Topological properties of activity orders for matroid bases

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Abstract

Las Vergnas [9] introduced several lattice structures on the bases of an ordered matroid M by using their external and internal activities. He also noted [10] that when computing the Möbius function of these lattices, it was often zero, although he had no explanation for that fact. The purpose of this paper is to provide a topological reason for this phenomenon. In particular, we show that the order complex of the external lattice $L(M)$ is homotopic to the independence complex of the restriction $M^*|T$ where M^* is the dual of M and T is the top element of $L(M)$. We then compute some examples showing that this latter complex is often contractible which forces all its homology groups, and thus its Möbius function, to vanish. A theorem of Björner [3] also helps us to calculate the homology of the matroid complex.

1 The external and internal orders

In September of 2001, there was a conference on Tutte Polynomials and Related Topics at the Centre de Recerca Matemàtica in Barcelona, Spain. At the meeting, Michel Las Vergnas gave a talk about three lattice structures which he had imposed on the bases of an ordered matroid using external and internal activity [9]. During the question and answer period that followed, one of us (Sagan), asked if Las Vergnas knew anything about the Möbius function of these lattices. Las Vergnas replied that he had computed some examples and noted that the value was often zero, but did not have an explanation for that fact.

In this paper, we will give a topological reason for Las Vergnas' observation. The rest of this section will be devoted to developing the definition and some basic properties of the external lattice, $L(M)$, of an ordered matroid M. In the next section, we derive some results about the structure of $L(M)$ which will be useful in working with its order complex $\Delta(M)$. In particular, we give a simpler formula for the join operator than was given by Las Vergnas. The third section contains our main theorem, showing that $\Delta(M)$ is homotopic to the independence complex IN of the restriction $M^*|T$ where M^* is the dual of M and T is the top element of $L(M)$. In section 4, we compute some examples showing that IN is often contractible which forces all its homology groups, and thus its Möbius function, to be zero. A characterization of the homology of IN due to Björner [3] is recalled in the next section and used for the calculation of yet more examples. The final section contains a couple of open problems.

Let M be a matroid on a finite set E. We denote the bases and independent sets of M by $\mathcal{B} = \mathcal{B}(M)$ and $\mathcal{I} = \mathcal{I}(M)$, respectively. We say that M is *ordered* if E is linearly ordered. From now on all matroids will be ordered.

Given a set $F \subseteq E$ we say that $e \in E$ is *active with respect to* F if there is a circuit $C(F; e) \subseteq F \cup \{e\}$ in which e is minimal with respect to the ordering on E. Let

$$\mathrm{Act}_M(F) = \{e \ : \ e \text{ is active with respect to } F\}.$$

Note that we include the possibility that $e \in F$. Note also that we will often write one-element sets without the set braces and drop M as a subscript if the matroid is clear from context.

For $F \subseteq E$ we define

$$\mathrm{Ext}_M(F) = \mathrm{Act}_M(F) - F.$$

The elements of $\mathrm{Ext}_M(F)$ are called *externally active with respect to* F. This coincides with the usual notion of externally active elements with respect to an element of $\mathcal{B}$.

Las Vergnas defined the external lattice of M in a manner equivalent to the following. For $A, B \in \mathcal{B}$, define

$$A \le_M^{\mathrm{ext}} B \text{ if and only if } A \subseteq B \cup \mathrm{Ext}_M(B).$$

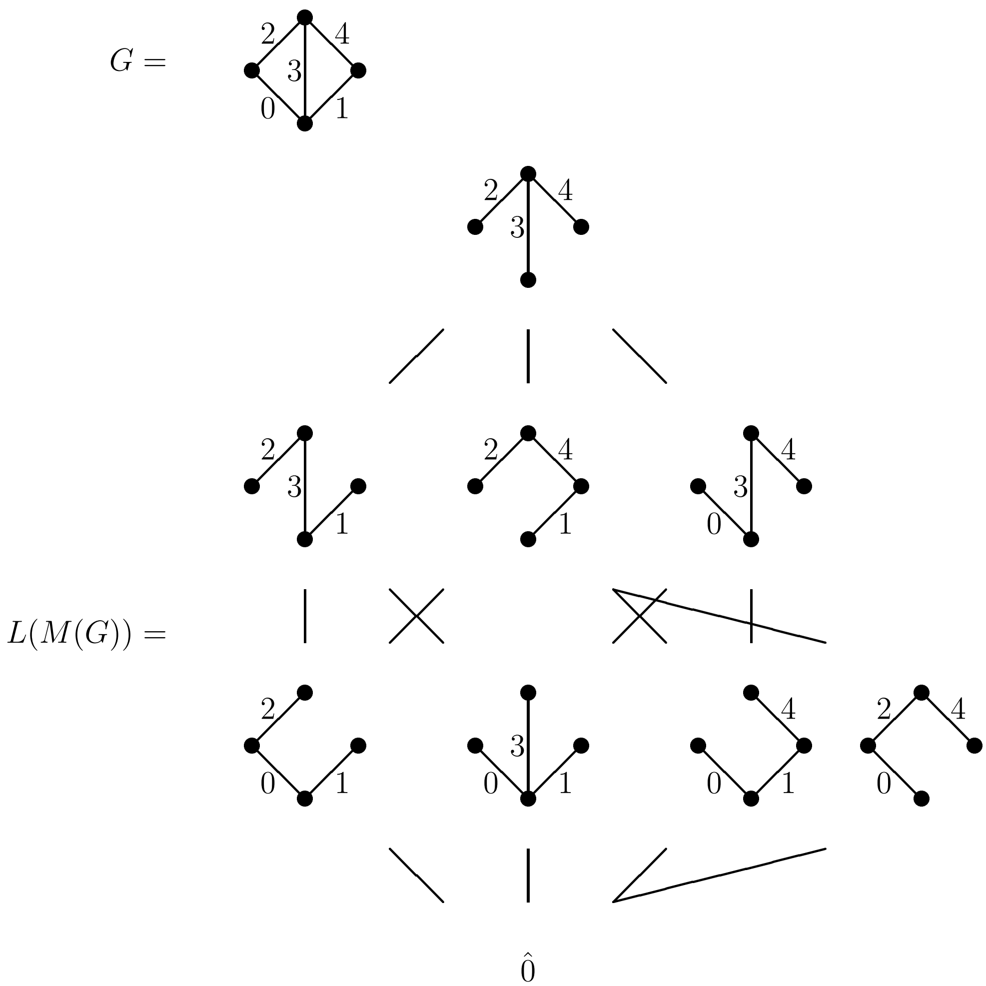

Figure 1: An example graph and its external lattice

It was proven in [9] that, when augmented with a minimum element $\hat{0}$, the resulting order is in fact a graded lattice with rank function

$$\rho_M(B) = |\operatorname{Ext}_M(B)| + 1. \tag{1}$$

We will denote this lattice by $L(M)$. It is important to remember that, even though our notation does not show it, this lattice structure depends on the ordering of the base set of M.

By way of illustration, let us construct an external lattice using the cycle matroid $M = M(G)$ of a graph $G = (V, E)$. Let G be the graph in Figure 1 with edges ordered as indicated. Then $L(M(G))$ has Hasse diagram as shown. So, for example, to compute the bases below the base $B = \{0, 3, 4\}$ note that $\text{Ext}_M(B) = \{1\}$ since the edge 1 is the smallest element in its fundamental circuit with B while 2 is not. It follows that any base contained in $B \cup \text{Ext}_M(B) = \{0, 1, 3, 4\}$ will be less than or equal to B. These bases are exactly those obtained by removing some element of the fundamental circuit of 1 from the union.

Returning to our general exposition, let M^* be the dual matroid of M. We turn M^* into an ordered matroid using the order already given on E. Las Vergnas [9] also defined another ordering $\leq_M^{\text{int}}$ on $\mathcal{B}(M)$ by

$$A \leq_M^{\text{int}} B \iff (E - B) \leq_{M^*}^{\text{ext}} (E - A). \tag{2}$$

We should note that one can also define $\leq_M^{\text{int}}$ using the internal activity of bases of M (which also eliminates the need to pass to M^*), but (2) will be more convenient for our purpose. When augmented with a maximum element $\hat{1}$, the resulting order is called the *internal order*. Directly from the definitions, we see that this structure is just the order-theoretic dual of $L(M^*)$. Since the dual of a lattice has the same homology as the original lattice, we will restrict ourselves to external orders. For that reason, we will also drop the ext superscript.

It will be useful in the sequel to have the following characterization, due to Las Vergnas [9, Proposition 3.1] of the external order.

Proposition 1.1 (Las Vergnas) *Let A, B be two bases of an ordered matroid M. Then $A \leq B$ if and only if B is the lexicographically maximum base of M contained in $A \cup B$ (where elements of a base are listed in increasing order).* ■

In the aforementioned paper it was shown that the number of elements at a given rank in $L(M)$ does not depend on the particular order on E, but that the lattice itself does. We wish to give some measure of how $L(M)$ depends on the order on E.

Proposition 1.2 *Let $\trianglelefteq$ and $\trianglelefteq'$ be linear orders on E. Given a matroid on E, let M and M' be the corresponding ordered matroids. Suppose that* $\text{Act}(M) = \text{Act}(M')$ *and that $\trianglelefteq$, $\trianglelefteq'$ when restricted to this set are same. Then*

$$L(M) \cong L(M').$$

Proof We prove that the identity map from $\mathcal{B}(M)$ to $\mathcal{B}(M')$ induces a lattice isomorphism of $L(M)$ with $L(M')$. So we need to show that for $A, B \in \mathcal{B}(M) = \mathcal{B}(M')$ we have $A \subseteq B \cup \text{Ext}_M(B)$ if and only if $A \subseteq B \cup \text{Ext}_{M'}(B)$. Clearly it suffices to have $\text{Ext}_M(B) = \text{Ext}_{M'}(B)$. We will show $\text{Ext}_M(B) \subseteq \text{Ext}_{M'}(B)$ and then the reverse inclusion follows by symmetry. Now take $a \in \text{Ext}_M(B)$ and let C be the unique cycle in $B \cup a$. So a is the $\trianglelefteq$-minimum in C and it suffices to show that it is also the $\trianglelefteq'$-minimum. Let a' be this $\trianglelefteq'$-minimum. Then $a, a' \in \text{Act}(M) = \text{Act}(M')$ with $a \trianglelefteq a'$ and $a' \trianglelefteq' a$. Since the two orderings agree on this set, $a = a'$ and we are done. ■

2 Sublattices and the join operator

Fix a subset $F \subseteq E$ and let $K = M|F$ be the restriction of M to F. Note that it is an ordered matroid with respect to the ordering induced on F by E. We will say that K is *spanning* if F is a spanning set of M, that is, F contains a base of M. We will show that the lattice for a spanning matroid is closely related to that of the parent matroid. But first we need a lemma.

Lemma 2.1 *Suppose that $F \subseteq E$ and $K = M|F$. Then for any $J \subseteq F$ we have*

(a) $\mathrm{Act}_K(J) = \mathrm{Act}_M(J) \cap F$, *and as a consequence*

(b) $\mathrm{Ext}_K(J) = \mathrm{Ext}_M(J) \cap F$.

Proof (a) The fact that $\mathrm{Act}_K(J) \subseteq \mathrm{Act}_M(J) \cap F$ is clear from the definitions. For the opposite inclusion, suppose $e \in \mathrm{Act}_M(J) \cap F$. Then there is a circuit $C \subseteq J \cup e$ in which e is minimal. But then $C \subseteq F$ and e is minimal with respect to the ordering induced on F so that $e \in \mathrm{Act}_K(J)$.

Part (b) follows immediately from part (a). ∎

Corollary 2.2 *Suppose that $K = M|F$ is spanning. Then the inclusion $\mathcal{B}(K) \subseteq \mathcal{B}(M)$ induces an inclusion*

$$L(K) \subseteq L(M).$$

Proof Suppose $A, B \in \mathcal{B}(K)$. We prove that $A \leq_M B$ if and only if $A \leq_K B$. By definition, $A \leq_M B$ if and only if $A \subseteq B \cup \mathrm{Ext}_M(B)$. Since $A, B \subseteq F$ this happens if and only if $A \subseteq B \cup (\mathrm{Ext}_M(B) \cap F)$. By the previous lemma, $B \cup (\mathrm{Ext}_M(B) \cap F) = B \cup \mathrm{Ext}_K(B)$. So we are done. ∎

Following Las Vergnas [9], for a spanning subset $A \subseteq E$ we define

$$\mathrm{MaxBas}\, A = A - \mathrm{Act}(A).$$

Alternatively, one can define this as the lexicographically maximum base of M contained in A, using the convention of Proposition 1.1. We obtain the maximum element of $L = L(M)$ as

$$T = \mathrm{MaxBas}\, E$$

and reserve the notation T for this top element. Las Vergnas gave a formula for the join operator $\vee$ for two elements of L using the MaxBas operator. Using Corollary 2.2 we give a slight but useful simplification of his result, at the same time extending it to the join of an arbitrary number of elements in L.

Corollary 2.3 *The join of elements $B_i \in \mathcal{B}(M)$ $(i = 1, 2, \ldots, m)$ in $L(M)$ is given by*

$$\bigvee_{i=1}^{m} B_i = \mathrm{MaxBas}\left(\bigcup_{i=1}^{m} B_i\right)$$

Proof Let $K = M|F$ where $F = \bigcup_{i=1}^{m} B_i$ and let $S = \mathrm{MaxBas}(F)$. We must prove that $S = \bigvee_{i=1}^{m} B_i$. First of all, for all i we have $B_i \leq_K S$ because S is the maximal element of $L(K)$. By Corollary 2.2 this means $B_i \leq_M S$ for all i.

Now suppose $T \in \mathcal{B}(M)$ satisfies $B_i \leq_M T$ for all i. Then $B_i \subseteq T \cup \mathrm{Ext}_M(T)$ so that $F = \bigcup_{i=1}^{m} B_i \subseteq T \cup \mathrm{Ext}_M(T)$. But $S \subseteq F \subseteq T \cup \mathrm{Ext}_M(T)$ and so by we have $S \leq_M T$. Thus $S = \bigvee_{i=1}^{m} B_i$. ■

We denote the set of atoms of $L(M)$ by $\mathcal{A}(M)$. By (1), these are precisely the bases B for M with $\mathrm{Ext}(B) = \emptyset$.

Corollary 2.4 *Let $\mathcal{A}' \subseteq \mathcal{A}(M)$. Then $\bigvee_{B \in \mathcal{A}'} B = T$ if and only if every element of T is contained in some element $B \in \mathcal{A}'$.*

Proof This follows from Corollary 2.3 and the following observation which is needed for the "if" direction. Suppose $T \subseteq F$ for some $F \subseteq E$. Then since $T \cap \mathrm{Act}(F) \subseteq T \cap \mathrm{Act}(E) = \emptyset$ we have $T \subseteq \mathrm{MaxBas}(F)$. Also, if F is spanning, then $\mathrm{MaxBas}(F)$ is a base for M. Since T is also a base for the matroid M, we find $T = \mathrm{MaxBas}(F)$. ■

The inclusion in Corollary 2.2 does not preserve the rank function in general. But it does under certain circumstances.

Lemma 2.5 *If $K = M|F$ is spanning and $B \in \mathcal{B}(K) \subseteq \mathcal{B}(M)$, then the following hold.*

(a) *We have $\rho_K(B) = \rho_M(B)$ if and only if $\mathrm{Ext}_M(B) \subseteq F$.*

(b) *If $F \supseteq E - T$, then the inclusion $L(K) \subseteq L(M)$ preserves rank.*

(c) *If $f < e$ for all $f \in F$ and $e \in E - F$, then the inclusion $L(K) \subseteq L(M)$ preserves rank.*

Proof (a) We have $\rho_K(B) = |\,\mathrm{Ext}_K(B)| + 1$ and $\rho_M(B) = |\,\mathrm{Ext}_M(B)| + 1$. Now Lemma 2.1 completes the proof.

(b) This follows from part (a) since for any $A \subseteq E$ we have $\mathrm{Ext}_M(A) \subseteq \mathrm{Act}_M(E) = E - T$.

(c) This also follows from part (a) since the assumption implies that no element of $E - F$ can be externally active with respect to any subset of F. ■

Given a subset $F \subseteq E$ and an ordering on F we can always define an ordering on E such that the condition in (c) of Lemma 2.5 holds. Thus we have proved the following observation.

Corollary 2.6 *Let K be an ordered matroid on a set F. If M is an unordered matroid on a set $E \supseteq F$ such that $K = M|F$ and K is spanning, then we can find an ordering on E inducing a rank-preserving inclusion $L(K) \subseteq L(M)$.*

In particular if K is the cycle matroid of a connected graph H with edge set F, then for M we can take the cycle matroid of the complete graph on the vertex set of H.

3 The homotopy equivalence

In this section we study the reduced homology of the order complex of the lattice $L(M)$. We will show that there is a homotopy equivalence between the order complex of $L(M)$ and the independence complex of M^* restricted to T. This will we used in the next section to explain Las Vergnas' observation about the Möbius function of $L(M)$.

Let L be a finite lattice with minimum and maximum elements $\hat{0}$ and $\hat{1}$, respectively. Note that L will be used when discussing an arbitrary lattice, whereas the symbol $L(M)$ will always be used when we wish to refer to the external lattice of a matroid. We denote by $\Delta(L)$ the *order complex* of L, that is, the abstract simplicial complex on the set $L - \{\hat{0}, \hat{1}\}$ whose faces are the nonempty chains in $L - \{\hat{0}, \hat{1}\}$ ordered by inclusion. If $L = L(M)$ for some matroid, then we will also use the notation $\Delta(M) = \Delta(L(M))$.

There is another abstract simplicial complex associated with a matroid. The *independence complex* of M, denoted $IN(M)$, is the simplicial complex of nonempty independent subsets of M. Our main theorem relates the two complexes we have defined. In it, $\tilde{\mathrm{H}}_i(\Delta)$ will denote the reduced i-dimensional homology group of a complex Δ with coefficients in $\mathbb{Z}$ (see e.g. Stanley [14, Ch.3]).

Theorem 3.1 *We have a homotopy equivalence*

$$\Delta(M) \simeq IN(M^*|T).$$

So, for all $i \geq -1$, we have an isomorphism in homology

$$\tilde{\mathrm{H}}_i(\Delta(M)) \cong \tilde{\mathrm{H}}_i(IN(M^*|T)).$$

Note that this result implies that the homotopy type of the order complex depends only on the maximum base T. We will prove Theorem 3.1 using the next two propositions.

Let L be an arbitrary lattice with atom set $\mathcal{A}$. Let $\mathcal{J} = \mathcal{J}(L)$ be the abstract simplicial complex of all subsets of $\mathcal{A}$ whose join is not $\hat{1}$. The following is a theorem of Lakser [8] later generalized by Björner [2] and Segev [13].

Proposition 3.2 *For any lattice* L

$$\Delta(L) \simeq \mathcal{J}(L). \qquad \blacksquare$$

Let $\mathcal{F}$ be an abstract simplicial complex on a finite set F. A *facet covering* of $\mathcal{F}$ is a multiset of facets $\mathcal{C} = \{F_0, F_1, \ldots, F_n\}$ such that every face of $\mathcal{F}$ is contained in some F_i. The *nerve* $\text{Nerv}(\mathcal{C})$ of the covering is the simplicial complex on the vertex set $I = \{0, 1, 2, \ldots, n\}$ where a subset $J \subseteq I$ is a face if and only if $\bigcap_{j\in J} F_j$ is a face of $\mathcal{F}$. As will be seen, the nerve of a certain covering of $\mathcal{J}(L)$ is isomorphic to $IN(M^*|T)$.

But first we must show that $\mathcal{F}$ and $\text{Nerv}(\mathcal{C})$ are the same up to homotopy. Note that every nonempty intersection of facets of $\mathcal{F}$ is again a face of $\mathcal{F}$. Thus the intersections $\bigcap_{j\in J} F_j$ are contractible as subspaces of $\mathcal{F}$ and hence are acyclic. Thus the hypotheses of the Nerve Theorem of Borsuk and Folkman are satisfied (see (10.6) in Björner [4]) and we obtain our second proposition.

Proposition 3.3 *Let $\mathcal{F}$ be a simplicial complex on a set F and let $\mathcal{C}$ be a facet covering. Then*

$$\mathcal{F} \simeq \text{Nerv}(\mathcal{C}). \qquad \blacksquare$$

The last link in our chain of homotopy equivalences will be provided by T', the set of elements of T which are independent as singleton sets in M^*. Then $IN(M^*|T) = IN(M^*|T')$. Note that the elements $e \in E$ which are not independent in M^* are precisely those which are contained in every base for M. We can now prove our main result.

Proof (of Theorem 3.1) Combining Propositions 3.2 and 3.3 for any facet covering $\mathcal{C}(L)$ of $\mathcal{J}(L)$ we have

$$\Delta(L) \simeq \mathcal{J}(L) \simeq \text{Nerv}(\mathcal{C}(L)).$$

So it suffices to show that we can find a facet covering $\mathcal{C} = \mathcal{C}(L(M))$ such that $\text{Nerv}(\mathcal{C})$ and $IN(M^*|T)$ are isomorphic as simplicial complexes.

We have $IN(M^*|T) = IN(M^*|T')$ and suppose $T' = \{t_0, t_1, \ldots, t_n\}$. For $0 \leq i \leq n$, define $F_i = \{A \in \mathcal{A} \ : \ A \subseteq E - \{t_i\}\}$. Then it follows from Corollary 2.4 that these are the facets of $\mathcal{J}(L(M))$, possibly with repetitions. Let $\mathcal{C}$ be the corresponding facet covering of $\mathcal{J}(L(M))$. We can now define a bijection $\phi : IN(M^*|T') \to \text{Nerv}(\mathcal{C})$ as follows. If $S \subseteq T'$ then let

$$\phi(S) = J = \{j \ : \ t_j \in S\}.$$

Clearly ϕ is a bijection between subsets of T' and subsets of I. We claim that ϕ restricts to a well-defined isomorphism between the respective complexes, that is, $\bigcap_{j\in\phi(S)} F_j \neq \emptyset$ if and only if S is independent in $M^*|T'$. This is because S is independent in $M^*|T'$ if and only if $E - S$ contains a base for M which, by Lemma 2.5(b), is equivalent to $E - S$ containing an atom for $L(M)$. This completes the proof of the isomorphism and of Theorem 3.1. $\blacksquare$

4 Applications

We are now ready to explain the empirical observation of Las Vergnas that the Möbius function μ of the external lattice $L(M)$ often satisfies $\mu(L(M)) = 0$. It is known that, given any finite lattice L with minimum element $\hat{0}$, maximum element $\hat{1}$, and Möbius function μ, one has

$$\mu(L) := \mu_L(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta) = \sum_{i=-1}^{\infty} (-1)^i \dim \tilde{\mathrm{H}}_i(\Delta) \tag{3}$$

where Δ is the order complex of L and $\tilde{\chi}$ is the reduced Euler characteristic. This equation together with Theorem 3.1 can be used to show that a number of external activity lattices have Möbius function zero. We will use the notation $\tilde{H}_i(M)$ and $\mu(M)$ for $\tilde{H}_i(\Delta(M))$ and $\mu(L(M))$, respectively. We will also use $\mathrm{rk}(M)$ for the rank of the matroid M. This should not be confused with the rank function ρ for the lattice $L(M)$.

Proposition 4.1 *Let M be an ordered matroid with maximum base T and rank $r = \mathrm{rk}(M) \geq 1$.*

(a) *Suppose that $M|(E - T)$ is spanning. Then*

$$\tilde{H}_i(M) = \{0\} \text{ for all } i \geq -1 \quad \text{and} \quad \mu(M) = 0.$$

(b) *Suppose that $M|(E - S)$ is spanning for all proper subsets $S \subset T$ but is not spanning for $S = T$. Then*

$$\tilde{H}_i(M) = \begin{cases} \mathbb{Z} & \text{if } i = r - 2, \\ \{0\} & \text{else,} \end{cases} \qquad \text{and} \quad \mu(M) = (-1)^{r-2}.$$

Proof Under the first (respectively, second) hypothesis, $IN(M^*|T)$ is homologically an $(r-1)$-ball (respectively, $(r-2)$-sphere). The conclusions now follow from Theorem 3.1 and equation (3). ■

As an example, consider the cycle matroid of a graph G where, as usual, the edge set $E = E(G)$ has been linearly ordered. In this case we will use G in our notation everywhere we used M before. In the following result a *star* is the complete bipartite graph $K_{1,n-1}$.

Corollary 4.2 *Let K_n be an ordered complete graph on n vertices, $n \geq 2$, and let T be its lexicographically maximal spanning tree.*

(a) *If T is not a star then*

$$\tilde{H}_i(K_n) = \{0\} \text{ for all } i \geq -1 \quad \text{and} \quad \mu(K_n) = 0.$$

(b) *If T is a star*

$$\tilde{H}_i(K_n) = \begin{cases} \mathbb{Z} & \textit{if } i = n-3, \\ \{0\} & \textit{else,} \end{cases} \qquad \textit{and} \quad \mu(K_n) = (-1)^{n-3}.$$

Proof If T is not a star, then $K_n - E(T)$ is connected and the hypotheses of Proposition 4.1 (a) are satisfied. If T is a star, then $K_n - E(S)$ is connected for all $S \subseteq T$, except for $S = T$. Thus the hypotheses of Proposition 4.1 (b) are fulfilled. ■

Note that this corollary lends support to Las Vergnas' remark cited in the introduction. In particular, almost all orderings of $E(K_n)$ give rise to a T which is not a star. To see this, note that T must always contain the two largest edges in the ordering since otherwise a larger base could be constructed by exchanging an element of T with one of these edges. So if the two largest edges are not adjacent in K_n then T cannot be a star. But the ratio of such orderings to the total number of orderings, counting edge choices from largest to smallest in the order, is

$$\frac{\binom{n}{2}\binom{n-2}{2}\left[\binom{n}{2}-2\right]!}{\binom{n}{2}!} = \frac{\binom{n-2}{2}}{\binom{n}{2}-1} \to 1$$

as $n \to \infty$.

Also as a result of this corollary, we can see that $\Delta(M)$ is not, in general, shellable (even though $IN(M^*|T)$ always is, see Björner [3, Theorem 7.3.3]). If Δ is any simplicial complex which is shellable and pure of dimension d, then Δ is topologically a wedge of d-spheres and so only has homology in dimension d. So if a finite lattice L graded of rank ρ is shellable, then it only has homology in dimension $\rho - 2$ (since we remove $\hat{0}$ and $\hat{1}$). But in $L(M)$ we have

$$\rho(L(M)) = \rho(T) = |\operatorname{Ext}(T)| + 1 = |E - T| + 1.$$

In particular

$$\rho(L(K_n)) = \binom{n}{2} - (n-1) + 1 = \binom{n-1}{2} + 1.$$

But from the previous corollary, if $T = K_{1,n-1}$ then $L(K_n)$ has homology in dimension $n - 3 < \binom{n-1}{2} - 1$ for $n \geq 4$.

Here is another family of matroids which have zero Möbius function.

Corollary 4.3 *Let M be an ordered matroid with maximum base T and suppose there is $t \in T$ such that* $\operatorname{rk}(E-T) = \operatorname{rk}((E-T)\cup t)$. *Then*

$$\tilde{H}_i(M) = \{0\} \textit{ for all } i \geq -1 \quad \textit{and} \quad \mu(M) = 0.$$

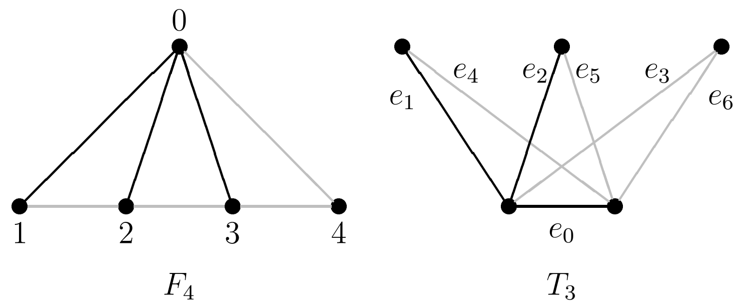

Figure 2: The fan F_4 and triangle graph T_3

Proof Suppose that $t \in T$ satisfies $\mathrm{rk}(E - T) = \mathrm{rk}((E - T) \cup t)$. This means that if a base $B \in \mathcal{B}(M)$ intersects T minimally, then $t \notin B$. That is, t is not contained in any base of the contraction $M.T$ and hence is contained in every base of $M^*|T$. Thus $IN(M^*|T)$ is a cone with vertex t. The result follows. ■

For application in our examples, note that for the cycle matroid of a graph G, the hypothesis of Corollary 4.3 just says that the edge $t \in T$ connects two vertices in the same component of $G - E(T)$. We first consider the *n-fan*, F_n, which is obtained from a path with n vertices by adding an additional vertex adjacent to every vertex of the path. More explicitly, $F_n = (V, E)$ where $V = \{0, 1, \ldots, n\}$ and

$$E = \{01, 02, \ldots, 0n\} \uplus \{12, 23, \ldots, (n-1)n\}$$

where $\uplus$ denotes disjoint union. We always write our edges with the smaller vertex first and order them lexicographically. Then

$$E(T) = \{0n, 12, 23, \ldots, (n-1)n\}.$$

Figure 2 contains a drawing of F_4 with the edges of T in gray. It is easy to see that if $n \geq 3$ then the edge $t = 12$ satisfies the component criterion of the first sentence in this paragraph.

Next consider the *n-triangle graph*, T_n, gotten by gluing together n copies of K_3 along a common edge. To set notation, let

$$E = \{e_0, e_1, \ldots, e_{2n}\}$$

where the ith triangle has edges $\{e_0, e_i, e_{n+i}\}$ and edges are ordered by their subscripts. Now

$$T = \{e_n, e_{n+1}, \ldots, e_{2n}\}$$

The graph T_3 is depicted in Figure 2. So if $n \geq 3$ then the edge $t = e_{n+1}$ will satisfy the component criterion. By Corollary 4.3, we have proved the following.

Proposition 4.4 *For the given orderings and $n \geq 3$ we have*

$$\tilde{H}_i(F_n) = \tilde{H}_i(T_n) = \{0\} \text{ for all } i \geq -1 \quad \text{and} \quad \mu(F_n) = \mu(T_n) = 0. \qquad \blacksquare$$

5 A theorem of Björner

A theorem of Björner [3, Theorem 7.8.1] characterizes the reduced homology of $IN(M)$ for any matroid M and can be used in conjunction with Theorem 3.1 for computations. To state it, we will need the lattice of flats of M which will be denoted $L_F(M)$ to distinguish it from the external activity lattice. Also, define the *reduced Möbius function* of M to be

$$\tilde{\mu}(M) = \begin{cases} |\mu(L_F(M))| & \text{if } M \text{ is loopless,} \\ 0 & \text{else.} \end{cases}$$

Theorem 5.1 (Björner) *If $r = \operatorname{rk}(M)$ then*

$$\tilde{H}_i(IN(M)) \cong \begin{cases} \mathbb{Z}^{\tilde{\mu}(M^*)} & \text{if } i = r-1, \\ \{0\} & \text{else.} \end{cases} \qquad \blacksquare$$

Now if $F \subseteq E$, consider $M.F$, the contraction of M to F. Our interest stems from the fact that $(M^*|F)^* = M.F$. An immediate corollary of the previous theorem and Theorem 3.1 is as follows.

Theorem 5.2 *If $r^* = \operatorname{rk}(M^*|T)$ then*

$$\tilde{H}_i(M) \cong \begin{cases} \mathbb{Z}^{\tilde{\mu}(M.T)} & \text{if } i = r^*-1, \\ \{0\} & \text{else.} \quad \blacksquare \end{cases}$$

Corollary 5.3 *If $r = \operatorname{rk}(M)$, then*

$$\mu(M) = \begin{cases} (-1)^{r-1}\mu(L_F(M.T)) & \text{if } M.T \text{ is loopless,} \\ 0 & \text{else.} \end{cases}$$

Proof Let $r^* = \operatorname{rk}(M^*|T)$. Viewing $\mu(M)$ as the reduced Euler characteristic of $\Delta(M)$ and using Theorem 5.2 we find $\mu(M) = (-1)^{r^*-1}\tilde{\mu}(M.T)$. So if $M.T$ has loops then $\mu(M) = 0$ by definition of $\tilde{\mu}$. Otherwise, since $M.T = (M^*|T)^*$ and $|T| = r$, the rank of $M.T$ and hence of $L_F(M.T)$ is $r - r^*$. As $L_F(M.T)$ is a geometric lattice, the sign of $\mu(L_F(M.T))$ is $(-1)^{r-r^*}$ and cancelling appropriate powers of -1 gives the desired conclusion. $\blacksquare$

Let us apply these results to some examples.

The uniform matroid Consider the *uniform matroid* $U_{n,k}$ on the n-set E whose collection of bases is

$$\mathcal{B}(U_{n,k}) = \{I \subseteq E \;:\; |I| = k\}.$$

The lattice of flats $L_F(U_{n,k})$ consists of the subsets of E of cardinality strictly less than k together with E itself, ordered by inclusion. Thus $L_F(U_{n,k})$ is obtained from the Boolean lattice B_n on E by deleting all elements of rank $l \geq k$, except the top element. We will call this poset the *truncated Boolean algebra* (see Zhang [17]). Using the fact that, for any two subsets $A \subseteq B \subseteq E$, the Möbius function of B_n satisfies

$$\mu(A, B) = (-1)^{|B-A|},$$

we find that

$$\mu(L_F(U_{n,k})) = -\sum_{i=0}^{k-1}(-1)^i\binom{n}{i} = (-1)^k\binom{n-1}{k-1}.$$

Now let $M = U_{n,k}$ for some $n > 0$, and order E linearly. The top element T of L is some k-subset of E. One verifies that $M^*|T$ is the uniform matroid U_{k,r^*}, where $r^* = \min\{k, n-k\}$, and that $M.T$ is the uniform matroid $U_{k,k-r^*}$.

Suppose $k \leq n/2$. Then $r^* = k$ and only the empty set is independent in $M.T$. Hence $M.T$ has loops, $\tilde{\mu}(M.T) = 0$, and we have $\tilde{\mathrm{H}}_i(\Delta) = \{0\}$ for all i, and $\mu(L) = 0$.

Suppose instead that $k > n/2$ so that $r^* = n - k$. Then $M.T$ has no loops and combining our computation of $\mu(L_F(U_{n,k}))$ with Theorem 5.2 and Corollary 5.3 we have the following result. In it, we assume that $\binom{j}{i} = 0$ if $i < 0$.

Proposition 5.4 *For any ordering of the uniform matroid $U_{n,k}$ we have*

$$\dim \tilde{H}_i(U_{n,k}) = \binom{k-1}{2k-n-1} \text{ if } i = n-k-1 \quad \text{and} \quad \mu(U_{n,k}) = (-1)^{n-k-1}\binom{k-1}{2k-n-1}. \qquad \blacksquare$$

Note that since $L(U_{n,k})$ has rank $n - k + 1$, the complex $\Delta(U_{n,k})$ is pure of dimension $n - k - 1$. Apparently $\Delta(U_{n,k})$ only has homology in the top dimension.

The wheel graph W_n Consider the *n-wheel graph*, W_n, obtained from an n-circuit C by adding a vertex v_0 adjacent to all vertices of the circuit. Let the edge set be ordered linearly and let T be the top element of $L(W_n)$.

Suppose first that some edge $t \in T$ satisfies Proposition 4.3 , i.e., t connects two vertices in the same component of $W_n - E(T)$. Then $\tilde{\mathrm{H}}_i(W_n) = \{0\}$ for all $i \geq -1$, and $\mu(W_n) = 0$.

If there is no such edge, then $W_n - E(T)$ is partitioned into connected components $C_0, C_1, \ldots, C_k$ as follows:

1. $k = 1$, $C_0 = \{v_0\}$ and $C_1 = C$, or

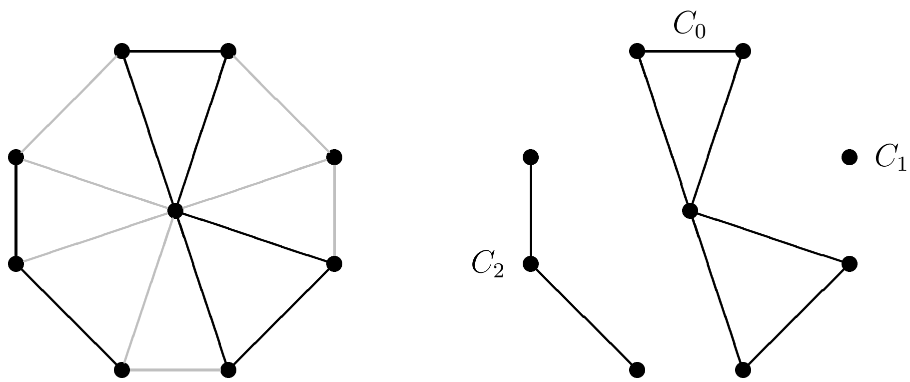

Figure 3: The wheel W_8 a spanning tree, and components

2. C_0 is the union of triangles intersecting only in v_0, the components C_1, C_2,..., C_l are paths, possibly of length 0, and every edge of T meets C_0 and C_i for some $i \geq 1$.

The graph W_8, a gray spanning tree T, and the corresponding components are shown in Figure 3.

Let T_i be the set of edges from T joining C_0 to C_i. Then by the above we have $T = \biguplus_{i=1}^{k} T_i$. Now $M.T$ is the cycle matroid of the graph with vertex set $\{C_0, C_1, \ldots, C_k\}$, where T_i represents a set of parallel edges joining the central vertex C_0 to C_i. Thus $M.T$ is the matroid of partial transversals of T with respect to the family $\{T_i\}_{i=1}^{k}$.

We now determine $L_F(M.T)$. The closed sets of $M.T$ are the unions of the sets T_i. Thus $L_F(M.T)$ is the Boolean algebra B_k on the set $\{T_i\}_{i=1}^{k}$. Hence we have $\mu(L_F(M.T)) = (-1)^k$. Clearly $M^*|T = (M.T)^*$ has rank $n-k$ and so, using Theorem 5.2 and its corollary, we obtain the following result.

Proposition 5.5 *Let T be the top element of $L(W_n)$ for some ordering of the edges of W_n.*

(i) If there is an edge $t \in T$ satisfying Proposition 4.3 then

$$\tilde{\mathrm{H}}_i(W_n) = \{0\} \text{ for all } i \geq -1 \quad \text{and} \quad \mu(W_n) = 0.$$

(ii) If there is no such edge, then

$$\dim \tilde{H}_i(W_n) = \begin{cases} 1 & \text{if } i = n-k-1, \\ 0 & \text{else,} \end{cases} \qquad \text{and} \quad \mu(W_n) = (-1)^{n-k-1}. \qquad \blacksquare$$

Note that since $L(W_n)$ has rank $n+1$, the complex $\Delta(W_n)$ is pure of dimension $n-1$. We have just shown that in case (ii) $\Delta(W_n)$ has homology in dimension $n-k-1$ and, since k cannot be zero, this complex is not shellable.

6 Comments and open problems

There are several comments and questions raised by our work which we address now.

I. We observed that the order complex for the uniform matroid has homology in the correct dimension for it to be shellable. We will now give an explicit shelling. This gives a way of rederiving Theorem 5.4.

First we recall some basic definitions. Given a finite poset P we let $\mathcal{C}(P)$ be the set of all pairs $(a,b) \in P^2$ such that a is covered by b, i.e., $a < b$ and there is no $c \in P$ with $a < c < b$. A *saturated* a_0-a_k *chain* is $C = (a_0, a_1, a_2, \ldots, a_k)$ where $(a_{i-1}, a_i) \in \mathcal{C}(P)$ for $1 \le i \le k$. Given a totally ordered set S, then a labelling (function) $l : \mathcal{C}(P) \to S$ induces a labelling of each saturated chain $l(C) = (l_1, l_2, \ldots, l_k)$ where $l_i = l(a_{i-1}, a_i)$. Any properties of the sequence $l(C)$, e.g., strictly increasing, will also be said to apply to C itself. We say that l is an *EL-labelling* and that P is *EL-shellable* if, for any $a \le b$ in P we have

1. There is a unique strictly increasing saturated a-b chain C.

2. Chain C is lexicographically smallest among all saturated a-b chains.

The fundamental theorem about this property is due to Björner [1].

Theorem 6.1 (Björner) *Let P be a finite, graded poset with a $\hat{0}$ and a $\hat{1}$. If P is EL-shellable then $\Delta(P)$ is homotopic to a wedge of spheres of dimension* $\dim \Delta(P)$ *and the number of spheres is just the number of weakly decreasing saturated chains from $\hat{0}$ to $\hat{1}$.* ■

Now consider the uniform matroid $U_{n,k}$ on the set $E = \{1, 2, \ldots, n\}$. If $B \in \mathcal{B}(U_{n,k})$ then $\mathrm{Act}(B) = \{1, 2, \ldots, \min(B) - 1\}$ and so B's rank in the lattice $L(U_{n,k})$ is $\rho(B) = \min(B)$. It follows that if B covers $A \in \mathcal{B}(U_{n,k})$ then we must have

$$B = A - \{\min(A)\} \uplus \{b\} \tag{4}$$

for some $b > \min(A)$ (where $b = \min(A) + 1$ iff $\min(A) + 1 \notin A$). So we can define a labelling of the covering pairs by

$$l(A,B) = \begin{cases} \text{the unique element of } B - A & \text{if } A, B \in \mathcal{B}(U_{n,k}) \\ \max(B) & \text{if } A = \hat{0} \text{ and } B \in \mathcal{B}(U_{n,k}). \end{cases} \tag{5}$$

Theorem 6.2 *The labelling (5) is an EL-labelling of $L(U_{n,k})$ where the labels on any saturated chain are all distinct. Furthermore, the number of strictly decreasing $\hat{0}$-$\hat{1}$ chains is $\binom{k-1}{n-k}$.*

Proof First consider $A \leq B$ where $A \neq \hat{0}$. Note that any saturated A-B chain has distinct labels. This is because in order for a label to be used twice it would have to be subtracted from one of the sets of the chain. But element l can only be subtracted when moving up from a set at rank l, and at higher ranks l is not permitted as an element. Furthermore, equations (4) and (5) show that a label sequence completely determines a corresponding chain, if one exists, since the element to be subtracted is predetermined by the rank. In addition, the restriction $A \neq \hat{0}$ and equation (4) ensure that any two saturated A-B chains use the same set of labels. So if a strictly increasing chain exists, then it is unique.

To show existence of a strictly increasing chain, we use the notion of an *inversion* in a sequence $(l_1, l_2, \ldots, l_k)$ which is a pair (l_i, l_j) such that $i < j$ and $l_i > l_j$. Let C be a saturated A-B chain that has the fewest number of inversions. If C is increasing then we are done. Otherwise C must have a descent, i.e., an inversion of the form (l_i, l_{i+1}). Suppose that the portion of C corresponding to this descent is A_{i-1}, A_i, A_{i+1}. Then $l_i > l_{i+1} \geq \rho(A_{i+1})$. Define $A'_i = A_i - \{l_i\} \uplus \{l_{i+1}\}$. From the inequalities just given it follows that $\rho(A'_i) = \min(A'_i) = \min(A_i) = \rho(A_i)$. So replacing A_i by A'_i in C gives a chain C' whose label sequence is $l(C)$ with l_i and l_{i+1} switched. Thus $l(C')$ has fewer inversions than $l(C)$, a contradiction. It is interesting to note that we have actually proved the stronger statement that if $A \neq \hat{0}$ then the interval from A to B has an S_n EL-labeling in the sense of McNamara [11] and McNamara and Thomas [12].

Now consider the case where $A = \hat{0}$. Much of what we have already proved goes through in this case. In particular, the labels on any saturated $\hat{0}$-B chain are distinct and a given sequence of labels determines a chain uniquely if it exists at all. (The latter is most easily seen by working down from B.) This time we explicitly construct the strictly increasing saturated $\hat{0}$-B chain. Consider the $\min(B)$ largest labels in the set $B \uplus \{1, 2, \ldots, \min(B) - 1\}$. Arranging these labels in increasing order shows that the desired chain exists since they are all sufficiently large to be added at the necessary point in the chain (or subtracted if one moves down).

To compute the number of decreasing $\hat{0}$-$\hat{1}$ chains, note that n must be a label on any saturated $\hat{0}$-$\hat{1}$ chain since it must be added at some point, and if it is added in the first cover then it is also the maximum. So for the chain to be decreasing the first label must be n. Similarly, the last label must be an element of $\hat{1} = \{n-k+1, n-k+2, \ldots, n\}$. So we need to pick $\rho(\hat{1}) - 1 = n - k$ labels from $|\hat{1} - \{n\}| = k - 1$ possible. As usual, each of these choices will produce a unique decreasing chain if it exists. But since all of the elements which could be chosen are at least as big as $\rho(\hat{1})$ they do indeed correspond to a chain. Thus there are $\binom{k-1}{n-k}$ such chains and we are done. ■

II. Forman [7] has introduced a discrete analogue of Morse theory as a way of studying CW complexes by collapsing them onto smaller, more tractable, complexes of critical cells. These techniques can be used to compute the homology of a complex even when it is not shellable. Are the nonshellable complexes which we have considered amenable to Forman's technique?

III. Las Vergnas defined a third ordering on the bases of an ordered matroid. Let the *pseudo-height* of a base $B \in \mathcal{B}(M)$ be

$$h_M(B) = |\operatorname{Ext}_M(B)| - |\operatorname{Int}_M(B)| + \operatorname{rk}(M)$$

where $\operatorname{Int}_M(B)$ is the set of internally active elements of B in M. Then from [9, Proposition 6.3] we obtain $h_M(A) < h_M(B)$ whenever either $A <_M^{\text{ext}} B$ or $A <_M^{\text{int}} B$. So there is a well-defined *external-internal order* $\leq_M^{\text{exin}}$ on $\mathcal{B}(M)$ given by

$$A \leq_M^{\text{exin}} B \text{ if and only if } A \leq_M^{\text{ext}} B \text{ or } A \leq_M^{\text{int}} B$$

with corresponding lattice $L_{\text{exin}}(M)$. We have been unable to find an analogue of Theorem 3.1 for this lattice. It would be very interesting to do so.

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