Point-line geometries with a generating set that depends on the underlying field

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Suppose Γ is a Lie incidence geometry defined over some field $\mathbb{F}$ having a Lie incidence geometry Γ_0 of the same type but defined over a subfield $\mathbb{F}_0 \leq \mathbb{F}$ as a subgeometry. We investigate the following question: how many points (if any at all) do we have to add to the point-set of Γ_0 in order to obtain a generating set for Γ? We note that if Γ is generated by the points of an apartment, then no additional points are needed. We then consider the long-root geometry of the group $\mathrm{SL}_{n+1}(\mathbb{F})$ and the line-grassmannians of the polar geometries associated to the groups $\mathrm{O}_{2n+1}(\mathbb{F})$, $\mathrm{Sp}_{2n}(\mathbb{F})$ and $\mathrm{O}^+_{2n}(\mathbb{F})$. It turns out that in these cases the maximum number of points one needs to add to Γ_0 in order to generate Γ equals the maximal number of roots one needs to adjoin to $\mathbb{F}_0$ in order to generate $\mathbb{F}$. We prove that in the case of the long-root geometry of the group $\mathrm{SL}_{n+1}(\mathbb{F})$ the point-set of Γ_0 does not generate Γ. As a by-product we determine the generating rank of the line grassmannian of the polar geometry associated to $\mathrm{Sp}_{2n}(\mathbb{F})$ ($n \geq 3$), if $\mathbb{F}$ is a prime field of odd characteristic.

1 Introduction

1.1 The problem studied in this paper

In a recent paper [10] Cooperstein determines the generating rank for the long-root geometries of $\mathrm{SL}_{n+1}(\mathbb{F})$, $\mathrm{O}^+_{2n}(\mathbb{F})$, $\mathrm{O}_{2n+1}(\mathbb{F})$ and $\mathrm{O}^-_{2n+2}(\mathbb{F})$ for a prime field $\mathbb{F}$ of any characteristic. The result of a simple computation on the long-root geometry Γ of the group $\mathrm{SL}_3(\mathbb{F})$, but now with $\mathbb{F}$ a proper extension over its prime field $\mathbb{F}_0$, was confirmed by an observation of Smith and Völklein (see Proposition 1.2 of [14]): the set considered by Cooperstein in [10] only generates the long-root subgeometry Γ_0 of $\mathrm{SL}_3(\mathbb{F}_0)$ but it does not generate Γ. However, by adding one well-chosen point to this set, one obtains a generating set for Γ.

In the present paper we investigate the following problem: suppose Γ is a Lie incidence geometry Γ defined over some field $\mathbb{F}$ having a Lie incidence geometry Γ_0 of the same type but defined over a subfield $\mathbb{F}_0 \leq \mathbb{F}$ as a subgeometry. How many points (if any at all) do we have to add to the point-set of Γ_0 in order to obtain a generating set for Γ? We are mainly concerned with the following Lie incidence geometries: the long-root geometry associated to the group $\mathrm{SL}_{n+1}(\mathbb{F})$ and the line-grassmannians of the polar geometries associated to the groups $\mathrm{O}_{2n+1}(\mathbb{F})$, $\mathrm{Sp}_{2n}(\mathbb{F})$ and $\mathrm{O}^+_{2n}(\mathbb{F})$ (these geometries are described in the following subsection).

1.2 Contents and main results

In Section 2, after having recalled a few basic notions on incidence geometries, we provide the general framework for the problem studied in this paper: for a Lie incidence geometry Γ defined over some field $\mathbb{F}$, we describe how one obtains a Lie incidence geometry Γ_0 of the same type as Γ, but defined over a subfield $\mathbb{F}_0 \leq \mathbb{F}$, as a subgeometry of Γ in the case that Γ is obtained from a building Δ associated to a non-twisted Chevalley group $G(\mathbb{F})$; Γ_0 is the Lie incidence geometry obtained from the sub-building Δ_0 of Δ associated to the Chevalley group $G(\mathbb{F}_0)$ viewed as a subgroup of $G(\mathbb{F})$.

In Section 3 we give a simple application of the results of Section 2, showing that if Γ is a Lie incidence geometry that is generated by the points of an apartment, then Γ_0 generates Γ.

In Section 4, Γ is the long-root geometry associated to the group $\mathrm{SL}_{n+1}(\mathbb{F})$ with $n \geq 2$. We regard the building Δ as the geometry of subspaces of $\mathrm{PG}(n, \mathbb{F})$. For Δ_0 we can take the geometry of subspaces of $\mathrm{PG}(n, \mathbb{F}_0)$ viewed as a subgeometry of $\mathrm{PG}(n, \mathbb{F})$ and Γ_0 becomes the subgeometry of Γ whose points belong to Δ_0. The geometries Γ and Γ_0 can be described as follows: the points of Γ and Γ_0 are the point-hyperplane flags of $\mathrm{PG}(n, \mathbb{F})$ and $\mathrm{PG}(n, \mathbb{F}_0)$. The lines of Γ and Γ_0 are the flags (X, Y) with $X \subseteq Y$ of codimension $n-2$ in Y; thus $X = Y$, when $n = 2$. A point (p, H) lies on a line (X, Y) whenever $p \subseteq X$ and $Y \subseteq H$.

According to the above description, we call Γ the *point-hyperplane geometry* of $\mathrm{PG}(n, \mathbb{F})$.

Theorem 1.1 *Given a field $\mathbb{F}$, let $\mathbb{F}_0$ be a subfield of $\mathbb{F}$. For $F = \mathbb{F}$ and $F = \mathbb{F}_0$, let Γ and Γ_0 be the long-root geometry of* $\mathrm{SL}_{n+1}(F)$*, viewed as the point-hyperplane geometry of* $\mathrm{PG}(n, F)$*. Then the following hold.*

(i) *The span $\langle \Gamma_0 \rangle_\Gamma$ of Γ_0 in Γ is the set of point-hyperplane flags (P, H) of* $\mathrm{PG}(n, \mathbb{F})$ *with $P \subseteq E \subseteq H$ for some element E of* $\mathrm{PG}(n, \mathbb{F}_0)$*.*

(ii) *Suppose S is a set of points of Γ with $\langle S \rangle_\Gamma \supseteq \Gamma_0$. Then $\mathbb{F}$ contains an extension $\mathbb{F}_1$ of $\mathbb{F}_0$ with the following property: the elements of* $\mathrm{PG}(n, \mathbb{F}_1)$ *are the elements E of* $\mathrm{PG}(n, \mathbb{F})$ *such that all $(P, H) \in \Gamma$ with $P \subseteq E \subseteq H$ are in $\langle S \rangle_\Gamma$.*

(iii) *In* (ii)*, if S is a generating set of Γ_0, then $\mathbb{F}_1 = \mathbb{F}_0$.*

Proof Part (i) is Theorem 4.5. Part (ii) is Theorem 4.7. Part (iii) is Theorem 4.6. □

Corollary 1.2 *With the notation of Theorem* 1.1*, if* $\mathbb{F}_0$ *is a proper subfield of* $\mathbb{F}$*, then* Γ_0 *does not generate* Γ.

Proof If, on the contrary, Γ_0 does generate Γ, then by part (ii) of Theorem 1.1, we have $\mathrm{PG}(n, \mathbb{F}_0) = \mathrm{PG}(n, \mathbb{F})$, which is clearly absurd. □

Problem Corollary 1.2 does not imply that the generating rank of Γ (see Subsection 2.1) is larger than the generating rank of Γ_0. Indeed, possibly there exists a generating set for Γ, not contained in Γ_0 and having as many points as a minimal generating set of Γ_0.

In Section 5 we consider the line-grassmannian Γ of the polar geometry associated to the group $\mathrm{O}_{2n+1}(\mathbb{F})$, $\mathrm{O}^+_{2n}(\mathbb{F})$ or $\mathrm{Sp}_{2n}(\mathbb{F})$ with $n \geq 3$. We view the building Δ as the polar geometry of subspaces of a vector space V of dimension $2n+1$ or $2n$ that are totally singular with respect to a non-degenerate quadratic form of Witt index n, or the polar geometry of subspaces of a vector space V of dimension $2n$ with respect to a non-degenerate symplectic form. The points (lines) of Γ are the lines (point-plane flags) of Δ and the incidence relation is inherited from the incidence relation between elements and flags of Δ.

In each of these cases, for a suitable choice of the quadratic or symplectic form involved, for Δ_0 we can take the geometry of elements of Δ that are defined over $\mathbb{F}_0$ and Γ_0 becomes the subgeometry of Γ whose points belong to Δ_0.

Note The line-grassmannian of the polar geometry associated with the group $\mathrm{O}_{2n+1}(\mathbb{F})$ or $\mathrm{O}^+_{2n}(\mathbb{F})$ is precisely the long-root geometry of that group. Also for the group $\mathrm{SL}_{n+1}(\mathbb{F})$, we study its long-root geometry here. The only long-root geometry studied by Cooperstein in [10] that is not considered here is the long-root geometry of the group $\mathrm{O}^-_{2n+2}(\mathbb{F})$, which is the line-grassmannian of the polar geometry associated to this group. The reason is that we do not want to go into the complications that arise in this case when constructing a subgeometry of the same type defined over a subfield, due to the fact that $\mathrm{O}^-_{2n+2}(\mathbb{F})$ is a twisted Chevalley group.

Theorem 1.3 *Given a field* $\mathbb{F}$*, let* $\mathbb{F}_0$ *be a subfield of* $\mathbb{F}$*. For* $F = \mathbb{F}$ *and* $F = \mathbb{F}_0$*, let* Γ *and* Γ_0 *be the line-grassmannian of the polar geometry* $\Delta(F)$ *associated to the group* $\mathrm{O}_{2n+1}(F)$*,* $\mathrm{Sp}_{2n}(F)$ *or* $\mathrm{O}^+_{2n}(F)$ *with* $n \geq 3$*. Let* S *be a set of points of* Γ *with* $\langle S \rangle_\Gamma \supseteq \Gamma_0$*. Then* $\mathbb{F}$ *contains an extension* $\mathbb{F}_1$ *of* $\mathbb{F}_0$ *such that the elements of* $\Delta(\mathbb{F}_1)$ *are precisely those elements of* $\Delta(\mathbb{F})$ *that are generated by* 1*-elements* E *with the following property:*

($*$) *all lines* L *of* $\Delta(\mathbb{F})$ *with* $L \subseteq E^\perp$ *and* $L \cap E \neq \{0\}$ *belong to* $\langle S \rangle_\Gamma$.

In fact, all elements E *of* $\Delta(\mathbb{F}_1)$ *satisfy* ($*$) *and conversely, the* i*-elements* E *of* $\Delta(\mathbb{F})$ *with* $i < n$ *having property* ($*$) *all belong to* $\Delta(\mathbb{F}_1)$.

The proof is given in Section 5.

Problems (1) Drop the condition $i < n$ in the last claim of Theorem 1.3.

(2) Computations in some cases with $n = 3$ suggest that if S is a generating set for Γ_0, then $\mathbb{F}_1 = \mathbb{F}_0$ and $\langle S \rangle_\Gamma$ is precisely the collection of points L of Γ for which there exists an i-element E of $\Delta(\mathbb{F}_0)$ such that $L \subseteq E^\perp$ (in $\Delta(\mathbb{F})$) and $L \cap E \neq \emptyset$. However, the arguments we have used in the examined cases do not seem to carry over to arbitrary n.

Theorem 1.4 *Let the field $\mathbb{F}$ be generated by adjoining k elements to some subfield $\mathbb{F}_0$. For $F = \mathbb{F}$ and $F = \mathbb{F}_0$, let Γ and Γ_0 be the long-root geometry of the group $\mathrm{SL}_{n+1}(F)$ or the line-grassmannian of the polar geometry associated to the group $\mathrm{O}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$ or $\mathrm{O}^+_{2n}(F)$. Then Γ can be generated by adding at most k points to Γ_0, viewed as a subgeometry.*

Proof This follows from Corollaries 4.8 and 5.4. □

The previous results entail several consequences for generating ranks, which are defined in Subsection 2.1. We first state the following proposition, which is similar to the results of Cooperstein [10], although the geometry considered here is not a long-root geometry.

Proposition 1.5 *For any prime field $\mathbb{F}$ with characteristic $\mathrm{Char}(\mathbb{F})$ different from 2, the line-grassmannian of the natural polar geometry for the group $\mathrm{Sp}_{2n}(\mathbb{F})$ has generating rank $2n^2 - n - 1$.*

The proof is given at the end of Section 5.

Corollary 1.6 *Given a prime power q and an integer $n \geq 3$, let Γ be the long-root geometry of the group $G = \mathrm{SL}_{n+1}(q)$ or the line-grassmannian of the polar geometry associated to the group $G = \mathrm{O}_{2n+1}(q)$, $\mathrm{Sp}_{2n}(q)$ or $\mathrm{O}^+_{2n}(q)$, with q odd when $G = \mathrm{Sp}_{2n}(q)$ or $\mathrm{O}^+_{2n}(q)$. Then Γ has generating rank r with $r_0 \leq r \leq r_0+1$ and r_0 as follows:*

G	$\mathrm{SL}_{n+1}(q)$	$\mathrm{O}_{2n+1}(q)$	$\mathrm{Sp}_{2n}(q)$	$\mathrm{O}^+_{2n}(q)$
r_0	$n^2 - 1$	$2n^2 + n$	$2n^2 - n - 1$	$2n^2 - n$

Proof Given a prime number p let Γ_0 be the long-root geometry associated to $G_0 = \mathrm{SL}_{n+1}(p)$ or the line-grassmannian of the polar geometry for $G_0 = \mathrm{O}_{2n+1}(p)$ or $\mathrm{O}^+_{2n}(p)$. By Theorems 4.1, 5.1 and 6.1 of Cooperstein [10], Γ_0 has a generating set of size $m = n^2 - 1$, $2n^2 + n$ or $2n^2 - n$, respectively. Furthermore, Γ_0 admits an embedding in $\mathrm{PG}(m-1, p)$, except possibly when $G_0 = \mathrm{O}^+_{2n}(2)$ with $n > 3$. Therefore, in these cases, Γ_0 has generating rank $r_0 = m$. The finite field of order q is at most a simple extension over its prime field. The result follows from Theorem 1.4.

By the same argument, except for exploiting Proposition 1.5 instead of the results of Cooperstein [10], we obtain the conclusion in the case of $G = \mathrm{Sp}_{2n}(q)$. □

In a forthcoming paper we will determine the precise generating rank for some of these geometries.

Note The case of $G = \mathrm{Sp}_{2n}(q)$ with q even is not considered in the above corollary, in view of the classical isomorphism between the groups $\mathrm{Sp}_{2n}(2^e)$ and $\mathrm{O}_{2n+1}(2^e)$ and the corresponding isomorphism between their polar geometries.

The case of $G = \mathrm{O}^+_{2n}(q)$ with q even is also missing in Corollary 1.6, but for a different reason. With the notation used in the proof of Corollary 1.6, let $G_0 = \mathrm{O}^+_{2n}(2)$. By Theorem 4.1 of Cooperstein [10], Γ_0 admits a generating set of size $m = 2n^2 - n$. Therefore, the generating rank of Γ_0 is at most m. However, no embedding of Γ_0 in $\mathrm{PG}(m-1, 2)$ is known when $n > 4$, whereas an embedding of Γ_0 in $\mathrm{PG}(m-2, 2)$, which we shall call the *natural embedding* of Γ_0, is induced by the natural embedding ε of the line-grassmannian of $\mathrm{PG}(2n-1, 2)$ in $\mathrm{PG}(m-1, 2)$. (Note that Γ_0 is a subgeometry of the polar geometry for $\mathrm{Sp}_{2n}(2)$ and the ε-image of the latter spans a hyperplane of $\mathrm{PG}(m-1, 2)$; compare [4, Section 3.3].) Accordingly, when $n > 4$ we can only claim the generating rank r_0 of Γ_0 is at least $m-1$ and at most m.

When $n = 3$, Γ_0 is isomorphic to the long-root geometry of $\mathrm{SL}_4(2)$, the generating rank of which is $r_0 = 15$ $(= m)$. When $n = 4$, we have three natural embeddings of Γ_0, permuted by the triality of the D_4-building, but none of them is universal, by the second claim of Corollary 5.6 of [1]; see also [2]. Consequently, the universal embedding of Γ_0, which exists by Ronan [12, Corollary 2], is 27-dimensional and so Γ_0 has generating rank $r_0 = 28$ $(= m)$.

Therefore, by Theorem 1.4, when q is even, the line-grassmannian of the polar geometry for $\mathrm{O}^+_{2n}(q)$ has generating rank r with $r_0 \le r \le r_0 + 1$ where r_0 is as follows:

n	r_0
3	$15\ (= 2n^2 - n)$
4	$28\ (= 2n^2 - n)$
$n > 4$	$2n^2 - n - 1 \le r_0 \le 2n^2 - n$

2 Preliminaries

2.1 Terminology

Following Tits [16], by a *geometry* Δ of *rank* n we will mean a quadruple $(\mathcal{O}, \star, \tau, I)$ consisting of a set $\mathcal{O}$ of elements, a symmetric relation $\star$ on $\mathcal{O}$ called the *incidence relation*, a set I of size n, called the *set of types*, and a surjective mapping $\tau\colon \mathcal{O} \longrightarrow I$ (called the *type-map*) with the property that, if x, y are distinct incident objects, then x and y have different type (namely, $\tau(x) \neq \tau(y)$).

A *flag* of a geometry Δ is a set of pairwise incident elements of Δ. The *type* of a flag F is its image $\tau(F)$ by τ. For $J \subseteq I$, the flags of type J are also called *J-flags*. The flags of type I are called *chambers*. The *panels* are the flags of type $I \setminus \{i\}$, for $i \in I$. Two flags F_1, F_2 are said to be *incident* when $F_1 \cup F_2$ is a flag.

Taking the flags as cells, we may also regard a geometry as a cell complex where every vertex is given a colour and type in such a way that no two vertices of the same colour or type belong to the same cell (compare Tits [15]).

A *subgeometry* of $\Delta = (\mathcal{O}, \star, \tau, I)$ is a geometry $\Delta_0 = (\mathcal{O}_0, \star_0, \tau_0, I_0)$, where $\mathcal{O}_0 \subseteq \mathcal{O}$, and for all $X, Y \in \mathcal{O}_0$ we have $\tau_0(X) = \tau(X)$, and $X \star_0 Y$ if and only if $X \star Y$.

A *point-line geometry* is a pair $\Gamma = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ is a set whose elements are called 'points' and $\mathcal{L}$ is a collection of subsets of $\mathcal{P}$ called 'lines' with the property that any two points belong to at most one line.

A *subgeometry* Γ_0 of Γ is a point-line geometry $(\mathcal{P}_0, \mathcal{L}_0)$ such that $\mathcal{P}_0 \subseteq \mathcal{P}$ and, for each $l_0 \in \mathcal{L}_0$ there is an $l \in \mathcal{L}$ such that $l_0 \subseteq l$.

A *subspace* of Γ is a subset $X \subseteq \mathcal{P}$ such that any line containing at least two points of X entirely belongs to X. Clearly, every subspace X can also be regarded as a subgeometry $(X, \mathcal{L}_X)$ of Γ, where $\mathcal{L}_X$ is the set of lines of Γ contained in X.

The *span* of a set $S \subseteq \mathcal{P}$ is the smallest subspace containing S; it is the intersection of all subspaces containing S and is denoted by $\langle S \rangle_\Gamma$. We say that S *spans* (or *generates*) $\langle S \rangle_\Gamma$ or that S is a *generating set* for $\langle S \rangle_\Gamma$.

The *generating rank* of Γ is the minimal size of a generating set of Γ.

Note Clearly, a point-line geometry is a geometry of rank 2. We may take the words 'point' and 'line' as its types, or the integers 0 and 1, or whatever pair of symbols we like.

It should also be noted that the above definition of geometry of rank n is weaker than other definitions existing in the literature. For instance, Buekenhout [6] also requires all maximal flags to be chambers. In Buekenhout [5] and Pasini [11] residual connectedness and firmness are also assumed. However, as we only need a terminological framework in this paper, where to put certain well-known structures (buildings, polar geometries, and so on), our lax definition is sufficient for our purposes.

2.2 The geometries considered in this paper

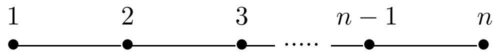
The point-line geometries considered in this paper are shadow geometries of certain buildings.

We will view a building as a geometry belonging to a Coxeter diagram, as in Tits [15]; see also Tits [16] or Pasini [Chapter 13][11]. For a description of a building as a chamber system see Tits [16] or Ronan [13]. The buildings we study in this paper are of irreducible spherical type, have rank $n \geq 3$ and arise from a (non-twisted) Chevalley group defined over a field $\mathbb{F}$.

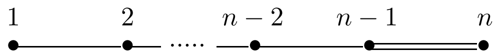
We recall that the (adjoint) non-twisted Chevalley groups of rank at least 3 are the following: $\mathrm{PSL}_{n+1}(\mathbb{F})$, $\mathrm{P}\Omega_{2n+1}(\mathbb{F})$, $\mathrm{PSp}_{2n}(\mathbb{F})$, $\mathrm{P}\Omega^+_{2n}(\mathbb{F})$, $E_6(\mathbb{F})$, $E_7(\mathbb{F})$, $E_8(\mathbb{F})$ and $F_4(\mathbb{F})$, with Dynkin diagrams A_n, B_n, C_n, D_n, E_6, E_7, E_8 and F_4, respectively. The corresponding buildings are said to be *defined over* $\mathbb{F}$ and to have type A_n, B_n, C_n, etc. Note that the type of a building, as defined above, is not simply the name of its Coxeter diagram, as the Dynkin diagrams B_n and C_n, which are different, correspond to the same Coxeter diagram, often called C_n in the literature.

In this paper, we are mainly interested in the diagrams A_n, B_n, C_n and D_n, but E_6 and E_7 will also be considered in Theorem 3.1. We take the positive integers $1, 2, 3, \ldots$ as types for these diagrams, labelling the nodes as follows:

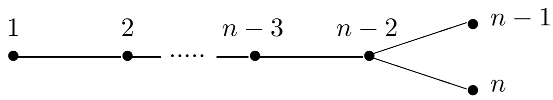
(A_n)



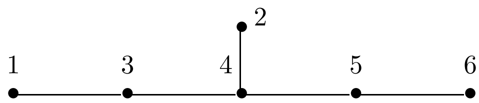
(B_n, C_n)



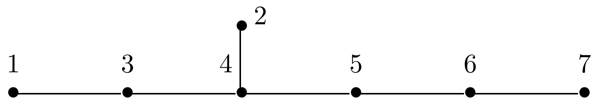
(D_n)



(E_6)



(E_7)



Henceforth, when referring to labels of the nodes of one of the above diagram, we shall mean the labels that diagram has been given here.

2.2.1 Shadow geometries

Given a building Δ, let M be its diagram and let I be its set of types. Given a non-empty $J \subseteq I$, we denote $\mathrm{Sh}_J(\Delta)$ the J-shadow geometry of Δ (Tits [15, Chapter 12]; see also Pasini [11, Chapter 5], where these geometries are called J-Grassmann geometries). We regard $\mathrm{Sh}_J(\Delta)$ as a point-line geometry: The J-flags of Δ are taken as points; for every $j \in J$ and every flag F of Δ of type $(J \cup M(j)) \setminus \{j\}$, with $M(j)$ denoting the set of types joined to j in M, the collection of J-flags of Δ incident to F is a line of $\mathrm{Sh}_J(\Delta)$ and all lines of $\mathrm{Sh}_J(\Delta)$ are defined in this way.

In particular, when $J = \{i\}$ the points of $\mathrm{Sh}_J(\Delta)$ are the i-elements of Δ and the lines of $\mathrm{Sh}_J(\Delta)$ correspond to the flags of Δ of type $M(i)$. If furthermore i is an end-node of M and j is the unique node joined to it in M (as when M is as in one of the previous pictures, $i = 1$ and $j = 2$), then the lines of $\mathrm{Sh}_i(\Delta)$ correspond to the j-elements of Δ.

With M and J as above, we say that $\mathrm{Sh}_J(\Delta)$ *is of type* M_J (of type M_i when $J = \{i\}$).

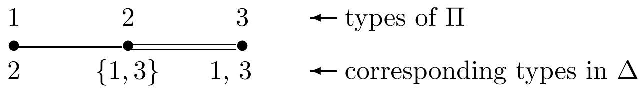
2.2.2 Geometries of type $A_{n,i}$ and $A_{n,\{1,n\}}$

Given a field $\mathbb{F}$, let Δ be the building of type A_n ($n \geq 3$) defined over $\mathbb{F}$. It is well known that Δ is isomorphic to the projective geometry $\mathrm{PG}(n, \mathbb{F})$. For

$i = 2, 3, ..., n-1$, the i-shadow geometry $\mathrm{Sh}_i(\Delta)$ is called the *i-grassmannian* of $\mathrm{PG}(n, \mathbb{F})$ (*line-grassmannian* when $i = 2$). Its points are the $(i-1)$-spaces of $\mathrm{PG}(n, \mathbb{F})$ (namely, the i-spaces of $V(n+1, \mathbb{F})$) and its lines are the collections of $(i-1)$-spaces contained in a given i-space and containing a given $(i-2)$-space.

Now let $J = \{1, n\}$. Then $\mathrm{Sh}_J(\Delta)$ is isomorphic to the *long-root geometry* of $\mathrm{SL}_{n+1}(\mathbb{F})$. Its points are the point-hyperplane pairs of $\mathrm{PG}(n, \mathbb{F})$. A line of $\mathrm{Sh}_J(\Delta)$ is the collection of point-hyperplane flags (p, H) where either $H = H_0$ and $p \in L$ for a given hyperplane H_0 and a given line L of H_0, or $H \supset S$ and $p = p_0$ for a given $(n-2)$-space S of $\mathrm{PG}(n, \mathbb{F})$ and a given point p_0 of S. Accordingly, two points (p_1, H_1) and (p_2, H_2) of $\mathrm{Sh}_J(\Delta)$ are collinear if and only if either $p_1 = p_2$ or $H_1 = H_2$.

In particular, let $n = 3$. The 2-grassmannian of $\Delta = \mathrm{PG}(3, \mathbb{F})$ is isomorphic to the polar geometry, say Π, associated to $\mathrm{O}_6^+(\mathbb{F})$, the elements of which are the subspaces of $V = V(6, \mathbb{F})$ that are totally singular for a given non-singular quadratic form of Witt index 3 over V:



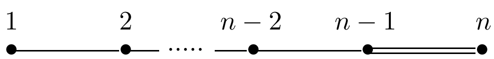
The $\{1, 3\}$-shadow geometry of Δ is isomorphic to the 2-shadow geometry of Π, which we shall call the *line-grassmannian* of Π.

2.2.3 Line-grassmannians of polar spaces

Let Δ be the building of type B_n ($n \geq 3$) defined over a given field $\mathbb{F}$. Namely, Δ is the geometry of subspaces of $V(2n+1, \mathbb{F})$ that are totally singular for a given non-singular quadratic form of Witt index n. The 2-shadow geometry $\mathrm{Sh}_2(\Delta)$ of Δ is a subgeometry of the 2-grassmannian of $\mathrm{PG}(2n, \mathbb{F})$. Its points are the lines (2-elements) of the polar space Δ and its lines correspond to the point-plane flags (namely, $\{1, 3\}$-flags) of Δ. We call $\mathrm{Sh}_2(\Delta)$ the *line-grassmannian* of Δ.

Now, let Δ be the building of type C_n ($n \geq 3$) defined over $\mathbb{F}$, namely the polar geometry of subspaces of $V(2n, \mathbb{F})$ that are totally isotropic for a given non-degenerate symplectic form of Witt index n. The 2-shadow geometry of Δ (type $C_{n,2}$) is called the *line-grassmannian* of Δ.

Finally, let Δ be the building of type D_n ($n \geq 4$) defined over $\mathbb{F}$. The 1-shadow geometry $\Pi = \mathrm{Sh}_1(\Delta)$ of Δ, viewed as a geometry of rank n, is the polar geometry associated to $\mathrm{O}_{2n}^+(\mathbb{F})$ (Tits [15, Chapter 7]), the elements of which are the subspaces of $V(2n, \mathbb{F})$ that are totally singular for a given non-singular quadratic form of Witt index n. As types for the elements of Π we take their dimensions as subspaces of $V(2n, \mathbb{F})$, thus obtaining the same labelling as for the Dynkin diagrams B_n and C_n.



Clearly, $\mathrm{Sh}_2(\Delta) \cong \mathrm{Sh}_2(\Pi)$. We call $\mathrm{Sh}_2(\Delta)$ the *line-grassmannian* of Δ (also, of Π).

Note When Δ is of type B_n or D_n, its line-grassmannian is isomorphic to the long-root geometry for $\mathrm{O}_{2n+1}(\mathbb{F})$ or $\mathrm{O}^+_{2n}(\mathbb{F})$, respectively.

2.3 Subgeometries defined over a subfield

The main purpose of this subsection is the following: given a field $\mathbb{F}$, a subfield $\mathbb{F}_0$ of $\mathbb{F}$, a Dynkin diagram M ($= A_n$, B_n, C_n, D_n, E_6, E_7, E_8, F_4 or G_2) and a building Δ of type M defined over $\mathbb{F}$, associated to a non-twisted Chevalley group G of type M, we want to define a sub-building Δ_0 of Δ defined over $\mathbb{F}_0$ that also has type M. To that goal, we regard Δ as obtained from the natural (B,N)-pair of G (see Tits [15, Chapter 3]; also Carter [8, Chapter 8]).

First some properties of (non-twisted) Chevalley groups will be discussed. Most information on Chevalley groups used below can be found in Carter [8].

Given a Dynkin diagram M defined over the set of types I, we denote by $M(\mathbb{F})$ and $\bar{M}(\mathbb{F})$) the adjoint and universal Chevalley group of type M defined over $\mathbb{F}$; thus, $A_n(\mathbb{F})$ stands for $PSL_{n+1}(\mathbb{F})$, $\bar{A}_n(\mathbb{F})$ for $SL_{n+1}(\mathbb{F})$, and so on. Let Φ_M be the root system with Dynkin diagram M. It is known (Carter [8, Theorem 12.1.1]) that, if $M \neq A_1$, the group $\bar{M}(\mathbb{F})$ is generated by a set of generators $\bar{x}_r(t)$, one for each $r \in \Phi_M$ and $t \in \mathbb{F}$, subject to the following relations:

$$(1) \quad \bar{x}_r(t_1)\bar{x}_r(t_2) \;=\; \bar{x}_r(t_1+t_2),$$

$$(2) \quad \bar{x}_s(u)\bar{x}_r(t) \;=\; \prod_{i,j>0} \bar{x}_{ir+js}(C_{ijrs}(-t)^i u^j)$$

for certain integers C_{ijrs} and

$$(3) \quad \bar{h}_r(t_1)\bar{h}_r(t_2) \;=\; \bar{h}_r(t_1 t_2)$$

for any non-zero elements $t_1, t_2 \in \mathbb{F}$ and with

$$\bar{h}_r(t) := \bar{n}_r(t)\bar{n}_r(-1), \qquad \bar{n}_r(t) := \bar{x}_r(t)\bar{x}_{-r}(-t^{-1})\bar{x}_r(t).$$

Furthermore, there is a surjective homomorphism $\bar{M}(\mathbb{F}) \longrightarrow M(\mathbb{F})$ sending a (B,N)-pair $(\bar{B},\bar{N})$ for $\bar{M}(\mathbb{F})$ to the (B,N)-pair (B,N) for $M(\mathbb{F})$ whose kernel is precisely the centre $\bar{Z}$ of $\bar{M}(\mathbb{F})$. Moreover, $\bar{Z} \leq \bar{B} \cap \bar{N}$. Hence, the buildings associated to $\bar{M}(\mathbb{F})$ and $M(\mathbb{F})$ (and all intermediate quotients) are the same.

Often, any quotient of $\bar{M}(\mathbb{F})$ by a subgroup of $\bar{Z}$ is referred to as a Chevalley group (defined over $\mathbb{F}$).

As the above discussion fails to hold when $M = A_1$, henceforth we assume $M \neq A_1$.

Lemma 2.1 *For every subfield $\mathbb{F}_0$ of $\mathbb{F}$, the Chevalley group $\bar{M}(\mathbb{F})$ contains $\bar{M}(\mathbb{F}_0)$ as a subgroup.*

Proof We put $\bar{G} := \bar{M}(\mathbb{F})$, $\bar{G}_0 := \bar{M}(\mathbb{F}_0)$ and denote all subgroups of $\bar{G}_0$ with a subscript 0. Define a map $\phi\colon \bar{G}_0 \longrightarrow \bar{G}$ by sending $\bar{x}_r(t) \in \bar{G}_0$ to the 'same' element in $\bar{G}$, for each $r \in \Phi_M$ and $t \in \mathbb{F}_0$. We can extend this to a homomorphism since the relations holding in $\bar{G}_0$ also hold in $\bar{G}$. Since $\bar{G}_0$ is an extension by its centre $\bar{Z}_0$ of the simple group $G(\mathbb{F}_0)$, the kernel of ϕ is contained in $\bar{Z}_0$. Put $\bar{H}_0 = \bar{B}_0 \cap \bar{N}_0$. Then $\bar{Z}_0 \leq \bar{H}_0$ (see the discussion above). In the proof of Theorem 12.1.1. of [8] it is proved that $\bar{H}_0$ is equal to the direct product $\prod_{i=1}^n \bar{H}_{0,p_i}(\mathbb{F}_0)$, where $\bar{H}_{0,p_i}(\mathbb{F}_0) = \langle \bar{h}_{0,p_i}(t) \mid t \in \mathbb{F}_0 \rangle$ and $\{p_i\}_{i=1}^n$ is a basis for Φ_M.

From the relations at the beginning of this subsection we can see that the kernel of $\phi\colon \bar{H}_{0,p_i} \longrightarrow \bar{H}_{p_i}$ is trivial, so $\phi\colon \bar{G}_0 \longrightarrow \bar{G}$ is an isomorphism onto its image. □

Lemma 2.2 *Let $\mathbb{F}_0$ be a subfield of $\mathbb{F}$, as in Lemma 2.1. For any non-empty $J \subseteq I$ let $\bar{P}_J(\mathbb{F})$ and $\bar{P}_J(\mathbb{F}_0)$ be the standard parabolic subgroup of type J of $\bar{M}(\mathbb{F})$ and $\bar{M}(\mathbb{F}_0)$ respectively and let $(\bar{B}, \bar{N})$ be the (B,N)-pair for $\bar{M}(\mathbb{F})$. Then*

$$\bar{B}\bar{P}_J(\mathbb{F}_0)\bar{B} = \bar{P}_J(\mathbb{F}).$$

Proof We use the notation as in the proof of Lemma 2.1. From the relations given at the beginning of this subsection it is clear that, for each $r \in \Phi_M$, all elements $\bar{n}_r(t)$ $(t \in \mathbb{F})$ belong to the same $\bar{H}$-coset in $\bar{N}$. Thus they represent the same element of the Weyl group W of Φ_M. Since every element of W is represented by some $\bar{n}_r(t)$ and $\bar{B}_0 \leq \bar{B}$ and $\bar{N}_0 \leq \bar{N}$, we can write

$$\bar{B}\bar{P}_{J0}\bar{B} = \bar{B}\bar{B}_0 W_J \bar{B}_0 \bar{B} = \bar{B} W_J \bar{B} = \bar{P}_J.$$

□

Definitions Let Δ and Δ_0 be the buildings associated to the groups $\bar{M}(\mathbb{F})$ and $\bar{M}(\mathbb{F}_0)$, respectively. Using Lemmas 2.1 and 2.2, we can identify the building Δ_0 with a sub-building of Δ. We call Δ_0 a sub-building of Δ *defined over* $\mathbb{F}_0$ or, an $\mathbb{F}_0$-*sub-building* of Δ, for short.

With I the type-set of M, let J be a non-empty subset of I and $\Gamma = \mathrm{Sh}_J(\Delta)$, $\Gamma_0 = \mathrm{Sh}_J(\Delta_0)$ (see 2.2). As Δ_0 can be regarded as a subgeometry of Δ, we may also view the points of Γ_0 as points of Γ and every line of Γ_0 is contained in a unique line of Γ. Thus we can regard Γ_0 as a subgeometry of Γ. We will call Γ_0 an $\mathbb{F}_0$-*subgeometry* of Γ.

Example Let $M = A_n$. Then $\bar{M}(\mathbb{F}) = \mathrm{SL}_{n+1}(\mathbb{F})$ and elements $\bar{x}_r(t)$ are precisely the standard transvection matrices with respect to some chosen basis. By taking only those elements $\bar{x}_r(t)$ with $t \in \mathbb{F}_0$ we get $\mathrm{SL}_{n+1}(\mathbb{F}_0)$. The building Δ_0 is the geometry of subspaces of $\mathrm{PG}(n, \mathbb{F}_0)$. By Lemmas 2.1 and 2.2, we can also view it as the sub-building of Δ containing only the subspaces of $\mathrm{PG}(n, \mathbb{F})$ that are $\mathbb{F}_0$-rational. Note that the latter description depends on the choice of a basis of $V(n+1, \mathbb{F})$.

2.4 Apartments and opposition

Given a Dynkin diagram M and a Chevalley group G of type M, let (B,N) be a (B,N)-pair of G and let Δ be the building associated to G, arising from

(B, N). Let W be the Weyl group of (B, N), with generator set $\{r_i\}_{i \in I}$. For $w \in W$, let $l(w)$ denote the length of a reduced expression for w with respect to $\{r_i\}_{i \in I}$; let w_0 be the longest word of W in this sense. For $J, K \subset I$, let $P_J = BW_JB$ and $P_K = BW_KB$ be the standard parabolic subgroups of G of type J and K. We recall that the left cosets of P_J and P_K correspond to the flags of Δ of type $I \setminus J$ and $I \setminus K$. The *distance* between flags gP_J and hP_K $(g, h \in G)$ is the shortest w such that $P_J g^{-1} h P_K = P_J w P_K$. These flags are called *opposite* when $P_J w P_K = P_J w_0 P_K$.

We recall that an *apartment* of Δ is the collection of elements of Δ that, viewed as left cosets of maximal parabolics, have a non-empty intersection with a given left coset of N. The apartments of Δ are isomorphic to the Coxeter complex of type M and each pair of flags is contained in some apartment (Tits [15]).

Lemma 2.3 (Tits [17]) *For any apartment A and any chamber c there exists a chamber d on A opposite to c.*

For a subfield $\mathbb{F}_0$ of the ground field $\mathbb{F}$ of G, let Δ_0 be the $\mathbb{F}_0$-sub-building of Δ. Then the apartments of Δ_0 are also apartments of Δ.

Corollary 2.4 *For any flag F of Δ and any non-empty $J \subseteq I$, there is a J-flag F_0 in Δ_0 opposite to F.*

Proof This follows from Lemma 2.3 by taking for A an apartment of Δ_0 and for c a chamber on F. □

3 Geometries generated by $\mathbb{F}_0$-subgeometries

Theorem 3.1 *For $M = A_n, B_n, C_n, D_n, E_6$ or E_7 and a field $\mathbb{F}$, let Δ be the building with diagram M defined over $\mathbb{F}$. Let $i \in I$ be the label of a node of M and $\mathbb{F}_0$ a subfield of $\mathbb{F}$. Then the i-shadow geometry of Δ is spanned by the points of its $\mathbb{F}_0$-subgeometry, if its type M_i is one of the following:*

(i) *$A_{n,i}$ with $i \in \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}_{\geq 1}$;*

(ii) *$D_{n,1}$, $D_{n,n-1}$, $D_{n,n}$ with $n \in \mathbb{N}_{\geq 3}$;*

(iii) *$E_{6,1}$, $E_{6,6}$, $E_{7,7}$;*

(iv) *$B_{n,n}$ with $n \in \mathbb{N}_{\geq 2}$, provided that* $\mathrm{Char}(\mathbb{F}) \neq 2$;

(v) *$C_{n,1}$, with $n \in \mathbb{N}_{\geq 2}$, provided that* $\mathrm{Char}(\mathbb{F}) \neq 2$.

Proof By Theorem 1 of Blok and Brouwer [3], in the above cases the i-shadow geometry Γ of Δ is spanned by the points of an apartment of Δ. The $\mathbb{F}_0$-subgeometry of Γ is the i-shadow geometry of the sub-building Δ_0 of Δ defined over $\mathbb{F}_0$. The points on an apartment of Δ_0 are also the points on an apartment of Δ. □

Proposition 3.2 *For $M = B_n, C_n$ ($n \in \mathbb{N}_{\geq 2}$) and a field $\mathbb{F}$, let Δ be the building of type M defined over $\mathbb{F}$. Let i be a type of M and $\mathbb{F}_0$ a subfield of $\mathbb{F}$. Then the i-shadow geometry of Δ is spanned by the points of its $\mathbb{F}_0$-subgeometry, if its type M_i is one of the following:*

(i) $B_{n,1}$;

(ii) $C_{n,1}$ *if* $\mathrm{Char}(\mathbb{F}) = 2$.

Proof We assume that the $B_{n,1}$ geometry is defined by the form f described in Section 5 and show that it is generated by its $\mathbb{F}_0$-subgeometry $B_{n,1}(\mathbb{F}_0)$ defined by the same form. Clearly, the $B_{n,1}$ geometry contains the $D_{n,1}$ geometry defined by the form described in Section 5 as a hyperplane induced by this embedding. By Theorem 3.1, this $D_{n,1}$ geometry is generated by the points of its $\mathbb{F}_0$-subgeometry $D_{n,1}(\mathbb{F}_0)$ which is a hyperplane of $B_{n,1}(\mathbb{F}_0)$. Thus $D_{n,1}(\mathbb{F}_0)$, together with any point p of $B_{n,1}(\mathbb{F}_0)$ outside $D_{n,1}(\mathbb{F}_0)$, generates $B_{n,1}$: if q is any other point outside $D_{n,1}(\mathbb{F}_0)$ then either it is collinear to p and we are done or we can find lines l, m on p, q respectively with $l^{\perp} \cap m = \{0\}$, where $\perp$ denotes collinearity. Now, every point of l is collinear to a unique point of m; so we can find a point r on l that is collinear to a point s of m outside $D_{n,1}(\mathbb{F}_0)$, assuming that $|\mathbb{F}_0| \geq 2$. Now we see that r, s and hence q is generated by p together with $D_{n,1}(\mathbb{F}_0)$.

The second assertion follows from the first, by noting that the isomorphism from $B_{n,1}$ to $C_{n,1}$, which is projection from the nucleus, carries $B_{n,1}(\mathbb{F}_0)$ to $C_{n,1}(\mathbb{F}_0)$. □

4 The long-root geometry of $\mathrm{SL}_{n+1}(\mathbb{F})$

In this section, $\mathbb{F}$ is a given field and Δ is the building of type A_n defined over $\mathbb{F}$, namely the projective geometry of linear subspaces of $V = V(n+1, \mathbb{F})$. Furthermore, $\Gamma = (\mathcal{P}, \mathcal{L})$ is the $\{1, n\}$-shadow geometry of Δ. As noted in Subsection 2.2, Γ is the long-root geometry associated to the group $\mathrm{SL}_{n+1}(\mathbb{F})$.

By the words 'point' and 'line' we will mean a point and a line of Γ, unless specified otherwise. The i-elements of Δ will be regarded as i-spaces of V and denoted by capital letters. Given a line l of Γ, we denote $\mathcal{P}(l)$ its set of points. We also denote the set of points of Γ incident to an element E of Δ by $\mathcal{P}(E)$. Hence, $\mathcal{P}(E)$ is the set of $\{1, n\}$-flags of Δ incident to E, usually called the $\{1, n\}$-*shadow* of E. We also extend this notation to $\{0\}$ and V, by setting $\mathcal{P}(\{0\}) = \mathcal{P}(V) = \mathcal{P}$, by convention.

Given a set $S \subseteq \mathcal{P}$ of points of Γ, we say that an element E of Δ is *full* with respect to S (also, *S-full*, for short) if $\mathcal{P}(E) \subseteq \langle S \rangle_{\Gamma}$. We denote the collection of elements that are full with respect to S as $\mathcal{F}(S)$ and the incidence structure induced on it by Δ as $\Delta(\mathcal{F}(S))$.

For a subfield $\mathbb{F}_0$ of $\mathbb{F}$, the sub-building Δ_0 of Δ defined over $\mathbb{F}_0$ is isomorphic to $\mathrm{PG}(n, \mathbb{F}_0)$ and its $\{1, n\}$-shadow geometry Γ_0 (which is the long-root geometry of $\mathrm{SL}_{n+1}(\mathbb{F}_0)$) is the $\mathbb{F}_0$-subgeometry of Γ.

The geometry Γ is preserved under the duality that interchanges i and $n+1-i$ elements for all $i = 1, 2, \ldots, n$. We will often make use of this fact, for instance by stating the dual part of a result but not proving it explicitly.

Lemma 4.1 *Let E_1, E_2 be elements of Δ.*

(i) *For a line l of Γ, if $\emptyset \neq \mathcal{P}(l) \cap \mathcal{P}(E_1) \neq \mathcal{P}(l) \cap \mathcal{P}(E_2) \neq \emptyset$, then $\mathcal{P}(l) \subseteq \mathcal{P}(E_1 \cap E_2) \cup \mathcal{P}(\langle E_1, E_2 \rangle_V)$.*

(ii) *Every point in $\mathcal{P}(E_1 \cap E_2) \cup \mathcal{P}(\langle E_1, E_2 \rangle_V) \setminus (\mathcal{P}(E_1) \cup \mathcal{P}(E_2))$ lies on a line meeting both $\mathcal{P}(E_1)$ and $\mathcal{P}(E_2)$.*

Proof Set $Z = E_1 \cap E_2$ and $Z' = \langle E_1, E_2 \rangle_V$.
(i) For $i = 1, 2$, let $(A_i, H_i) \in \mathcal{P}(l) \cap \mathcal{P}(E_i)$ be distinct. By duality we may assume that $A_1 = A_2 = A$. Then, $Z \neq \{0\}$ and $A \subseteq Z \subseteq H_1, H_2$, so $\mathcal{P}(l) \subseteq \mathcal{P}(Z)$.
(ii) Let $(A, H) \in \mathcal{P}(Z') \setminus (\mathcal{P}(E_1) \cup \mathcal{P}(E_2))$. Then A lies on some 2-space L of V meeting E_1 and E_2 in 1-spaces A_1 and A_2 respectively, and $H \supseteq E_1, E_2$. Clearly, (L, H) is the required line of Γ. The case $(A, H) \in \mathcal{P}(Z) \setminus (\mathcal{P}(E_1) \cup \mathcal{P}(E_2))$ is settled dually. □

Lemma 4.2 *Let X be a subspace of Γ.*

(i) *Given an n-element H of Δ, suppose that H is spanned by 1-spaces $A_1, A_2, ..., A_n$ of V such that $(A_i, H) \in X$ for every $i = 1, 2, ..., n$. Then H is full with respect to X. Dually, let A be a 1-element of Δ with $A = \cap_{i=1}^{n} H_i$ for n-spaces H_1, H_2,..., H_n of V such that $(A, H_i) \in X$ for $i = 1, 2, ..., n$. Then A is full with respect to X.*

(ii) *If an element E of Δ is spanned by 1-spaces of V that are full with respect to X, then E is full with respect to X. Dually, if an element E of Δ is the intersection of n-elements that are full with respect to X, then E is full with respect to X.*

(iii) *Let L and H be a 2- and an n-element of Δ such that $L \not\subseteq H$ and let $A = L \cap H$. Suppose that L is full with respect to X and $(A, H) \in X$. Then A is full with respect to X.*

Proof (i) This follows from the fact that $\mathcal{P}(H)$ and $\mathcal{P}(A)$ are subspaces (see Lemma 4.1(i)) isomorphic to the projective geometry $\mathrm{PG}(n-1, \mathbb{F})$.
(ii) For every n-element $H \supseteq E$ of Δ and every $j = 1, 2, ..., i$, $(A_j, H) \in X$ because A_j is full. Hence $(A, H) \in X$ for any 1-element $A \subseteq E$, by (i). Therefore, E is X-full.
(iii) Let $H_1, H_2, ..., H_{n-1}$ be n-elements with $L = \cap_{i=1}^{n-1} H_i$. Then $A = H_1 \cap ... \cap H_{n-1} \cap H$. Furthermore, $(A, H) \in X$ by assumption and $(A, H_i) \in X$ for every $i = 1, 2, ..., n-1$ because L is X-full. The conclusion follows from (i). □

Lemma 4.3 *Let X be a subspace of Γ containing Γ_0.*

(i) *Every element of* Δ_0 *is full with respect to* X.

(ii) *Let* A_0, A_1, A_2 *be distinct* 1*-elements of* Δ *not contained in the same* 2*-element. Suppose that both* A_1 *and* A_2 *are full with respect to* X. *Then there exists an* n*-element of* Δ *that is full with respect to* X *and contains both* A_1 *and* A_2 *but not* A_0.

Proof (i) We first prove the statement for a 1-element A of Δ_0. Clearly, A is the intersection of n-elements $H_1, H_2, ..., H_n$ of Δ_0 and $(A, H_i) \in \Gamma_0$ for all $i = 1, 2, ..., n$. By Lemma 4.2(i), A is X-full. The conclusion now follows from Lemma 4.2(ii).

(ii) Clearly, there exist 1-elements $A_3, A_4, ..., A_n$ of Δ_0 which, together with A_1 and A_2, span a hyperplane H of V not containing A_0. The elements A_1, A_2 are X-full by assumption whereas $A_3, A_4, ..., A_n$ are X-full by (i). Hence H is X-full, by Lemma 4.2(i). □

Lemma 4.4 *Let* X *be a subspace of* Γ *containing* Γ_0. *Then, for every element* E *of* $\Delta \setminus \Delta_0$ *full with respect to* X, *there is a chamber* C *of* Δ *containing* E *and such that all elements of* C *are full with respect to* X *and do not belong to* Δ_0.

Proof Let E be as in the hypotheses of the lemma and let i be its dimension in V. Suppose $i > 1$. We shall show that there exists an $(i-1)$-element E' of $\Delta \setminus \Delta_0$ that is also X-full.

Take an $(n+1-i)$-element $E_0 \in \Delta_0$ opposite to E, that is, such that $E \cap E_0 = \{0\}$. By Corollary 2.4, E_0 exists. Clearly $E = \langle H \cap E \rangle_V$, where H ranges over an independent set of n-elements of Δ_0 containing E_0. As $E \notin \Delta_0$, one of these n-elements, say H', satisfies $H' \cap E \notin \Delta_0$. By Lemma 4.3(i), H' is full. Hence $E' = H' \cap E$ is the required $(i-1)$-element, by Lemma 4.1(ii).

The case $i < n$ is handled dually. The conclusion now follows by repeating the above arguments $n-1$ times. □

Theorem 4.5 $\langle \Gamma_0 \rangle_\Gamma = \bigcup_{E \in \Delta_0} \mathcal{P}(E)$.

Proof As for every two elements E_1, E_2 of Δ_0 also $E_1 \cap E_2$ and $\langle E_1, E_2 \rangle_V$ are elements of Δ_0, it follows from the first part of Lemma 4.1 that the right hand side of the above formula is a subspace of Γ. This proves '$\subseteq$'.

By Lemma 4.3(i), every 1-element of Δ_0 is full with respect to Γ_0. An i-element $(i > 1)$ of Δ_0 is spanned in V by an $(i-1)$-element and a 1-element of Δ_0. Thus, by inductively applying the second part of Lemma 4.1, we obtain that $\mathcal{P}(E) \subseteq \langle \Gamma_0 \rangle_\Gamma$. □

Theorem 4.6 $\Delta(\mathcal{F}(\Gamma_0)) = \Delta_0$.

Proof The inclusion '$\supseteq$' in Theorem 4.5 implies the inclusion '$\supseteq$' here.

Now suppose there exists an i-element H that is full with respect to Γ_0, but is not in Δ_0. We assume that $i = n$, using Lemma 4.4. As H is full, for every 1-element $A \subseteq H$, the point (A, H) of Γ belongs to $\langle \Gamma_0 \rangle_\Gamma$. For any such 1-element A there exists $E \in \Delta_0$ with $(A, H) \in \mathcal{P}(E)$ or, equivalently, $A \subseteq E \subseteq H$, by Theorem 4.5. Hence $H \in \Delta_0$, a contradiction. □

Theorem 4.7 *Suppose X is a subspace of Γ containing Γ_0. Then $\mathbb{F}$ contains an extension $\mathbb{F}_1$ of $\mathbb{F}_0$ such that $\Delta(\mathcal{F}(X)) = \Delta(\mathbb{F}_1)$.*

Proof For $i = 1, 2, ..., n$, let $\mathcal{F}(X)_i$ denote the collection of X-full i-elements of Δ. We will show that $\mathcal{F}(X)_1$ and $\mathcal{F}(X)_2$ are the collections of points and lines of a non-degenerate projective space. Note first that, by Lemma 4.2(ii), if $A, B \in \mathcal{F}(X)_1$ and L is the 2-space of V spanned by A and B, then $L \in \mathcal{F}(X)_2$.

We now prove that the point-line geometry $\Pi = (\mathcal{F}(X)_1, \mathcal{F}(X)_2)$, with the natural incidence relation (inherited from Δ) satisfies the axiom of Pasch. Given $A \in \mathcal{F}(X)_1$, let L_1, L_2 be distinct 2-elements on A and let B_1, C_1 and B_2, C_2 be distinct 1-elements in L_1 and L_2, different from A and belonging to $\mathcal{F}(X)_1$. Let L be the 2-element on B_1 and B_2, and let L' be the 2-element on C_1 and C_2. Then $L_1, L_2, L, L' \in \mathcal{F}(X)_2$, as each of them is spanned by pairs of full 1-elements. Let $P = L \cap L'$. Clearly, $P = H \cap L$ for any n-element H on L' not containing L. By Lemma 4.3(ii), such an H can be chosen in $\mathcal{F}(X)_n$. Hence $P \in \mathcal{F}(X)_1$, by Lemma 4.2(iii). That is, L and L' meet in Π.

Thus, Π is a (possibly degenerate) projective space and, using Lemma 4.2(ii), we find that $\Delta(\mathcal{F}(X))$ contains the geometry of subspaces of Π. Since $\Gamma_0 \subseteq X$, the dimension of Π is n. We now show that Π is not degenerate. Given $L \in \mathcal{F}(X)_2$, take an $(n-1)$-element $E_0 \in \Delta_0$ opposite to L. That is, $L \cap E_0 = \{0\}$. Then the n-elements of Δ_0 containing E_0 intersect L in distinct X-full 1-elements, by Lemmas 4.3(i) and 4.2(iii). However, there are at least three such n-elements. Hence L has at least three points in Π.

Now note that Π is a non-degenerate projective space contained in the Desarguesian projective space Δ and containing the Desarguesian projective space Δ_0. It is easy to see that Π is Desarguesian as well. Since a Desarguesian projective geometry determines the algebraic structure of the field over which it is defined, Π is defined over some field $\mathbb{F}_1$ with $\mathbb{F}_0 \leq \mathbb{F}_1 \leq \mathbb{F}$. Now the second part of Lemma 4.2 implies '$\supseteq$'.

Next we prove the inclusion '$\subseteq$'. From Lemma 4.4 applied with $\mathbb{F}_0 = \mathbb{F}_1$, it follows that if there is an X-full i-element in $\Delta \setminus \Delta(\mathbb{F}_1)$, then there is a 1-element also exists with the same properties. However, this contradicts the fact that all X-full 1-elements belong to $\Delta(\mathbb{F}_1)$. □

Theorem 4.7 shows that $X \supseteq \Gamma(\mathbb{F}_1)$. However in general we do not have equality here.

Corollary 4.8 *Suppose that $\mathbb{F}$ can be generated by adjoining k elements to $\mathbb{F}_0$. Then Γ can be generated by adding at most k points to its subgeometry Γ_0.*

Proof Let $k = 1$. We shall show that we can choose a point x of Γ such that there is a 1-element A of Δ not defined over $\mathbb{F}_0$, but full with respect to $\Gamma_0 \cup \{x\}$.

For clarity, let $\mathbb{F}_0$ be the prime field of $\mathbb{F}$. By Theorem 4.6, $\langle \Gamma_0 \rangle_\Gamma$ is a proper subspace of Γ. Let $x = (A, H)$ where $A = \langle (\alpha, 1, \ldots, 1) \rangle_V$ and

H is any n-element from $\Delta \setminus \Delta_0$ containing A. Since the 2-element $L = \langle (1,0,\ldots,0),(0,1,\ldots,1)\rangle_V \in \Delta_0$ is full with respect to Γ_0 and A is the intersection of H and L, so A is full with respect to $X = \Gamma_0 \cup \{x\}$, by Lemma 4.2(iii).

By Theorem 4.7, $\Delta(\mathcal{F}(X)) = \Delta(\mathbb{F}_1)$ for some intermediate field $\mathbb{F}_0 \le \mathbb{F}_1 \le \mathbb{F}$, but since $A \notin \Delta_0$, apparently we have $\mathbb{F}_1 = \mathbb{F}$. The corollary now follows by induction on k. □

5 The line-grassmannians of the buildings of type B_n, C_n and D_n

Henceforth, $\mathbb{F}$ is a given field, M is the Dynkin diagram of rank $n \ge 3$ and type B_n, C_n or D_n and $V = V(m,\mathbb{F})$, with $m = 2n+1, 2n$ or $2n$ respectively. When $M = B_n$ or C_n, we denote by Δ the building of type M defined over $\mathbb{F}$. When $M = D_n$ and $n \ge 3$, we change the notation used at the end of Subsection 2.2, denoting by Δ the polar geometry associated to a non-singular quadratic form of Witt index n over V. Thus, in any case, the i-elements of Δ are i-spaces of V.

Note As noted in Subsection 2.2, when $M = D_3$ the line-grassmannian of Δ is isomorphic to the long-root geometry for $SL_4(\mathbb{F})$, considered in the previous section. Thus, the case of $M = D_3$ might be omitted here, but we prefer to keep it, for the sake of uniformity of exposition.

We recall that, after choosing a suitable basis $\{e_i\}_{i=1}^m$ of V and rescaling (in the B_n case), the (quadratic or symplectic) form f on V that gives rise to Δ can be expressed as follows:

$$
\begin{array}{lll}
(B_n) & f(x_1,x_2,...,x_{2n+1}) & = \sum_{i=1}^n x_i x_{i+n} + x_{2n+1}^2; \\
(C_n) & f((x_1,...,x_{2n}),(y_1,...,y_{2n})) & = \sum_{i=1}^n (x_i y_{i+n} - y_i x_{i+n}); \\
(D_n) & f(x_1,x_2,...,x_{2n}) & = \sum_{i=1}^n x_i x_{i+n}.
\end{array}
$$

The set $\{e_i\}_{i=1}^{2n}$ is called a *hyperbolic set* of vectors. In all cases, the apartments of Δ arise from the hyperbolic sets of vectors: given a hyperbolic set of vectors $\{e_i\}_{i=1}^{2n}$, the elements of the corresponding apartment are the elements of Δ spanned by subsets of that set.

From now on, we assume to have chosen a basis $B = \{e_i\}_{i=1}^m$ of V as above. We call B a *hyperbolic basis* because it contains a hyperbolic set.

With B as above, the form f is defined over the prime subfield of $\mathbb{F}$. For any subfield $\mathbb{F}_0 \le \mathbb{F}$, for Δ_0 we can simply take the building whose elements are the elements of Δ that are defined over $\mathbb{F}_0$ with respect to the basis B. Clearly, Δ_0 is a building of the same type as Δ, but defined over the subfield $\mathbb{F}_0$ of $\mathbb{F}$.

In each case, we denote by Γ the line-grassmannian of Δ; see Subsection 2.2. Accordingly, Γ_0 is the line-grassmannian of Δ_0. Clearly Γ_0 is a subgeometry (but not a subspace) of Γ.

Henceforth, the words 'point' and 'line' will refer to points and lines of Γ, unless otherwise specified. The set of points of Γ will be denoted $\mathcal{P}$. We shall now denote 1-elements and 2-elements of Δ by small letters, keeping capital letters for the remaining elements of Δ or for a generic element of Δ.

Let $\perp$ be the orthogonality relation defined by f on V. For any element E of Δ, we set

$$\mathcal{P}^*(E) := \{l \in \mathcal{P} \mid l \subseteq E^\perp \text{ and } l \cap E \neq \{0\}\}.$$

Note that $\mathcal{P}^*(E) \supseteq \mathcal{P}(E)$ and that equality holds only when E is a 1-element. Note also that $\mathcal{P}^*(E)$ is a subspace of Γ.

Given $S \subseteq \mathcal{P}$, we say that an element E of Δ is *full* with respect to S (also, *S-full*, for short) if $\mathcal{P}^*(E) \subseteq \langle S \rangle_\Gamma$. We denote the collection of elements that are full with respect to S as $\mathcal{F}(S)$ and the incidence structure induced on it by Δ as $\Delta(\mathcal{F}(S))$.

Lemma 5.1 *Let $S \subseteq \mathcal{P}$ and suppose that $E_1, E_2 \in \Delta$ are full with respect to S.*

(i) *If E_1 is a 1-element and E_2 is an i-element with $1 < i < n$, then $E_1^\perp \cap E_2$ ($\neq \{0\}$) is full with respect to S.*

(ii) *If $E_1 \perp E_2$, then $\langle E_1, E_2 \rangle_V$ is full with respect to S.*

Proof (i) If $E_1^\perp \cap E_2 = E_2$ there is nothing to prove. Suppose this is not the case and $E_3 = E_1^\perp \cap E_2$. Then E_3 is an $(i-1)$-element of Δ. Let $l \in \mathcal{P}^*(E_3)$. If $l \subseteq E_3$, then $l \subseteq E_2$ and we are done.

Suppose $l \not\subseteq E_3$ and let $p = l \cap E_3$. The family $E_3^\perp/p$ of elements of Δ containing p and contained in $E_3^\perp$ is a (possibly degenerate) polar space Π of rank $r = n-1$. The radical $R(\Pi)$ of Π is the family of 2-elements of Δ contained in E_3 and containing p. Thus, $R(\Pi) = \emptyset$ if and only if $p = E_3$ and, if $R(\Pi) \neq \emptyset$, the rank of $R(\Pi)$ in Π is $r_0 = i-2$. As $r - r_0 = n-1-(i-2) = n-i+1 \geq 2$, if E and E' are elements of Δ containing E_3 (hence contained in $E_3^\perp$) with dimensions i and 2 respectively in V and $E' \not\subseteq E^\perp$, then Π is spanned by its points contained $E^\perp \cup E'$. The elements $E = E_2$ and $E' = \langle p, E_1 \rangle_V$ satisfy the previous hypotheses. Therefore, the 2-elements of Δ containing p and contained in $E_2^\perp \cup \langle E_1, p \rangle_V$, viewed as points of Π, span Π. However, since E_1 and E_2 are full, all those 2-elements belong to $\langle S \rangle_\Gamma$. Furthermore, two points l_1 and l_2 of Π are collinear in Π if and only if $\langle l_1, l_2 \rangle_V$ is a 3-element of Δ. It is now clear that two points of Π are collinear in Π if and only if they are collinear as points of Γ. As Π is spanned by points belonging to $\langle S \rangle_\Gamma$, all points of Π belong to $\langle S \rangle_\Gamma$. In particular, $l \in \langle S \rangle_\Gamma$. This shows that E_3 is S-full.

(ii) Suppose $E_1 \perp E_2$. Let $E_3 = \langle E_1, E_2 \rangle_V$ and let $l \in \mathcal{P}^*(E_3) \setminus (\mathcal{P}^*(E_1) \cup \mathcal{P}^*(E_2))$. Suppose p is a 1-element in $l \cap E_3$ and $q \neq p$ is a 1-element in $l \cap E_3^\perp = E_1^\perp \cap E_2^\perp$.

Since $l \notin \mathcal{P}^*(E_1) \cup \mathcal{P}^*(E_2)$ we have $p \subseteq E_3 \setminus (E_1 \cup E_2)$. Hence there are 1-elements $p_1 \subseteq E_1$ and $p_2 \subseteq E_2$ such that $p \subseteq \langle p_1, p_2 \rangle_V$. Then $(q, \langle q, p_1, p_2 \rangle_V)$ is a line of Γ containing $l_1 = \langle p_1, q \rangle_V \in \mathcal{P}^*(E_1)$, $l_2 = \langle p_2, q \rangle_V \in \mathcal{P}^*(E_2)$ and l. Furthermore, $l_1, l_2 \in \langle S \rangle_\Gamma$, because E_1 and E_2 are full, by assumption. Hence $l \in \langle S \rangle_\Gamma$; we are done. □

Lemma 5.2 *Let X be a subspace of Γ containing Γ_0. Then all elements of Δ_0 are X-full.*

Proof Let p be a 1-element of Δ_0. Then $p^\perp$ is a degenerate polar space of rank n, with radical $R(\Pi) = p$. Accordingly, the systems Π and Π_0 of all elements of Δ and Δ_0 contained in $p^\perp$ and properly containing p is a non-degenerate polar space of rank $n-1 \geq 2$. Let $\mathcal{L}$ be a spanning set of points of Π_0. As Δ and Δ_0 have the same Dynkin type, $\mathcal{L}$ also spans Π. However, for every 3-element E of Δ on p, the flag (p, E) is a line of Γ. As X is a subspace of Γ containing Γ_0, X contains every point of Π. That is, p is X-full.

Thus, we have proved that all 1-elements of Δ_0 are X-full. As every element of Δ_0 is spanned (as an element of Δ) by 1-elements belonging to Δ_0, the conclusion follows from the above and Lemma 5.1(ii). □

Lemma 5.3 *Let X be a subspace of Γ containing Γ_0. Then, for every $i = 2, 3, ..., n-1$ and every i-element E of $\Delta \setminus \Delta_0$, full with respect to X, there is a flag $F = (E_1, E_2, ..., E_i = E)$ of Δ such that, for every $j = 1, 2, ..., i$, the element E_j has type j, is full with respect to X and does not belong to Δ_0.*

Proof Let $E = E_i$ be an i-element of $\Delta \setminus \Delta_0$, full with respect to X, with $1 < i < n$. We shall show that there exists an $(i-1)$-element E' of $\Delta \setminus \Delta_0$ that is also X-full. Take an i-element $E_0 \in \Delta_0$ opposite to E, that is, such that $E_0^\perp \cap E = \{0\}$. Such an element exists, by Corollary 2.4. Clearly $E = \langle a^\perp \cap E \rangle_V$, where a ranges over a set of 1-elements of Δ_0 contained in E_0 and spanning E_0. As $E \notin \Delta_0$, one of these 1-elements, say a', satisfies ${a'}^\perp \cap E \notin \Delta_0$. Since a' is X-full (Lemma 4.2), $E' = {a'}^\perp \cap E$ is the required $(i-1)$-element, by Lemma 5.1(i). □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3

For $i = 1, 2, \ldots, n$ let $\mathcal{F}(S)_i$ denote the collection of S-full i-elements of Δ. We will show first that $\mathcal{F}(S)_1$ and $\mathcal{F}(S)_2$ are the collections of points and lines of a non-degenerate polar geometry, using the well-known Buekenhout–Shult axioms [7]. Throughout the proof we refer to 1-elements and 2-elements as 'points' and 'lines' respectively.

Let $l \in \mathcal{F}(S)_2$. Assume that there exists $q \in \mathcal{F}(S)_1$ such that $q^\perp \not\supseteq l$. Then $p = l \cap q^\perp \in \mathcal{F}(S)_1$ by Lemma 5.1(i). Hence we are dealing with a possibly degenerate polar geometry. We call this polar geometry Π.

We now show that Π is not degenerate. We will use the fact that all elements of $\Delta(\mathbb{F}_0)$ are S-full (Lemma 5.2) and the following principle:

(1) if l and m are opposite lines each containing at least two S-full points, then there is a one-to-one correspondence between the S-full points on l and the S-full points on m given by collinearity.

Proof of (1): First note that since l contains two elements from $\mathcal{F}(S)_1$ it belongs to $\mathcal{F}(S)_2$; see Lemma 5.1. Let p be a point on l and let q be the unique point on m collinear to p. As we saw above, if q is S-full then $p = q^\perp \cap l$ is S-full. Thus (1) is proved.

Given a line l of Π there is a line $m \in \Delta(\mathbb{F}_0)$ opposite to l (such an element exists by Corollary 2.4). Since m contains at least three S-full points, it follows from principle (1) that also l contains at least three S-full points. Thus, Π is non-degenerate.

Since the points and lines of Π are points and lines of Δ with the same incidence relation, the i-elements (singular subspaces of rank i) of Π can be identified as those i-elements of Δ that are generated by points from Π.

Recall that we have chosen a basis B such that the form f that gives rise to Δ is defined over the prime field of $\mathbb{F}$ (and hence over any subfield). Assume that $\Delta(\mathbb{F}_0)$ is the $\mathbb{F}_0$-sub-building of Δ defined by f. We will show that $\Pi = \Delta(\mathbb{F}_1)$, where the latter is the $\mathbb{F}_1$-sub-building of Δ defined by f, for some subfield $\mathbb{F}_1$ of $\mathbb{F}$ containing $\mathbb{F}_0$.

We first prove $\Pi \subseteq \Delta(\mathbb{F}_1)$ by showing that every point p of Π belongs to $\Delta(\mathbb{F}_1)$.

Consider $E = \langle e_1, e_2, \ldots, e_n\rangle$. This is an n-element of Π because all points of $\Delta(\mathbb{F}_0)$ belong to Π. The points of Π on E form a projective space containing the points of $\Delta(\mathbb{F}_0)$ on E. Hence there exists a subfield $\mathbb{F}_1$ of $\mathbb{F}$ containing $\mathbb{F}_0$ that coordinatizes all points of Π on E. Thus we have the property:

(2) all points and other elements of Π on E belong to $\Delta(\mathbb{F}_1)$.

Now let E' be an n-element of $\Delta(\mathbb{F}_0)$ disjoint from (opposite to) E and let p be a point of Π on E'. Then $p^\perp \cap E$ is an element of Π and hence belongs to $\Delta(\mathbb{F}_1)$. Since E, E' and f are defined over $\mathbb{F}_0$, also p must belong to $\Delta(\mathbb{F}_1)$. Thus (2) holds for E' as well.

Consider the graph on the n-elements of $\Delta(\mathbb{F}_0)$ in which two such elements are joined by an edge whenever they are disjoint as subspaces of V. This graph is connected and so we have (2) for every n-element of $\Delta(\mathbb{F}_0)$.

Now let p be any point of Π. For every n-element E of the polar space Π also $p^\perp \cap E$ belongs to Π. By the preceding, if E belongs to $\Delta(\mathbb{F}_0)$ then $p^\perp \cap E$ belongs to $\Delta(\mathbb{F}_1)$. We prove that $p = \cap_E (p^\perp \cap E)^\perp$, where E runs over all n-elements of $\Delta(\mathbb{F}_0)$. Then it follows that p belongs to $\Delta(\mathbb{F}_1)$, as f is defined over $\mathbb{F}_1$.

Let $(p_i)_{i=1}^m$ and $(q_i)_{i=1}^m$ be non-zero vectors in p and q and suppose that $p^\perp \cap E = q^\perp \cap E$ for every n-element of $\Delta(\mathbb{F}_0)$. We prove that $p = q$. First we note that it follows from the assumption that $p^\perp \cap E = q^\perp \cap E$ for every element of $\Delta(\mathbb{F}_0)$. We use the fact that for each element e of the hyperbolic basis B there is a unique element $\widehat{e}$ that is not orthogonal to it; for $i \in \{1, 2 \ldots, 2n+1\}$ let $\hat{\imath}$ be such that $\widehat{e_i} = e_{\hat{\imath}}$.

By considering $p^\perp \cap E$ and $q^\perp \cap E$ for all 2-elements E of the form $\langle e_i, e_j\rangle$, $\langle e_i, e_{n+i}\rangle$, $\langle e_{n+i}, e_{n+j}\rangle$ with $i, j \in \{1, 2 \ldots, n\}$, we find in the D_n case that $p_k \neq 0$ if and only if $q_k \neq 0$, and moreover that $p_k/p_l = q_k/q_l$ whenever $p_k \neq 0 \neq p_l$ (and $q_k \neq 0 \neq q_l$) ($k, l \in \{1, 2, \ldots, m\}$). Thus we conclude that $p = q$. We reach the same conclusion in the C_n case by considering, in addition, all 2-elements of the form $\langle e_i + e_j, e_{\hat{\imath}} - e_{\hat{\jmath}}\rangle$ and also in the B_n case by considering, in addition, the 2-elements of the form $\langle e_i, e_j - e_{\hat{\jmath}} + e_{2n+1}\rangle$ ($i, j \in \{1, 2, \ldots, n\}$).

This concludes the proof that $\Pi \subseteq \Delta(\mathbb{F}_1)$.

Next, we prove $\Pi \supseteq \Delta(\mathbb{F}_1)$ by showing that all points of $\Delta(\mathbb{F}_1)$ belong to Π. Suppose p and q are points of $\Pi \cap \Delta(\mathbb{F}_1)$ lying on a line l of $\Delta(\mathbb{F}_1)$.

If l belongs to $\Delta(\mathbb{F}_0)$, then we already know that the points of Π on l are precisely the points of $\Delta(\mathbb{F}_1)$ on l. If l is any other line, we can find a line m of $\Delta(\mathbb{F}_0)$ opposite to l. Then the one-to-one correspondence between the points of l and m given by collinearity relates the points of $\Delta(\mathbb{F}_1)$ on l with the points of $\Delta(\mathbb{F}_1)$ on m. By principle (1), the same holds for the points of Π on l and m. It follows that the points of Π on l are precisely the points of $\Delta(\mathbb{F}_1)$ on l.

As the points of $\Delta(\mathbb{F}_0)$ are contained in Π and generate the geometry of points and lines of $\Delta(\mathbb{F}_1)$ (see Theorem 3.1 and Proposition 3.2), all points of $\Delta(\mathbb{F}_1)$ belong to Π. This concludes the proof that $\Pi \supseteq \Delta(\mathbb{F}_1)$.

Next, we prove the last part of the theorem. All elements of $\Delta(\mathbb{F}_1)$ are S-full because they are generated by S-full 1-elements; see Lemma 5.1. As for the converse, we must show that every S-full i-element E of Δ with $i < n$ belongs to $\Delta(\mathbb{F}_1)$. The case $i = 1$ was proved above. Now let $1 < i < n$. From Lemma 5.3 applied with Δ replaced by $\Delta(\mathbb{F}_1)$, it follows that if there is an S-full i-element in $\Delta \setminus \Delta(\mathbb{F}_1)$, then there is a 1-element with the same properties. But this contradicts the fact that all S-full 1-elements belong to $\Delta(\mathbb{F}_1)$. $\square$

Corollary 5.4 *Suppose that $\mathbb{F}$ is generated by adjoining k elements to its subfield $\mathbb{F}_0$. Then Γ can be generated by adding at most k points to its subgeometry Γ_0.*

Proof Let $k = 1$ and suppose that $\mathbb{F} = \mathbb{F}_0(\alpha)$ for some element $\alpha \in \mathbb{F}$. Take a point l of Γ_0 containing 1-elements $a, b \in \Delta_0$ and let $p \subset l$ not be in Δ_0 (for example, $\langle \alpha e_1 + e_2 \rangle_V$). Let l_0 be a point in Γ_0 opposite to l and let q be a 1-element on l_0 such that $p = q^\perp \cap l$. Put $x = \langle p, q \rangle_V$. Since $l \in \Delta_0$ is full with respect to Γ_0, p is full with respect to $X = \Gamma_0 \cup \{x\}$. By Theorem 1.3, the X-full points of Δ are precisely those of $\Delta(\mathbb{F}_1)$ for some extension $\mathbb{F}_1$ of $\mathbb{F}_0$ contained in $\mathbb{F}$. However, since $p \notin \Delta_0$, we have $\mathbb{F}_1 = \mathbb{F}$. The corollary now follows by induction on k. $\square$

Proof of Proposition 5

Suppose Δ is of type C_n and let $B = \{e_i, e_{n+i}\}_{i=1}^n$ be the chosen hyperbolic basis. For $J, K \subseteq I = \{1, 2, ..., n\}$, put $E_{J,K} = \langle e_j, e_{n+k} \mid j \in J, k \in K \rangle_V$. The collection of totally isotropic subspaces of this form is an apartment A.

Let e be a 1-element of Δ contained in $E_{I,\emptyset}$ but not in $E_{J,\emptyset}$ for any $J \subset I$. Let S be the collection of 2-elements of A, together with any $n-1$ 2-elements on e that span an n-element meeting $E_{I,\emptyset}$ only in e. Then S is a generating set for Γ.

First we recall that, if $\mathrm{Char}(\mathbb{F}) \neq 2$, then the 1-elements of Δ contained in a given apartment span Δ (see Theorem 3.1; also Cooperstein and Shult [9]). As this also applies to the polar space formed by the elements of Δ on a given 1-element, we get that all 1-elements of A are S-full. In particular, all 2-elements

$\langle e_i, e\rangle_V$ are in the span of S, whence e is S-full. Similarly, every 1-element of Δ contained in $E_{i,i}$ is S-full, for every $i \in I$.

Consider the geometry Θ of pairs (p, H), where p is a 1-element and H is an $(n-1)$-element with $p \subset H \subset E_{I,\emptyset}$. This geometry is isomorphic to the long-root geometry associated to $\mathrm{SL}_n(\mathbb{F})$. It is not difficult to see that it is also isomorphic to the subgeometry of Γ containing those 2-elements of Δ that meet both $E_{I,\emptyset}$ (in p) and $E_{\emptyset,I}$ (in $H^\perp \cap E_{\emptyset,I}$).

Note that the 1-elements of A in $E_{I,\emptyset}$ together with e form a spanning set for Θ (compare Cooperstein [10, Theorem 4.1]). Hence, for any 1-element $p \subset E_{I,\emptyset}$, all 2-elements on p meeting $E_{\emptyset,I}$ are in the span of S. By an argument similar to the one used to prove that e is S-full, it follows that all 1-elements in $E_{I,\emptyset}$ are S-full. The remainder of the proof is based on the following principle (see the proof of Theorem 1.3, principle (1)):

(∗) Suppose that a 2-element l contains two S-full 1-elements. If there exists a 2-element m opposite to l such that all 1-elements in m are S-full, then all 1-elements in l are S-full.

As all 1-elements in $E_{I,\emptyset}$ are S-full, it follows from (∗) that, if a 2-element l meets $E_{i,i}$ and $E_{j,j}$ in 1-elements different from $E_{i,\emptyset}$ and $E_{j,\emptyset}$ for distinct $i, j \in I$, then all 1-elements of l are S-full. (The 2-element $\langle e_i, e_j\rangle_V$ can be given the role of m in (∗).)

By considering now the 2-elements $\langle e_i + e_{n+i}, e_j + e_{n+j}\rangle_V$, for any distinct $i, j \in I$, it follows in turn that all 1-elements contained in a 2-element of A are S-full. As the 1-elements of A span Δ and every 2-element of Δ is opposite to some 2-element of A, we find that in fact all 1-elements of Δ are full.

Thus, Γ is spanned by $2n^2 - n - 1$ points if $\mathbb{F}$ is a prime field of odd characteristic. It has a natural embedding of dimension $2n^2 - n - 1$ into a hyperplane of the natural embedding for the line-grassmannian of the ambient projective space $\mathrm{PG}(2n-1, \mathbb{F})$. It follows that its generating rank is $2n^2 - n - 1$. □

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References

[1] N. Batens and A. Pasini, On certain hyperplanes and projective embeddings of D_4-buildings, *Int. J. of Math. Game Theory Algebra* **8** (1998), 137–166.

[2] N. Batens and A. Pasini, On certain hyperplanes and projective embeddings of D_4-buildings (Errata), *Int. J. of Math. Game Theory Algebra*, to appear.

[3] R. J. Blok and A. E. Brouwer, Spanning point-line geometries in buildings of spherical type, *J. Geom.* **62** (1998), 26–35.

[4] R.J. Blok and A. Pasini, On absolutely universal embeddings, submitted.

[5] F. Buekenhout, The basic diagram of a geometry, *Geometries and Groups* (eds. M. Aigner and D. Jungnickel), Springer, Berlin, 1981.

[6] F. Buekenhout, Foundations of incidence geometry, Chapter 3 of *Handbook of Incidence Geometry* (ed. F. Buekenhout), North-Holland, Amsterdam, 1995.

[7] F. Buekenhout and E.E. Shult, On the foundations of polar geometry, *Geom. Dedicata* **3** (1974), 155–170.

[8] R.W. Carter, *Simple Groups of Lie Type*, Wiley, London, 1972.

[9] B.N. Cooperstein and E.E. Shult, Frames and bases of Lie incidence geometries, *J. Geom.* **60** (1997), 17–46.

[10] B. N. Cooperstein, Generating long root subgroup geometries of classical groups over finite prime fields, *Bull. Belg. Math. Soc. Simon Stevin* **5** (1998), 531–548.

[11] A. Pasini, *Diagram Geometries*, Oxford University Press, Oxford, 1994.

[12] M.A. Ronan, Embeddings and hyperplanes of discrete geometries, *European J. Combin.* **8** (1987), 179–185.

[13] M. A. Ronan, *Lectures on Buildings*, Academic Press, New York, 1989.

[14] S.D. Smith and H. Völklein, A geometric presentation for the adjoint module of $\mathrm{SL}_3(k)$, *J. Algebra* **127** (1989), 127–138.

[15] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, First edition, Springer, Berlin, 1974.

[16] J. Tits, A local approach to buildings, *The Geometric Vein* (eds. C. Davis et al.), Springer, Berlin, 1981, 519–547.

[17] J. Tits, Twin buildings and groups of Kac–Moody type, *Groups, Combinatorics and Geometry* (eds. M.W. Liebeck and J. Saxl), Cambridge University Press, Cambridge, 1992, 249–286.