On absolutely universal embeddings

R. J. Blok* and A. Pasini

Abstract

It is well known that, given a point-line geometry Γ and a projective embedding $\varepsilon : \Gamma \to PG(V)$, if $dim(V)$ equals the size of a generating set of Γ, then ε is not derived from any other embedding. Thus, if Γ admits an absolutely universal embedding, then ε is absolutely universal. In this paper, without assuming the existence of any absolutely universal embedding, we give sufficient conditions for an embedding ε as above to be absolutely universal.

1 Introduction and main results

1.1 Basic definitions

In view of the purposes of this paper, we define a *point-line geometry* as a pair $\Gamma = (P, \mathcal{L})$ where P (the set of *points*) is a nonempty set and $\mathcal{L}$ (the set of *lines*) is a nonempty collection of distinguished proper subsets of P such that:

(G1) (**linearity**) any two distinct lines have at most one point in common;

(G2) (**connectedness**) the relation 'being *collinear*' (namely, 'being in the same line') defines a connected graph on P, called the *collinearity graph* of Γ;

(G3) (**order**) all lines have precisely $s + 1$ points, for a given cardinal number $s > 0$ called the *order* of Γ (and with the usual convention that $s + 1 = s$ when s is infinite).

Remark. The above conditions look rather restrictive if compared with other definitions in the literature. However, all embeddable geometries satisfy (G1)–(G3) and we are only interested in such geometries here. Thus, our definition is suitable for this paper.

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Subspaces, spanning sets and subgeometries. A *subspace* of a point-line geometry $\Gamma = (P, \mathcal{L})$ is a subset X of P containing any line of Γ that meets X in at least two points. Clearly, the intersection of arbitrary families of subspaces is a subspace. Thus, given a subset X of P, the intersection of all subspaces of Γ containing X is the minimal subspace of Γ containing X. We call it the *span* of X. If the span of X is P, then we say that X *generates* (also, *spans*) Γ.

A *subgeometry* of Γ is a pair $\Xi = (X, \mathcal{X})$ where X is a nonempty subset of P (possibly not a subspace) and $\mathcal{X} \subseteq \mathcal{L}$ is a nonempty collection of lines of Γ such that

(S1) every $L \in \mathcal{X}$ is contained in X and

(S2) condition (G2) holds in Ξ (whence, Ξ is a point-line geometry).

In particular, given a subspace X of Γ and denoting by $\mathcal{X}$ the set of lines of Γ contained in X, the pair $(X, \mathcal{X})$ is a subgeometry of Γ if and only if X properly contains a line of Γ and it is *connected*, namely the collinearity graph of Γ induces a connected graph on X. If that is the case, then we say that X *supports* a subgeometry of Γ and we call $(X, \mathcal{X})$ the *induced subgeometry supported* by X.

Some terminology for the collinearity graph. Given two distinct collinear points a, b of a point-line geometry Γ, the line through a and b will be called the *support* of the edge $\{a, b\}$ of the collinearity graph of Γ. Given a path γ of that graph, suppose that all lines of Γ supporting edges of γ belong to a given subgeometry Σ of Γ. Then we say that γ *belongs* to Σ.

1.2 Basics on embeddings

Given a point-line geometry $\Gamma = (P, \mathcal{L})$, a *projective embedding* of Γ (an *embedding*, for short) is an injective mapping ε from P into the set of points of the projective geometry $PG(V)$ of linear subspaces of a vector space V such that

(E1) the set $\varepsilon(P)$ spans $PG(V)$ and

(E2) the image $\varepsilon(L)$ of L is the full set of points of a line of $PG(V)$, for every line L of Γ.

To fix ideas, we always assume that the vector space V is a right vector space. Furthermore, if K is the division ring over which V is defined and we want to keep a record of this fact in our terminoloy, then we may say that ε is *defined over* K.

Note that, if the embedding $\varepsilon : \Gamma \to PG(V)$ is defined over K and s is the order of Γ, then $s = |K|$. When $s < \infty$, the equality $s = |K|$ forces s to be a prime power and, on the other hand, it uniquely determines $|K|$ once the prime power s is given. On the contrary, when s is infinite Γ might possibly admit embeddings defined over different division rings. (This is the case for infinite grids, for instance.) However, when speaking of embeddings of a given point-line geometry, people implicitly assume that K is given once for all and all embeddings to consider are defined over K. We will also follow this habit.

Morphisms and equivalence of embeddings. Given a point-line geometry Γ and two embeddings $\varepsilon : \Gamma \to PG(V)$ and $\eta : \Gamma \to PG(W)$, a *morphism* from ε to η is a semilinear mapping $\varphi : V \to W$ such that

(M) $\quad \varphi\varepsilon = \eta$

(with φ regarded as a mapping from the set of linear subspaces of V to the set of linear subspaces of W). Note that (M) forces $\varepsilon(p) \cap Ker(\varphi) = \{0\}$ for every point p of Γ. Also, (M) and (E1) together force the semilinear mapping φ to be surjective. If, furthermore, φ is injective, then we call it an *isomorphism* from η to ε and we write $\eta \equiv \varepsilon$.

It easily follows from (E1), (E2) and from the connectedness of Γ that condition (M) uniquely determines φ modulo a scalar. Thus, if morphisms of emdeddings are taken modulo scalars, the category of embeddings of Γ is a preorder. We denote it $\mathbf{Emb}(\Gamma)$. Accordingly (and following Kasikova and Shult [4]) if there is a morphism from η to ε (and $\eta \not\equiv \varepsilon$) then we write $\varepsilon \leq \eta$ (resp. $\varepsilon < \eta$) and we say that ε is (*properly*) *derived* from η.

Automorphisms arising from embeddings. Given a projective embedding $\varepsilon : \Gamma \to PG(V)$, the composite $\varepsilon\alpha$ is also an embedding, for every $\alpha \in Aut(\Gamma)$. If $\varepsilon \equiv \varepsilon\alpha$ then $\hat{\alpha}\varepsilon = \varepsilon\alpha$ for a uniquely determined automorphism $\hat{\alpha}$ of $PG(V)$ and we say that α is *induced by* $\hat{\alpha}$ *via* ε, or that it *arises* from ε, for short.

Universal embeddings. An embedding ε of Γ is said to be *relatively universal* (*dominant* in Tits [9]) if it is maximal in the preorder $\mathbf{Emb}(\Gamma)$, namely it is not properly derived from any other embedding of Γ. If furthermore every embedding of Γ is derived from ε, then we say that ε is *absolutely universal*. Clearly, the absolutely universal embedding, if it exists, is unique modulo isomorphisms. Furthermore, if Γ admits the absolutely universal embedding, then all automorphisms of Γ arise from it.

However, embeddable geometries exist where not all automorphisms arise from embeddings. Those geometries do not admit any absolutely universal embedding. An example of this kind is examined in the next paragraph.

Remarks on grids. Let Γ be a grid of order s, with s a prime power in the finite case. It is well known that Γ is embeddable and that, if $\varepsilon : \Gamma \to PG(V)$ is an embedding of Γ, then $dim(V) = 4$ and $\varepsilon(\Gamma)$ is a hyperbolic quadric of $PG(V)$ (see Hirschfeld and Thas [3, §26.5]).

Given a point p of Γ, let L_1 and L_2 be the two lines of Γ on p and let α_1 and α_2 be permutations of $L_1 \setminus \{p\}$ and $L_2 \setminus \{p\}$, respectively. There is a unique automorphism α of Γ stabilizing p, L_1 and L_2 and acting as α_1 and α_2 on $L_1 \setminus \{p\}$ and $L_2 \setminus \{p\}$. However, α arises from an embedding of Γ only if, for a suitable division ring K, there are bijections $\beta_i : L_i \setminus \{p\} \to K$ such that $\beta_i \alpha_i \beta_i^{-1} \in A\Gamma L(1, K)$ for $i = 1, 2$ and, furthermore, $\beta_1 \alpha_1 \beta_1^{-1}$ and $\beta_2 \alpha_2 \beta_2^{-1}$ belong to the same coset of $AGL(1, K)$ in $A\Gamma L(1, K)$.

It is easily seen that, when $s > 3$, we can always choose α_1 and α_2 in such a way that the above fails to hold. If so, α does not arise from any embedding.

Universal hulls. Given an embedding ε of a point-line geometry Γ, the *universal hull* of ε is an embedding $\tilde{\varepsilon}$ such that $\eta \leq \tilde{\varepsilon}$ for any embedding $\eta \geq \varepsilon$. The following is well known:

Theorem 1.1 (Ronan [6]) *Every embedding admits a universal hull.*

Clearly, the universal hull of a given embedding is uniquely determined modulo isomorphisms and it is relatively universal. Also, an embedding is relatively universal if and only if it is its own universal hull. Thus, the preorder $\mathbf{Emb}(\Gamma)$ has the following structure: we have a set $\mathbf{U}$ of relatively universal embeddings and, if the members of $\mathbf{U}$ are taken modulo isomorphism, every embedding of Γ is derived by exactly one member of $\mathbf{U}$. The geometry Γ admits an absolutely universal embedding if and only if $\mathbf{U}$ has just one member (modulo isomorphisms).

1.3 A theorem of Kasikova and Shult

The existence of absolutely universal embeddings has been known for a long time in a number of special cases: for instance, classical polar spaces of finite rank $n \geq 3$ (Tits [9, Chapter 8, Lemma 8.18]) and geometries of order 2 (Ronan [6]). Sufficient conditions for the existence of the absolutely universal embedding of a geometry Γ are given by Shult [7, Theorem 7], in terms of properties of the hyperplanes of Γ. However, hyperplanes are often very difficult to study. A different, far reaching criterion has recently been found by Kasikova and Shult [4]. Their main theorem is reported below, but we split it in two separate statements, according to its two main cases. As in the previous subsection, all embeddings considered in the sequel are understood to be defined over a given division ring K.

Theorem 1.2 (Kasikova and Shult [4]) *Let Γ be a point-line geometry admitting an embedding. Suppose furthermore that Γ possesses a family $\mathcal{S}$ of subspaces such that:*

(1) *every member of $\mathcal{S}$ supports a subgeometry of Γ;*

(2) *every line of* Γ *belongs to a member of* $\mathcal{S}$*;*

(3) *for every* $A \in \mathcal{S}$*, the induced subgeometry supported by* A *admits an absolutely universal embedding.*

Let Σ *be the graph with* $\mathcal{S}$ *as the vertex set and adjacency relation defined as follows: Two distinct members of* $\mathcal{S}$ *are adjacent if and only if their intersection contains a line and is connected.*

Assume the following:

(4) *The graph* Σ *is simply connected.*

(5) *For every* 3*-clique* $\{A, B, C\}$ *of* Σ *with* $A \cap B \cap C = \emptyset$*, there is a subspace* X *supporting a subgeometry of* Γ *such that* X *meets each of* $A \cap B$*,* $B \cap C$ *and* $C \cap A$ *non-trivially, each of the intersections* $X \cap A$*,* $X \cap B$ *and* $X \cap C$ *is connected and the induced subgeometry supported by* X *admits an absolutely universal embedding.*

(6) *For each point* p *of* Γ*, the graph* Σ *induces a connected graph on the set of members of* $\mathcal{S}$ *containing* p*.*

Then Γ *admits an absolutely universal embedding.*

Theorem 1.3 (Kasikova and Shult [4]) *Let* Γ *be a point-line geometry admitting an embedding and suppose that* Γ *possesses a family* $\mathcal{S}$ *of subspaces satisfying the conditions* (1)–(3) *of Theorem* 1.2 *but now assume that no intersection of distinct members of* $\mathcal{S}$ *contains a line. Let* Σ *be the graph with* $\mathcal{S}$ *as the vertex set and adjacency relation defined as follows: Two distinct members of* $\mathcal{S}$ *are adjacent if and only if they meet at a single point.*

Still assuming (4) *and* (5) *of Theorem* 1.3*, replace* (6) *with the following:*

($6'$) *If* $\{A, B\}$ *is an edge of* Σ *and* $p = A \cap B$*, then there exists a subspace* X *containing* p*, supporting a subgeometry of* Γ *and such that each of the intersections* $X \cap A$ *and* $X \cap B$ *contains a line on* p *and the induced subgeometry supported by* X *admits the absolutely universal embedding.*

Then Γ *admits an absolutely universal embedding.*

Many applications of the previous two theorems are given in [4]. However, some interesting cases escape them. The following is an example of that kind.

A 'bad' example. Given a field K of order $|K| > 3$, let Φ be the polar space for $O_6^+(K)$ and let Γ be its grassmannian of lines. Namely, the points of Γ are the lines of Φ and the lines of Γ are the point-plane flags of Φ.

Looking for subspaces supporting subgeometries of Γ, planes and points of Φ are the first things one might think of (and they are likely to be the only proper subspaces of Γ that support subgeometries). Precisely, if A is the set of lines of Φ contained in a plane of Φ, then A is a subspace of Γ and it supports a subgeometry isomorphic to $PG(2, K)$. On the other hand, if A is the set of lines of Φ containing a given point of Φ, then A is again a subspace and it supports a subgeometry of Γ, but the latter is now a grid of order $s = |K|$. Let us denote $\mathcal{S}_1$ and $\mathcal{S}_2$ these two families of subspaces, where $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) is the family corresponding to the planes (points) of Γ.

As noticed in Subsection 1.2, grids of order $s > 3$ do not admit any absolutely universal embedding. Therefore, in view of (3) of Theorem 1.2, if we want to apply Theorems 1.2 or 1.3 to Γ taking one of $\mathcal{S}_1$ or $\mathcal{S}_2$ as $\mathcal{S}$, the former is the family to choose. Thus, let $\mathcal{S} = \mathcal{S}_1$. We recall that the lines of Γ are point-plane flags of Φ. Therefore, no two distinct members of $\mathcal{S}$ meet in a line, and so Theorem 1.3 is what we should exploit here. Accordingly, we should check if (6') of Theorem 1.3 holds. As $\mathcal{S} = \mathcal{S}_1$, the subspace X mentioned in (6') of Theorem 1.3 could only be a member of $\mathcal{S}_2$. However, if $X \in \mathcal{S}_2$, then the subgeometry of Γ supported by X, being a grid, does not admit any absolutely universal embedding. Thus (6') fails to hold and Theorem 1.3 does not apply.

Note that, in view of the Klein correspondence, Γ is isomorphic to the long root geometry for $SL(4, K)$: its points are the point-plane flags of $PG(3, K)$ and its lines are the point-line and the line-plane flags of $PG(3, K)$. More generally, if we want to apply the theorems of Kasikova and Shult to the long root geometry for $SL(n, K)$, then we face the same difficulties as above.

Our purpose in this paper is to find a way to overcome difficulties like these.

1.4 The main result of this paper

We begin with a few simple remarks. Given an embedding $\varepsilon : \Gamma \to PG(V)$ of a point-line geometry Γ, every spanning set of Γ is sent by ε to a (possibly dependent) spanning set of $PG(V)$. If $\varepsilon(B)$ is a basis of $PG(V)$ for some spanning set B of Γ, then ε is relatively universal. If furthermore Γ admits the absolutely universal embedding, then ε is absolutely universal.

The information on the existence of the absolutely universal embedding of Γ is essential for the above conclusion. However, the main theorem of this paper will give sufficient conditions to obtain the same conclusion on ε without knowing in advance if Γ admits the absolutely universal embedding.

We need a few definitions before to state our theorem.

Definitions. As in the previous subsections, all embeddings considered in the sequel are assumed to be defined over the same division ring K.

Given a point-line geometry Ξ, two embeddings $\varepsilon : \Xi \to PG(V)$ and $\eta : \Xi \to PG(W)$ and a line L of Ξ, we denote by $\eta\varepsilon_L^{-1}$ the bijection induced by $\eta\varepsilon^{-1}$ from $\varepsilon(L)$ to $\eta(L)$. We say that $\eta\varepsilon_L^{-1}$ is *semilinear* if it is induced by a semilinear mapping from $\varepsilon(L)$ to $\eta(L)$ (where $\varepsilon(L)$ and $\eta(L)$ are regarded as 2-dimensional linear subspaces of V and W, respectively). Clearly, if $\eta\varepsilon_L^{-1}$ is semilinear, then the semilinear mapping representing it is uniquely determined modulo a scalar.

We say that ε and η are *locally isomorphic* if $\eta\varepsilon_L^{-1}$ is semilinear for every line L of Ξ. If ε and η are locally isomorphic, then we write $\varepsilon \sim \eta$. Clearly, the relation $\sim$ is an equivalence relation and the relation $\equiv$ (isomorphism of embeddings) is a refinement of $\sim$.

We say that Ξ is *locally rigid* with regard to embeddings if any two embeddings of Ξ are locally isomorphic (in short, $\sim$ is the trivial relation). On the other hand, if the relations $\sim$ and $\equiv$ are the same (namely, two embeddings ε and η of Ξ are locally isomorphic only if they are isomorphic), then we say that Ξ is *almost rigid* (with regard to embeddings).

We say that Ξ is *rigid* (with regard to embeddings) if, modulo isomorphisms, it admits a unique embedding. Clearly, Ξ is rigid if and only if it is both locally rigid and almost rigid.

Examples. Projective spaces are rigid. Many classical polar spaces are rigid, too (see Tits [9, Lemma 8.18]).

All geometries admitting absolutely universal embeddings are locally rigid. Indeed, let Ξ admit an absolutely universal embedding $\tilde{\varepsilon}$. Then, for any two embeddings ε and η of Ξ and every line L, both $\eta\tilde{\varepsilon}_L^{-1}$ and $\varepsilon\tilde{\varepsilon}_L^{-1}$ are semilinear. However, $\eta\varepsilon_L^{-1} = \eta\tilde{\varepsilon}_L^{-1}(\varepsilon\tilde{\varepsilon}_L^{-1})^{-1}$. Hence $\eta\varepsilon_L^{-1}$ is semilinear. Therefore, $\varepsilon \sim \eta$.

We are now ready to state our main theorem.

Theorem 1.4 *Let Γ be a point-line geometry admitting a finite spanning set B and an embedding $\varepsilon : \Gamma \to PG(V)$ with $dim(V) = |B|$. Suppose furthermore that:*

(1) *every line of Γ belongs to a locally rigid subgeometry;*

(2) *every closed path of the collinearity graph of Γ splits in closed paths belonging to almost rigid subgeometries;*

(3) *for any two lines L, M of Γ, the collinearity graph of Γ contains a closed path γ such that both L and M support edges of γ and γ has no repeated vertices.*

Then the embedding ε is absolutely universal.

We will prove this theorem in Section 2. It will be clear from the proof that, when the division ring K has no non-trivial automorphisms, there is no need of the above condition (3) to obtain the conclusion of Theorem 1.4. Therefore,

Corollary 1.5 *Let Γ be a point-line geometry admitting a finite spanning set B and an embedding $\varepsilon : \Gamma \to PG(V)$ with $dim(V) = |B|$. Assume the following:*

(1) *every line of* Γ *belongs to a locally rigid subgeometry;*

(2) *every closed path of the collinearity graph of* Γ *splits in closed paths belonging to almost rigid subgeometries;*

(3) *the division ring over which* ε *is defined only admits the identity automorphism.*

Then the embedding ε *is absolutely universal.*

An application of our theorem will be given in Section 3, where the long root geometry for $SL(n, K)$ will be considered. It is well known that this geometry admits an embedding in $PG(n^2 - 2, K)$. With the aid of Corollary 1.5 and a result of Cooperstein [2] we shall prove that, when $K = GF(p)$ with p prime or K is the field of rational numbers, that embedding is absolutely universal.

In particular, turning to the 'bad' example discussed at the end of Subsection 1.3, if K is as above, then the embedding of that geometry in $PG(14, K)$ is absolutely universal. This will also enable us to apply Theorem 1.2 to certain point-line geometries related to buildings of type D_n and E_n (Section 4).

2 Proof of Theorem 1.4 and Corollary 1.5

Proof of Theorem 1.4. Given Γ, B and ε as in the hypotheses of Theorem 1.4, let $\eta : \Gamma \to PG(W)$ be another embedding of Γ. Then $\varepsilon \sim \eta$ by condition (1). (Note that, for every subgeometry Ξ of Γ, the restrictions of ε and η to Ξ are embeddings of Ξ.)

For every line L of Γ, let φ_L be the a semilinear transformation representing $\eta\varepsilon_L^{-1}$. We first need to prove that we can choose those mappings in a consistent way. In order to make this statement more precise, we need to fix some notation.

Let P be the set of points of Γ. For every $X \subseteq P$ we denote $[X]_V$ (resp. $[X]_W$) the set of non-zero vectors of V (of W) that represent points of $\varepsilon(X)$ (of $\eta(X)$).

We will define a mapping $\varphi_P : [P]_V \to [P]_W$ such that, for every line L of Γ, the restriction of φ_P to $[L]_V$ is the above mentioned mapping φ_L. First we choose a point p_0 of Γ and a vector $v_p \in [p]_V$ for every point p of Γ, writing v_0 for v_{p_0}. We also choose $w_0 \in [p_0]_W$. Then, given a point p of Γ, we choose a path $\gamma = (p_0, p_1, ..., p_n = p)$ from p_0 to p and, if L_i is the line supporting the edge $\{p_{i-1}, p_i\}$ of γ, we choose φ_i inductively, as follows:

(a.1) φ_1 is the unique representative of $\eta\varepsilon_{L_1}^{-1}$ that maps v_0 onto w_0;

(a.2) for $i = 2, 3, ..., n$, φ_i is the unique representative of $\eta\varepsilon_{L_i}^{-1}$ that agrees with φ_{i-1} on v_{p_i}.

For $v \in [p]_V$, we set

$$(*) \quad \varphi_P(v) := \varphi_n(v).$$

Lemma 2.1 *The above definition does not depend on the choice of the path* γ.

Proof. Let $\gamma' = (p_0, p'_1, ..., p'_m = p)$ be another path from p_0 to p and let $\varphi'_1, \varphi'_2, ..., \varphi'_m$ be defined according to the above clauses (a.1) and (a.2) but with γ replaced by γ'. If the closed path $\gamma^{-1}\gamma'$ belongs to an almost rigid subgeometry of Γ, then φ_n and φ'_m are induced by the same semilinear mapping and we have $\varphi_n(v) = \varphi'_m(v)$ for any $v \in [p]_V$. Otherwise, we apply the hypothesis (2) of Theorem 1.4 and we get the same conclusion. □

Thus, φ_P is well defined and it is clear from the definition $(*)$ that $\varphi_P(v) \in [p]_W$ for every point p of Γ and every vector $v \in [p]_V$. For every line L of Γ, the restriction of φ_P to $[L]_V$ is the representative φ_L we choose for $\eta\varepsilon^{-1}$.

With L as above, let σ_L be the automorphism of K involved in φ_L, namely $\varphi_L(vt) = \varphi_L(v)t^{\sigma_L}$ for every $v \in [L]_V$ and every scalar $t \in K$. Then,

Lemma 2.2 *We have* $\sigma_L = \sigma_M$ *for any two lines* L, M *of* Γ.

Proof. If L and M belong to the same almost rigid subgeometry of Γ, then φ_L and φ_M are induced by the same semilinear mapping and, therefore, $\sigma_L = \sigma_M$. Otherwise, we apply the hypotheses (3) and (2) of Theorem 1.4 and we get the conclusion. □

Accordingly, we set $\sigma := \sigma_L$ and we call σ the automorphism of K *involved* in φ_P.

Given another point p of Γ and a vector $v \in [p]_V$, if we define the mapping φ_P according to (a.1), (a.2) and $(*)$ but replacing p_0, v_0 and w_0 with p, v and $\varphi_P(v)$ respectively, then we obtain just the same mapping φ_P as above. Thus, we may always assume to have chosen $p_0 \in B$.

If $p_1, p_2, ..., p_n$ are the other points of B, we set $v_i := v_{p_i}$ for every $i = 1, 2, ..., n$. Thus, $\{v_i\}_{i=0}^n$ is a basis of V. For every vector $v = \sum_{i=0}^n v_i t_i$ of V we set

$$(**) \quad \varphi(v) := \sum_{i=0}^{n} \varphi_P(v_i) t_i^{\sigma}$$

where σ is the automorphism of K involved in φ_P.

Lemma 2.3 *We have* $\varphi(v) = \varphi_P(v)$ *for every* $v \in [P]_V$.

Proof. We first define the *distance* $\delta_B(v)$ of v from B. We define it inductively, as follows:

(b.1) If $v \in [B]_V$, then $\delta_B(v) := 0$.

(b.2) Let $V_i := \{v \mid \delta_B(v) \leq i\}$ and let $v \in [P]_V \setminus V_i$. Suppose that v represents a point of a line L of Γ spanned by points p, p' with $[p]_V \cup [p']_V \subseteq V_i$. Then $\delta_B(v) := i + 1$.

(Note that, in (b.2), at least one of the sets $[p]_V$ and $[p']_V$ is contained in $V_i \setminus V_{i-1}$, otherwise $v \in V_i$, contrary to the assumptions made on v.)

As B spans Γ, the above definition gives every vector $v \in [P]_V$ a finite distance from B. Hence we can prove the lemma by induction on $\delta_B(v)$.

If $\delta_B(v) = 0$, there is nothing to prove. Let $\delta_B(v) = i + 1$. Then, according to (b.2), we have $v = x + y$ for suitable vectors $x, y \in V_i \cap [L]_V$ and a suitable line L of Γ. Therefore,

$$\begin{array}{rcll} \varphi(v) & = & \varphi(x) + \varphi(y) & \text{(as } \varphi \text{ is semilinear)} \\ & = & \varphi_P(x) + \varphi_P(y) & \text{(by the inductive hypothesis)} \\ & = & \varphi_L(x) + \varphi_L(y) & \text{(by the definition of } \varphi_P\text{)} \\ & = & \varphi_L(x + y) = \varphi_L(v) & \text{(as } \varphi_L \text{ is semilinear)} \\ & = & \varphi_P(v) & \text{(again by definition of } \varphi_P\text{)} \end{array}$$

The lemma is proved. □

Thus, φ_P is induced by a semilinear mapping $\varphi : V \to W$ and latter is a morphism from ε to η. Theorem 1.4 is proved. □

Proof of Corollary 1.5. The condition (3) of Theorem 1.4 has only been exploited in the proof of Lemma 2.2. However, when K only admits the trivial automorphism, the statement of that Lemma is empty. This remark is enough to prove Corollary 1.5. □

3 An application of Corollary 1.5

In this Section, we apply Corollary 1.5 to the long root geometry for $SL_n(K)$, where K is either a field of prime order or the field of rational numbers. In view of this, we need some preliminary results on embeddings of grids.

3.1 On embeddings of grids

Given a field K, let $s = |K|$ be its order and let Ξ be a grid of order s. Let $\varepsilon : \Xi \to PG(V)$ be an embedding of Ξ defined over K. Then, as remarked in Subsection 1.2, $dim(V) = 4$ and $\varepsilon(\Xi)$ is a hyperbolic quadric. Thus, modulo isomorphisms, we may assume that $W = V$ and $\eta(\Xi) = \varepsilon(\Xi)$ for any other embedding $\eta : \Xi \to PG(W)$ defined over K.

Lemma 3.1 *Given an equivalence class of* $\sim$ *(notation as in Subsection* 1.4*), the classes of* $\equiv$ *contained in it bijectively correspond to the elements of* $Aut(K)$.

Proof. Let $\varepsilon : \Xi \to PG(V)$ and $\eta : \Xi \to PG(W)$ be embeddings defined over K and such that $\varepsilon \sim \eta$. As remarked above, we may assume that $V = W$ and $\varepsilon(\Xi) = \eta(\Xi) = Q$, for a given hyperbolic quadric Q of $PG(V)$. We may also choose a hyperbolic basis $B = (v_0, v_1, v_2, v_3)$ of Q in V, ordered in such a way that $v_0 \perp v_1 \perp v_2 \perp v_3 \perp v_0$ (with $\perp$ denoting the orthogonality relation with respect to Q). For $i = 0, 1, 2, 3$, let p_i be the point of $PG(V)$ represented by v_i and let L_i be the line of $PG(V)$ through p_i and p_{i+1} (indices being taken modulo 4). We may assume that $\eta\varepsilon^{-1} : Q \to Q$ fixes each of the points p_0, p_1, p_2, p_3, all points of L_0 and the point $p \in L_1 \setminus \{p_1, p_2\}$ represented by the vector $v_1 + v_2$. Furthermore, the semilinear mappings φ_0, φ_1, φ_2 and φ_3 induced by $\eta\varepsilon^{-1}$ on L_0, L_1, L_2 and L_3 can be chosen in such a way that $\varphi_0(v_0) = v_0$ and $\varphi_i(v_i) = \varphi_{i-1}(v_i)$ for $i = 1, 2, 3$. Therefore, φ_0 and φ_2 are the identity mapping whereas φ_1 and φ_3 act as follows, for some automorphims σ of K:

$$\varphi_1(v_1t_1 + v_2t_2) = v_1t_1^\sigma + v_2t_2^\sigma, \qquad \varphi_3(v_3t_1 + v_0t_2) = v_3t_1^\sigma + v_0t_2^\sigma.$$

The conclusion is obvious. □

Corollary 3.2 *If $Aut(K) = 1$, then Ξ is almost rigid.*

(Clear, by Lemma 3.1.)

3.2 On the long root geometry for $SL_n(K)$

Given a field K and a positive integer $n > 3$, let $V_0 := V(n, K)$ and $\Pi := PG(V_0)$. Let Γ be the long root geometry for $SL(n, K)$, namely:

- The points of Γ are the point-hyperplane flags of Π;
- The lines of Γ are the line-hyperplane flags of Π and the flags (x, X) of Π where x is a point and X a subspace of codimension 2.
- A point (x, X) of Γ and a line-hyperplane flag (Y, Z) of Π are incident in Γ if and only if $x \in Y$ and $X = Z$.
- Given a point (x, X) of Γ and a line (y, Y) of Γ with y a point of Π and Y a subspace of Π of codimension 2, (x, X) and (y, Y) are incident in Γ if and only if $x = y$ and $Y \subset X$.

(With the terminology of [5], Γ is the point-line system of the J-Grassmann geometry $Gr_J(\Pi)$ of Π, where J is the pair of types of Π corresponding to points and hyperplanes.)

It is well known that Γ admits and embedding $\varepsilon : \Gamma \to PG(V)$, where V is the adjoint module for $SL_n(K)$. Precisely, with V_0^* denoting the dual of V_0 and regarding the tensor product $V_0^* \otimes V_0$ as the vector space of n-by-n matrices over K, the elements of V are the traceless n-by-n matrices and, for every point

(a, A) of Γ, if $(a_i)_{i=1}^n$ and $(\alpha_i)_{i=1}^n$ are n-tuples of homogeneous and Plűcker coordinates for a and A respectively, relative to a given basis of V_0, then $\varepsilon(a, A)$ is represented by the matrix $(\alpha_i a_j)_{i,j=1}^n$. Clearly, $dim(V) = n^2 - 1$.

Theorem 3.3 *Suppose that K has no proper subfield (that is, either $K = GF(p)$ for a prime p or K is the field of rational numbers). Then the above embedding ε is absolutely universal.*

Proof. Given a point a (a hyperplane A) of Π, the set of point-hyperplane flags (a, H) (resp. (x, A)) of Π is a subspace of Γ supporting a subgeometry isomorphic to $PG(n-2, K)$. As projective spaces are rigid, (1) of Corollary 1.5 holds in Γ: every line belongs to a locally rigid subspace.

Given a line X of Π, let Y be a subspace of Π of codimension 2 containing X (when $n = 4$, $Y = X$). Then the set of point-hyperplane flags (a, A) of Π with $a \in X \subseteq Y \subset A$ is a subspace of Γ supporting a geometry isomorphic to a grid of order p. We call these subgeometries *grids* of Γ. By Corollary 3.2, the grids of Γ are almost rigid. Furthermore, as Π is simply connected, the geometry Γ equipped with its grids is also simply connected [5, Chapter 12]. Consequently, every closed path of the collinearity graph of Γ splits in closed paths belonging to grids [5, Theorem 12.64]. Therefore, (2) of Corollary 1.5 also holds in Γ.

Furthermore, according to Cooperstein [2], the geometry Γ admits a spanning set B with $|B| = n^2 - 1 = dim(V)$. Hence ε is absolutely universal, by Corollary 1.5. □

Remark. When K is algebraic over the rationals or perfect of characteristic $ch(K) \neq 0$, then ε is relatively universal (Smith and Vőlklein [8] and Vőlklein [10]).

3.3 A remark on the case of n = 4

Let $n = 4$. Then Γ is isomorphic to the grassmannian of lines of the polar space Φ for $O_6^+(K)$ (compare Subsection 1.3) and another embedding η of Γ is implicit in this isomorphism, seemingly different from the embedding ε considered in Theorem 3.3.

Description of η. We recall that, if $W_0 = V(6, K)$ ($\cong V_0 \wedge V_0$ with $V_0 = V(4, K)$), the points of Φ are the points of $PG(W_0)$ that are singular for a given non-singular quadratic form of Witt index 3. Let $W := W_0 \wedge W_0$ ($\cong V(15, K)$; note that 15 is also the dimension of the space V of traceless 4-by-4 matrices). The grassmannian of lines of $PG(W_0)$ admits an embedding η into $PG(W)$ sending a line $L = \langle (x_i)_{i=1}^6, (y_i)_{i=1}^6 \rangle$ to the point of $PG(W_0 \wedge W_0)$ represented by the vector $(x_{i,j})_{1 \leq i < j \leq 6}$, where

$$(1) \quad x_{i,j} := \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

We may always assume that the quadratic form f defining Φ is expressed as follows with respect to the basis we have chosen in W_0:

$$f(x_1, x_2, ..., x_6) = \sum_{i=1}^{3} x_i x_{i+3}.$$

Accordingly, given L as above, the line L belongs to Φ if and only if

$$(2) \qquad \sum_{i=1}^{3} (x_i y_{i+3} + y_i x_{i+3}) = 0$$

Let W_1 be the subspace of W spanned by the images by η of the lines of Φ. Then η can be regarded as an embedding of Γ into $PG(W_1)$. If $ch(K) \neq 2$, then no linear condition on the coordinates $x_{i,j}$ (defined in (1)) is implicit in (2) and we have $W_1 = W$. On the other hand, if $ch(K) = 2$ then (2) can be read as follows:

$$(3) \qquad \sum_{i=1}^{3} x_{i,i+3} = 0.$$

In this case W_1 is the hyperplane of W represented by the linear equation (3).

Proposition 3.4 *If $ch(K) \neq 2$, then $\eta \equiv \varepsilon$. If $ch(K) = 2$, then $\eta < \varepsilon$.*

Proof. Given a point-plane flag (a, A) of $PG(V_0)$, let b, c be two points of A such that $\{a, b, c\}$ spans A. Let $(a_i)_{i=1}^4$, $(b_i)_{i=1}^4$ and $(c_i)_{i=1}^4$ be vectors of V_0 representing a, b and c, respectively, and let $(\alpha_i)_{i=1}^4$ be the 4-tuple of Plűcker coordinates of A. Then we may assume that

$$(4) \qquad \alpha_i = (-1)^i \begin{vmatrix} a_j & a_k & a_h \\ b_j & b_k & b_h \\ c_j & c_k & c_h \end{vmatrix}$$

for $\{i, j, k, h\} = \{1, 2, 3, 4\}$ with $j < k < h$. On the other hand, η sends (a, A) to the point represented by the vector $(a \wedge b) \wedge (a \wedge c)$ (where we take the liberty to denote the vectors representing the points a, b, c by the same letters a, b, c used for those points). The coordinates of $(a \wedge b) \wedge (a \wedge c)$ are as follows, where $i, j, k, h \in \{1, 2, 3, 4\}$ with $i < j$, $k < h$ and either $i < k$ or $i = k$ and $j < h$,

$$t_{i,j;k,h} = \epsilon_{i,j;k,h} \left((a_i b_j - a_j b_i)(a_k c_h - a_h c_k) - (a_i c_j - a_j c_i)(a_k b_h - a_h b_k)\right)$$

and $\epsilon_{i,j;k,h} = \pm 1$ according to a rule which we are not going to fix here (since it has no relevance for the sequel). By easy computations we get:

$$(5) \qquad t_{i,j;k,h} = \epsilon_{i,j;k,h} \left(a_i \begin{vmatrix} a_k & a_j & a_h \\ b_k & b_j & b_h \\ c_k & c_j & c_h \end{vmatrix} - a_j \begin{vmatrix} a_k & a_i & a_h \\ b_k & b_i & b_h \\ c_k & c_i & c_h \end{vmatrix} \right).$$

By comparing (4) and (5) we see that, if $\{i,j\} \cap \{k,h\} = \emptyset$ then

(6.a) $\quad t_{i,j;k,h} = \alpha_i a_i + \alpha_j a_j.$

Otherwise, $\{i,j,k,h\}$ has size 3 and we have

(6.b) $\quad t_{i,j;k,h} = \pm\alpha_\lambda a_\mu$

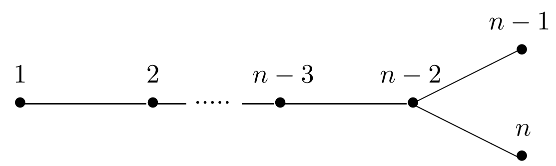
where $\{\mu\} = \{i,j\} \cap \{k,h\}$, $\{\lambda\} = \{1,2,3,4\} \setminus \{i,j,k,h\}$ and the sign $+$ or $-$ depends on the relative positions of the indices i,j,k,h. A morphism $\varphi : \varepsilon \to \eta$ is implicit in (6.a) and (6.b) (recall that $\varepsilon(a,A)$ is represented by the matrix $(\alpha_i a_j)_{i,j=1}^4$). If $ch(K) \neq 2$, then φ is an isomorphism. Otherwise, $Ker(\varphi)$ is the 1-dimensional subspace of V spanned by the identity matrix. □

4 More applications

In this Section we combine the theorem of Kasikova and Shult (Theorem 1.2) with our Theorem 3.3 to prove that certain point-line geometries related to buildings of type D_n and E_n admit the absolutely universal embedding.

4.1 On certain geometries related to D_n-buildings

Given a field K and an integer $n \geq 3$, let Δ be the building of type D_n defined over K. We take the integers $1, 2, ..., n-1, n$ as types, as follows:

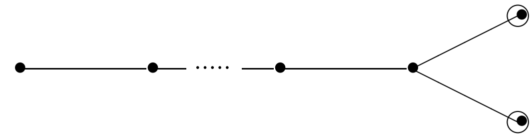


Let Γ be the point-line geometry of the $\{n-1,n\}$–Grassmann geometry of Δ [5, Chapter 5]. That is, the points of Γ are the flags of Δ of type $\{n-1,n\}$ and the lines of Γ are the flags of Δ of type $\{n-2,n-1\}$ and $\{n-2,n\}$, two flags of type $\{n-1,n\}$ and $\{n-2,i\}$ ($i = n-1$ or n) being incident as elements of Γ when they are incident as flags of Δ. As distinct flags of Δ of type $\{n-2,i\}$ ($i = n-1$ or n) are incident with distinct sets of $\{n-1,n\}$-flags, Γ is indeed a point-line geometry in the sense of this paper. When $n = 3$, Γ is just the long root geometry for $SL(4,K)$.

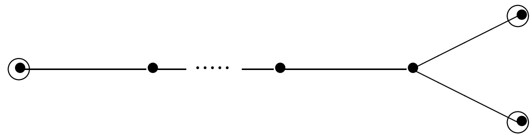
When $n > 3$, we also consider the point-line geometry of the $\{1,n-1,n\}$-Grassmann geometry of Δ, where the points are the $\{1,n-1,n\}$-flags of Δ and the lines are the flags of Δ of type $\{2,n-1,n\}$, $\{1,n-2,n-1\}$ and $\{1,n-2,n\}$. We denote this geometry Φ.

Finally, when $n > 4$, we denote Ψ the point-line geometry of the $\{1,n-1\}$-Grassmann geometry of Δ, where the points are the $\{1,n-1\}$-flags of Δ and the lines are the flags of Δ of type $\{1,n-2\}$ and $\{2,n-1\}$.

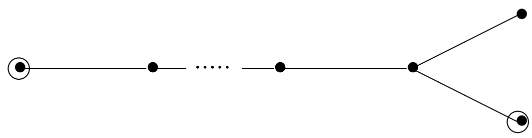
(Γ)



(Φ)



(Ψ)



Let Δ_1 be the polar space with the 1- and 2-elements of Δ as point and lines respectively. Then Γ can be regarded as the grassmannian of singular subspaces of Δ_1 of (projective) dimension $n-2$.

For $i = n-1$ or n, we denote Δ_i the corresponding half-spin geometry of Δ, with the i-elements of Δ as points and the $(n-2)$-elements as lines. Furthermore, when $n > 3$, we denote Δ_{n-2} the $(n-2)$-Grassmann geometry of Δ, where the elements of Δ of type $n-2$ are taken as points and the flags of Δ of type $\{n-3, n-1, n\}$ are the lines.

Note that Δ_{n-2} can also be regarded as the grassmannian of $(n-3)$-dimensional singular subspaces of the polar space Δ_1. In particular, when $n = 4$, the geometry $\Delta_{n-2} = \Delta_2$ is the *metasymplectic space* of Δ (Tits [9, 10.14]) and Φ is isomorphic to the grassmannian of lines of Δ_2.

Lemma 4.1 *Each of* Γ, Φ *and* Ψ *is embeddable.*

Proof. A projective embedding of Γ can be constructed in at least two ways (compare Subsection 3.3).

1) (*Via tensor products.*) It is well known that, for $i = n-1, n$, Δ_i admits an embedding $\varepsilon_i : \Delta_i \to PG(V)$ where V is the 2^{n-1}-dimensional half-spin module. If we send every $\{n-1, n\}$-flag $\{a, b\}$ of Δ to the point $\varepsilon_{n-1}(a) \otimes \varepsilon_n(b)$ of $PG(V \otimes V)$, then we get an embedding Γ into a subspace of $PG(V \otimes V)$.

2) (*Via exterior powers.*) As previously noticed, Γ is isomorphic to the grassmannian of $(n-2)$-dimensional singular subspaces of Δ_1. The latter is embedded in $PG(V)$ where $V = V(2n, K)$ and the grassmannian of $(n-2)$-dimensional subspaces of $PG(V)$ is embedded in $PG(W)$ with $W = \wedge^{n-1} V$. Accordingly, Γ is embedded in a subspace of $PG(W)$.

Turning to Φ, let ε_1 be the natural embedding of Δ_1 in $PG(V_1)$, with $V_1 = V(2n, K)$. With ε_{n-1} and ε_n defined as above, we can send every $\{1, n-1, n\}$-flag $\{a, b, c\}$ of Δ to the point $\varepsilon_1(a) \otimes \varepsilon_{n-1}(b) \otimes \varepsilon_n(c)$ of $PG(V_1 \otimes V \otimes V)$. Thus, we obtain an embedding of Φ into a subspace of $PG(V_1 \otimes V \otimes V)$.

Similarly, by taking tensor products $\varepsilon_1(a) \otimes \varepsilon_{n-1}(b)$ with $\{a, b\}$ a $\{1, n-1\}$-flag, we obtain an embedding for Ψ. □

Remark. When $n = 4$, an embedding of Φ can also be constructed via exterior powers. With ε_1 and V_1 as in the proof of the previous Lemma, let $n = 4$. As previously noticed, Δ_2 is the grassmannian of lines of Δ_1. Accordingly, the embedding of the grassmannian of lines of $PG(V_1)$ in $PG(V_1 \wedge V_1)$ induces an embedding ε_2 of Δ_2 in $PG(V_2)$, for a suitable subspace V_2 of $V_1 \wedge V_1$. (Actually, $V_2 = V_1 \wedge V_1$ except when $ch(K) = 2$; if $ch(K) = 2$, then V_2 is a hyperplane of $V_1 \wedge V_1$.) In its turn, the grassmannian of lines of $PG(V_2)$ is embeddable in $PG(V_2 \wedge V_2)$. Accordingly, the grassmannian of lines of Δ_2 is embedded in a subspace of $PG(V_2 \wedge V_2)$. As Φ is isomorphic to the grassmannian of lines of Δ_2, we obtain an embedding of Φ into a subspace of $PG(V_2 \wedge V_2)$.

Warning. The second author of this paper feels compelled to warn the reader that the final results of [1], where the above embedding ε_2 is discussed, are wrong, because of mistakes occurring in §3.6.2 and in the proof of Theorem 4.4 of [1]. Consequently, Theorem 4.4 of [1] is false and Theorem 4.5 of [1] is (correct but) empty, Lemma 5.4 and Theorem 5.5 are wrong when $ch(K) \neq 2$ (but correct when $ch(K) = 2$) and Corollary 5.6 is false when $p \neq 2$ (but true when $p = 2$).

Theorem 4.2 *If K has no proper subfield, then each of Γ, Φ and Ψ admits the absolutely universal embedding.*

Proof. We first consider Γ, then we shall turn to Φ and Ψ.

1) In the case of Γ we exploit induction on n. When $n = 3$, the conclusion follows from Theorem 3.3. Let $n > 3$. Given a point x of Δ_1, the set of $\{n-1, n\}$-flags of Δ incident to x is a subspace of Γ supporting a subgeometry Γ_x of the same kind as Γ, but defined in the residue of x in Δ, which is a building of type D_{n-1}. By the inductive hypothesis, Γ_x admits the absolutely universal embedding.

Let $\mathcal{S}$ be the family of the above subspaces of Γ and let Σ be the graph defined on $\mathcal{S}$ as in Theorem 1.2. Clearly, every line of Γ belongs to a member of $\mathcal{S}$. Furthermore, Σ is the collinearity graph of the polar space Δ_1. Hence Σ is simply connected.

So far, we have proved that conditions (1)–(4) of Theorem 1.2 hold. Turning to condition (5), let $\{A, B, C\}$ be a 3-clique of Σ. The points of Δ_1 corresponding to A, B and C are incident to a common plane of Δ_1. However, every plane of Δ_1 is incident to an $\{n-1, n\}$-flag of Δ. Thus, condition (5) of Theorem 1.2 is empty for Γ.

Finally, the members of $\mathcal{S}$ containing a given point of Γ form a complete subgraph of Σ. Hence (6) of Theorem 1.2 also holds. Furthermore, Γ is embeddable, as stated in Lemma 4.1. Hence Γ admits the absolutely universal embedding, by Theorem 1.2.

2) Turning to Φ, let x be an element of Δ of type $i = 1, n-1$ or n. Then the set of $\{1, n-1, n\}$-flags of Δ incident to x is a subspace of Φ supporting a subgeometry Φ_x of Φ. If $i = 1$, then Φ_x is just like Γ, but of rank $n-1$. By the above, Φ_x admits the absolutely universal embedding. Otherwise, Φ_x is isomorpic to the long root geometry for $SL(n, K)$. By Theorem 3.3, Φ_x admits the universal embedding.

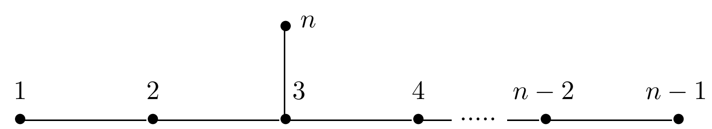
Let $\mathcal{S}$ be the family of the above subspaces of Φ and let Σ be the graph defined on $\mathcal{S}$ as in Theorem 1.2. Then every line of Φ belongs to a member of $\mathcal{S}$ and Σ is the collinearity graph of the dual of Δ_{n-2}. Therefore, Σ is connected and, as Δ_{n-2} is simply connected, Σ is also simply connected [5, Theorem 12.56].

Thus, conditions (1)–(4) of Theorem 1.2 hold. Condition (5) of Theorem 1.2 is empty and condition (6) holds, like in the proof of Theorem 4.2. (We leave the details for the reader.) According to Lemma 4.1, Φ admits an embedding. Therefore, by Theorem 1.2, Φ admits the absolutely universal embedding.

3) Finally, we consider Ψ. We now take the n-elements of Δ as members of $\mathcal{S}$. The subgeometry of Ψ corresponding to such an element is isomorphic to the long root geometry for $SL(n, K)$. Hence it admits the absolutely universal embedding, by Theorem 3.3. The graph Σ, defined on $\mathcal{S}$ as in Theorem 1.2, is isomorphic to the collinearity graph of Δ_n. Hence it is simply connected. The remaining hypotheses of Theorem 1.2 are readily seen to hold in this case, too. □

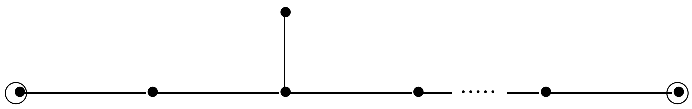
4.2 On certain geometries related to E_n-buildings

Given a field K, let Δ be the building of type E_n defined over K, where $n = 6, 7$ or 8. We take the integers $1, 2, ..., n$ as types, as follows:

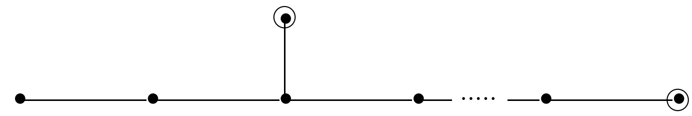


We define the geometries Γ, Φ, Φ' and Ψ as follows.

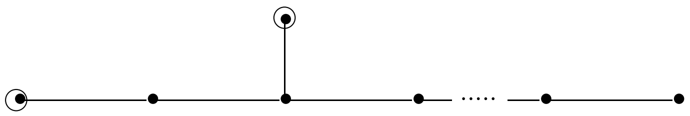
Γ is the point-line geometry of the $\{1, n-1\}$–Grassmann geometry of Δ. That is, the points of Γ are the flags of Δ of type $\{1, n-1\}$ and the lines of Γ are the flags of Δ of type $\{2, n-1\}$ and $\{1, n-2\}$.

(Γ) 

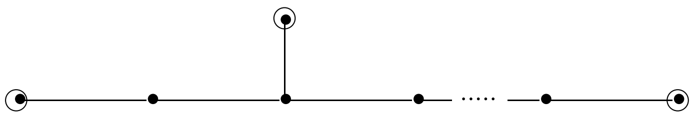
Φ is the point-line geometry of the $\{n-1,n\}$-Grassmann geometry of Δ, where the points are the $\{n-1,n\}$-flags and the lines are the flags of type $\{3,n-1\}$ and $\{n-2,n\}$.

(Φ) 

Φ' is the point-line geometry of the $\{1,n\}$-Grassmann geometry of Δ, where the points are the $\{1,n\}$-flags and the lines are the flags of type $\{1,3\}$ and $\{2,n\}$. (Note that $\Phi' \cong \Phi$ when $n=6$.)

(Φ′) 

Ψ is the point-line geometry of the $\{1,n-1,n\}$-Grassmann geometry of Δ, where the points are the $\{1,n-1,n\}$-flags and the lines are the flags of type $\{2,n-1,n\}$, $\{1,n-2,n\}$ and $\{1,3,n-1\}$.

(Ψ) 

Lemma 4.3 *Each of the geometries* Γ, Φ, Φ' *and* Ψ *is embeddable.*

Proof. For $i=1,n-1,n$, let Δ_i be the i-Grassmann geometry of Δ, with the i-elements of Δ as points and the elements of type j as lines, where $j=2,n-2$ or 3 according to whether $i=1,n-1$ or n. For $i=1,n-1,n$, it is well known that Δ_i admits a projective embedding, say ε_i. Projective embeddings of Γ, Φ, Ψ and Ψ' can be obtained via tensor products from the embeddings ε_1, ε_{n-1} and ε_n, as in the proof of Lemma 4.1. □

Theorem 4.4 *If* K *has no proper subfield, then each of* Γ, Φ, Φ' *and* Ψ *admits the absolutely universal embedding.*

Proof. The proof is similar to that of Theorem 4.2. We leave the details for the reader. We only explain how to choose $\mathcal{S}$ in each case.

Consider Γ, first. In this case we may take the n-elements of Δ as members of $\mathcal{S}$. The subspaces corresponding to those elements support subgeometries isomorphic to the long root geometry for $SL(n, K)$, which is known to admit the absolutely universal embedding by Theorem 3.3.

Turning to Φ, we take the 1-elements of Δ as members of $\mathcal{S}$. The corresponding subspaces support geometries isomorphic to those called Ψ in Theorem 4.2. By that Theorem, those geometries admit the absolutely universal embedding.

The case of Φ' can be settled by an inductive argument. When $n = 6$, we have $\Phi' \cong \Phi$. As the geometry Φ has been considered above, let $n > 6$ and take the $(n-1)$-elements of Δ as members of $\mathcal{S}$. Then we get subgeometries of the same kind of Ψ', but defined in the E_{n-1}-building. By an inductive hypotheses, these geometries admit the absolutely universal embedding.

Finally, in the geometry Ψ, the members of $\mathcal{S}$ are the elements of Δ of type 1, $n-1$ and n. The elements of type n support subgeometries isomorphic to the long root geometry for $SL(n, K)$ and those of type 1 and $n-1$ support geometries of the same kind as Φ and Φ', respectively. In any case, the supported geometries admit the absolutely universal embedding. □

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Address of the authors

Rieuwert Blok
Department of Mathematics
Michigan State University
East Lansing MI 48829
U.S.A.
e-mail: blokr@member.ams.org

and

Antonio Pasini
Department of Mathematics
University of Siena
Via del Capitano 15
I-53100 Siena
Italy
e-mail: pasini@unisi.it