0 + s \geq 3h_0 + r_2,$$

hence

$$3g_0 + r_1 \geq 3h_0.$$

With (3) this gives

$$\begin{aligned} 3g_0 + 2(h_0 - 1) - 2n(g_0 - 1) &\geq 3h_0, \\ g_0(3 - 2n) + 2n &\geq h_0 + 2 \geq g_0 + 2, \\ g_0 &< \frac{n}{n-1} \leq 2, \end{aligned}$$

a contradiction.

Case 2: $g_0 = 1$

If $r = 0$ then by (1) we have $g = 1$, contrary to the assumption. Thus, $r > 0$. Now (3) gives $2(h_0 - 1) \geq r_1$. If $h_0 > 1$ then (Hyp) gives $r_1 \geq 3(h_0 - 1)$ (as in Case 1), a contradiction. If $h_0 = 1$ then $r_1 = 0$, hence (Hyp) gives $r_2 = s \geq nr_2$; contradiction, since $r_2 = r > 0$.

We have now proved $g_0 = 0$. Thus, if $h_0 > 0$ then $r_1 \geq 3$. Further, we can choose $\gamma_i \in C_i$ with $\gamma_1 \cdots \gamma_r = 1$ and $G = \langle \gamma_1, \dots, \gamma_r \rangle$ (by (2)). Then c_i is the order of γ_i . Let p_1, \dots, p_r (resp., q_1, \dots, q_s) be the points of X/G (resp., X/H) that are ramified in X . We choose the labelling such that p_i corresponds to C_i and

$$(4) \quad c_1 \leq \dots \leq c_r.$$

The orbits of γ_i on G/H correspond to the points of X/H over p_i (where the length of the orbit equals the ramification index of the corresponding point over p_i). Let $\text{Ind}(C_i)$ ($= \text{Ind}(\gamma_i)$) be n minus the number of those orbits. The orbits that correspond to points $\notin \{q_1, \dots, q_s\}$ have length $c_i \geq 2$. Thus, the total number of orbits of all γ_i is $\leq s + (nr - s)/2$. Hence Riemann-Hurwitz for $X/H \rightarrow X/G$ yields

$$(5) \quad 2(h_0 - 1 + n) = \sum_{i=1}^r \text{Ind}(C_i) \geq nr - \frac{nr + s}{2} = \frac{nr - s}{2}.$$

Case 3: $h_0 \geq 2$

Here (Hyp) gives $r = 3h_0 + s$. Plugging this into (5) gives

$$(6) \quad 2(h_0 - 1 + n) \geq \frac{n(3h_0 + s) - s}{2} = \frac{3}{2}h_0n + \frac{s(n-1)}{2} \geq \frac{3}{2}h_0n.$$

This yields $n \leq 2(h_0 - 1)/(\frac{3}{2}h_0 - 2) \leq 2$ since $h_0 \geq 2$. Plugging $n = 2$ into (6) gives $h_0 \leq 2$. Thus, $h_0 = 2$ and the inequalities in (6) are equalities, which implies $s = 0$. This is case (I) in the lemma.

Case 4: $h_0 = 1$

If $s = 0$ then $g = 1$, contrary to the assumption. Thus, $s > 0$, and so (Hyp) gives $r - 3 = s$. Thus, $r \geq 4$. Plugging $s = r - 3$ into (5) gives

$$(7) \quad 2n \geq \frac{r(n-1) + 3}{2},$$

which implies $r \leq 4 + \frac{1}{n-1}$. If $n = 2$ then we are in case (II) (resp. (III).a) of the lemma. Assume now $n > 2$. Then $r = 4$ and $s = 1$. If all $c_i = 2$ then (7) would have to hold with equality, which is not possible. Thus $c_4 \geq 3$ (by (4)).

Let p_{i_0} be the image in X/G of q_1 . If $i_0 < 4$ then Riemann-Hurwitz yields $2n \geq \frac{3n-1}{2} + \frac{2}{3}n$. Hence $n = 3$ and $c_1 = c_2 = c_3 = 2$, a contradiction, since n is divisible by those c_i with $i \neq i_0$. Thus, $i_0 = 4$. If $n = 3$ then $c_1 = c_2 = c_3 = 3$, contradicting Riemann-Hurwitz. Thus $n > 3$. Riemann-Hurwitz yields $2n \geq 3\frac{n}{2} + \frac{2}{3}(n-1)$, hence $n = 4$, $c_1 = c_2 = c_3 = 2$ and $c_4 = 3$. Thus γ_1, γ_2 and γ_3 act as double transpositions on the 4-set G/H , and γ_4 acts as a 3-cycle. However, three double transpositions in S_4 cannot have a 3-cycle as their product (since the double transpositions together with the identity form a normal subgroup), a contradiction.

Case 5: $h_0 = 0$

If $r \leq 2$ or $s \leq 2$ then $g = 0$, a contradiction. Thus, we have $r \geq 3$ and $s \geq 3$. Hence (Hyp) gives $r = s$. Plugging this into (5) gives $r \leq 4$.

Case 5.a: $r = s = 4$

Here (5) holds with equality, hence the q_j 's are unramified over X/G , and all other points of X/H over some p_i have ramification index $2 = c_i$. If all $c_i = 2$ then (1) gives $g = 1$, a contradiction. Thus, some p_i , say p_{i_0} , is unramified in X/H . If two p_i 's are unramified in X/H then $s \geq 2n$, hence $n \leq s/2 = 2$; this is case III.b in the lemma.

Now suppose p_i is unramified in X/H only for $i = i_0$. Then $c_i = 2$ for $i \neq i_0$, and n is even (since not every p_i , $i \neq i_0$, can have a q_j over it). Also, $n \leq s = 4$ and $n \neq 2$ (because $X/H \rightarrow X/G$ has 3 branch points). Hence $n = 4$ and the γ_i with $i \neq i_0$ act as double transpositions on the 4-set G/H (and γ_{i_0} acts trivially). This is case III.c.

Case 5.b: $r = s = 3$

This is the last case to be considered. Now X/H and X/G have genus zero, and the cover $X/H \rightarrow X/G$ (resp., $X \rightarrow X/H$) is ramified at 3 points of X/G (resp., of X/H). This gives the following condition on the action of γ_1, γ_2 and γ_3 on the n -set G/H : Apart from exactly three orbits of γ_1, γ_2 and γ_3 , all orbit lengths of γ_i equal c_i ; and if one of the three exceptional orbits belongs to γ_i then its length is $< c_i$. From (1) we also get $\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} < 1$. Finally, we have the genus zero condition $2(n-1) = \sum_{i=1}^3 \text{Ind}(\gamma_i)$. Triples of permutations with these properties (and with product 1, and generating a transitive group) have been classified by Malle [Ma1]. His Table 2 gives the cycle types of the three permutations plus the group they generate (plus the c_i). This is part (IV) of the Lemma. (Actually, Malle's situation is slightly more general than ours, since he doesn't require c_i to be the order of γ_i ; thus two of his cases, where $n = 10$ and $n = 18$, do not occur here).

5 The full automorphism group

Now we change perspective and fix the finite group H . An H -curve X is called *generic* if the following holds: The curve X/H , together with its points q_1, \dots, q_s that are ramified in X , defines a generic point of $\mathcal{M}_{h,s}$. Here h is the genus of X/H , and 'generic' means generic over $\overline{\mathbb{Q}}$ (the algebraic closure of \mathbb{Q}).

Let $\text{Aut}(X)$ denote the automorphism group of the curve X (without regard of H -action).

Theorem 5.1 *Suppose X is a generic H -curve of genus ≥ 2 . Then $[\text{Aut}(X) : H] \leq 24$. Let h be the genus of X/H , and s the number of points of X/H ramified in X . Then $\text{Aut}(X) = H$ unless (h, s) equals $(2, 0)$, $(1, 2)$, $(1, 1)$, $(0, 4)$ or $(0, 3)$. If $h > 0$ we have $[\text{Aut}(X) : H] \leq 2$, and in the case $(0, 4)$ we have H normal in $\text{Aut}(X)$ and $[\text{Aut}(X) : H] \leq 4$.*

Proof : Let $G = \text{Aut}(X)$. Let X be of type (g, H, \mathbf{D}) when viewed as an H -curve, and of type (g, G, \mathbf{C}) when viewed as a G -curve.

Note that all maps and spaces in section 3 are defined over $\overline{\mathbb{Q}}$. Hence the condition that X is a generic H -curve means that its corresponding point in \mathcal{M}_g is a generic point of a component of the locus $\mathcal{M}(g, H, \mathbf{D})$. This point lies in the sublocus $\mathcal{M}(g, G, \mathbf{C})$, hence the dimensions are the same: $\delta(g, G, \mathbf{C}) = \delta(g, H, \mathbf{D})$. Now the claim follows from Lemma 4.1.

Lemma 5.2 (i) *Let Σ be a group with a presentation on generators $\sigma_1, \dots, \sigma_6$ and relations*

$$\sigma_1^2 = \dots = \sigma_6^2 = 1 = \sigma_1 \cdots \sigma_6$$

Let Π be the kernel of the map $\Sigma \rightarrow \{\pm 1\}$, $\sigma_i \mapsto -1$. Then Π is generated by

$$\begin{aligned} \alpha_1 &= \sigma_2 \sigma_1 & \alpha_2 &= \sigma_5 \sigma_4 \\ \beta_1 &= \sigma_2 \sigma_3 & \beta_2 &= \sigma_5 \sigma_6, \end{aligned}$$

which gives a presentation for Π with the single relation

$$[\alpha_1, \beta_1][\alpha_2, \beta_2] = 1.$$

We have $\Sigma = \Pi \rtimes \langle \sigma_2 \rangle$ where σ_2 acts on Π via

$$(9) \quad \begin{aligned} \alpha_1 &\mapsto \alpha_1^{-1} & \alpha_2 &\mapsto (\alpha_2^{-1})^{\alpha_2 \beta_2 \alpha_1^{-1} \beta_1^{-1}} \\ \beta_1 &\mapsto \beta_1^{-1} & \beta_2 &\mapsto (\beta_2^{-1})^{\alpha_2 \beta_2 \alpha_1^{-1} \beta_1^{-1}}. \end{aligned}$$

(ii) *Let $\Sigma = \langle \sigma_1, \dots, \sigma_5 \rangle$ subject to the relations*

$$\sigma_1^2 = \dots = \sigma_4^2 = 1 = \sigma_1 \cdots \sigma_5.$$

Let Π be the kernel of the map $\Sigma \rightarrow \{\pm 1\}$, $\sigma_i \mapsto -1$ for $i = 1, \dots, 4$ and $\sigma_5 \mapsto 1$. Then Π is generated by

$$\begin{aligned} \alpha &= \sigma_2 \sigma_1, & \gamma_1 &= \sigma_5^{\sigma_4} \\ \beta &= \sigma_2 \sigma_3, & \gamma_2 &= \sigma_5. \end{aligned}$$

which gives a presentation for Π with the single relation

$$[\alpha, \beta] \gamma_1 \gamma_2 = 1.$$

Now σ_2 acts on Π via

$$(10) \quad \begin{aligned} \alpha &\mapsto \alpha^{-1} & \gamma_1 &\mapsto \gamma_2^{\alpha^{-1} \beta^{-1}} \\ \beta &\mapsto \beta^{-1} & \gamma_2 &\mapsto \gamma_1^{\beta^{-1} \alpha^{-1}}. \end{aligned}$$

(iii) Let $\Sigma = \langle \sigma_1, \dots, \sigma_4 \rangle$ subject to the relations

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 = \sigma_1 \cdots \sigma_4.$$

Let Π be the kernel of the map $\Sigma \rightarrow \{\pm 1\}$, $\sigma_i \mapsto -1$ (resp., the normal subgroup of Σ generated by the conjugates of σ_4). Then Π is generated by

$$\begin{aligned} \alpha &= \sigma_2 \sigma_1 & \gamma &= \sigma_4^2 & \text{resp.}, & \gamma_1 &= \sigma_4 & \gamma_3 &= \sigma_4^{\sigma_4^2} \\ \beta &= \sigma_2 \sigma_3 & & & & \gamma_2 &= \sigma_4^{\sigma_4}, & \gamma_4 &= \sigma_4^{\sigma_4^3}, \end{aligned}$$

which gives a presentation for Π with the single relation

$$[\alpha, \beta] \gamma = 1 \quad \text{resp.}, \quad \gamma_1 \cdots \gamma_4 = 1.$$

Now σ_2 acts on Π by inverting α and β (resp., σ_2 switches γ_1, γ_3 and switches γ_2, γ_4 , and σ_1 switches γ_1, γ_2 and maps $\gamma_3 \mapsto \gamma_4^{\gamma_1}, \gamma_4 \mapsto \gamma_3^{\gamma_2^{-1}}$, and σ_3 switches γ_1, γ_4 and maps $\gamma_2 \mapsto \gamma_3^{\gamma_4}, \gamma_3 \mapsto \gamma_2^{\gamma_1^{-1}}$).

Proof : (i) All straightforward computation. One computes that the relation among the α_j, β_j implies the relations among the σ_i , hence the former yields a presentation for Π since $\Sigma = \Pi \rtimes \langle \sigma_2 \rangle$. The proof of (ii) and (iii) is similar. \blacksquare

We say that $\alpha_1, \beta_1, \dots, \alpha_h, \beta_h, \gamma_1, \dots, \gamma_s$ form a *genus g generating system* of H if these elements generate H and satisfy the basic relation (2), where h is given by Riemann-Hurwitz

$$(8) \quad \frac{2(g-1)}{|G|} = 2(h-1) + \sum_{i=1}^s \left(1 - \frac{1}{\text{ord}(\gamma_i)}\right).$$

Call a genus g curve X *exceptional* if $X/\text{Aut}(X)$ has genus 0 and exactly three of its points are ramified in X .

Theorem 5.3 Fix $g \geq 2$. A finite group H is isomorphic to the full automorphism group of a non-exceptional genus g curve if and only if it has a genus g generating system $\alpha_1, \beta_1, \dots, \alpha_h, \beta_h, \gamma_1, \dots, \gamma_s$ with $(h, s) \neq (0, 3)$ satisfying:

- (a) If $(h, s) = (2, 0)$ (resp., $(1, 2)$) then there is no automorphism of H acting on the generators via (9) (resp., (10)).
- (b) If $(h, s) = (1, 1)$ (resp., $(0, 4)$) then there is no automorphism of H inverting α_1 and β_1 (resp., acting on the generators like σ_1, σ_2 or σ_3 in (iii) of the lemma).

For the proof we need the following set-up. Let Y be a curve of genus h , and let q_1, \dots, q_s be distinct points of Y such that the pair (h, s) is as in (a) or (b). Then there is a Galois cover $Y \rightarrow \mathbb{P}^1$ such that the group $A = \text{Aut}(Y/\mathbb{P}^1)$ permutes q_1, \dots, q_s ; further, $|A| = 2$ unless $(h, s) = (0, 4)$ in which case A is a Klein four-group whose non-trivial elements act as double transpositions on $\{q_1, \dots, q_4\}$. If (Y, q_1, \dots, q_s) corresponds to a generic point of $\mathcal{M}_{h,s}$ then A is the full stabilizer of $\{q_1, \dots, q_s\}$ in $\text{Aut}(Y)$.

Let $\{p_1, \dots, p_r\}$ be the subset of \mathbb{P}^1 consisting of the points ramified in Y plus the images of q_1, \dots, q_s . Let t be the identity function on \mathbb{P}^1 , so that $\mathbb{C}(t)$ is the function field of \mathbb{P}^1 . View $\mathbb{C}(t)$ as a subfield of $\mathbb{C}(Y)$ naturally. Let \mathcal{M} be a maximal (algebraic) extension of $\mathbb{C}(t)$ unramified outside the places $t = p_1, \dots, p_r$ and

containing $\mathbb{C}(Y)$. Its Galois group $\tilde{\Sigma}$ is a profinite group generated by elements $\tilde{\sigma}_1, \dots, \tilde{\sigma}_r$ subject to the single relation $\tilde{\sigma}_1 \cdots \tilde{\sigma}_r = 1$; here $\tilde{\sigma}_i$ can be chosen to be a distinguished inertia group generator over the place $t = p_i$. Let Ω be the smallest (closed) normal subgroup of $\tilde{\Sigma}$ containing those $\tilde{\sigma}_i^2$ for which there is no q_j over p_i . Then $\Sigma := \tilde{\Sigma}/\Omega$ is the Galois group of $\mathcal{M} := \tilde{\mathcal{M}}^\Omega$ over $\mathbb{C}(t)$, and $\mathbb{C}(t) \subset \mathbb{C}(Y) \subset \mathcal{M}$. Let $\Pi = G(\mathcal{M}/\mathbb{C}(Y))$, and let $\sigma_1, \dots, \sigma_r$ be the images of the $\tilde{\sigma}_i$ in Σ . Then Σ (resp., Π) is the profinite completion of the group denoted by the same symbol in the lemma; the cases $(h, s) = (2, 0), (1, 2), (1, 1)$ and $(0, 4)$ correspond to cases (i), (ii) and the two cases in (iii), respectively.

Proof : (of Theorem) First we prove that the condition is necessary. So assume $H = \text{Aut}(X)$ for a non-exceptional genus g curve X . Let h be the genus of $Y := X/H$, and q_1, \dots, q_s the points of Y ramified in X . It is well-known that then H has a genus g generating system with parameters h and s . So we may assume that (h, s) is as in (a) or (b). Then we are in the set-up discussed before this proof, with $\mathbb{C}(t) \subset \mathbb{C}(Y) \subset \mathbb{C}(X) \subset \mathcal{M}$. The generators of Π from the lemma yield generators of H via the natural surjection $\Pi = G(\mathcal{M}/\mathbb{C}(Y)) \rightarrow G(\mathbb{C}(X)/\mathbb{C}(Y)) = H$. It remains to show that these generators satisfy the condition in (a) and (b).

Let f be the automorphism of Π induced by conjugation action of σ_2 unless $(h, s) = (0, 4)$ when f could be induced by any of σ_1, σ_2 and σ_3 . If H has an automorphism \bar{f} making the following diagram commutative

$$\begin{array}{ccc} \Pi & \xrightarrow{f} & \Pi \\ \downarrow & & \downarrow \\ H & \xrightarrow{\bar{f}} & H \end{array}$$

then the kernel Γ of $\Pi \rightarrow H$ is f -invariant. Then the normalizer Δ of Γ in Σ is strictly bigger than Π ; thus Δ/Γ is strictly bigger than $\Pi/\Gamma \cong H$. But Δ/Γ acts faithfully on X (since $\Gamma = G(\mathcal{M}/\mathbb{C}(X))$). This contradicts the assumption $H = \text{Aut}(X)$. Thus there is no such commutative diagram, which implies (a) and (b).

Assume now H has a genus g generating system as in the Theorem. It is well-known that then there exists a generic H -curve X of genus g with parameters h and s . If (h, s) is not as in (a) or (b) then $H = \text{Aut}(X)$ by Theorem 5.1 and we are done. Now assume (h, s) is as in (a) or (b). Choose data (Y, q_1, \dots, q_s) as in the discussion before this proof, corresponding to a generic point of $\mathcal{M}_{h,s}$. Mapping the generators of Π from the Lemma to the given generators of H defines a surjection $\Pi \rightarrow H$. Let K be the fixed field (in \mathcal{M}) of the kernel of this surjection. Then $K = \mathbb{C}(X)$ for a generic H -curve X of genus g . Thus H is normal in $G := \text{Aut}(X)$ by Theorem 5.1. Hence G/H embeds into A , and so $\mathbb{C}(t) \subset K^G \subset \mathbb{C}(Y)$. Thus the map $\Pi \rightarrow H$ extends to a map from a group between Π and Σ onto G . Thus if $G \neq H$ then there is an element of G that acts on the given generators of H in the same way that σ_2 (resp., σ_1, σ_2 or σ_3 in case $(h, s) = (0, 4)$) acts on the generators of Π (see the lemma). However, this is excluded by the condition in (a) resp. (b). Hence $G = H$ and we are done. ■

Remark 5.4 *The proof shows more: Given a signature (g, H, \mathbf{e}) (resp., a ramification type (g, H, \mathbf{E})), the existence of a generating system of H as in the Theorem with $\text{ord}(\gamma_i) = e_i$ (resp., $\gamma_i \in E_i$) is equivalent to the existence of a non-exceptional genus g curve X with $\text{Aut}(X) \cong H$ such that the resulting H -curve is of given signature (resp., type).*

Part III

Classification of automorphism groups

Recall that a group G acts faithfully on a genus g curve if and only if it has a genus g generating system (see (8) above). For g up to 48, all such groups and the signatures of all their genus g generating systems have been listed by Breuer [Br]. More precisely, for each $g \leq 48$, he produced a list containing all *signature-group pairs* in genus g , *i.e.*, pairs consisting of a group G together with the signature of a genus g generating system of G .

If G acts on X_g then so does each subgroup of G . This shows that Breuer's lists have to be long, and contain some redundancy. Part II of our paper was written in order to eliminate those signature-group pairs that do not yield the full automorphism group of a curve. This can also be done by the results of [Si2] and [Ri] (as we learned later). It turns out that the larger g gets, the larger is the ratio of entries in Breuer's lists that do occur as full automorphism group in genus g . That can already be seen from the fact (see Lemma 4.1) that if a signature-group pair does not yield the full automorphism group of a curve, then its δ -invariant (dimension of corresponding locus in \mathcal{M}_g) is at most 3.

For small g , a relatively large portion of those groups do not occur as full automorphism group in genus g . Among those that do occur, we distinguish those that occur for a particularly simple class of curves: Call a group *homocyclic* if it is a direct product of isomorphic cyclic groups. A cyclic (resp., homocyclic) cover $X \rightarrow \mathbb{P}^1$ is defined to be a Galois cover with cyclic (resp., homocyclic) Galois group C . We call it an *invariant cyclic (resp., homocyclic) cover of \mathbb{P}^1* if C is normal in $G = \text{Aut}(X)$. Then $\bar{G} := G/C$ embeds as a finite subgroup of $PSL(2, \mathbb{C})$ and it is easy to write down an equation for X (e.g., in the cyclic case, X has an equation of the form $y^c = f(x)$, where $c = |C|$). To illustrate this, we work out the equations in the genus 3 case. The hyperelliptic case (cyclic with $|C| = 2$) is the simplest case and can be done as in the genus 2 case (see [ShV]); see also [Bra] for the cyclic case (in an arbitrary characteristic). Equations for genus 3 curves with given automorphism group were found in the two long papers [KuKo1], [KuKo2] by other methods, but there are several errors which we correct here.

5.1 Genus 3

For $g = 3$, there are 49 signature-group pairs. Applying Theorem 5.3 we obtain that 23 of them correspond to G -curves having G as full automorphism group. For each of these 23 signature-group pairs, we find that the corresponding curves form an irreducible locus in \mathcal{M}_3 , and we compute the equation of a general curve in that locus. There is only two groups (namely C_2 and V_4) that occur with more than one signature, and as soon as $|G| > 2$, then the orbit genus (genus of X_3/G) is always 0.

Each row in Tables 1, 2 and 3 corresponds to an irreducible family of curves X_3 with automorphism group G (given in the first column). We let K denote the function field of X_3 over the algebraically closed field k of characteristic 0 (and identify G with $\text{Aut}(K/k)$). We display an equation for K . For almost all values (Zariski-open set) of the parameters in these equations, the equation describes a genus 3 curve with G as its full automorphism group. For the exceptional values, the automorphism group may be larger, or the curve may degenerate. (We do not specify these exceptional values). Most of these equations agree with those found in [KuKo1], [KuKo2], but they missed two cases: the hyperelliptic case with $\bar{G} = D_{12}$, and the 3-dimensional hyperelliptic case with $\bar{G} = V_4$. Further, they display an equation for the hyperelliptic case with $\bar{G} = S_4$ which actually belongs to the case $\bar{G} = D_{12}$. Some of the loci they describe with more parameters than the dimension requires. The cases with $g_0 = 1$ and group of order 3 or 4 should be omitted in their list, because the full automorphism group of these curves is larger (as shown by our Theorem 5.3 above). They note this for two of these cases in the Errata at the end of their paper, but again missed one case.

Now we show how to obtain these equations quickly from our group-theoretic data.

Homocyclic non-hyperelliptic case:

$G = C_4^2 \rtimes S_3$: The fixed field of C_4^2 is of genus 0, call it $k(u)$. The extension $K/k(u)$ is ramified at 3 places of $k(u)$, say $u = 0, 1, \infty$, and the corresponding inertia groups are cyclic of order 4. The fixed fields of these 3 inertia groups extend $k(u)$ with two ramified places each, hence they are of genus 0; call them $k(x)$, $k(y)$ and $k(z)$. We may assume $x^4 = u - 1$, $y^4 = u$. Thus $K = k(x, y)$ with $y^4 = u = x^4 + 1$. The equation $y^4 = x(x^2 - 1)$ given by [KuKo1] is also correct. (Indeed, the four roots of $x^4 + 1$ have j -invariant 1728, same as ∞ plus the 3 roots of $x(x^2 - 1)$).

$G = (48, 33)$ (this identifies G as a member of the GAP library of small groups [BE]; in particular, the order of G is 48): Here G is a central extension of C_4 by quotient A_4 . Let $H \cong C_3$ be a Sylow 3-subgroup of G . The fixed field of H is a $k(x)$, and $K/k(x)$ is ramified at 5 places of $k(x)$. Let h_1, \dots, h_5 be the corresponding distinguished inertia group generators in H . Their product is 1, hence we may assume $h_1 = \dots = h_4 = h_5^{-1}$. Take the place with generator h_5 to be $x = \infty$, and the other 4 ramified places to be the roots of $f(x)$. Then $K = k(x, y)$ with $y^3 = f(x)$. Further, the C_4 acts on $k(x)$ permuting the 4 roots of $f(x)$. We can choose x such that C_4 acts on $k(x)$ by the maps $x \mapsto \zeta x$, $\zeta^4 = 1$. Then $f(x) = x^4 - b$. We can get each $b \neq 0$ by further normalization of x . This agrees with [KuKo1].

$G = (16, 13)$: Here G is a central extension of $H \cong C_4$ by a quotient V_4 . The fixed field of H is a $k(x)$, and $K/k(x)$ is ramified at 4 places p_1, \dots, p_4 of $k(x)$. Let h_1, \dots, h_4 be the corresponding inertia elements in H . The Klein four-group G/H permutes p_1, \dots, p_4 . If this action were not transitive, then p_1, \dots, p_4 could not be four general points in \mathbb{P}^1 , contradicting the fact that the covers under consideration form a 1-dimensional family. Thus G/H permutes p_1, \dots, p_4 transitively, hence also permutes h_1, \dots, h_4 transitively (by conjugation). Thus $h_1 = \dots = h_4$ since H is central in G . (Alternatively, this can be shown by the algorithm before Lemma 4.1). Hence $K = k(x, y)$ with f of degree 4. The generic curve of this type has automorphism group of order 16, because there is always a Klein four-group in $\text{Aut}(k(x))$ permuting the four roots of f .

$G = C_9, C_6$ or C_3 : Let H be the subgroup of G of order 3. The fixed field of H is a $k(x)$, and $K/k(x)$ is ramified at 5 places p_1, \dots, p_5 of $k(x)$. Let h_1, \dots, h_5 be the corresponding inertia elements. They have product 1, hence we may assume $h_1 = \dots = h_4 = h_5^{-1}$. The group G/H permutes p_1, \dots, p_5 , fixing p_5 . Take p_1, \dots, p_5 to be the places $x = 0, 1, s, t, \infty$, respectively. Then $K = k(x, y)$ with $y^3 = x(x-1)(x-s)(x-t)$. Let α be the automorphism of $k(x)$ induced by a generator of G/H . If it has order 3 then we may assume that $x = 0$ is the other fixed point of α (besides $x = \infty$); then $\alpha(x) = \zeta_3 x$ and the equation becomes $y^3 = x(x^3 - 1)$. Now assume α has order 2. We may assume it switches 0 and 1. Then $\alpha(t) = 1 - t$, again as claimed.

$G = S_4, D_8$ or V_4 : Let H be a normal Klein four-group in G . The fixed field of H is a $k(w)$, and $K/k(w)$ is ramified at 6 places p_1, \dots, p_6 of $k(w)$. Let h_1, \dots, h_6 be the corresponding inertia elements. If they do not comprise all involutions in H then there is $h \in H$ that occurs four times among the h_i ; then K^h extends $k(w)$ with two branch points, hence K^h has genus 0 and so K is hyperelliptic. We excluded that case. Thus we may assume $h_1 = h_2, h_3 = h_4, h_5 = h_6$ and $\{h_1, h_3, h_5\} = H \setminus \{1\}$.

Let u and v be independent transcendentals over k with $w = u/v$. The places of $k(w)$ correspond to homogeneous degree 1 polynomials in u and v , modulo scalar multiples. Let P, Q and R be homogeneous degree 2 polynomials in $k[u, v]$ such that the degree 1 factors of P (resp., Q , resp., R) correspond to p_1, p_2 (resp., p_3, p_4 , resp., p_5, p_6). The quotients $P/Q, R/Q, P/R$ are naturally elements of $k(w)$. Then $K^{h_3} = k(w, x)$ with $x^2 = P/Q$, and $K^{h_1} = k(w, y)$ with $y^2 = R/Q$, and $K^{h_2} = k(w, z)$ with $z^2 = P/R$. Thus $K = k(w, x, y)$. We claim $K = k(x, y)$. Assume false. Then $[K : k(x, y)] = 2$, since w satisfies the equation $P(w, 1) - x^2 Q(w, 1) = 0$ over $k(x, y)$. Let g_0 be the generator of $G(K/k(x, y))$. Then g_0 is not in H , since none of the h_i fixes both x and y . But each h_i maps x (resp., y) to $\pm x$ (resp., $\pm y$), hence H leaves $k(x, y)$ invariant and thus centralizes g_0 . Hence $\langle H, g_0 \rangle \cong C_2^3$, contradiction (since G contains no such subgroup). This proves that $K = k(x, y)$.

The six homogeneous polynomials $P^2, Q^2, R^2, PQ, PR, QR$ of degree 4 are linearly dependent in $k[u, v]$. This gives a non-trivial relation

$$aPR + bPQ + cQR + dP^2 + eQ^2 + fR^2 = 0.$$

If $f = 0$ then dividing by Q^2 gives an equation for y over $k(x)$ of y -degree ≤ 2 and so K is hyperelliptic or of genus 0, a contradiction. Thus f (and similarly d, e) are non-zero. Replacing P, Q, R by suitable scalar multiples we get a relation

$$P^2 + Q^2 + R^2 + aPR + bPQ + cQR = 0.$$

with $a, b, c \in k$. Dividing by Q^2 gives

$$x^4 + 1 + y^4 + ax^2y^2 + bx^2 + cy^2 = 0.$$

This settles the case $G = V_4$.

Suppose now $G = D_8$. Then G/H induces an involutory automorphism α of $k(w)$ switching (without loss) the sets $\{p_1, p_2\}$ and $\{p_5, p_6\}$. We may assume $\alpha(w) = 1/w$. Extend it to $k(u, v)$ by $\alpha(u) = v, \alpha(v) = u$. Then α fixes Q and switches P, R up to scalar multiples. Since $\alpha^2 = \text{id}$, we have $\alpha(Q) = \pm Q$. Applying α to the relation between P, Q, R yields that $\alpha(P) = \pm R$. Replacing R by $-R$ if necessary we may assume $\alpha(P) = R$ (resp., $\alpha(P) = -R$) if $\alpha(Q) = Q$ (resp., $\alpha(Q) = -Q$). Then $b = c$. Finally, suppose $G = S_4$. Then there is another automorphism like α , fixing R and switching P, Q up to scalar multiples. Replacing Q by $-Q$ if necessary yields $a = b = c$.

Hyperelliptic case: Here G has a central subgroup C of order 2 whose fixed field in K is a $k(x)$. There are 8 places of $k(x)$ that ramify in K , say $x = a_1, \dots, a_8$. Then $\bar{G} = G/C$ embeds into $PGL(2, k)$ as a subgroup permuting $a_1, \dots, a_8 \in \mathbb{P}^1$. Let σ_m (resp., τ) be the element of $PGL(2, k)$ mapping x to $\zeta_m x$ (resp., $1/x$), where ζ_m is a primitive m -th root of 1 in k . Then each subgroup of $PGL(2, k)$ isomorphic to C_m resp., D_{2m} is conjugate to $\langle \sigma_m \rangle$ resp., $\langle \sigma_m, \tau \rangle$. We identify C_m (resp., D_m) with $\langle \sigma_m \rangle$ (resp., $\langle \sigma_m, \tau \rangle$). In particular, $D_2 = V_4 = \langle \sigma_2, \tau \rangle$.

$\bar{G} = S_4$: Let $i \in k$ with $i^2 = -1$. We may assume \bar{G} contains D_2 and $\rho : x \mapsto (x+i)/(x-i)$. (ρ is of order 3, permuting the six fixed points of the involutions in V_4). The two fixed points of ρ are the roots of $X^2 - (i+1)X - i = 0$. The group $\langle V_4, \rho \rangle \cong A_4$ has exactly one orbit of length 8 on \mathbb{P}^1 , containing those fixed points of ρ . Hence the orbit consists of the roots of $(X^2 - (i+1)X - i)(X^2 + (i+1)X - i)(-iX^2 - (i+1)X + 1)(-iX^2 + (i+1)X + 1) = X^8 + 14X^4 + 1$.

$\bar{G} = D_{16}, D_{12}, C_7$: In the dihedral case, \bar{G} leaves exactly one set of length 8 in \mathbb{P}^1 invariant, consisting of the roots of $X^8 - 1$, resp., the roots of $X^6 - 1$ plus 0 and ∞ . In the third case, we may assume $a_1 = 1$ by a coordinate change; then a_1, \dots, a_8 comprise the roots of $X^7 - 1$ plus either 0 or ∞ . (The latter two choices

are conjugate under the map τ). The case $\bar{G} = D_{12}$ is missing in [KuKo1] and the corresponding equation $y^2 = x(x^6 - 1)$ is erroneously associated with the case $\bar{G} = S_4$.

$\bar{G} = D_8, S_3$: Here a_1, \dots, a_8 comprise a regular orbit of D_8 resp., 0 and ∞ plus a regular orbit. Thus in the first case, a_1, \dots, a_8 are the roots of $(X^4 - t)(X^4 - 1/t) = X^8 - (t + 1/t)X^4 + 1$. The other case is similar.

$\bar{G} = V_4$: Here a_1, \dots, a_8 either comprise two regular orbits of \bar{G} , or a regular orbit plus two of three orbits of length 2. The orbits of length 2 are all conjugate under S_4 , so we may take $\{0, \infty\}$ and $\{\pm 1\}$ as the two orbits of length 2. Now the claim follows as in the previous case.

$\bar{G} = C_2$: Here a_1, \dots, a_8 either comprise 4 regular orbits, or 0 and ∞ plus 3 regular orbits. We may assume $a_1 = 1$ by a coordinate change.

Remaining cases, collected in Table 1:

Here we take X_3 in its canonical embedding in the projective plane associated with the dual of the 3-dimensional (faithful) G -module Ω of holomorphic differential forms on X_3 . Let x, y, z be coordinates of this module Ω^* . Let $f(x, y, z) = 0$ be the equation of X_3 in these coordinates. Then f spans a 1-dimensional fixed space for the representation of G on the space of quartic homogeneous polynomials in x, y, z .

To identify the representation of G on Ω , we use that the space of fixed points of G on Ω has dimension equal to the genus g_0 of X_3/G (see [Br],...; this fixed point space can be identified with $\Omega(X_3/G)$).

$G = L_3(2)$: This is the well-known Hurwitz group of order 168. The $(2, 3, 7)$ triple is weakly-rigid in G , so there is exactly one associated curve found already by Klein.

$G = S_3, g_0 = 0$: The group S_3 has only one faithful 3-dimensional representation with trivial fixed space: It is the natural permutation representation tensored with the 1-dimensional sign-representation. Thus G acts in the projective plane associated with Ω^* just by permuting the homogeneous coordinates x, y, z (for suitable choice of such coordinates). Thus the quartic polynomial $f(x, y, z)$ defining X_3 is either symmetric in x, y, z , or has the property that it is invariant under a cyclic permutation of x, y, z , but changes sign under a transposition. The only possibility (up to scalar multiples) for the latter case is $f = x^3y + y^3z + z^3x - y^3x - z^3y - x^3z$, but this polynomial is not irreducible (product of linear factors). Hence f is symmetric, thus is of the form $f = a(x^4 + y^4 + z^4) + b(x^2y^2 + x^2z^2 + y^2z^2) + c(x^2yz + y^2xz + z^2xy) + d(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y)$. The centralizer of S_3 in $GL_3(\mathbb{C})$ consists of the matrices $tI + sJ$, where I (resp., J) is the identity matrix (resp., the all 1's matrix). By a coordinate change with such a matrix we can make $d = 0$, but not uniquely: Generically, there is always 4 triples (a, b, c) (up to scalar multiples) that give the same curve (up to isomorphism).

$G = C_2, g_0 = 1$: We may assume that the involution in G acts as $x \mapsto -x, y \mapsto y, z \mapsto z$. Thus f either contains only even powers of x (*i.e.*, f is fixed by G) or f contains only odd powers of x (*i.e.*, f is mapped to $-f$ by the involution in G). In the latter case, f is reducible (product of x and a genus 1 equation). Thus f contains only even powers of x , hence

$$f = a_0x^4 + x^2(a_1y^2 + a_2yz + a_3z^2) + a_4y^4 + a_5y^3z + a_6y^2z^2 + a_7yz^3 + a_8z^4.$$

By a linear change of the variables y, z we can make $a_1 = 1$ and $a_2 = 0$. (If $a_1 = a_2 = a_3 = 0$ then X_3 has an automorphism of order 4, namely multiplying x by ζ_4 , so we don't consider this case here). Further we can replace x and z by scalar multiples to get the normalizations from Table 1. (If either $a_0 = 0$ or $a_7 = a_8 = 0$ then the curve is hyperelliptic).

Table 1: $\text{Aut}(X_3)$ for X_3 not an invariant homocyclic cover of \mathbb{P}^1

G	signature	orbit genus	proj. equation	Group ID
$L_3(2)$	$(2, 3, 7)$	0	$x^3y + y^3z + z^3x = 0$	$(168, 42)$
S_3	$(2, 2, 2, 2, 3)$	0	$a(x^4 + y^4 + z^4) + b(x^2y^2 + x^2z^2 + y^2z^2) +$ $c(x^2yz + y^2xz + z^2xy) = 0$	$(6, 1)$
C_2	$(2, 2, 2, 2)$	1	$x^4 + x^2(y^2 + az^2) + by^4 + cy^3z + dy^2z^2 + eyz^3 + gz^4 = 0$ either $e = 1$ or $g = 1$	$(2, 1)$

Table 2: $\text{Aut}(X_3)$ for invariant homocyclic covers of \mathbb{P}^1

G	C	G/C	signature	equation	Group ID
V_4	V_4	$\{1\}$	(2^6)	$x^4 + y^4 + ax^2y^2 + bx^2 + cy^2 + 1 = 0$	$(4, 2)$
D_8	V_4	C_2	(2^5)	take $b = c$	$(8, 3)$
S_4	V_4	S_3	$(2, 2, 2, 3)$	take $a = b = c$	$(24, 12)$
$C_4^2 \rtimes S_3$	V_4	S_4	$(2, 3, 8)$	take $a = b = c = 0$ or $y^4 = x(x^2 - 1)$	$(96, 64)$
16	C_4	V_4	$(2, 2, 2, 4)$	$y^4 = x(x-1)(x-t)$	$(16, 13)$
48	C_4	A_4	$(2, 3, 12)$	$y^4 = x^3 - 1$	$(48, 33)$
C_3	C_3	$\{1\}$	(3^3)	$y^3 = x(x-1)(x-s)(x-t)$	$(3, 1)$
C_6	C_3	C_2	$(2, 3, 3, 6)$	take $s = 1 - t$	$(6, 2)$
C_9	C_3	C_3	$(3, 9, 9)$	$y^3 = x(x^3 - 1)$	$(9, 1)$

Table 3: $\text{Aut}(X_3)$ for hyperelliptic X_3

G	G/C	signature	dim. of locus	equation $y^2 =$	Group ID
48	S_4	$(2, 4, 6)$	0	$x^8 + 14x^2 + 1$	$(48, 48)$
32	D_{16}	$(2, 4, 8)$	0	$x^8 - 1$	$(32, 9)$
24	D_{12}	$(2, 4, 12)$	0	$x(x^6 - 1)$	$(24, 5)$
C_{14}	C_7	$(2, 7, 14)$	0	$x^7 - 1$	$(14, 2)$
16	D_8	$(2, 2, 2, 4)$	1	$x^8 + ax^4 + 1$	$(16, 11)$
D_{12}	S_3	$(2, 2, 2, 6)$	1	$x(x^6 + ax^3 + 1)$	$(12, 4)$
$C_2 \times C_4$	V_4	$(2, 2, 4, 4)$	1	$x(x^2 - 1)(x^4 + ax^2 + 1)$	$(8, 2)$
C_2^3	V_4	(2^5)	2	$(x^4 + ax^2 + 1)(x^4 + bx^2 + 1)$	$(8, 5)$
C_4	C_2	$(2, 2, 2, 4, 4)$	2	$x(x^2 - 1)(x^4 + ax^2 + b)$	$(4, 1)$
V_4	C_2	(2^6)	3	$(x^2 - 1)(x^6 + ax^4 + bx^2 + c)$	$(4, 2)$
C_2	$\{1\}$	(2^8)	5	$x(x-1)(x^5 + ax^4 + bx^3 + cx^2 + dx + e)$	$(2, 1)$

Table 4: Components of the Hurwitz loci $\mathcal{M}(g, G, \mathbf{C})$ for large G

#	dim	aut	sig	subs	#	dim	aut	sig	subs
Genus 4									
1	0	(120,34)	(2, 4, 5)		2	0	(72,42)	(2, 3, 12)	
3	0	(72,40)	(2, 4, 6)		4	0	(40,8)	(2, 4, 10)	
5	0	(36,12)	(2, 6, 6)		6	0	(32,19)	(2, 4, 16)	
7	0	(24,3)	(3, 4, 6)		8	0	(18,2)	(2, 9, 18)	
9	0	(15,1)	(3, 5, 15)		10	1	(36,10)	(2, 2, 2, 3)	3
11	1	(24,12)	(2, 2, 2, 4)	1, 2	12	1	(20,4)	(2, 2, 2, 5)	4
13	1	(18,3)	(2, 2, 3, 3)	2, 5	14	1	(16,7)	(2, 2, 2, 8)	6
Genus 5									
1	0	(192,181)	(2, 3, 8)		2	0	(160,234)	(2, 4, 5)	
3	0	(120,35)	(2, 3, 10)		4	0	(96,195)	(2, 4, 6)	
5	0	(64,32)	(2, 4, 8)		6	0	(48,14)	(2, 4, 12)	
7	0	(48,30)	(3, 4, 4)		8	0	(40,5)	(2, 4, 20)	
9	0	(30,2)	(2, 6, 15)		10	0	(22,2)	(2, 11, 22)	
11	1	(48,48)	(2, 2, 2, 3)	1, 4	12	1	(32,43)	(2, 2, 2, 4)	
13	1	(32,28)	(2, 2, 2, 4)	1	14	1	(32,27)	(2, 2, 2, 4)	2, 4, 5
15	1	(24,14)	(2, 2, 2, 6)	6	16	1	(24,8)	(2, 2, 2, 6)	4
17	1	(24,13)	(2, 2, 3, 3)	3, 7	18	1	(20,4)	(2, 2, 2, 10)	3, 8
Genus 6									
1	0	(150,5)	(2, 3, 10)		2	0	(120,34)	(2, 4, 6)	
3	0	(72,15)	(2, 4, 9)		4	0	(56,7)	(2, 4, 14)	
5	0	(48,6)	(2, 4, 24)		6	0	(48,29)	(2, 6, 8)	
7	0	(48,15)	(2, 6, 8)		8	0	(39,1)	(3, 3, 13)	
9	0	(30,1)	(2, 10, 15)		10	0	(26,2)	(2, 13, 26)	
11	0	(21,2)	(3, 7, 21)		12	1	(60,5)	(2, 2, 2, 3)	2
13	1	(28,3)	(2, 2, 2, 7)	4	14	1	(24,12)	(2, 2, 3, 4)	2
15	1	(24,8)	(2, 2, 3, 4)	3	16	1	(24,6)	(2, 2, 2, 12)	5
17	1	(24,6)	(2, 2, 3, 4)	7					
Genus 7									
1	0	(504,156)	(2, 3, 7)		2	0	(144,127)	(2, 3, 12)	
3	0	(64,41)	(2, 4, 16)		4	0	(64,38)	(2, 4, 16)	
5	0	(56,4)	(2, 4, 28)		6	0	(54,6)	(2, 6, 9)	
7	0	(54,6)	(2, 6, 9)		8	0	(54,3)	(2, 6, 9)	
9	0	(48,32)	(3, 4, 6)		10	0	(42,4)	(2, 6, 21)	
11	0	(32,11)	(4, 4, 8)		12	0	(32,10)	(4, 4, 8)	
13	0	(30,4)	(2, 15, 30)		14	1	(48,48)	(2, 2, 2, 4)	
15	1	(48,41)	(2, 2, 2, 4)	2	16	1	(48,38)	(2, 2, 2, 4)	
17	1	(36,10)	(2, 2, 2, 6)		18	1	(32,43)	(2, 2, 2, 8)	
19	1	(32,42)	(2, 2, 2, 8)	3	20	1	(32,39)	(2, 2, 2, 8)	4
21	1	(28,3)	(2, 2, 2, 14)	5					

Table 4: (Cont.)

#	dim	aut	sig	subs	#	dim	aut	sig	subs
Genus 8									
1	0	(336,208)	(2, 3, 8)		2	0	(336,208)	(2, 3, 8)	
3	0	(84,7)	(2, 6, 6)		4	0	(84,7)	(2, 6, 6)	
5	0	(72,8)	(2, 4, 18)		6	0	(64,53)	(2, 4, 32)	
7	0	(60,8)	(2, 6, 10)		8	0	(48,25)	(2, 6, 24)	
9	0	(48,17)	(2, 8, 12)		10	0	(48,28)	(3, 4, 8)	
11	0	(40,10)	(2, 10, 20)		12	0	(34,2)	(2, 17, 34)	
13	1	(42,1)	(2, 2, 3, 3)	1, 2, 3, 4	14	1	(36,4)	(2, 2, 2, 9)	5
15	1	(32,18)	(2, 2, 2, 16)	6	16	1	(30,3)	(2, 2, 3, 5)	7
Genus 9									
1	0	(320,1582)	(2, 4, 5)		2	0	(192,194)	(2, 3, 12)	
3	0	(192,990)	(2, 4, 6)		4	0	(192,955)	(2, 4, 6)	
5	0	(128,138)	(2, 4, 8)		6	0	(128,136)	(2, 4, 8)	
7	0	(128,134)	(2, 4, 8)		8	0	(128,75)	(2, 4, 8)	
9	0	(120,35)	(2, 5, 6)		10	0	(120,34)	(2, 5, 6)	
11	0	(96,187)	(2, 4, 12)		12	0	(96,186)	(2, 4, 12)	
13	0	(96,13)	(2, 4, 12)		14	0	(80,14)	(2, 4, 20)	
15	0	(72,5)	(2, 4, 36)		16	0	(64,6)	(2, 8, 8)	
17	0	(57,1)	(3, 3, 19)		18	0	(48,5)	(2, 8, 24)	
19	0	(48,4)	(2, 8, 24)		20	0	(48,30)	(4, 4, 6)	
21	0	(42,3)	(2, 14, 21)		22	0	(40,12)	(4, 4, 10)	
23	0	(38,2)	(2, 19, 38)		24	1	(96,193)	(2, 2, 2, 3)	3
25	1	(96,227)	(2, 2, 2, 3)	4	26	1	(64,190)	(2, 2, 2, 4)	
27	1	(64,177)	(2, 2, 2, 4)	5	28	1	(64,140)	(2, 2, 2, 4)	
29	1	(64,138)	(2, 2, 2, 4)	4	30	1	(64,135)	(2, 2, 2, 4)	1, 3, 6
31	1	(64,134)	(2, 2, 2, 4)	7	32	1	(64,128)	(2, 2, 2, 4)	
33	1	(64,73)	(2, 2, 2, 4)	2, 8	34	1	(48,43)	(2, 2, 2, 6)	13
35	1	(48,38)	(2, 2, 2, 6)		36	1	(48,15)	(2, 2, 2, 6)	3
37	1	(48,48)	(2, 2, 2, 6)	4, 12	38	1	(48,48)	(2, 2, 2, 6)	11
39	1	(40,13)	(2, 2, 2, 10)	14	40	1	(40,8)	(2, 2, 2, 10)	
41	1	(36,4)	(2, 2, 2, 18)	15					

Table 4: (Cont.)

#	dim	aut	sig	subs	#	dim	aut	sig	subs
Genus 10									
1	0	(432,734)	(2, 3, 8)		2	0	(432,734)	(2, 3, 8)	
3	0	(360,118)	(2, 4, 5)		4	0	(324,160)	(2, 3, 9)	
5	0	(216,92)	(2, 3, 12)		6	0	(216,158)	(2, 4, 6)	
7	0	(216,87)	(2, 4, 6)		8	0	(216,153)	(3, 3, 4)	
9	0	(180,19)	(2, 3, 15)		10	0	(168,42)	(2, 4, 7)	
11	0	(162,14)	(2, 3, 18)		12	0	(144,122)	(2, 3, 24)	
13	0	(108,25)	(2, 6, 6)		14	0	(108,15)	(2, 4, 12)	
15	0	(88,7)	(2, 4, 22)		16	0	(80,6)	(2, 4, 40)	
17	0	(72,28)	(2, 6, 12)		18	0	(72,23)	(2, 6, 12)	
19	0	(72,42)	(3, 4, 6)		20	0	(63,3)	(3, 3, 21)	
21	0	(60,10)	(2, 6, 30)		22	0	(42,6)	(2, 21, 42)	
23	0	(42,2)	(3, 6, 14)		24	0	(42,2)	(3, 6, 14)	
25	1	(108,40)	(2, 2, 2, 3)	4, 6	26	1	(108,17)	(2, 2, 2, 3)	1, 2, 7
27	1	(72,43)	(2, 2, 2, 4)	5	28	1	(72,40)	(2, 2, 2, 4)	1, 2, 6
29	1	(72,15)	(2, 2, 2, 4)		30	1	(60,5)	(2, 2, 2, 5)	3, 9
31	1	(54,8)	(2, 2, 2, 6)	11, 14	32	1	(54,5)	(2, 2, 3, 3)	5, 8, 13
33	1	(48,29)	(2, 2, 2, 8)	1, 2, 12	34	1	(44,3)	(2, 2, 2, 11)	15
35	1	(40,6)	(2, 2, 2, 20)	16					

References

- [Ac1] R. ACCOLA, On the number of automorphisms of a closed Riemann surface, *Trans. Amer. Math. Soc.* **131** (1968), 398–408.
- [Ac2] R. ACCOLA, Two theorems on Riemann surfaces with noncyclic automorphism groups, *Proc. Amer. Math. Soc.* **25** (1970), 598–602.
- [Ba] W. BAILY, On the automorphism group of a generic curve of genus > 2 , *J. Math. Kyoto Univ.* **1** (1961/1962), 101–108; correction, 325.
- [Be] JOSÉ BERTIN, Compactification des schémas de Hurwitz, *C.R. Acad. Sci. Paris I*, vol. 322 (1996), 1063–1066.
- [BE] H. BESCHE AND B. EICK, The groups of order at most 1000 except 512 and 768, *J. Symbolic Comput.* **27** (1999), no. 4, 405–413.
- [BS] R. BRANDT AND H. STICHTENOTH, Die Automorphismengruppen hyperelliptischer Kurven, *Man. Math.* **55** (1986), 83–92.
- [Bra] R. BRANDT, Über die Automorphismengruppen von algebraischen Funktionenkörpern, PhD thesis. Universität-Gesamthochschule Essen, 1988.
- [Br] TH. BREUER, Characters and automorphism groups of compact Riemann surfaces, *London Math. Soc. Lect. Notes* **280**, Cambridge Univ. Press 2000.
- [BGG] E. BUJULANCE, J. GAMBOA, AND G. GROMADZKI, The full automorphism groups of hyperelliptic Riemann surfaces, *Manuscripta Math.* **79** (1993), no. 3-4, 267–282.
- [Co] M. CONDER, Hurwitz groups: a brief survey. *Bull. Amer. Math. Soc. (N.S.)* **23** (1990), no. 2, 359–370.
- [Du] A. DUMA, Automorphisms of compact Riemann surfaces. *Confer. Sem. Mat. Univ. Bari*, No. 207 (1985).
- [DT] A. DUMA, C. TANASI, Groups of automorphisms of a family of compact Riemann surfaces of genus four. *Ricerche Mat.* **34** (1985), no. 2, 377–390.
- [FK] M. FARKAS AND I. KRA, *Riemann Surfaces*, Springer-Verlag, 1992.
- [Fr] R. FRICKE, *Lehrbuch der Algebra*, Vieweg, Braunschweig, 1926.
- [FV] M. FRIED AND H. VÖLCKLEIN, The inverse Galois problem and rational points on moduli spaces, *Math. Annalen* **290** (1991), 771–800.
- [GAP] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.2; 2000. (<http://www.gap-system.org>)
- [Ha] W. HARVEY, Cyclic groups of automorphisms of a compact Riemann surface, *Quart. J. Math. Oxford Ser. (2)* **17** (1966), 86–97.
- [Ho] M. HOMMA, Automorphisms of prime order of curves. *Manuscripta Math.* **33** (1980/81), no. 1, 99–109.
- [Hu] A. HURWITZ, Über algebraische Gebilde mit eindeutigen Transformationen in sich, *Math. Ann.* **41** (1893), 403–442.
- [Ki] W. KILEY, Automorphism groups of compact Riemann surfaces. *Trans. Amer. Math. Soc.* **150** (1970), 557–563.
- [Kl] F. KLEIN, Über die Transformationen siebenter Ordnung der elliptischen Functionen, *Math. Annalen* **14**, (1879), 428–471.
- [Ku] R. S. KULKARNI, Riemann surfaces admitting large automorphism groups. Extremal Riemann surfaces (San Francisco, CA, 1995), 63–79, *Contemp. Math.* **201**, Amer. Math. Soc., Providence, RI, 1997.
- [KH] A. KURIBAYASHI AND K. HAYAKAWA, On the Hurwitz group. *Bull. Fac. Sci. Engrg. Chuo Univ. Ser. I Math.* **37** (1994), 1–23 (1995).

- [KuKi] A. KURIBAYASHI AND H. KIMURA, Automorphism groups of compact Riemann surfaces of genus five, *J. Algebra* **134** (1990), 80–103.
- [KuKo1] A. KURIBAYASHI, K. KOMIYA, On Weierstraßpoints and automorphisms of curves of genus three. In: Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), pp. 253–299, *Lecture Notes in Math.* **732**, Springer, Berlin, 1979.
- [KuKo2] A. KURIBAYASHI, K. KOMIYA, On the structure of the automorphism group of a compact Riemann surface of genus 3. *Bull. Fac. Sci. Engrg. Chuo Univ.* **23** (1980), 1–34.
- [KK1] A. KURIBAYASHI AND I. KURIBAYASHI, On subgroups of automorphism groups of hyperelliptic Riemann surfaces of genus three. *Bull. Fac. Sci. Engrg. Chuo Univ.* **24** (1981), 21–37.
- [KK2] A. KURIBAYASHI AND I. KURIBAYASHI, On subgroups of automorphism groups of hyperelliptic Riemann surfaces of genus three. II. *Bull. Fac. Sci. Engrg. Chuo Univ.* **25** (1982), 5–43.
- [KK3] A. KURIBAYASHI AND I. KURIBAYASHI, Automorphism groups of compact Riemann surfaces of genus three and four, *J. Pure Appl. Algebra* **65** (1990), 277–292.
- [KKO] A. KURIBAYASHI, A. KOBAYASHI, AND S. OHMORI, Automorphism groups of compact Riemann surfaces. Travaux de la Conférence Internationale d’Analyse Complexe et du 7e Séminaire Roumano-Finlandais (1993). *Rev. Roumaine Math. Pures Appl.* **40** (1995), no. 1, 65–75.
- [LN] J. LEHNER AND M. NEWMAN, On Riemann surfaces with maximal automorphism groups, *Glasgow Math. J.* **8** (1967), 102–112.
- [Le] S. LEVY, The eightfold way. The beauty of Klein’s quartic curve. Ed. Silvio Levy. *Mathematical Sciences Research Institute Publications* **35**, Cambridge University Press, Cambridge, 1999.
- [Mb] A. M. MACBEATH, On a curve of genus 7. *Proc. London Math. Soc. (3)* **15** (1965), 527–542.
- [Mb2] A. M. MACBEATH, On a theorem of Hurwitz. *Proc. Glasgow Math. Assoc.* **5** (1961), 90–96.
- [Mc1] C. MACLACHLAN, Abelian groups of automorphisms of compact Riemann surfaces, *Proc. London Math. Soc. (3)* **15** (1965), 699–712.
- [Mc2] C. MACLACHLAN, A bound for the number of automorphisms of a compact Riemann surface, *J. London Math. Soc. (2)* **44** (1969), 265–272.
- [MSV] K. MAGAARD, S. SPHECTOROV AND H. VÖLKLEIN, Computing braid group action, in preparation.
- [Ma1] G. MALLE, Genus zero translates of three point ramified Galois extensions, *Manuscripta Math.* **71** (1991), 97–111.
- [Ma2] G. MALLE, Hurwitz groups and $G_2(q)$, *Canad. Math. Bull.* **33** (1990), no. 3, 349–357.
- [Mt] T. MATSUNO, Compact Riemann surfaces with large automorphism groups, *J. Math. Soc. Japan* **51** (1999), no. 2, 309–329.
- [Ri] J. RIES, Subvarieties of moduli space determined by finite groups acting on surfaces, *Trans. Amer. Math. Soc.* **335** (1993), no. 1, 385–406.
- [Si1] D. SINGERMAN, Subgroups of Fuchsian groups and finite permutation groups, *Bull. London Math. Soc.* **2** (1970), 319–323.
- [Si2] D. SINGERMAN, Finitely maximal Fuchsian groups, *J. London Math. Soc.* **6** (1972), 29–38.
- [Si3] D. SINGERMAN, Symmetries of Riemann surfaces with large automorphism group, *Math. Ann.* **210** (1974), 17–32.
- [ShV] T. SHASKA AND H. VÖLKLEIN, Elliptic Subfields and automorphisms of genus fields, to appear in: *Proceeding of the Conference on Algebra and Algebraic Geometry with Applications: The celebration of the seventieth birthday of Professor S.S. Abhyankar*, Springer-Verlag, 2001.

- [Ta] N. TAKIGAWA, Compact Riemann surfaces with an automorphism group of order p , *Math. Rep. Toyama Univ.* **4** (1981), 115–127.
- [V] H. VÖLKLEIN, *Groups as Galois Groups – an Introduction*, Cambr. Studies in Adv. Math. **53**, Cambridge Univ. Press 1996.
- [V1] H. VÖLKLEIN, Moduli spaces for covers of the Riemann sphere, *Israel J. Math.* **85** (1994), 407–430.
- [W1] A. WIMAN, Über die hyperelliptischen Curven und diejenigen vom Geschlechte $p = 3$, welche eindeutige Transformationen in sich zulassen, *Bihang Kongl. Svenska Vetenskaps-Akademiens Handlingar* (1895), no. 21 (1), 1–23.
- [W2] A. WIMAN, Über die hyperelliptischen Curven vom den Geschlechte $p = 4, 5$, und 6 , welche eindeutige Transformationen in sich besitzen, *Bihang Kongl. Svenska Vetenskaps-Akademiens Handlingar* (1895), no. 21 (3), 1–41.
- [W1] R. WILSON, The Monster is a Hurwitz group. *J. Group Theory* **4** (2001), no. 4, 367–374.