The generating rank of the symplectic grassmannians: hyperbolic and isotropic geometry

Rieuwert J. Blok
Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403
U.S.A.
blokr@member.ams.org

Accepted: March 7 2006

Key Words: symplectic geometry, grassmannian, generating rank, hyperbolic geometry

AMS subject classification (2000): Primary 51A50; Secondary 51A45, 05A15.

Proposed running head:

Symplectic grassmannians

Send proofs to:

Rieuwert J. Blok
Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403

Tel.: +1 (419) 372-7455
FAX : +1 (419) 372-6092
Email: blokr@member.ams.org

Abstract

Exploiting the interplay between hyperbolic and isotropic geometry, we prove that the grassmannian of totally isotropic k-spaces of the polar space associated to the symplectic group $\mathrm{Sp}_{2n}(\mathbb{F})$ has generating rank $\binom{2n}{k} - \binom{2n}{k-2}$ when $\mathrm{Char}(\mathbb{F}) \neq 2$.

1 Introduction

Generating sets of point-line geometries serve both theoretical and computational purposes. For instance, the generating rank of a geometry equals the dimension of the universal embedding of that geometry, if it exists. On the other hand, minimal generating sets may serve in creating computer models of point-line geometries.

Until now, the building of type A_n of all linear subspaces of a vector space is the only one of which the generating rank is known for each of its k-shadow spaces (projective k-Grassmannians) (see Cooperstein and Shult [7] and Blok and Brouwer [4]). In this paper we deal with the symplectic building of type C_n over a field of odd characteristic and exhibit minimal generating sets for all symplectic k-Grassmannians.

For a few of these geometries a minimal generating set has already been found. Let $\mathbb{F}$ be a field of characteristic not 2. It was proved by Cooperstein and Shult [7] and Blok and Brouwer [4] that the generating rank of the symplectic polar space associated to the group $\mathrm{Sp}_{2n}(\mathbb{F})$ has generating rank $2n = \binom{2n}{1} - \binom{\mathbf{2}n}{-1}$. In [5] Blok and Pasini prove that the line-grassmannian of the symplectic polar space associated to the group $\mathrm{Sp}_{2n}(\mathbb{F})$, if $\mathbb{F}$ is a prime field, has generating rank $\binom{2n}{2} - \binom{2n}{0}$. In Cooperstein [12] and Blok [3] it is proved that this geometry has that generating rank over an arbitrary field of characteristic not 2. In [10] Cooperstein proved that the dual polar space associated to $\mathrm{Sp}_{2n}(\mathbb{F})$ has generating rank $\binom{2n}{n} - \binom{2n}{n-2}$. The generating sets presented in the above papers vary greatly in structure.

It is one of our main goals to describe a minimal generating set for an arbitrary symplectic grassmannian in a unified way and to prove the following.

Theorem 1 *The k-grassmannian of the polar space associated to* $\mathrm{Sp}_{2n}(\mathbb{F})$ *has generating rank* $\binom{2n}{k} - \binom{2n}{k-2}$ *if* $\mathbb{F}$ *is a field with* $\mathrm{Char}(\mathbb{F}) \neq 2$.

We mention that in even characteristic the geometry can be viewed as orthogonal rather than symplectic through the isomorphism $\mathrm{Sp}_{2n}(\mathbb{F}) \cong \mathrm{O}_{2n+1}(\mathbb{F})$ induced by projection from the nucleus of the bilinear form. Generating results for such geometries were obtained by Brouwer and Blokhuis [6], Cooperstein [8], and Li [13].

More on generating sets can be found in Cooperstein [11] and Blok [2].

2 Preliminaries

A *point-line geometry* is a pair $\Gamma = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ is a set whose elements are called 'points' and $\mathcal{L}$ is a collection of subsets of $\mathcal{P}$ called 'lines' with the property that any two points belong to at most one line. If $\mathcal{P}$ and $\mathcal{L}$ are not mentioned explicitly, the sets of points and lines of a point-line geometry Γ are denoted $\mathcal{P}(\Gamma)$ and $\mathcal{L}(\Gamma)$.

A *subspace* of Γ is a subset $X \subseteq \mathcal{P}$ such that any line containing at least two points of X entirely belongs to X. A *hyperplane* of Γ is a proper subspace that meets every line.

Projective embeddings and Generating sets The *span* of a set $\mathcal{S} \subseteq \mathcal{P}$ is the smallest subspace containing $\mathcal{S}$; it is the intersection of all subspaces containing $\mathcal{S}$ and is denoted by $\langle \mathcal{S} \rangle_\Gamma$. We say that $\mathcal{S}$ is a *generating set* (or *spanning set*) for Γ if $\langle \mathcal{S} \rangle_\Gamma = \mathcal{P}$.

For a vector space W over some field $\mathbb{F}$, the *projective geometry* associated to W is the point-line geometry $\mathbb{P}(W) = (\mathcal{P}(W), \mathcal{L}(W))$ whose points are the 1-spaces of W and whose lines are the sets of 1-spaces contained in some 2-space.

A *projective embedding* of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is a pair (ϵ, W), where ϵ is an injective map $\mathcal{P} \xrightarrow{\epsilon} \mathcal{P}(W)$ that sends every line of $\mathcal{L}$ onto a line of $\mathcal{L}(W)$, and with the property that

$$\langle \epsilon(\mathcal{P}) \rangle_{\mathbb{P}(W)} = \mathcal{P}(W).$$

In the literature, this is often referred to as a *strong projective embedding.* The *dimension* of the embedding is the dimension of the vector space W. In this paper we will assume both $\dim(W)$ and $|\mathcal{S}|$ to be finite. Then, since $\epsilon(\langle \mathcal{S} \rangle_\Gamma) \subseteq \langle \epsilon(\mathcal{S}) \rangle_W$, for any generating set $\mathcal{S}$ and any embedding (ϵ, W) we have

$$\dim(W) \leq |\mathcal{S}|.$$

In case of equality, $\mathcal{S}$ is a minimal generating set and W is an embedding of maximal dimension. We then call $\dim(W) = |\mathcal{S}|$ the *generating rank* of Γ.

We briefly describe the particular geometries we will discuss in this paper.

The Projective Grassmannian Let V be a vector space over some field $\mathbb{F}$. For any k with $1 \leq k \leq \dim(V)$, the *projective k-grassmannian* associated to V is the point-line geometry $\mathrm{Gr}(V, k)$ whose points are the k-spaces of V and whose lines are sets of the form

$$\{K \text{ a } k\text{-space in } V \mid D \subseteq K \subseteq U\}$$

for some $(k-1)$-space D and $(k+1)$-space $U \supseteq D$.

The Symplectic Grassmannian Let V be a vector space of dimension $2n$ over a field $\mathbb{F}$ endowed with a non-degenerate symplectic form $\mathsf{s}(\cdot, \cdot)$. For a subspace U of V we define

$$U^\perp = \{v \in V \mid \mathsf{s}(u, v) = 0 \, \forall u \in U\}.$$

We write $U \perp W$ if $W \subseteq U^\perp$. The *radical* of a subspace W is

$$\mathrm{Rad}(W, \mathsf{s}) = W^\perp \cap W.$$

A subspace U of V is called *totally isotropic* (t.i.) with respect to the form $\mathsf{s}(\cdot,\cdot)$ if $U \subseteq U^{\perp}$. It is called non-degenerate if $\mathrm{Rad}(U) = \{0\}$.

The *symplectic polar space* is the point-line geometry Γ_1^{s} whose points are the t.i. 1-spaces of V and whose lines are sets of 1-spaces of the form

$$\{P \text{ a t.i. 1-space of } V \mid P \subseteq L\}$$

for some t.i. 2-space L. We sometimes call t.i. 3-spaces *planes*. A *hyperbolic line* H is a 2-space in V such that s restricted to H is non-degenerate.

The *symplectic k-grassmannian* is the point-line geometry $\Gamma_k^{\mathsf{s}}(V) = (\mathcal{P}_k, \mathcal{L}_k)$ whose points are the t.i. k-spaces and whose lines are the sets of the form

$$\{K \text{ a t.i. } k\text{-space in } V \mid D \subseteq K \subseteq U\}$$

for some t.i. $(k-1)$-space D and t.i. $(k+1)$-space $U \supseteq D$. In case $k = n$, the lines are of the form

$$\{K \text{ a t.i. } k\text{-space in } V \mid D \subseteq K\}$$

for some t.i. $(n-1)$-space D. We will call the points and lines of $\Gamma_k^{\mathsf{s}}(V)$ Points and Lines to distinguish them from the points and lines of $\Gamma_1^{\mathsf{s}}(V)$. Whenever V or s is clear from the context, we'll drop it from the notation.

We denote the group of linear transformations preserving the form s on V by $\mathrm{Sp}(V) \cong \mathrm{Sp}_{2n}(\mathbb{F})$. Clearly $\mathrm{Sp}(V)$ is an automorphism group of Γ_k^{s} for all $k = 1, 2, \ldots, n$ in that it preserves points and lines and the incidence between them.

We will need the following result from Blok and Brouwer [4] and Cooperstein and Shult [7] on the generating rank of Γ_1.

Theorem 2.1 *Let Γ_1 be the symplectic polar space associated to* $\mathrm{Sp}(V)$ *where* $\mathrm{Char}(\mathbb{F}) \neq 2$. *Then Γ_1 has generating rank $2n$ and it is generated by the $2n$ points of an apartment i.e. a hyperbolic basis.*

3 Hyperbolic geometry and hyperbolic $2n$-gons

Let V be a vector space of dimension $2n$ over a field $\mathbb{F}$ of odd characteristic endowed with a non-degenerate symplectic form $\mathsf{s}(\cdot,\cdot)$. For any subspace $W \subseteq V$ with the form induced by s, the *hyperbolic geometry of W* is the point-line geometry $\Gamma_{\mathrm{hyp}}(W) = (\mathcal{P}_{\mathrm{hyp}}(W), \mathcal{L}_{\mathrm{hyp}}(W))$ with

$$\begin{aligned} \mathcal{P}_{\mathrm{hyp}} &= \{\text{1-spaces in } W - \mathrm{Rad}(W)\} \\ \mathcal{L}_{\mathrm{hyp}} &= \{\text{hyperbolic 2-spaces in } W - \mathrm{Rad}(W)\} \end{aligned}$$

We set $\Gamma_{\mathrm{hyp}} = \Gamma_{\mathrm{hyp}}(V)$, $\mathcal{P}_{\mathrm{hyp}} = \mathcal{P}_{\mathrm{hyp}}(V)$, $\mathcal{L}_{\mathrm{hyp}} = \mathcal{L}_{\mathrm{hyp}}(V)$. Note that, if W is non-degenerate, then $\Gamma_1(W)$ and $\Gamma_{\mathrm{hyp}}(W)$ have the same point-set.

Definition 3.1 Let $W \leq V$ be a subspace of dimension m such that

$$\dim(\mathrm{Rad}(W, \mathsf{s})) = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

A *tight hyperbolic sequence* or THS for W is an ordered set of points in $\Gamma_{\mathrm{hyp}}(W)$ $p_1, p_2, \ldots, p_m \subseteq W - \mathrm{Rad}(W, \mathsf{s})$, such that setting $W_i = \langle p_1, p_2, \ldots, p_i\rangle_W$, we have

(THS1) $\dim(W_i) = i$,

(THS2) $p_{i+1}^{\perp} \cap \mathrm{Rad}(W_i, \mathsf{s}) = 0$, and

(THS3)

$$\dim(\mathrm{Rad}(W_i, \mathsf{s})) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Note that in fact (THS3) follows from (THS1)+(THS2).

Lemma 3.2 *Let $W \leq V$ be a subspace as in Definition 3.1. Then,*

(a) *W has a tight hyperbolic sequence,*

If, in addition, $|\mathbb{F}| \geq 3$, then

(b) *the points on a tight hyperbolic sequence generate $\Gamma_{\mathrm{hyp}}(W)$.*

Proof (a) We prove this by induction on m. For $m = 2$ we have $\mathrm{Rad}(W, \mathsf{s}) = 0$ and so any two points, in any order, form a THS. Now let $m > 2$.

If m is even, then $\dim(\mathrm{Rad}(W, \mathsf{s})) = 0$ and any subspace W_{m-1} of dimension $m-1$ has $\dim(\mathrm{Rad}(W_{m-1}, \mathsf{s})) = 1$. If m is odd then $\dim(\mathrm{Rad}(W, \mathsf{s})) = 1$ and any W_{m-1} of dimension $m-1$ not containing $\mathrm{Rad}(W, \mathsf{s})$ has $\dim(\mathrm{Rad}(W_{m-1}, \mathsf{s})) = 0$. By induction there exists a THS $p_1, p_2, \ldots, p_{m-1}$ for W_{m-1} so we only need to find p_m.

If $m-1$ is even, then, $\mathrm{Rad}(W_{m-1}, \mathsf{s}) = 0$ and $\dim(W_{m-1}^{\perp}) = 1$. So we can choose $p_m \subseteq W - W_{m-1} - W_{m-1}^{\perp}$ and we are done. If $m-1$ is odd, then $\dim(\mathrm{Rad}(W_{m-1}, \mathsf{s})) = 1$ and $\dim(\mathrm{Rad}(W, \mathsf{s})) = \dim(\mathrm{Rad}(W_{m-2}, \mathsf{s})) = 0$. We find that $W_{m-2}^{\perp} \cap W$ is a hyperbolic line containing $\mathrm{Rad}(W_{m-1}, \mathsf{s})$. Choosing p_m also in this hyperbolic line, but different from $\mathrm{Rad}(W_{m-1}, \mathsf{s})$ we ensure that $p_m^{\perp} \cap \mathrm{Rad}(W_{m-1}, \mathsf{s}) = 0$ and $\langle W_{m-1}, p_m\rangle_W = W$. This concludes the proof of part (a).

(b) Again, we prove this by induction on m. For $m = 2$ any THS is a set of two points which generate the line of $\Gamma_{\mathrm{hyp}}(W)$ that is W itself.

Now let $m > 2$ and let $p_1, p_2, \ldots, p_m$ be a THS for $W = W_m$. By induction $\langle p_1, p_2, \ldots, p_{m-1}\rangle_{\Gamma_{\mathrm{hyp}}}$ contains all points of $W_{m-1} - \mathrm{Rad}(W_{m-1}, \mathsf{s})$.

Note that every 2-space containing p_m and a point of $W_{m-1} - \mathrm{Rad}(W_{m-1}, \mathsf{s}) - (p_m^\perp \cap W_{m-1})$ is hyperbolic. Therefore $\langle p_1, p_2, \ldots, p_m \rangle_{\Gamma_{\mathrm{hyp}}}$ contains all points on such hyperbolic 2-spaces.

It remains to show that all points (except $\mathrm{Rad}(W, \mathsf{s})$) on some 2-space containing p_m and a point of $\mathrm{Rad}(W_{m-1}, \mathsf{s}) \cup (p_m^\perp \cap W_{m-1})$ are in $\langle p_1, p_2, \ldots, p_m \rangle_{\Gamma_{\mathrm{hyp}}}$.

We first deal with the special case $m = 3$. By definition of a THS, $p_3^\perp \not\supseteq W_2$. Using that Γ_{hyp} is a dual affine plane, one easily checks that p_3 together with any two points on W_2 generates Γ_{hyp}.

Now let $m > 3$. First consider any point $x \in p_m^\perp \cap W_{m-1}$. In view of (THS2) we know that $x \in W_{m-1} - \mathrm{Rad}(W_{m-1}, \mathsf{s}) = \langle p_1, p_2, \ldots, p_{m-1} \rangle_{\Gamma_{\mathrm{hyp}}}$. Let $y \in L = \langle x, p_m \rangle_W$ with $y \neq \mathrm{Rad}(W, \mathsf{s})$. We show that there exists a point

$$z \in W_{m-1} - \mathrm{Rad}(W_{m-1}, \mathsf{s}) - (p_m^\perp \cup x^\perp \cup y^\perp).$$

Then, x, z, p_m is a THS for $W' = \langle x, z, p_m \rangle_W$ and $y \notin \mathrm{Rad}(W', \mathsf{s})$. It then follows from case $m = 3$ that $y \in \langle x, z, p_m \rangle_{\Gamma_{\mathrm{hyp}}}$ and we are done since $x, z, p_m \in \langle p_1, p_2, \ldots, p_m \rangle_{\Gamma_{\mathrm{hyp}}}$.

We have two cases.

(1) $x^\perp \cap W_{m-1} = p_m^\perp \cap W_{m-1}$. In this case first note that $x^\perp \cap W_{m-1} = \{x, p_m\}^\perp \cap W_{m-1} = y^\perp \cap W_{m-1}$. Now let z be any point in $W_{m-1} - \mathrm{Rad}(W_{m-1}, \mathsf{s}) - (p_m^\perp \cap W_{m-1})$. Such a point exists otherwise $x = \mathrm{Rad}(W_{m-1}, \mathsf{s}) \subseteq p_m^\perp$ contradicting the THS property for p_m.

(2) $x^\perp \cap W_{m-1} \neq p_m^\perp \cap W_{m-1}$. In this case the hyperplanes $t^\perp \cap W_{m-1}$ with $t \in L = xp_m$ are all distinct and intersect in the codimension-2 subspace $L^\perp \cap W_{m-1}$. Let L' be any 2-space of W_{m-1} missing $L^\perp$. Then, the points $t^\perp \cap L'$ with $t \in L$ are all distinct. In particular, since $|\mathbb{F}| \geq 3$ we can find a point z on $L' - (x^\perp \cup y^\perp \cup p_m^\perp)$. Clearly z has the required properties.

Having handled cases (1) and (2) we have shown that all points on all lines meeting p_m and $W_{m-1} - \mathrm{Rad}(W_{m-1}, \mathsf{s})$ except $\mathrm{Rad}(W, \mathsf{s})$ itself are in $\langle p_1, p_2, \ldots, p_m \rangle_{\Gamma_{\mathrm{hyp}}}$.

The only case left is $m > 3$ even and y a point on the line $L = p_m r$ where $r = \mathrm{Rad}(W_{m-1}, \mathsf{s})$. Note that $L^\perp \cap W_{m-1} = t^\perp \cap W_{m-1}$ for all $t \in L - \{r\}$. By definition of a THS, $r \notin p_m^\perp$ so L is a line of Γ_{hyp}. Take any line $L' \subseteq W_{m-1}$ on r. Since $|\mathbb{F}| \geq 3$ we can find two points $z, z' \in L' - \{r\} - p_m^\perp$. Then z, p_m, z' is a THS for $W' = \langle z, z', p_m \rangle_V$. Note that $y \in W'$ and $y \neq \mathrm{Rad}(W', \mathsf{s})$ so we can apply case $m = 3$. Thus, since $z, z', p_m \in \langle p_1, p_2, \ldots, p_m \rangle_{\Gamma_{\mathrm{hyp}}}$ also $y \in \langle p_1, p_2, \ldots, p_m \rangle_{\Gamma_{\mathrm{hyp}}}$. Since y is allowed to equal r, we are done. □

Definition 3.3 Let U be a vector space of dimension $2l$ over a field $\mathbb{F}$ endowed with a non-degenerate symplectic form $\mathsf{s}(\cdot, \cdot)$. A *hyperbolic k-path* is a set of points $P - \{p_1, p_2, \ldots, p_k\}$ in $\Gamma_{\mathrm{hyp}}(U)$ with the property that $\dim(\langle p_1, p_2, \ldots, p_k \rangle) = k$

and the geometry induced on P by $\Gamma_{\text{hyp}}(U)$ is a k-path. A *hyperbolic k-cycle* is a hyperbolic $(k-1)$-path together with a point that is collinear in $\Gamma_{\text{hyp}}(U)$ to both endpoints of that path. A *hyperbolic $2l$-gon* for U is a set of points $T = \{p_1, p_2, \ldots, p_{2l}\}$ in $\Gamma_{\text{hyp}}(U)$ with the property that $\langle p_1, p_2, \ldots, p_{2l}\rangle_U = U$ and such that the geometry induced on T by $\Gamma_{\text{hyp}}(U)$ is a $2l$-gon. That is, p_i and p_j are collinear in $\Gamma_{\text{hyp}}(U)$ if and only if $|i-j| = 1 \bmod 2l$.

Lemma 3.4 (a) *V has a hyperbolic $2n$-gon,*

(b) *a hyperbolic $2n$-gon can be ordered as a tight hyperbolic sequence for V,*

(c) *the points on a hyperbolic $2n$-gon generate $\Gamma_{\text{hyp}}(V)$.*

Proof (a) Let $\{e_i, f_i \mid i = 1, 2, \ldots, n\}$ be a basis for V so that

$$\begin{aligned} \mathsf{s}(e_i, f_j) &= \delta_{ij} \\ \mathsf{s}(e_i, e_j) = \mathsf{s}(f_i, f_j) &= 0 \end{aligned}$$

for all $i, j \in \{1, 2, \ldots, n\}$. That is, it is a *hyperbolic basis.*

Let

$$\begin{array}{lll} p_{2j-1} & = \langle e_j\rangle & \forall j = 1, 2, \ldots, n \\ p_{2j} & = \langle f_j - f_{j+1}\rangle & \forall j = 1, 2, \ldots, n-1 \\ p_{2n} & = \langle cf_n - f_1\rangle & \text{with } c \neq 0 \end{array}$$

In addition, we require that $c \neq 1$ if $n = 2$.

One checks that $\mathsf{s}(p_i, p_j) = 0$ except when $|i-j| \equiv 1 \bmod 2n$. Thus $T = \{p_1, p_2, \ldots, p_{2n}\}$ is a $2n$-gon in Γ_{hyp}. Clearly $\langle T\rangle_V = \langle e_i, f_i \mid i = 1, 2, \ldots, n\rangle_V = V$ so T is a hyperbolic $2n$-cycle, which is a hyperbolic $2n$-gon unless $n = 2$ and $c = 1$. Note that in case $n = 2$ we need to be able to choose $c \in \mathbb{F} - \{0, 1\}$. In our case this is possible since $\mathbb{F} \neq \mathbb{F}_2$.

(b) Given a hyperbolic $2n$-gon $T = \{p_1, p_2, \ldots, p_{2n}\}$, let $V_i = \langle p_1, p_2, \ldots, p_i\rangle_V$.

(THS1) This follows from the fact that $\langle T\rangle_V = V$.

(THS2) We show by induction on $i = 2, 3, 4, \ldots, 2n-1$, that $p_{i+1}^{\perp} \cap \mathrm{Rad}(V_i, \mathsf{s}) = \{0\}$. Consider the subspaces $U = \langle p_1, p_3, \ldots, p_{2k-1}\rangle_V$ and $U' = \langle p_2, p_4, \ldots, p_{2k}\rangle_V$ of V_{2k} These are disjoint and totally isotropic. One can extend $\{e_j\}_{j=1}^k$ to a hyperbolic basis for V_{2k} using vectors in U'. Thus, $\dim(\mathrm{Rad}(V_{2k})) = 0$.

Now let $1 \leq k < n$. Then we see that $p_{2k+1}^{\perp} \cap \mathrm{Rad}(V_{2k}) = p_{2k+1}^{\perp} \cap \{0\} = \{0\}$. Also, $p_{2k+2}^{\perp} \cap \mathrm{Rad}(V_{2k+1}) \subseteq \mathrm{Rad}(V_{2k+2}) = \{0\}$, showing that $p_{2k+2}^{\perp} \cap \mathrm{Rad}(V_{2k+1}) = \{0\}$ as well.

(c) This follows from (b) and Lemma 3.2. □

Definition 3.5 The hyperbolic $2n$-cycle defined in Lemma 3.4 is called the *standard hyperbolic $2n$-cycle of type* (n, c). We call it the *standard hyperbolic $2n$-gon of type* (n, c) if it is a hyperbolic $2n$-gon, that is, if $(n, c) \neq (2, 1)$.

In the remainder of this section we will exploit the interaction between long-root subgroups, the hyperbolic geometry Γ_{hyp} and a certain graph on the set of hyperbolic $2n$-gons to obtain the generation tool presented in Proposition 3.20.

Recall that the *long-root group* U_p of a t.i. 1-space p of V is the transvection group with center p and axis $p^\perp$.

Lemma 3.6 *Let p, q, and x denote t.i. 1-spaces in V.*

(a) *For all p, U_p is a subgroup of $G = \mathrm{Sp}(V)$ isomorphic to $\mathbb{F}^+$, the additive group of the field $\mathbb{F}$.*

(b) *For any p and q, there are precisely two possibilities*

  (h) $U_{p,q} = \langle U_p, U_q \rangle \cong \mathrm{SL}_2(\mathbb{F})$, *$\langle p, q \rangle_V$ is a hyperbolic line, $U_x \leq U_{p,q}$ if and only if $x \in \langle p, q \rangle_V$, any U_x is transitive on all points of $\langle p, q \rangle_V - \{x\}$ and all long-root groups in $U_{p,q} - U_x$ are conjugate under U_x, or*

  (s) U_p and U_q commute, $\langle p, q \rangle_V$ is a totally isotropic line, and the only root groups contained in $U_{p,q}$ are U_p and U_q.

Proof This is essentially 12.1 of Aschbacher and Seitz [1]. □

We denote the element of U_p corresponding to $t \in \mathbb{F}^+$ by $U_p(t)$.

Lemma 3.7 *Let X be a set of points in Γ_{hyp} and let $U_X = \langle U_x \mid x \in X \rangle_G$. Then for every $y \in \langle X \rangle_{\Gamma_{\mathrm{hyp}}}$ we have $U_y \leq U_X$ and there is $u \in U_X$ and $x \in X$ such that $ux = y$.*

Proof Consider the sequence $X = X_0 \subseteq X_1 \subseteq \cdots$ and let $X_\infty = \bigcup_{i=0}^{\infty} X_i$. Here X_{i+1} consists of all points in X_i and all points y on some line of Γ_{hyp} meeting X_i in at least two points. Then we have $\langle X \rangle_{\Gamma_{\mathrm{hyp}}} = X_\infty$.

We now prove the following statement by induction on i.

> For every $y \in X_i$ we have $U_y \leq U_X$ and there is $u \in U_X$ and $x \in X$ such that $ux = y$.

For $i = 0$ take $x = y$ and $u = \mathrm{id}$. Now let $i > 0$. Choose $y \in X_{i+1}$. Then there exist $y', y'' \in X_i$ such that y lies on the line $\langle y', y'' \rangle_V$ of Γ_{hyp}. By assumption there are $x', x'' \in X$ and $u', u'' \in U_X$ with $u'x' = y'$ and $u''x'' = y''$. Now by Lemma 3.6 case (h) we have $U_y \leq \langle U_{y'}, U_{y''} \rangle_G \cong \mathrm{SL}_2(\mathbb{F})$ and there exists $v \in U_{y''}$ such that $vy' = y$ and $U_{y'}^{v^{-1}} = U_y$. Let $u = vu'$. By induction $U_{y'}, U_{y''} \leq U_X$ and so $U_y = U_{y'}^{v^{-1}} \leq U_X$ and $ux' = vu'x' = y$ where $u = vu' \in U_X$, $x' \in X$, as desired. □

Corollary 3.8 *Let $T = \{p_1, p_2, \ldots, p_{2n}\}$ be a hyperbolic $2n$-gon. Then $U_T = \langle U_{p_i} \mid i = 1, 2, \ldots, 2n\rangle$ is transitive on the points of Γ_{hyp} and contains U_p for every point p. Hence, $U_T = \mathrm{Sp}(V)$.*

Proof By Lemma 3.4 $\langle T\rangle_{\Gamma_{\text{hyp}}} = \Gamma_{\text{hyp}}$ so the first statement follows from Lemma 3.7. It is well-known that $\mathrm{Sp}(V)$ is generated by its long-root groups and we find $U_T = \mathrm{Sp}(V)$. □

Definition 3.9 Let $\mathcal{T}$ be the collection of all hyperbolic $2n$-gons in V and let $\Theta = (\mathcal{T}, \sim)$ be the graph whose vertices are the elements of $\mathcal{T}$ and in which $T \sim T'$ if and only if there exists a point $p \in T$ and an element $u \in U_p$ such that $uT = T'$. For any $T \in \mathcal{T}$, let $C(T)$ be the connected component of Θ containing T.

Lemma 3.10 *Let T be a hyperbolic $2n$-gon. Then,*

(a) $U_T = \mathrm{Sp}(V)$,

(b) $\mathrm{Sp}(V) \leq \mathrm{Aut}(\Theta)$,

(c) *In fact,* $\mathrm{Sp}(V) \leq \mathrm{Stab}_{\mathrm{Aut}(\Theta)}(C(T))$.

(d) $\mathrm{Sp}(V)$ *is transitive on* $C(T)$.

(e) *The orbits of* $\mathrm{Sp}(V)$ *on* Θ *are the connected components* $C(T)$, $T \in \mathcal{T}$.

Proof (a) This is part of Corollary 3.8.

(b) First note that $U_T \leq \mathrm{Sp}(V)(\mathbb{F}) \leq \mathrm{Aut}(\Gamma_{\text{hyp}})$ so U_T preserves $\mathcal{T}$ as a set. Let $T', T'' \in \mathcal{T}$ be adjacent in Θ. That is, there is $p' \in T'$ and $u' \in U_{p'}$ such that $u'T' = T''$. Now let $p \in T$ and $u \in U_p$. Then $uT'' = uu'T' = uu'u^{-1}uT'$, and $uu'u^{-1} \in U_{p'}^{u^{-1}} = U_{up'}$ where $up' \in uT'$. Thus again uT' and uT'' are adjacent in Θ and it follows that $U_p \leq \mathrm{Aut}(\Theta)$. Hence $U_T \leq \mathrm{Aut}(\Theta)$, as desired.

(c) Let $p \in T$ and $u \in U_p$. Then by (a) $uC(T) = C(T')$, where $T' = uT$. By definition of Θ, though, $T' \sim T$ so that $C(T') = C(T)$.

(d) Any two hyperbolic $2n$-gons T and T' are adjacent whenever there is an element $u \in U_p$ for some point p on T such that $uT = T'$. Therefore $\mathrm{Sp}(V)$ is transitive on $C(T)$.

(e) This is immediate from (c) and (d). □

We can in fact label and count the orbits of $\mathrm{Sp}(V)$ on Θ.

Definition 3.11 An *edge marked hyperbolic* $2n$*-cycle* is a pair (T, e), where T is a hyperbolic $2n$-cycle and e is a directed edge on T that is the first edge of a hyperbolic

$(2n-1)$-path in T. In particular, an *edge marked hyperbolic $2n$-gon* (T, e) is an edge marked hyperbolic $2n$-cycle, where T is a hyperbolic $2n$-gon. Note that in this case, we can take any edge for e. We calculate the *class number* of (T, e) as follows. We number the points of T as $p_1, p_2, \ldots, p_{2n}$ such that $p_1, p_2, \ldots, p_{2n-1}$ is a hyperbolic $(2n-1)$-path and $e = p_1p_2$. The *class number* of (T, e) is

$$c(T, e) = \frac{\Pi_{i=1}^{n} \mathsf{s}(u_{2i-1}, u_{2i})}{\Pi_{i=1}^{n} \mathsf{s}(u_{2i}, u_{2i+1})}, \tag{3.1}$$

where indices are taken modulo $2n$ and, for all $j = 1, 2, \ldots, 2n$, $u_j \in p_j$ is an arbitrary non-zero vector. In the case $n = 2$, even if $p_4 = p_2$, we do not require that $u_4 = u_2$.

Lemma 3.12 (a) *With the above notation, $c(T, e)$ does not depend on the choice of $\{u_j\}_{j=1}^{2n}$.*

(b) *Now let T be a hyperbolic $2n$-gon and let f be another directed edge of T. Then,*

$$c(T, f) = \begin{cases} c(T, e) & \text{if } f = p_{2i-1}p_{2i} \quad \text{for some } 1 \le i \le n, \\ c(T, e) & \text{if } f = p_{2i}p_{2i-1} \quad \text{for some } 1 \le i \le n, \\ c(T, e)^{-1} & \text{else.} \end{cases}$$

Proof (a) Any other non-zero vector in p_j looks like λu_j for some $\lambda \in \mathbb{F}$. Since s is bilinear, the value of 3.1, in which u_j appears once above and below, is unchanged.

(b) If $f = p_{2i-1}p_{2i}$, then all indices j above an below in 3.1 are replaced by $j + 2i - 2$. This clearly leaves the value unaffected. If $f = p_{2i}p_{2i-1}$ then effectively both the numerator and denominator of 3.1 are multiplied by -1^n. This clearly does not affect the value. In all other cases the numerator and denominator of 3.1 are interchanged. □

Definition 3.13 Let T be a hyperbolic $2n$-gon. The *class number of* T is $c(T) = \{c(T, e), c(T, e)^{-1}\}$, where e is some directed edge on T. By Lemma 3.12 this is well-defined.

Example 3.14 Let T be the standard hyperbolic $2n$-cycle of type (n, c) and let $e = p_1p_2$. Then, $c(T, e) = c$.

Definition 3.15 An isomorphism of edge-marked hyperbolic $2n$-cycles (T_i, e_i), $i = 1, 2$, is a graph isomorphism $\varphi: T_1 \to T_2$ such that $\varphi(e_1) = e_2$ as a directed edge. We call (T_1, e_1) and (T_2, e_2) *isometric* if there is some element $g \in \mathrm{Sp}(V)$ that induces an isomorphism between (T_1, e_1) and (T_2, e_2). We write $(T_1, e_1) \equiv (T_2, e_2)$. We call T_1 and T_2 *isometric* if (T_1, e_1) and (T_2, e_2) are isometric for some directed edges e_1 and e_2. We write $T_1 \equiv T_2$. The same notation applies to edge-marked hyperbolic $2n$-gons.

Proposition 3.16 *For $i = 1, 2$, let (T_i, e_i) be an edge-marked hyperbolic $2n$-cycle. Then,*

(a) $(T_1, e_1) \equiv (T_2, e_2)$ *if and only if* $c(T_1, e_1) = c(T_2, e_2)$,

Now let T_1 and T_2 be hyperbolic $2n$-gons. Then,

(b) $T_1 \equiv T_2$ *if and only if* $c(T_1) = c(T_2)$,

(c)

$$\mathrm{Stab}_{\mathrm{Sp}(V)}(T) \cong \begin{cases} D_{2n} & \text{if } c(T) \neq \{1\} \\ D_{4n} & \text{if } c(T) = \{1\} \end{cases}$$

Proof (a) The "only if" part is obvious since if there is $g \in \mathrm{Sp}(V)$ sending (T_1, e_1) to (T_2, e_2), then that g preserves s and hence the value of 3.1.

Now let (T_1, e_1) and (T_2, e_2) be any two edge-marked hyperbolic $2n$-gons. Number the points cyclically

$$\begin{aligned} T_1 &= \{p_1, p_2, \ldots, p_{2n}\}, \\ T_2 &= \{q_1, q_2, \ldots, q_{2n}\}, \end{aligned}$$

such that $e_1 = p_1p_2$ and $e_2 = q_1q_2$. Let

$$\begin{aligned} A_1 &= \langle p_1, p_3, \ldots, p_{2n-1} \rangle, \\ A_2 &= \langle q_1, q_3, \ldots, q_{2n-1} \rangle, \\ B_1 &= \langle p_2, p_4, \ldots, p_{2n} \rangle, \\ B_2 &= \langle q_2, q_4, \ldots, q_{2n} \rangle. \end{aligned}$$

Then $V = A_1 \oplus B_1 = A_2 \oplus B_2$ where A_i and B_i are disjoint maximal totally isotropic subspaces for $i = 1, 2$. By choosing hyperbolic bases from $A_1 \cup B_1$ and $A_2 \cup B_2$ we can see that there exists an element $g \in G$ with $gA_1 = A_2$ and $gB_1 = B_2$. Thus assume that $A_1 = A_2$ and $B_1 = B_2$. Then $\mathrm{Stab}_{\mathrm{Sp}(V)}(A, B) \cong \mathrm{GL}(A)$. Thus it contains an element h such that $hp_{2i-1} = q_{2i-1}$ for $i = 1, 2, \ldots, n$. Thus assume in addition that $p_{2i-1} = q_{2i-1}$ for $i = 1, 2, \ldots, n$. Note that $p_{2i} = B \cap \{r_{2i}, p_1, \ldots, \hat{p_{2i-1}}, \hat{p_{2i+1}}, p_{2n-1}\}^{\perp}$, where indices are taken modulo $2n$ and r_{2i} is a unique point on $\langle p_{2i-1}, p_{2i+1} \rangle - \{p_{2i-1}, p_{2i+1}$.

Then $\mathrm{Stab}_{\mathrm{Sp}(V)}(p_1, p_3, \ldots, p_{2n-1}, B) \cong \mathrm{GL}_1(\mathbb{F})^n$ is simultaneously transitive on the points of $\langle p_{2i-1}, p_{2i+1} \rangle - \{p_{2i-1}, p_{2i+1}\}$ for all $i = 1, 2, \ldots, n-1$. The kernel of this action is the center $Z(\mathrm{GL}(A))$. Thus modulo the kernel of the action of $\mathrm{Sp}(V)$ on $\mathrm{PG}(V)$, there is a unique element $g \in \mathrm{Sp}(V)$ sending p_j to q_j for $j = 1, 2, \ldots, 2n-1$.

We now claim that $gp_{2n} = q_{2n}$ if and only if $c(T_1, e_1) = c(T_2, e_2)$. To this end choose nonzero $u_j \in p_j$ and $v_j \in q_j$ for all $j = 1, 2, \ldots, 2n$. From the preceding discussion and Lemma 3.12 it is clear that we may assume that $u_j = v_j$ for all $i = 1, 2, \ldots, 2n-1$. Moreover, replacing v_{2n} by λv_{2n} we may also assume that $\mathsf{s}(u_{2n-1}, u_{2n}) = \mathsf{s}(v_{2n-1}, v_{2n})$. We now show that $c(T_1, e_1) = c(T_2, e_2)$ if and only if $\mathsf{s}(u_{2n}, u_1) = \mathsf{s}(v_{2n}, v_1)$ if and only if $u_{2n} = v_{2n}$. For the first equivalence, we note that we can choose $u_1, u_2, \ldots, u_{2n}$ such that $\mathsf{s}(u_j, u_{j+1}) = 1$ for all $j = 1, 2, \ldots, 2n-1$. Clearly, then $c(T_1, e_1) = \mathsf{s}(u_{2n}, u_1)^{-1}$ and the first equivalence follows. For the second equivalence, we note that u_{2n} and v_{2n} belong to $X = \{p_3, \ldots, p_{2n-1}\}^{\perp} \cap B$, which has dimension 2. The equations $\mathsf{s}(u_{2n-1}, u_{2n}) = \mathsf{s}(v_{2n-1}, v_{2n})$ and $\mathsf{s}(u_{2n}, u_1) = \mathsf{s}(v_{2n}, v_1)$ determine two independent affine hyperplanes of X that intersect in a single vector. Hence $u_{2n} = v_{2n}$.

(b) Clearly $T_1 \equiv T_2$ if and only if $(T_1, f_1) \equiv (T_2, f_2)$ for some directed edges f_1, f_2. By (a) this happens precisely if $c(T_1, f_1) = c(T_2, f_2)$ for some directed edges f_1, f_2. Given preselected edges e_1, e_2, by Lemma 3.12 this happens precisely if $c(T_1, e_1) \in \{c(T_2, e_2), c(T_2, e_2)^{-1}\}$, that is if $c(T_1) = c(T_2)$.

(c) Fix $e \in T$ and let $g \in \mathrm{Stab}_{\mathrm{Sp}(V)}(T)$. Let $g(e) = f$. Then we have $c(T, e) = c(T, f)$. If $c(T, e) \neq 1$, then by Lemma 3.12, there are exactly $2n$ such directed edges f lying on n undirected edges. If $c(T, e) = 1$, then $c(T, e)^{-1} = c(T, e)$ and so there are exactly $4n$ such directed edges lying on $2n$ undirected edges. The isomorphisms with the dihedral groups are obvious. □

Corollary 3.17 *Hyperbolic $2n$-gons of class number $c \in \mathbb{F}^*$ exist except if $(n, c) = (2, 1)$.*

Proof If $n \geq 3$, then for any $c \in \mathbb{F}^*$, the standard hyperbolic $2n$-cycle of type (n, c) is in fact a hyperbolic $2n$-gon. Now let $n = 2$ and let T be a hyperbolic $2n$-gon. By Proposition 3.16 part (a) such a hyperbolic $2n$-gon is isometric to the standard $2n$-cycle of type (n, c). However, this is a hyperbolic $2n$-gon if and only if $c \neq 1$. □

For any $c \in \mathbb{F}^*$, let $\Theta_c = \{T \in \Theta, \mid c(T) = \{c, c^{-1}\}\}$.

Corollary 3.18 (a) *The orbits of* $\mathrm{Sp}(V)$ *on* Θ *are of the form*

$$C(T(n, c)) = \Theta_c = \Theta_{c^{-1}} = C(T(n, c^{-1})),$$

where c runs over $\mathbb{F}^* - \{1\}$ if $n = 2$ and over $\mathbb{F}^*$ if $n \geq 3$.

(b) *In particular, if* $\mathbb{F} = \mathbb{F}_q$*, then there are* N *orbits, where*

$$N = \begin{cases} \frac{q-3}{2} + 1 & \text{if } n = 2, q \text{ odd,} \\ \frac{q-2}{2} & \text{if } n = 2, q \text{ even,} \\ \frac{q-3}{2} + 2 & \text{if } n \geq 3, q \text{ odd,} \\ \frac{q-2}{2} + 1 & \text{if } n \geq 3, q \text{ even.} \end{cases}$$

Proof (a) By Proposition 3.16 the sets Θ_c are the orbits of $\mathrm{Sp}(V)$ on Θ. By Lemma 3.10 every orbit is a connected component $C(T)$ for some $T \in \mathcal{T}$. Thus $\Theta_c = C(T)$ for some hyperbolic $2n$-gon T with $c(T) = c$. By Corollary 3.17, $c \in \mathbb{F}^*$ can be anything so long as $(n, c) \neq (2, 1)$. Clearly we may choose $T = T(n, c)$, the standard hyperbolic $2n$-gon of type (n, c). By definition $\Theta_c = \Theta_{c^{-1}}$ and (a) follows.

(b) We count the sets $\{(n, c), (n, c^{-1})\}$ such that there exists a hyperbolic $2n$-gon of type (n, c). Noting that the only $c \in \mathbb{F}^*$ such that $c = c^{-1}$ are $c = \pm 1$ and that (n, c) can be any pair except $(2, 1)$ the number N follows. □

We now focus again on the points of Γ_{hyp}.

Lemma 3.19 *Given a hyperbolic* $2n$*-gon* T*, every point of* Γ_{hyp} *belongs to some hyperbolic* $2n$*-gon in* $C(T)$*.*

Proof Let y be a point of Γ_{hyp}. Then by Lemma 3.7 there exists $u \in U_T$ and a point $x \in T$ such that $ux = y$. Clearly y belongs to the hyperbolic $2n$-gon uT and by Lemma 3.10 $uT \in C(T)$. □

Proposition 3.20 *Let* V *be a vector space of dimension* $2n$ *over a field* $\mathbb{F}$ *of odd characteristic endowed with a non-degenerate symplectic form. Suppose* X *is a set of points in the hyperbolic geometry* $\Gamma_{\mathrm{hyp}}(V)$ *such that*

(1) X *contains a hyperbolic* $2n$*-gon* T*,*

(2) *for any hyperbolic* $2n$*-gon* $T' \subseteq X$ *and any hyperbolic line* L *with* $|L \cap T'| \geq 2$ *we have* $p \in X$ *for all* $p \in L$*,*

then, $X = \Gamma_{\mathrm{hyp}}(V)$*.*

Proof Let $T' \subseteq X$ and let T'' be adjacent to T' in Θ. That is, there is $x \in T'$ and $u \in U_x$ such that $uT' = T''$. We claim that $T'' \subseteq X$. Namely, let L and L' be the two lines on x meeting T' in points $y, y' \in T'$ respectively. Then $z = uy \in L$ and $z' = uy' \in L'$ by Lemma 3.6 case (h). By assumption (2) also $z, z' \in X$. Moreover, since $x \perp (T' - \{y, y'\})$ it follows from Lemma 3.6 case (s), that $ux' = x'$ for all points in $T' - \{y, y'\}$. Hence $T'' \subseteq X$. Thus we see that all hyperbolic $2n$-gons in $C(T)$ are contained in X. By Lemma 3.19 we have $X = \Gamma_{\mathrm{hyp}}(V)$. □

We will apply Proposition 3.20 in the proof of Proposition 6.3, where X is a set of $\mathcal{S}$-full 1-spaces with respect to some set of points $\mathcal{S}$ in Γ_k. We recall the definition of $\mathcal{S}$-full.

Definition 3.21 Let V and s be as before and let $\mathcal{S} \subseteq \mathcal{P}_k$ be some Point set of the symplectic k-grassmannian $\Gamma_k^{\mathsf{s}}(V)$. For any l-space $Y \subseteq V$ that is t.i. w.r.t. s, let $\Gamma_k^{\mathsf{s}}(V;Y) = (\mathcal{P}_k(V;Y), \mathcal{L}_k(V;Y))$ be the subgeometry of $\Gamma_k^{\mathsf{s}}(V)$ of all Points and Lines incident to Y. We call Y *$\mathcal{S}$-full* if

$$\mathcal{P}_k(V;Y) \subseteq \langle \mathcal{S} \rangle_{\Gamma_k(V)}.$$

As usual, we'll drop s and V from the notation, if these are clear from the context. If $\{Y_\alpha\}_{\alpha\in A}$ is a collection of t.i. subspaces, then we interpret $\Gamma_k^{\mathsf{s}}(V;\{Y_\alpha\}_{\alpha\in A})$ as $\Gamma_k^{\mathsf{s}}(V;\langle Y_\alpha \mid \alpha \in A\rangle_V)$. This is done typically if $\{Y_\alpha\}_{\alpha\in A}$ is a set of t.i. points on some hyperbolic $2n$-gon.

We will apply Proposition 3.20 in combination with the inductive methods given by Lemmas 3.24 and 3.25.

Definition 3.22 Given a hyperbolic $2n$-gon T and a point p we define a set T^p as follows. Let $T = \{p_1, p_2, \ldots, p_{2n}\}$ and without loss of generality assume $p = p_1$. Take q to be the unique point on $p_1^\perp \cap \langle p_{2n}, p_2\rangle_V$. Then $T^p = \{q, p_3, p_4, \ldots, p_{2n-1}\}$. If the points are labeled $T = \{p_1, p_2, \ldots, p_{2n}\}$ and $p = p_i$ we will write T^i and label $q = q_i$. Note that T^p is a hyperbolic $2(n-1)$-cycle.

Lemma 3.23 *Let (T, e) be an edge-marked hyperbolic $2n$-gon. Suppose that p is a point of T such that e is an edge of T^p. Then, $c(T, e) = c(T^p, e)$. In particular $c(T) = c(T^p)$.*

Proof (a) We number the points of T as $p_1, p_2, \ldots, p_{2n}$ such that p_i is collinear to p_j for all $1 \le i, j \le 2n$, such that $|i-j| \equiv 1 (\mathrm{mod}\ 2n)$ and such that $e = p_1p_2$. We will pick $p = p_4$, the other cases are completely similar. Now T^p has points $p_1, p_2, q_4, p_6, \ldots, p_{2n}$. Choose non-zero $u_j \in p_j$ for all $j = 1, 2, \ldots, 2n$. Note that q_4 is the unique point on $\langle p_3, p_5\rangle - \{p_3, p_5\}$ that is orthogonal to p_4. Thus q_4 contains the non-zero vector $v_4 = u_3\mathsf{s}(u_4, u_5) - u_5\mathsf{s}(u_4, u_3)$ since this vector is clearly orthogonal to $u_4 \in p_4$ and belongs to $\langle p_3, p_5\rangle$. Now

$$\frac{c(T,e)}{c(T^p,e)} = \frac{\mathsf{s}(u_3,u_4)\mathsf{s}(u_5,u_6)}{\mathsf{s}(u_2,u_3)\mathsf{s}(u_4,u_5)} \Big/ \frac{\mathsf{s}(v_4,u_6)}{\mathsf{s}(u_2,v_4)}.$$

However

$$\begin{aligned}
\mathsf{s}(v_4,u_6) &= \mathsf{s}(u_3\mathsf{s}(u_4,u_5)-u_5\mathsf{s}(u_4,u_3),u_6)\\
&= \mathsf{s}(u_3,u_6)\mathsf{s}(u_4,u_5)-\mathsf{s}(u_5,u_6)\mathsf{s}(u_4,u_3)\\
&= \mathsf{s}(u_3,u_4)\mathsf{s}(u_5,u_6)\\
\text{and} & \\
\mathsf{s}(u_2,v_4) &= \mathsf{s}(u_2,u_3\mathsf{s}(u_4,u_5)-u_5\mathsf{s}(u_4,u_3))\\
&= \mathsf{s}(u_2,u_3)\mathsf{s}(u_4,u_5)-\mathsf{s}(u_2,u_5)\mathsf{s}(u_4,u_3)\\
&= \mathsf{s}(u_2,u_3)\mathsf{s}(u_4,u_5).
\end{aligned}$$

so the quotient is 1 and we're done. □

Lemma 3.24 *Let $n \geq 3$ and let T be a hyperbolic $2n$-gon with $c(T) \neq 1$. Then, for any point $p \in T$, the set T^p is a hyperbolic $2(n-1)$-gon for $p^\perp/p$.*

Proof We already saw that T^p is a hyperbolic $2(n-1)$-cycle. By Corollary 3.17 therefore it is a hyperbolic $2n$-gon except if $n-1=2$ and $c(T^p,e)=1$ for some edge e on T^p. By part (a) and the assumption on $c(T)$ this exception does not occur. □

Lemmas 3.24 and 3.25 play a role in the inductive definition of the generating set $\mathcal{S}_{n,k}$.

Lemma 3.25 *Let Y be a t.i. l-space and X a t.i. m-space of V such that $l,m<k$. Then,*

(1) *there is an isomorphism*

$$\begin{array}{lll}
\Gamma^{\mathsf{s}}_k(V;Y) & \to \Gamma^{\mathsf{s}'}_{(k-l)}((Y^\perp\cap V)/Y) & \\
K & \mapsto K/Y & \textit{for any Point } K\\
(D,U) & \mapsto (D/Y,U/Y) & \textit{for any Line } (D,U),
\end{array}$$

where s' is the form induced by s on V/Y, and

(2) *if $Y \leq X$, the isomorphism in (1) restricts to an isomorphism*

$$\Gamma^{\mathsf{s}}_k(V;X) \to \Gamma^{\mathsf{s}'}_{(k-l)}(V/Y;X/Y).$$

Proof This is a direct consequence of the quotient properties of polar spaces. □

4 Definition of the generating set $\mathcal{S}_{n,k}$.

Given a vector space V of dimension $2n$ (with $n \geq 2$) over a field $\mathbb{F}$ endowed with a non-degenerate symplectic form s. For $1 \leq k \leq n$ and any hyperbolic $2n$-gon T of V with $c(T) \neq 1$ we define a set $\mathcal{S}_{n,k}(T,V)$ of points of $\Gamma^{\mathsf{s}}_k(V)$. In case T or V are clear from the context we'll drop them from the notation.

Let us label $T = \{p_1, p_2, \ldots, p_{2n}\}$. We call a set $I \subseteq \mathcal{P}_1$ *totally isotropic* (t.i.) if $\langle I\rangle_V$ is totally isotropic. Given a set $\mathcal{S}$ of points of Γ_k and any $I \subseteq T$ we let

$$\begin{aligned} \mathcal{S}_I &= \{K \in \mathcal{S} \mid K \cap T = I\} \\ \overline{\mathcal{S}}_I &= \biguplus_{I \subseteq J \subseteq T} \mathcal{S}_J. \end{aligned}$$

We note that $\mathcal{S}_I = \overline{\mathcal{S}}_I = \emptyset$ if I is not totally isotropic.

Lemma 4.1

$$\begin{aligned} \mathcal{S} &= \biguplus_{I \subseteq T, t.i.} \mathcal{S}_I \\ \overline{\mathcal{S}}_I &= \biguplus_{I \subseteq J \subseteq T, t.i.} \mathcal{S}_J = \{K \in \mathcal{S} \mid K \supseteq I\} \end{aligned}$$

□

We will now define $\mathcal{S}_{n,k}(T,V)$ by induction on k. For $n \geq 1$ and $k = 1$ define:

$$\mathcal{S}_{n,1}(T,V) = T.$$

In the sequel we'll write $\mathcal{S}_{n,k}$ for $\mathcal{S}_{n,k}(T,V)$. Now let $k \geq 2$ and, by induction, assume that, for $i = 1, 2, \ldots, 2n$, we have already defined a set

$$\mathcal{S}_{n-1,k-1}(T^i, p_i^\perp/p_i),$$

where $T^i = (T \cap p_i^\perp) \cup \{q_i\}$ is as in Definition 3.22. Note that by Lemmas 3.23 and 3.24 and the fact that $c(T) \neq 1$, the set T^i is a hyperbolic $2(n-1)$-gon with $c(T^i) \neq 1$. This set induces a subset of Γ_k defined as follows.

$$\widehat{\mathcal{S}}^{p_i}_{n,k}(T,V) \;= \{\langle p_i, K'\rangle_V \mid K' \in \mathcal{S}_{n-1,k-1}(T^i, p_i^\perp/p_i)\}.$$

In the sequel we'll write $\widehat{\mathcal{S}}^{p_i}_{n,k}$ for $\widehat{\mathcal{S}}^{p_i}_{n,k}(T,V)$ and will replace p_i by i. By Lemma 3.25 with $Y = p_i$, if $\mathcal{S}_{n-1,k-1}(T^i, p_i^\perp/p_i)$ generates $\Gamma_{k-1}(p_i^\perp/p_i)$, then the set $\widehat{\mathcal{S}}^i_{n,k}$ generates $\Gamma_k(V; p_i)$. In that case, if $\widehat{\mathcal{S}}^i_{n,k} \subseteq \mathcal{S}_{n,k}$, then p_i is $\mathcal{S}_{n,k}$-full. From Proposition 6.3 one can conclude that the set $\bigcup_{i=1}^{2n} \widehat{\mathcal{S}}^i_{n,k}$ in fact generates Γ_k, but it is not at all minimal.

From the same proposition it follows that, given any t.i. subset $I \subseteq T$, every set $\widehat{\mathcal{S}}^i_{n,k}$ with $p_i \in I$ in fact contains a generating set for $\Gamma_k(V; \langle I \rangle_V)$. To address this redundancy we use the notation introduced for Lemma 4.1. One verifies that, for $p_i \in I$, we have

$$\begin{aligned} \widehat{\mathcal{S}}^i_{n,k,I} &= \{\langle p_i, K' \rangle_V \mid K' \in \mathcal{S}_{n-1,k-1,I-\{p_i\}}(T^i, p_i^{\perp}/p_i)\} \\ &\uplus \{\langle p_i, K' \rangle_V \mid K' \in \mathcal{S}_{n-1,k-1,(I-\{p_i\})\cup\{q_i\}}(T^i, p_i^{\perp}/p_i)\}, \end{aligned}$$

where $I - \{p_i\} \subseteq T^i - \{q_i\} \subseteq T$ is t.i. and q_i is the point of Definition 3.22. We recall that if $(I - \{p_i\}) \cup \{q_i\}$ is not a t.i. set, then $\mathcal{S}_{n-1,k-1,(I-\{p_i\})\cup\{q_i\}} = \emptyset$.

Note 4.2 Note that $\widehat{\mathcal{S}}^i_{n,k,I}$ is mapped bijectively onto $\mathcal{S}_{n-1,k-1,I-\{p_i\}}(T^i, p_i^{\perp}/p_i) \cup \mathcal{S}_{n-1,k-1,(I-\{p_i\})\cup\{q_i\}}(T^i, p_i^{\perp}/p_i)$ under the isomorphism $\Gamma_k(p_i) \cong \Gamma_{k-1}(p_i^{\perp}/p_i)$.

Abusing notation slightly, for $I \subseteq T$, let $\min I$ be the minimal element in $\{i \mid p_i \in I\}$. Our proposed generating set for Γ_k is defined as follows. We set

$$\mathcal{S}_{n,k} = \begin{cases} T & \text{if } k = 1 \\ \left(\biguplus_{\emptyset \subsetneq I \subseteq T \text{ t.i. and } |I| \leq k} \widehat{\mathcal{S}}^{\min I}_{n,k,I} \right) \setminus \{L\} & \text{if } k = 2 \\ \biguplus_{\emptyset \subsetneq I \subseteq T \text{ t.i. and } |I| \leq k} \widehat{\mathcal{S}}^{\min I}_{n,k,I} & \text{if } k \geq 3 \end{cases}$$

with the following stipulation

(†) In case $k = 2$, L is any one element from

$$\biguplus_{\emptyset \subsetneq I \subseteq T \text{ t.i. and } |I| \leq 2} \widehat{\mathcal{S}}^{\min I}_{n,2,I}$$

meeting T in a single point p, that is, $\{L\} = \widehat{\mathcal{S}}^l_{n,2,\{p_l\}}$ for some fixed $l \in [2n]$. We'll call p the *redundancy point*.

In case $k = 3$, for any $i \in [2n]$, in creating $\mathcal{S}_{n,k,\{p_i\}}$ from $\mathcal{S}_{n-1,k-1}(T^i, p_i^{\perp}/p_i)$, the redundancy point is chosen to be $q_i \in T^i - T$ (see Definition 3.22).

For any $i \in [2n]$, we have $\widehat{\mathcal{S}}^i_{n,2,\{p_i\}} = \{p_i q_i\}$ and $\widehat{\mathcal{S}}^i_{n,2,\{p_i,p_j\}} = \{p_i p_j\}$. As a consequence of Stipulation (†) therefore,

$$|\mathcal{S}_{n,2,\{p_i\}}| = \begin{cases} 0 & \text{if } p_i \text{ is the redundancy point} \\ 1 & \text{else.} \end{cases}$$

Also, any element from $\mathcal{S}_{n,3,\{p_i\}}$ meets T in at least two points.

Other elementary properties of the sets $\mathcal{S}_{n,k}$ are listed in the next lemma. The significance of Stipulation (†) is explained by part (3).

Lemma 4.3 *For $n \geq 2$ we have*

(1a) $\mathcal{S}_{n,k,\emptyset} = \emptyset$ *for all* $k \geq 1$,

(1b) $\mathcal{S}_{n,1,\{p_i\}} = \{p_i\}$ *for all* $i = 1, 2, \ldots, 2n$,

(2a) $\mathcal{S}_{n,2,\{p_i\}} = \{\langle p_i, q_i \rangle_V\}$ *for all* $i \in [2n] \setminus \{l\}$, *and* $\mathcal{S}_{n,2,\{p_l\}} = \emptyset$, *where* p_l *is the redundancy point,*

(2b) $\mathcal{S}_{n,2,\{p_i,p_j\}} = \{\langle p_i, p_j \rangle_V\}$, *for all t.i.* $I = \{p_i, p_j\}$.

(3) $\mathcal{S}_{n,k,I} = \emptyset$ *for all t.i.* $I \subseteq T$ *with* $|I| < 2$ *and all* $k \geq 3$.

Part (1a) says that all elements of $\mathcal{S}_{n,k}$ meet T, and by part (3) if $k \geq 3$, then they meet T in at least two points.

Proof (1a): For $k = 1$ this is because $\mathcal{S}_{n,1} = T$. For $k \geq 2$ this is because, for all $i = 1, 2, \ldots, 2$ and all $K \in \widehat{\mathcal{S}}^i_{n,k}$ we have $p_i \in K$.

(1b,2a,2b): This follows immediately from the definitions.

(3): By part (1a), we only need to show that $\mathcal{S}_{n,k,I} = \emptyset$ for $k \geq 3$ and $|I| = 1$. Let $p_i \in T$, for some $i = 1, 2, \ldots, 2n$ and let $I = \{p_i\}$ so that $I - \{p_i\} = \emptyset$. Then, by definition $\widehat{\mathcal{S}}^i_{n,k,I} = \emptyset$ if and only if

$$\mathcal{S}_{n-1,k-1,\emptyset}(T^i, p_i^{\perp}/p_i) = \emptyset = \mathcal{S}_{n-1,k-1,\{q_i\}}(T^i, p_i^{\perp}/p_i).$$

By (1a), the first equality always holds.

For $k = 3$, also the second equality holds by Stipulation (†) and for $k > 3$ the equality follows by induction on k. □

Example 4.4 Let $n = 4$ and let $|\mathbb{F}| \geq 3$ of characteristic not 2. Then for $k = 1, 2, 3$ we describe the set $\mathcal{S}_{n,k}$ explicitly. First let $T = T(4, -1) = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$ be the standard hyperbolic 8-gon with $c = -1$. On the standard hyperbolic basis this means:

$$\begin{array}{llll} p_1 = \langle e_1 \rangle, & p_3 = \langle e_2 \rangle, & p_5 = \langle e_3 \rangle, & p_7 = \langle e_4 \rangle, \\ p_2 = \langle f_1 - f_2 \rangle, & p_4 = \langle f_2 - f_3 \rangle, & p_6 = \langle f_3 - f_4 \rangle, & p_8 = \langle -f_4 - f_1 \rangle. \end{array}$$

All indices below will be taken modulo 8.

The set $\mathcal{S}_{4,1}$: This is the set T.

The set $\mathcal{S}_{4,2}$: This is the set

$$\{\langle p_i, p_j\rangle, \langle p_i, q_i\rangle \mid 1 \le i, j \le 8, \text{ with } i \ne 8 \text{ and } j \ne i \pm 1\},$$

where $q_i = p_i^\perp \cap \langle p_{i-1}, p_{i+1}\rangle$. On the standard hyperbolic basis this means:

$$\begin{array}{ll} q_1 = \langle f_4 + f_2\rangle, & q_2 = \langle e_1 + e_2\rangle, \\ q_3 = \langle f_1 - f_3\rangle, & q_4 = \langle e_2 + e_3\rangle, \\ q_5 = \langle f_2 - f_4\rangle, & q_6 = \langle e_3 + e_4\rangle, \\ q_7 = \langle f_3 + f_1\rangle, & q_8 = \langle e_1 - e_4\rangle. \end{array}$$

Note that q_8 is not used in the construction of $\mathcal{S}_{4,2}$. The set $\mathcal{S}_{4,2}$ has 27 elements.

The set $\mathcal{S}_{4,3}$: We use the fact that $\mathcal{S}_{4,3} = \biguplus_{i=1}^{6}(\biguplus_{i+1<j\le 8} \widehat{\mathcal{S}}^i_{4,3,\{i,j\}})$. These 3-spaces are of the form $P = \langle p_i, p_j, x\rangle$, where $\langle p_j, x\rangle \in \mathcal{S}_{3,2}(T^i)$ and p_i is the lowest labeled point of P on T.

We describe $\mathcal{S}_{4,3}$ by exhibiting $\biguplus_{i+1<j\le 8} \widehat{\mathcal{S}}^i_{4,3,\{i,j\}}$ for each $1 \le i \le 6$. That is, we first list the points of the hyperbolic 6-gon T^i. Then we describe the set of elements $\langle p_j, y\rangle \in \mathcal{S}_{3,2}(T^i)$ with $i+1 < j \le 8$ so that $\langle p_i, p_j, y\rangle$ belongs to $\mathcal{S}_{4,3}$. In this example only we'll call this set $\widehat{\mathcal{S}}^i$. By Stipulation †, one of these elements is $\langle p_j, x\rangle$, where x is the unique point on the isotropic line spanned by the neighbors of p_j in T^i that is perpendicular to p_j. One easily verifies that if neither of these neighbors is q_i, then $x = q_j$. Otherwise we call this point q_{ij}.

We have $T^1 = \{p_3, p_4, p_5, p_6, p_7, q_1\}$ and $q_{13} = \langle f_4 + f_3\rangle$, $q_{17} = \langle f_3 + f_2\rangle$.

$$\widehat{\mathcal{S}}^1 = \left\{ \begin{array}{llll} \langle p_3, p_5\rangle, & \langle p_3, p_6\rangle, & \langle p_3, p_7\rangle, & \langle p_3, q_{13}\rangle, \\ \langle p_4, p_6\rangle, & \langle p_4, p_7\rangle, & \langle p_4, q_1\rangle, & \langle p_4, q_4\rangle, \\ \langle p_5, p_7\rangle, & \langle p_5, q_1\rangle, & \langle p_5, q_5\rangle, & \\ \langle p_6, q_1\rangle, & \langle p_6, q_6\rangle, & & \\ \langle p_7, q_{17}\rangle & & & \end{array} \right\}.$$

We have $T^2 = \{p_4, p_5, p_6, p_7, p_8, q_2\}$ and $q_{24} = \langle e_1 + e_2 + e_3\rangle$, $q_{28} = \langle e_1 + e_2 - e_4\rangle$.

$$\widehat{\mathcal{S}}^2 = \left\{ \begin{array}{llll} \langle p_4, p_6\rangle, & \langle p_4, p_7\rangle, & \langle p_4, p_8\rangle, & \langle p_4, q_{24}\rangle, \\ \langle p_5, p_7\rangle, & \langle p_5, p_8\rangle, & \langle p_5, q_2\rangle, & \langle p_5, q_5\rangle, \\ \langle p_6, p_8\rangle, & \langle p_6, q_2\rangle, & \langle p_6, q_6\rangle, & \\ \langle p_7, q_2\rangle, & \langle p_7, q_7\rangle & & \\ \langle p_8, q_{28}\rangle & & & \end{array} \right\}.$$

We have $T^3 = \{p_5, p_6, p_7, p_8, p_1, q_3\}$ and $q_{35} = \langle f_1 - f_4\rangle$.

$$\widehat{\mathcal{S}}^3 = \left\{ \begin{array}{llllll} \langle p_5, p_7\rangle, & \langle p_5, p_8\rangle, & \langle p_5, q_{35}\rangle, & \langle p_6, p_8\rangle, & \langle p_6, q_3\rangle, & \langle p_6, q_6\rangle, \\ \langle p_7, q_3\rangle, & \langle p_7, q_7\rangle, & \langle p_8, q_3\rangle, & \langle p_8, q_8\rangle & & \end{array} \right\}.$$

We have $T^4 = \{p_6, p_7, p_8, p_1, p_2, q_4\}$ and $q_{46} = \langle e_2 + e_3 + e_4 \rangle$.

$$\widehat{\mathcal{S}}^4 = \{\langle p_6, p_8\rangle, \langle p_6, q_{46}\rangle, \langle p_7, q_4\rangle, \langle p_7, q_7\rangle, \langle p_8, q_4\rangle, \langle p_8, q_8\rangle\}.$$

We have $T^5 = \{p_7, p_8, p_1, p_2, p_3, q_5\}$ and $q_{57} = \langle f_1 + f_2 \rangle$.

$$\widehat{\mathcal{S}}^5 = \{\langle p_7, q_{57}\rangle, \langle p_8, q_5\rangle, \langle p_8, q_8\rangle\}.$$

We have $T^6 = \{p_8, p_1, p_2, p_3, p_4, q_6\}$ and $q_{68} = \langle e_1 - e_3 - e_4 \rangle$.

$$\widehat{\mathcal{S}}^6 = \{\langle p_8, q_{68}\rangle\}.$$

The set $\mathcal{S}_{4,3}$ has $14 + 14 + 10 + 6 + 3 + 1 = 48 = \binom{8}{3} - \binom{8}{1}$ elements.

For $q = 3$ this geometry has 918400 points and 11939200 lines. A computation with the computer algebra package GAP shows that $\mathcal{S}_{4,3}$ generates the geometry rather efficiently.

5 The set $\mathcal{S}_{n,k}$ has size $\binom{2n}{k} - \binom{2n}{k-2}$.

In this section we count the elements in $\mathcal{S}_{n,k} = \mathcal{S}_{n,k}(T, V)$ as defined in Section 4. Before we do so, we need some preparation.

Lemma 5.1 *Let*

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_k \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ \cdots \\ n_k \end{bmatrix}, \textit{ and } \quad \mathbf{A} = \left(\quad a_{ij} \quad \right),$$

where

$$a_{ij} = \binom{j}{i}.$$

If

$$\mathbf{c} = A\mathbf{n}, \tag{5.1}$$

then

$$\sum_{i=1}^{k} (-1)^{i+1} c_i = \sum_{j=1}^{k} n_j.$$

Proof We need to show that

$$\mathbf{v}^t\mathbf{c} = \mathbf{u}^t\mathbf{n}, \tag{5.2}$$

where $\mathbf{u}^t = (1, 1, \ldots, 1)$ and $\mathbf{v}^t = (1, -1, 1, -1, \ldots, (-1)^{k+1})$. In view of Equation 5.1 it suffices to show that

$$\mathbf{v}^t A = \mathbf{u}^t, \tag{5.3}$$

since multiplying this by $\mathbf{n}$ yields Equation 5.2. But the j-th entry of $\mathbf{v}^t A$ is

$$\sum_{i=1}^{j}(-1)^{i+1}\binom{j}{i} = -((1-1)^j - 1) = 1.$$

We are done. □

We will count the elements in $\mathcal{S}_{n,k}$ by grouping them into subsets $\mathcal{S}_{n,k,I}$ where $I \subseteq T$ is an i-set generating a totally isotropic i-space. That is, I is a coclique in the $2n$-gon that is T. Clearly we will need to know how many such i-cocliques T possesses.

Lemma 5.2 (a) *The number of cocliques of size i in a $2n$-gon equals*

$$\frac{2n}{i}\binom{2n-i-1}{i-1}.$$

(b) *For any $1 \leq k$, the number of k-sets in a $2n$-set equals*

$$\binom{2n}{k} = \frac{2n}{k}\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\binom{2n-i-1}{k-1}.$$

Proof (a): Let $A_{m,l}$ be the number of l-subsets L of the m-set $[m] = \{1, 2, \ldots, m\}$ such that L contains no two consecutive elements. We will prove that

$$A_{m,l} = \binom{m-l+1}{l}.$$

Clearly $A_{m,l}$ is the number of ways to arrange l black balls and $m-l$ white balls in a sequence so that no two black balls touch. In every such sequence every black ball, except possibly the last, is followed by at least one white ball. Viewing the first $l-1$ black ball-white ball pairs and the last 1 black ball as separators for $l+1$ slots, we see that the number of desired sequences is equal to the number of ways to distribute the remaining $m-(2l-1)$ white balls over $l+1$ slots. Thus there are really $\binom{m-(2l-1)+l}{l} = \binom{m-l+1}{l}$ choices for L.

It now follows that the number of cocliques of size i in a $2n$-gon containing a given vertex equals $A_{2n-3,i-1} = \binom{2n-i-1}{i-1}$. The result now follows.

(b): Fix a k-set $K_0 \subseteq [2n]$ with $1 \notin K_0$. Let

$$c_i = |\{(I, K) \mid |I| = i, I \subseteq K_0, |K| = k, 1 \in K \subseteq [2n], I \subseteq K_0 - K\}|,$$

and

$$n_j = |\{(J, K) \mid |J| = j, J \subseteq K_0, |K| = k, 1 \in K \subseteq [2n], J = K_0 - K\}|.$$

One verifies that

$$\begin{aligned} c_i &= \tbinom{k}{i}\tbinom{2n-(i+1)}{k-1} \\ \text{and} \qquad n_j &= \tbinom{k}{k-j}\tbinom{2n-(k+1)}{j-1}. \end{aligned}$$

Note that $\sum_{j=1}^{k} n_j$ is the number of all k-sets K containing 1 since each such K determines J (as in the definition of n_j) uniquely. It now follows that

$$\binom{2n}{k} = \frac{2n}{k} \sum_{j=1}^{k} n_j.$$

Hence in order to prove part (b) we need

$$\sum_{j=1}^{k} n_j = \sum_{i=1}^{k} (-1)^{i+1} c_i.$$

This follows from Lemma 5.1 and the observation that

$$c_i = \sum_{j=1}^{k} \binom{j}{i} n_j.$$

This, in turn, is not hard to see if we interpret $\binom{j}{i} = |\{I \mid |i| = i, I \subseteq J\}|$ for the j-set J. Clearly each of the pairs (I, K) counted by c_i occurs exactly once since K determines J (as in the definition of n_j) uniquely. □

We shall now count the elements of $\mathcal{S}_{n,k}$. Our inductive argument follows the inductive definition of $\mathcal{S}_{n,k}$.

Proposition 5.3 (a) *For $n \geq 2$ and $k = 1, 2$ we have*

$$|\mathcal{S}_{n,k}| = \binom{2n}{k} - \binom{2n}{k-2}.$$

Now let $n \geq 3$.

(b) *For $k \geq 3$ and any $I \subseteq T$ with $1 \leq i = |I| \leq k$ we have*

$$|\overline{\mathcal{S}}_{n,k,I}| = \binom{2(n-i)}{k-i} - \binom{2(n-i)}{k-i-2}.$$

(c) *For $k \geq 3$ we have*

$$|\mathcal{S}_{n,k}| = \binom{2n}{k} - \binom{2n}{k-2}.$$

Proof (a): Clearly

$$\begin{aligned} |\mathcal{S}_{n,1}| &= 2n &= \tbinom{2n}{1} - \tbinom{2n}{-1} \\ \text{and} \qquad |\mathcal{S}_{n,2}| &= \tfrac{2n(2n-3)}{2} + 2n - 1 &= \tbinom{2n}{2} - \tbinom{2n}{0}. \end{aligned}$$

We prove (b) and (c) by induction on $k \geq 3$. Let us refer to the statement (b) for fixed k as (b,k), and likewise for (c). We first show that (b, k)$\Longrightarrow$(c, k): We claim that

$$|\mathcal{S}_{n,k}| = \sum_{i=1}^{k} (-1)^{i+1} \frac{2n}{i} \binom{2n-i-1}{i-1} \left(\binom{2(n-i)}{k-i} - \binom{2(n-i)}{k-i-2} \right). \tag{5.4}$$

To this end, let

$$c_i = |\{(K, I) \mid |I| = i, K \in \mathcal{S}_{n,k}, I \subseteq K \cap T\}|$$

and

$$n_j = |\{(K, J) \mid |J| = j, K \in \mathcal{S}_{n,k}, J = K \cap T\}|$$

According to Lemma 5.2, the number of t.i. i-sets $I \subseteq T$ is equal to

$$\frac{2n}{i} \binom{2n-i-1}{i-1}.$$

Therefore, it follows from our assumption of (b), that

$$c_i = \frac{2n}{i} \binom{2n-i-1}{i-1} \left(\binom{2(n-i)}{k-i} - \binom{2(n-i)}{k-i-2} \right).$$

By Lemma 4.3 (1a), for every $K \in \mathcal{S}_{n,k}$ we have $K \cap T = J \neq \emptyset$. Hence,

$$|\mathcal{S}_{n,k}| = \sum_{j=1}^{k} n_j.$$

Hence in order to prove our claim 5.4, we need

$$\sum_{j=1}^{k} n_j = \sum_{i=1}^{k} (-1)^{i+1} c_i.$$

This follows from Lemma 5.1 and the observation that

$$c_i = \sum_{j=1}^{k} \binom{j}{i} n_j.$$

This, in turn, is not hard to see if we interpret $\binom{j}{i} = |\{I \mid |i| = i, I \subseteq J\}|$ for the j-set J. Clearly each of the pairs (I, K) counted by c_i occurs exactly once since K determines J (as in the definition of n_j) uniquely.

One verifies that

$$\frac{2n}{i}\binom{2n-i-1}{i-1}\binom{2(n-i)}{k-i} = \frac{2n}{k}\binom{k}{i}\binom{2n-i-1}{k-1}$$

simply by writing $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ and cancelling what appears left and right. Recalling that $k \geq 3$, this enables us to rewrite Equation 5.4 as

$$|\mathcal{S}_{n,k}| = \sum_{i=1}^{k} (-1)^{i+1} \left(\frac{2n}{k}\binom{k}{i}\binom{2n-i-1}{k-1} - \frac{2n}{k-2}\binom{k-2}{i}\binom{2n-i-1}{k-3} \right).$$

We split this as the difference of two sums in the obvious way and observe that in the second sum the terms with $i > k-2$ are zero since $\binom{k-2}{i} = 0$. Then, using Lemma 5.2 part (b) we find that

$$|\mathcal{S}_{n,k}| = \binom{2n}{k} - \binom{2n}{k-2},$$

as desired.

We now prove (b) by induction on $k \geq 3$. First note that the following equation holds for $I \subseteq T$ t.i. and any $p_r \in I$, where either $k = 3$ and $1 \leq i = |I| \leq 3$ or $k \geq 4$ and $2 \leq i = |I| \leq k$:

$$|\overline{\mathcal{S}}_{n-1,k-1,I-\{p_r\}}(T^r, p_r^{\perp}/p_r)| = \binom{2(n-i)}{k-i} - \binom{2(n-i)}{k-i-2}. \tag{5.5}$$

For $k = 3$ and $i = 1$ this is ensured by part (a). For $k = 3$ and $i = 2$, Stipulation (†) ensures that the redundancy point is $q_r \notin I \subseteq T$ so that the set has size $2n-4$ rather than $2n-5$. For $k = 3$ and $i = 3$ we have 1 on either side. For $k \geq 4$ this is the induction hypothesis.

To simplify notation, for $J \subseteq T$ t.i. with $2 \leq |J| \leq k$ and $p_r \in J$, let

$$\begin{aligned} s_{J-p_r} &= |\mathcal{S}_{n-1,k-1,J-\{p_r\}}(T^r, p_r^\perp/p_r)| \\ s_{J-p_r+q_r} &= |\mathcal{S}_{n-1,k-1,J-\{p_r\}\cup\{q_r\}}(T^r, p_r^\perp/p_r)|. \end{aligned}$$

We now prove that for $J \subseteq T$ t.i. with $2 \leq |J| \leq k$ and any two points $p_r, p_s \in J$, we have

$$s_{J-p_r} + s_{J-p_r+q_r} = s_{J-p_s} + s_{J-p_s+q_s} \tag{5.6}$$

That is,

$$|\widehat{\mathcal{S}}^r_{n,k,J}| = |\widehat{\mathcal{S}}^s_{n,k,J}|. \tag{5.7}$$

We prove this by induction on $k - j$, where $j = |J|$. For $j = k$ we have

$$s_{J-p_r} = 1 = s_{J-p_s} \text{ and } s_{J-p_r+q_r} = 0 = s_{J-p_s+q_s}$$

and we are done.

Now let $2 \leq j < k$. We have

$$\begin{aligned} \sum_{J\subseteq H} s_{H-p_r} + s_{H-p_r+q_r} &= |\overline{\mathcal{S}}_{n-1,k-1,J-\{p_r\}}(T^r, p_r^\perp/p_r)| && \text{by Lemma 4.1,} \\ &= \binom{2(n-j)}{k-j} - \binom{2(n-j)}{k-j-2} && \text{by Equation 5.5,} \end{aligned}$$

where the sum is over those t.i. H satisfying $J \subseteq H \subseteq T$ and $|H| \leq k$. In particular,

$$\sum_{J\subseteq H} s_{H-p_r} + s_{H-p_r+q_r} = \sum_{J\subseteq H} s_{H-p_s} + s_{H-p_s+q_s}.$$

Now Equation 5.6 follows since by induction

$$\sum_{J\subsetneq H} s_{H-p_r} + s_{H-p_r+q_r} = \sum_{J\subsetneq H} s_{H-p_s} + s_{H-p_s+q_s}.$$

As a consequence, for any t.i. $I \subseteq T$ with $1 \leq |I| \leq k$ we have

$$\begin{aligned} |\overline{\mathcal{S}}_{n,k,I}| &= \sum_{I\subseteq J\subseteq T \text{ t.i.}} |\mathcal{S}_{n,k,J}| && \text{by Lemma 4.1} \\ &= \sum_{I\subseteq J\subseteq T \text{ t.i.}} |\widehat{\mathcal{S}}^{\min J}_{n,k,J}| && \text{by definition of } \mathcal{S}_{n,k} \\ &= \sum_{I\subseteq J\subseteq T \text{ t.i.}} |\widehat{\mathcal{S}}^r_{n,k,J}| \text{ for any } p_r \in I && \text{by Equation 5.7} \end{aligned}$$

Note for the latter equality, that this is automatic if $|I| = 1$ and $I = J$ so that $\min J = r$ already. By definition of $\widehat{\mathcal{S}}$ the latter sum equals

$$\sum_{I \subseteq J \subseteq T \text{ t.i.}} |\mathcal{S}_{n-1,k-1,J-\{p_r\}}(T^r, p_r^\perp/p_r)| + |\mathcal{S}_{n-1,k-1,J-\{p_r\}\cup\{q_r\}}(T^r, p_r^\perp/p_r)|. \qquad (5.8)$$

Now in case $2 \leq |I|$ this is

$$\begin{aligned} &= |\overline{\mathcal{S}}_{n-1,k-1,I-\{p_r\}}(T^r, p_r^\perp/p_r)| && \text{by definition of } \overline{\mathcal{S}} \\ &= \binom{2(n-i)}{k-i} - \binom{2(n-i)}{k-i-2} && \text{by Equation 5.5} \end{aligned}$$

In case $|I| = 1$, note that sum 5.8 is in fact a sum over all t.i. subsets of T^r. Hence it equals

$$|\mathcal{S}_{n-1,k-1}(T^r, p_r^\perp/p_r)|$$

which is equal to

$$\binom{2(n-1)}{k-1} - \binom{2(n-1)}{k-3},$$

as desired. For $k = 3$ this follows from (a) and, for $k \geq 4$ it follows from the induction hypothesis together with the implication (b, $k-1$)$\Longrightarrow$(c, $k-1$). This proves (b, k). By the first part of this proof, we then also have (c, k). □

6 The set $\mathcal{S}_{n,k}$ generates Γ_k.

Let V be a vector space of dimension $2n$ over a field $\mathbb{F}$ of odd characteristic. Let s be a non-degenerate symplectic form on V. Fix a hyperbolic $2n$-gon T for V and let $\mathcal{S}_{n,k}(T, V)$ be as defined in Section 4.

Lemma 6.1 *For $n \geq 2$, the set $\mathcal{S}_{n,1}$ generates Γ_1.*

Proof Recall $\mathcal{S}_{n,1} = T$ and $L = \langle p_1, p_2 \rangle_V$. Using Proposition 3.20 it suffices to prove that every point x on a line L of Γ_{hyp} with $|L \cap T| = 2$ belongs to $\langle T \rangle_{\Gamma_1}$.

We proceed by induction on n. Let $n = 2$. Let $T = \{p_1, p_2, p_3, p_4\}$. Clearly the singular lines p_1p_3 and p_2p_4 belong to $\langle T \rangle_{\Gamma_1}$. Among the points on these two lines one finds the set of points on an apartment, a quadrangle in Γ_1 and it is well-known that such a set of points generates the geometry if $\text{Char}(\mathbb{F}) \neq 2$ (compare Theorem 2.1). In particular $x \in \langle T \rangle_{\Gamma_1}$.

Now let $n > 2$. Then $L' = \langle p_{n+1}, p_{n+2} \rangle_V$ satisfies $L \perp L'$. Now $p_1, p_2, p_{n+1}, p_{n+2}$ are the points of an apartment for the $\text{Sp}_4(\mathbb{F})$ polar space on $\langle p_1, p_2, p_{n+1}, p_{n+2} \rangle_V$. These points generate the $\text{Sp}_4(\mathbb{F})$ polar space. In particular all points on L, including x, are in $\langle p_1, p_2, p_{n+1}, p_{n+2} \rangle_{\Gamma_1}$. □

Lemma 6.2 *For $n \geq 2$, the set $\mathcal{S}_{n,2}$ generates Γ_2.*

Proof Let $\mathcal{S} = \mathcal{S}_{n,2}$ and let $\mathcal{F}$ be the set of $\mathcal{S}$-full points of Γ_1. Recall that a hyperbolic basis for V is a basis $\mathcal{E} = \{e_{i,\emptyset}, e_{\emptyset,i} \mid i \in [n]\}$ such that, for $I, J \subseteq [n]$, the linear subspace

$$e_{I,J} = \langle e_{i,\emptyset}, e_{\emptyset,j} \mid i \in I, j \in J\rangle_V$$

is totally isotropic if and only if $I \cap J = \emptyset$.

We will prove that, for some hyperbolic basis $\mathcal{E}$, the span $\langle \mathcal{S}\rangle_{\Gamma_2}$ contains a set of t.i. 2-spaces of the following form:

(1) All t.i. 2-spaces $e_{i,j}$ with $1 \leq i \neq j \leq n$,

(2) $(n-1)$ linearly independent 2-spaces lying on a point e contained in a totally isotropic n-space, where $e \in e_{[n],\emptyset}$, but $e \notin e_{J,\emptyset}$ for any $J \subsetneq [n]$.

The main result in Blok [3] says that such a set generates Γ_2 and we are done.

An easy consequence is the following:

(A) If $\mathcal{F}$ contains a hyperbolic basis $\mathcal{E}$ and a point e such that

$$\mathsf{s}(e, e_{i,\emptyset}) = 0 \neq \mathsf{s}(e, e_{\emptyset,i})$$

for all $i = 1, 2, \ldots, n$, then $\mathcal{S}$ is a generating set for Γ_2.

Namely, the equalities ensure that e belongs to the maximal t.i. subspace $e_{[n],\emptyset}$ and the inequalities ensure that $e \notin e_{[n],\emptyset} \cap e_{\emptyset,[n]-J}^{\perp} = e_{J,\emptyset}$ for any $\emptyset \neq J \subseteq [n]$. Hence, $\mathcal{S}$ contains a generating set as in (1) and (2).

We first consider the case $n = 2$. Note that we have no t.i. 3-spaces. Let $H = \langle p_1, p_2\rangle_V$. Then $U = H^{\perp}$ is a hyperbolic 2-space also. We claim that all points on $U \cup H$ are $\mathcal{S}$-full. Namely, for any point $u \in U$ the t.i. 2-spaces p_1u and p_2u are in $\langle \mathcal{S}\rangle$. Let $e_{1,\emptyset} = p_1$, $e_{\emptyset,1} = p_2$, $e_3 = p_1p_3 \cap p_2^{\perp} \in H$, and $e_4 \in H - e_3$, and let $e = p_3$. Then, A holds and we are done.

We prove the cases $n \geq 3$ by induction on n. First, let $n = 3$. Let $e_{1,\emptyset} = p_1$, $e_{2,\emptyset} = p_5$ $e_{\emptyset,1} = p_2$ $e_{\emptyset,2} = p_4$ Setting $H = \langle p_1, p_2, p_4, p_5\rangle^{\perp}$, let $e_{3,\emptyset} = p_3^{\perp} \cap H$ and let $e_{\emptyset,3} \in H - e_{3,\emptyset}$. Finally, let $e = p_3$. We claim that $\mathcal{E} = \{e_{i,\emptyset}, e_{\emptyset,i} \mid i \in [3]\}$ together with e is a set of $\mathcal{S}$-full points as described in (A).

To this end, we first show that every point $h \in H$, in particular $e_{3,\emptyset}$ and $e_{\emptyset,3}$, is $\mathcal{S}$-full. Namely, since p_1, p_2, p_4, p_5 are $\mathcal{S}$-full, these points contain a t.i. 2-space on h. Since, by Theorem 2.1 the points p_1, p_2, p_4, p_5 generate $\Gamma_1(\langle p_1, p_2, p_4, p_4\rangle_V)$, the point h is $\mathcal{S}$-full. We only have to show that

$$\mathsf{s}(e, e_{i,\emptyset}) = 0 \neq \mathsf{s}(e, e_{\emptyset,i})$$

for all $i = 1, 2, 3$. But this is true by definition of T and by choice of $e_{3,\emptyset}$ and $e_{\emptyset,3}$.

Now let $n \geq 4$. Let $T' = p_1, p_2, \ldots, p_{2n-3}, q_{2n-1}$. This is a hyperbolic $(2n-2)$-gon for $U = \langle T' \rangle_V$. Let $H = U^\perp$. Clearly U is non-degenerate of dimension $2n-2$ and H is a hyperbolic 2-space. We can take q_{2n-1} as the redundancy point for T' and so, by induction, the t.i. 2-spaces of $\mathcal{S}$ on the points of T' generate $\Gamma_2(U)$. In particular, all t.i. 2-spaces on q_{2n-1} contained in U are in $\langle \mathcal{S} \rangle_{\Gamma_2}$. In addition, the t.i. 2-spaces $\langle q_{2n-1}, p_{2n-1} \rangle_V$ and $\langle q_{2n-1}, p_{2n-2} \rangle_V = \langle p_{2n-2}, p_{2n} \rangle_V$ are in $\mathcal{S}$. It follows that q_{2n-1} is $\mathcal{S}$-full also.

We now show that each point $h \in H$ is $\mathcal{S}$-full. Namely, for each of the points $p \in T'$ the t.i. 2-space $\langle p, h \rangle_V$ belongs to $\langle \mathcal{S} \rangle_{\Gamma_2}$. Since T' generates $\Gamma_1(U)$ and $n > 2$ we find that h is $\mathcal{S}$-full. Combining this with the earlier proved fact that $\Gamma_2(U) \subseteq \langle \mathcal{S} \rangle_{\Gamma_2}$, it follows that all points of U are $\mathcal{S}$-full, also.

Let $e = p_{2n-2}$. As $e \notin U$ and $e \not\perp p_{2n-3}$ also $e \notin H$. It follows that in U we can choose a hyperbolic basis $\mathcal{E}(U) = \{e_{i,\emptyset}, e_{\emptyset,i} \mid i \in [n-1]\}$ such that

$$\mathsf{s}(e, e_{i,\emptyset}) = 0 \neq \mathsf{s}(e, e_{\emptyset,i})$$

for all $i \in [n-1]$. In addition, let $e_{n,\emptyset} = e^\perp \cap H$ and let $e_{\emptyset,n} \in H - e^\perp$. Now $\mathcal{E} = \mathcal{E}(U) \cup \{e_{n,\emptyset}, e_{\emptyset,n}\}$ and e form an $\mathcal{S}$-full set as in (A) and we are done. $\square$

Proposition 6.3

For $n \geq 2$, and $k \geq 1$ we have

(a) $|\mathcal{S}_{n,k}| = \binom{2n}{k} - \binom{2n}{k-2}$,

(b) $\mathcal{S}_{n,k}$ *generates* Γ_k.

For $n, k \geq 3$ and any t.i. $I \subseteq T$ with $1 \leq i = |I| \leq k$, we have

(c) $|\overline{\mathcal{S}}_{n,k,I}| = \binom{2(n-i)}{k-i} - \binom{2(n-i)}{k-i-2}$

(d) $\overline{\mathcal{S}}_{n,k,I}$ *generates* $\Gamma_k(I)$.

In the proof we will use the following notation.

$$\overline{\widehat{\mathcal{S}}}^i_{n,k,I} = \{\langle p_i, K' \rangle_V \mid K' \in \overline{\mathcal{S}}_{n-1,k-1,I-\{p_i\}}(T^i, p_i^\perp/p_i)\}$$

We note that using Lemma 4.1, one verifies that $\overline{\widehat{\mathcal{S}}}^i_{n,k,I} = \biguplus_{I \subseteq J \subseteq T} \widehat{\mathcal{S}}^i_{n,k,J}$.

Proof Part (a) and (c) follow immediately from Proposition 5.3, but we include them here for convenience.

Let us refer to statement (b) for fixed k and arbitrary $n \geq k$ as (b, k). We'll do likewise for statement (d).

We first prove (b, k) by induction on $k \geq 1$. For $k = 1, 2$, this is Lemma 6.1 and Lemma 6.2.

We now prove (b, k) for $k \geq 3$ assuming (d, k) holds. Call $\mathcal{S} = \mathcal{S}_{n,k}$ and let $\mathcal{F}$ be the set of $\mathcal{S}$-full points of Γ_1. Using Proposition 3.20 we must prove

(1) $\mathcal{F}$ contains a hyperbolic $2n$-gon T,

(2) given a hyperbolic $2n$-gon $T' \subseteq \mathcal{F}$ and a line L of Γ_{hyp} with $|L \cap T'| = 2$, then every point x on L belongs to $\mathcal{F}$.

Namely, taking $|I| = 1$ in (d, k) we find that $T \subseteq \mathcal{F}$ and so (1) holds.

We now prove (2) by induction on $n-k$ with $n \geq k$. Let $T' = \{p_1, p_2, \dots, p_{2n}\} \subseteq \mathcal{F}$ be a hyperbolic $2n$-gon and let H be a line of Γ_{hyp} with $H \cap T' = \{p_1, p_2\}$ and let $x \in H$ be an arbitrary point. We will show that x is $\mathcal{S}$-full as well.

To this end, let $W = H^\perp$. Note that, since p_1 and p_2 are $\mathcal{S}$-full, all t.i. k-spaces on p_1 or p_2 and meeting W in a $(k-1)$-space K' are in $\langle \mathcal{S} \rangle_{\Gamma_k}$.

Case $n = k$: In this case, the collection of t.i. k-spaces on K' forms a line of Γ_k and it follows that the k-space $\langle x, K' \rangle_V$ also lies in $\langle \mathcal{S} \rangle_{\Gamma_k}$. Thus x is $\mathcal{S}$-full and we are done.

Case $3 \leq k < n$: Let $T'_{1,2} = H^\perp \cap T' = \{p_4, p_5, \dots, p_{2n-1}\}$. This can be completed to a hyperbolic $2(n-1)$-gon T'_W by adding two points r_1 and r_2 that are collinear in Γ_{hyp}. Consider the set $\mathcal{S}_{n-1,k}(T'_W, W)$. By induction on n, this is a generating set for $\Gamma_k(W)$. Since $k \geq 3$, if follows from Lemma 4.3 that all elements of $\mathcal{S}_{n-1,k}(T'_W, W)$ meet $T'_{1,2}$ in at least one point. Since all points of $T'_{1,2}$ are $\mathcal{S}$-full, we have $\mathcal{S}_{n-1,k}(T'_W, W) \subseteq \langle \mathcal{S} \rangle_{\Gamma_k}$.

At this point we make the following observation: For a set $\mathcal{S}(W)$ of points in $\Gamma_k(W)$ we have

$$\langle \mathcal{S}(W) \rangle_{\Gamma_k(W)} \subseteq \langle \mathcal{S}, \mathcal{S}(W) \rangle_{\Gamma_k}. \tag{6.1}$$

This is self-evident, except if the lines of the geometry $\Gamma_k(W)$ are not lines of Γ_k, that is, if $k = n-1$. However, in that case, let K_3 and K_4 be collinear points of $\Gamma_k(W)$. Then $K_3 = \langle s_3, L \rangle_V \leq W$ and $K_4 = \langle s_4, L \rangle_V \leq W$ for some $(k-1)$-space L and certain t.i. 1-spaces $s_3, s_4 \subseteq W$. Note that also $K_1 = \langle p_1, L \rangle_V$ and $K_2 = \langle p_2, L \rangle_V$ are t.i. k-spaces in $\langle \mathcal{S} \rangle_{\Gamma_k}$. Now $\Gamma_k(L) \cong \Gamma_1(L^\perp/L)$ is isomorphic to $\mathrm{Sp}_4(\mathbb{F})$ and is generated by p_1, p_2, s_3, s_4 since $\langle p_1, p_2 \rangle_V$ and $\langle s_3, s_4 \rangle_V$ are orthogonal hyperbolic 2-spaces. As a consequence, for any t.i. 1-space $s \leq \langle s_3, s_4 \rangle_V$ the t.i. k-space $\langle s, L \rangle_V$ is in $\langle \mathcal{S}, \mathcal{S}(W) \rangle_{\Gamma_k}$, as desired.

Now applying (d, k) we see that $\overline{\mathcal{S}}_{n-1,k,\{r_i\}}(T'_W, W)$ $(i = 1, 2)$ generates $\Gamma_k(r_i)$ in $\Gamma_k(W)$. That is, r_1 and r_2 are $\mathcal{S}_{n-1,k}(T'_W, W)$-full in $\Gamma_k(W)$ and by Equation 6.1 all t.i. k-spaces of W on r_1 are in $\langle \mathcal{S} \rangle_{\Gamma_k}$. In turn, since all points of T'_W are $\mathcal{S}_{n-1,k}(T'_W, W)$-full, again using Equation 6.1 we find that all t.i. k-spaces of $\Gamma_k(W)$ belong to $\langle \mathcal{S} \rangle_{\Gamma_k}$.

Now consider an arbitrary k-space K on x. Then $K \cap W$ is a $(k-1)$-space L. Then by the previous, all t.i. k-spaces on L, both those in W and those on p_1 and p_2 are in $\langle \mathcal{S} \rangle_{\Gamma_k}$. We find that these t.i. k-spaces generate $\Gamma_k(L) \cong \Gamma_1(L^\perp/L)$. In particular $\langle x, L \rangle_V \in \langle \mathcal{S} \rangle_{\Gamma_k}$. This completes the proof of (2) and establishes (d, k)$\Longrightarrow$(b, k).

We will now prove (d, k) by induction on $k \geq 3$. When considering case $k = 3$, we keep in mind that (d, 1) holds for any I and that (d, 2) holds for all $I \neq \{p\}$, where p is the redundancy point. When considering case $k \geq 4$, we assume that (d, $k-1$) holds by induction. Since we proved that (b, 2) holds and that (d, $k-1$) implies (b, $k-1$) for $k \geq 4$, in all cases $k \geq 3$ we may assume that (b, $k-1$) holds as well, that is, $\mathcal{S}_{n,k-1}$ generates Γ_{k-1}.

Note that, for $I \subseteq T$ t.i. with $2 \leq |I| \leq k$ and any $p_r \in I$ we have

$$\langle \overline{\mathcal{S}}_{n-1,k-1,I-\{p_r\}}(T^r, p_r^\perp/p_r) \rangle = \Gamma_{k-1}(I - \{p_r\}) \tag{6.2}$$

For $k = 3$ this is ensured by (d, 2); Note that (d, 2) only fails if somehow $I - \{p_r\} = \{q\}$, where q is the redundancy point of T^r. This does not occur since by Stipulation (†), $q = q_r \notin T$ and $I \subseteq T$. For $k \geq 4$ Equation 6.2 follows directly from the induction hypothesis.

In order to apply induction it will be necessary to show that, for any $p_r \in I$,

$$\left\langle \biguplus_{I \subseteq J \subseteq T \text{ t.i. and } |J| \leq k} \widehat{\mathcal{S}}^r_{n,k,J} \right\rangle = \left\langle \biguplus_{I \subseteq J \subseteq T \text{ t.i. and } |J| \leq k} \widehat{\mathcal{S}}^{\min J}_{n,k,J} \right\rangle \tag{6.3}$$

We now prove Equation 6.3 by induction on $i = |I|$. Clearly the statement is true for $i = k$ since then left and right we have only one element, namely $\langle I \rangle_V$. Now assume that the statement is true for $i+1, i+2, \ldots, k$. That is,

$$\left\langle \biguplus_{I \subsetneq J \subseteq T \text{ t.i. and } |J| \leq k} \widehat{\mathcal{S}}^r_{n,k,J} \right\rangle = \left\langle \biguplus_{I \subsetneq J \subseteq T \text{ t.i. and } |J| \leq k} \widehat{\mathcal{S}}^{\min J}_{n,k,J} \right\rangle$$

Notice that lifting Equation 6.2 to Γ_k tells us that for any $p_r \in I$, we have

$$\langle \overline{\widehat{\mathcal{S}}}^r_{n,k,I} \rangle_{\Gamma_k} = \Gamma_k(I) \tag{6.4}$$

Namely,

$$\begin{aligned}
\langle \overline{\widehat{\mathcal{S}}}^r_{n,k,I} \rangle_{\Gamma_k} &= p_r \oplus \langle \overline{\mathcal{S}}_{n-1,k-1,I-\{p_r\}}(T^r, p_r^\perp/p_r) \rangle && \text{by } \Gamma_k(I) \cong \Gamma_{k-1}(I - \{p_r\}) \\
&= p_r \oplus \Gamma_{k-1}(I - \{p_r\}) && \text{by Equation 6.2} \\
&= \Gamma_k(I) && \text{by } \Gamma_k(I) \cong \Gamma_{k-1}(I - \{p_r\})
\end{aligned}$$

Then,

$$
\begin{aligned}
\langle \biguplus_{I\subseteq J\subseteq T \text{ t.i. and } |J|\leq k} \widehat{\mathcal{S}}^{r}_{n,k,J}\rangle &= \langle \overline{\widehat{\mathcal{S}}}^{r}_{n,k,I}\rangle_{\Gamma_k} \\
&= \langle \overline{\widehat{\mathcal{S}}}^{\min I}_{n,k,I}\rangle_{\Gamma_k} && \text{Equation 6.4} \\
&= \langle \biguplus_{I\subseteq J\subseteq T \text{ t.i. and } |J|\leq k} \widehat{\mathcal{S}}^{\min I}_{n,k,J}\rangle \\
&= \langle \widehat{\mathcal{S}}^{\min I}_{n,k,I} \cup \biguplus_{I\subsetneq J\subseteq T \text{ t.i. and } |J|\leq k} \widehat{\mathcal{S}}^{\min I}_{n,k,J}\rangle \\
&= \langle \widehat{\mathcal{S}}^{\min I}_{n,k,I} \cup \biguplus_{I\subsetneq J\subseteq T \text{ t.i. and } |J|\leq k} \widehat{\mathcal{S}}^{\min J}_{n,k,J}\rangle && \text{induction on } i \\
&= \langle \biguplus_{I\subseteq J\subseteq T \text{ t.i. and } |J|\leq k} \widehat{\mathcal{S}}^{\min J}_{n,k,J}\rangle
\end{aligned}
$$

This proves Equation 6.3.

As a consequence, for any t.i. $I \subseteq T$ with $1 \leq |I| \leq k$ we have

$$
\begin{aligned}
\langle \overline{\mathcal{S}}_{n,k,I}\rangle_{\Gamma_k} &= \langle \biguplus_{I\subseteq J\subseteq T \text{ t.i.}} \mathcal{S}_{n,k,J}\rangle_{\Gamma_k} && \text{by Lemma 4.1} \\
&= \langle \biguplus_{I\subseteq J\subseteq T \text{ t.i.}} \widehat{\mathcal{S}}^{\min J}_{n,k,J}\rangle_{\Gamma_k} && \text{by definition of } \mathcal{S}_{n,k} \\
&= \langle \biguplus_{I\subseteq J\subseteq T \text{ t.i.}} \widehat{\mathcal{S}}^{r}_{n,k,J}\rangle_{\Gamma_k} \text{ for any } p_r \in I && \text{by Equation 6.3}
\end{aligned}
$$

Note for the latter equality, that this is automatic if $|I| = 1$ and $I = J$ so that $\min J = r$ already. By definition of $\widehat{\mathcal{S}}$ the latter span equals

$$
\left\langle p_r \oplus K' \mid K' \in \biguplus_{J} \mathcal{S}_{n-1,k-1,J-\{p_r\}}(T^r, p_r^{\perp}/p_r) \uplus \mathcal{S}_{n-1,k-1,J-\{p_r\}\cup\{q_r\}}(T^r, p_r^{\perp}/p_r)\right\rangle_{\Gamma_k} \tag{6.5}
$$

where the union is taken over all t.i. J with $I \subseteq J \subseteq T$.

Now in case $2 \leq |I|$ this is

$$
\begin{aligned}
&\langle p_r \oplus K' \mid K' \in \overline{\mathcal{S}}_{n-1,k-1,I-\{p_r\}}(T^r, p_r^{\perp}/p_r)\rangle_{\Gamma_k} && \text{definition of } \overline{\mathcal{S}} \\
&= p_r \oplus \langle \overline{\mathcal{S}}_{n-1,k-1,I-\{p_r\}}(T^r, p_r^{\perp}/p_r)\rangle_{\Gamma_{k-1}} && \Gamma_k(I) \cong \Gamma_{k-1}(I-\{p_r\}) \\
&= p_r \oplus \Gamma_{k-1}(I-\{p_r\}) && \text{Equation 6.2 and } 2 \leq |I| \\
&= \Gamma_k(I) && \Gamma_k(I) \cong \Gamma_{k-1}(I-\{p_r\})
\end{aligned}
$$

In case $|I| = 1$, that is, $I = \{p_r\}$, note that the span 6.5 is in fact the span over all t.i. subsets of T^r. Hence it equals

$$\begin{aligned}&\langle p_r \oplus K' \mid K' \in \mathcal{S}_{n-1,k-1}(p_r^\perp/p_r)\rangle_{\Gamma_k}\\ &= p_r \oplus \langle \mathcal{S}_{n-1,k-1}(p_r^\perp/p_r)\rangle_{\Gamma_{k-1}(p_r^\perp/p_r)} \quad \text{since } \Gamma_k(p_r) \cong \Gamma_{k-1}(p_r^\perp/p_r)\end{aligned}$$

which, by (b, $k-1$) is equal to

$$p_r \oplus \Gamma_{k-1}(p_r^\perp/p_r) \quad = \Gamma_k(p_r)$$

as desired. This proves (d, k). □

7 Embeddings and the proof of the main theorem

Lemma 7.1 *The geometry Γ_k has an embedding of dimension $\binom{2n}{k} - \binom{2n}{k-2}$.*

Proof The embedding is afforded by the Lie algebra module whose highest weight is the k-th fundamental dominant weight. We will sketch the construction here. For a more detailed description, see Blok [2]. The symplectic group $G = \mathrm{Sp}(V)$ is the group of linear transformations of V preserving s. Let $I = \{1, 2, \ldots, n\}$ and let $\mathcal{A} = \{e_i, f_i \mid i \in I\}$ be a hyperbolic basis/apartment for V. Furthermore, for $i \in I$, let $c_i = \langle e_1, e_2, \ldots, e_i\rangle$, a t.i. i-space and consider the chamber $c = (c_1, c_2, \ldots, c_n)$. Now $B = \mathrm{Stab}_G(c)$, $N = \mathrm{Stab}_G(\mathcal{A})$ is a (B, N)-pair for G. Then $(W, \{r_i\}_{i\in I})$, where $W = \langle r_i \mid i \in I\rangle \cong 2^n \cdot \mathrm{Sym}(n)$ and

$$\begin{aligned} r_i\colon\ & e_i \leftrightarrow e_{i+1} \quad (1 \le i \le n-1)\\ & f_i \leftrightarrow f_{i+1}\\ r_n\colon\ & e_n \leftrightarrow f_n \end{aligned}$$

is the Coxeter system of type $M = C_n$ associated to (B, N). For any $J \subseteq I$ we have a Coxeter subsystem $(W_J = \langle r_j \mid j \in J\rangle, \{r_j\}_{j\in J})$ of type M_J. Using Bruhat decomposition we may define the standard parabolic subgroup of type J as

$$P_J = \biguplus_{w\in W_J} BwB.$$

Now Γ_k can be presented as having the coset space $G/P_{I-\{k\}}$ as points and the set $gP_k \in G/P_k$ as lines, identifying gP_k with $\{ghP_{I-\{k\}} \mid h \in P_k\}$.

Let $M = M(\lambda_k)$ be the Lie algebra module over $\mathbb{F}$ whose highest weight is the fundamental weight λ_k corresponding to the node k of the C_n-diagram (see Figure 1). Then there is a natural action $G \to \mathrm{GL}(M)$ for which the stabilizer of the highest weight subspace $\langle v^+\rangle$ is exactly $P_{I-\{k\}}$ so that the mapping

$$\begin{aligned}\varphi\colon\ & \Gamma_k \hookrightarrow \mathbb{P}(M)\\ & gP \mapsto g\langle v^+\rangle\end{aligned}$$

injects the point-set of Γ_k into the set of 1-spaces of M. Moreover, since $P_k \cong \mathrm{SL}_2(\mathbb{F})$ has a 2-dimensional submodule, φ sends the lines of Γ_k to full 2-spaces of M. Finally, since M is a cyclic module by construction we have

$$\langle \varphi(\Gamma_k) \rangle_M = M.$$

Thus φ is a full projective embedding. The module M has dimension $\binom{2n}{k} - \binom{2n}{k-2}$.
□

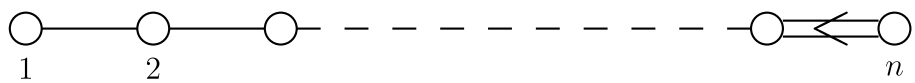

Figure 1: The Dynkin diagram of type C_n

The embedding $M(\lambda_k)$ is obtained as follows. Recall that Γ_k is obtained as the geometry of t.i. k-subspaces of V (of dimension $2n$) with respect to the symplectic form s. It is well-known that the map sending $K \mapsto \wedge^k K$, where $K \subseteq V$ is a k-subspace yields an embedding of the projective k-grassmannian into $\mathbb{P}(\bigwedge^k V)$. Consider the map

$$\begin{aligned} \theta\colon \bigwedge^{k} V &\to \bigwedge^{k-2} V \\ x_1 \wedge x_2 \wedge \cdots \wedge x_k &\mapsto \sum_{1 \le i < j \le k} (-1)^{i+j} \mathsf{s}(x_i, x_j) x_1 \wedge x_2 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_k. \end{aligned}$$

Then $M(\lambda_k) = \ker\theta$. It is clear that a k-space in V is in $\ker\theta$ if and only if it is totally isotropic with respect to s. Thus, the 1-spaces generated by pure vectors in $\ker\theta$ correspond bijectively to the t.i. k-spaces in V and Γ_k has $\ker\theta$ as a projective embedding.

Proof (of Theorem 1). Let as before Γ_k denote the grassmannian of totally isotropic k-spaces of a vector space of dimension $2n$ with respect to a non-degenerate symplectic form. In Proposition 6.3 we found a generating set $\mathcal{S}_{n,k}$ for Γ_k of size $\binom{2n}{k} - \binom{2n}{k-2}$. By Lemma 7.1 Γ_k has an embedding W of dimension $\binom{2n}{k} - \binom{2n}{k-2}$. Since $\dim(W) \le |\mathcal{S}|$ for any embedding W and generating set $\mathcal{S}$ we are done. □

References

[1] M. Aschbacher and Gary M. Seitz, Involutions in Chevalley groups over fields of even order, *Nagoya Math. J.*, **63** (1976), 1–91.

[2] Blok, Rieuwert J., On geometries related to buildings, Doctorate Thesis, Delft University of Technology (1999).

[3] R. J. Blok, The generating rank of the symplectic line-grassmannian, *Beiträge zur Algebra und Geometrie* **44:2** (2003), 575-580.

[4] R. J. Blok and A. E. Brouwer, Spanning point-line geometries in buildings of spherical type, *J. Geometry* **62** (1998), 26–35.

[5] R. J. Blok and A. Pasini, Point-line geometries with a generating set that depends on the underlying field, in *Finite Geometries*, Proceedings of the Fourth Isle of Thorns Conference (2000), 1–25.

[6] A. Blokhuis and A. E. Brouwer, The universal embedding dimension of the binary symplectic dual polar space, *Discrete Math.***264:1-3** (2003), 3–11.

[7] B. N. Cooperstein and E. E. Shult, Frames and bases of Lie incidence geometries, *J. Geometry* **60** (1997), 17–46.

[8] B. N. Cooperstein, B. N., On the generation of some dual polar spaces of symplectic type over GF(2), *European J. Combin.***18: 7** (1997), 741–749.

[9] B. N. Cooperstein, Generating long root subgroup geometries of classical groups over finite prime fields, *Bull. Belg. Math. Soc.* **5** (1998), 531–548.

[10] B.N. Cooperstein, On the generation of dual polar spaces of symplectic type over finite fields, *J. Combin. Theory Ser. A*, **83:2** (1998), 221–232.

[11] B.N. Cooperstein, Generation of embeddable incidence geometries: a survey, in *Topics in diagram geometry* 29–57, (2003).

[12] B. N. Cooperstein, personal communication.

[13] P. Li, On the universal embedding of the $\mathrm{Sp}_{2n}(2)$ dual polar space, *J. Combin. Theory Ser. A* **94:1** (2001), 100–117.