A Curtis-Tits-Phan theorem for the twin-building of type $\widetilde{A}_{n-1}$

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Abstract

The Curtis-Tits-Phan theory as laid out originally by Bennett and Shpectorov describes a way to employ Tits' lemma to obtain presentations of groups related to buildings as the universal completion of an amalgam of low-rank groups. It is formulated in terms of twin-buildings, but all concrete results so far were concerned with spherical buildings only. We describe an explicit flip-flop geometry for the twin-building of type $\widetilde{A}_{n-1}$ associated to $\mathrm{k}[t, t^{-1}]$ on which a unitary group $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$, related to a certain non-degenerate hermitian form β, acts flag-transitively and obtain a presentation for this group in terms of a rank-2 amalgam consisting of unitary groups. This is the most natural generalization of the original result by Phan for the unitary groups.

1 Introduction

In the revision of the classification of finite simple groups one of the important steps requires one to prove that if a simple group G (the minimal counterexample) contains a certain amalgam of subgroups that one normally finds in a known simple group H then G is isomorphic to H. A geometric approach to recognition theorems was initiated in [BGHS03, BeSh04, GHS03]. It was formulated in terms of twin-buildings, but so far the only concrete results deal with spherical buildings. In the present paper we obtain an explicit amalgamation result for the unitary group $\mathrm{SU}_n(\mathrm{k}[t,t^{-1}],\beta)$ associated to a unitary flip of the affine twin-building of type $\widetilde{A}_{n-1}$ over $\mathrm{k}[t,t^{-1}]$. Here k is a field of order at least 16, and β is a non-degenerate σ-hermitian form, for a suitable $\mathrm{k}(t)$-involution σ. Similar groups have been considered in [Ca05], [GlGrHa] and [KaPe85] for complex numbers.

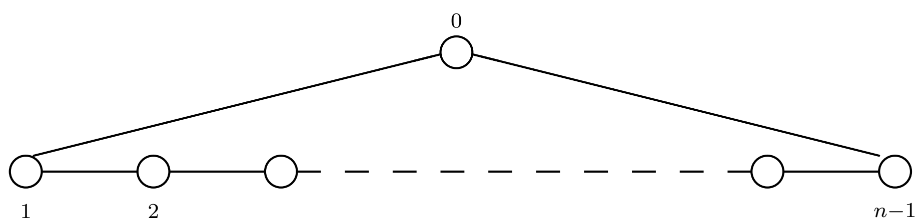

Figure 1: The diagram $\widetilde{A}_{n-1}$.

Let us describe the general geometric approach. We consider a group G which is either semi-simple of Lie type or a Kac-Moody group. Let $\Delta = (\Delta_+, \Delta_-)$ be the twin building associated to G via its twin BN-pair. We first define a *flip* to be an involutory automorphism φ of Δ that interchanges the two halves, preserves distances and co-distances and takes at least one chamber to an opposite.

Given a flip φ, construct Δ_φ as the chamber system whose chambers are the pairs of opposite chambers (c, c^φ) of Δ. Let G_φ be the fixed subgroup under the φ-induced automorphism of G. We refer to [BGHS03] for details on the construction. In most cases, the pre-geometry Γ_φ is transversal and residually connected. A result by Tits then asserts that the geometry Γ_φ is simply connected if and only if G_φ is the universal completion of the amalgam of maximal parabolic subgroups for its action on Γ_φ.

An induction procedure allows us to replace this amalgam by a related amalgam $\mathcal{A}_{(2)}$ of rank at most two groups without altering its universal completion. In the present paper we describe this amalgam $\mathcal{A}_{(2)}$ in Section 10. Our main result is then the following.

Theorem 1 *Let $n \geq 4$. Let* k *be infinite or* $\mathrm{k} = \mathbb{F}_{q^2}$ *with $q \geq 4$. Then, the group* $\mathrm{SU}_n(\mathrm{k}[t,t^{-1}],\beta)$ *is the universal completion of the amalgam $\mathcal{A}_{(2)}$.*

For completeness we will describe a Phan system for the amalgam $\mathcal{A}_{(2)}$. We will say that subgroups U_1 and U_2 of $\mathrm{SU}_3(\mathrm{k})$ form a standard pair whenever each U_i is the stabilizer in $\mathrm{SU}_3(\mathrm{k})$ of a non-singular vector v_i (v_i is then unique up to a scalar factor) and, furthermore, v_1 and v_2 are perpendicular. By Witt's theorem, standard pairs are exactly the conjugates of the pair formed by the two subgroups $\mathrm{SU}_2(\mathrm{k})$ arising from the 2×2 blocks on the main diagonal. Standard pairs in $\mathrm{PSU}_3(\mathrm{k})$ will be defined as the images under the natural homomorphism of the standard pairs from $\mathrm{SU}_3(\mathrm{k})$. We say that a group G possesses a weak Phan system of type $\widetilde{A}_{n-1}$ if G contains subgroups $U_i \cong \mathrm{SU}_2(\mathrm{k})$, where $i = 0, 1, \ldots, n-1$ and $U_{i,j}$, for $0 \leq i < j \leq n-1$ so that the following hold (taking subscripts modulo n).

(wP1) if $j - i \neq 1$ then $U_{i,j}$ is a central product of U_i and U_j,

(wP2) for $i = 0, 1, \ldots, n-1$, U_i and U_{i+1} are contained in $U_{i,i+1}$, which is isomorphic to $\mathrm{SU}_3(\mathrm{k})$ or $\mathrm{PSU}_3(\mathrm{k})$. Moreover, U_i and U_{i+1} form a standard pair in $U_{i,i+1}$, and

(wP3) the subgroups $U_{i,j}$ with $0 \leq i < j \leq n-1$ generate G.

Using this terminology, Theorem 1 can be restated as follows.

Theorem 2 *The group $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ has a weak Phan system of type $\widetilde{A}_{n-1}$ when $n \geq 4$ and k is infinite or $\mathrm{k} = \mathbb{F}_{q^2}$ with $q \geq 4$. Moreover any group admitting a weak Phan system isomorphic (as an amalgam) with the one in $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ is a quotient of $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$.*

The paper is organized as follows. In Section 2 we review some basic notions on geometries, automorphism groups, simple connectedness and amalgams, twin-buildings and (geometric) flips. In Section 3 we describe the affine (twin-) building Δ of type $\widetilde{A}_{n-1}$ over the field of rational functions $\mathrm{k}(t)$, over some field k in terms of a concrete lattice-chain model. In Section 4 we describe a flip φ for Δ using a σ-hermitian form β. We assume that $\sigma \neq \mathrm{id}|_{\mathrm{k}}$ and prove that in this case, φ is what we call geometric. In Section 5 we characterize the flip-flop geometry Γ_φ for φ and in Section 6 we describe its residues. We prove that in most cases, Γ_φ is transversal, connected, and residually connected. In Section 7 we show that the geometry Γ_φ and all residues of rank at least 3 are simply connected with few exceptions. This is where the requirement that k has order at least 16 and $n \geq 4$ comes in.

In Section 9 we describe the flag-transitive action of $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ on Γ_φ. In Section 10, we describe the amalgam $\mathcal{A}_{(2)}$ of parabolic subgroups of rank 2 for this action in some detail. In the Section 8 we reprove some of the needed results from Bennett and Shpectorov [BeSh04] and Gramlich et al. [GHMS], but now for infinite fields.

2 Preliminaries

2.1 Geometries

For our viewpoint on geometries we'll use the following definitions from Buekenhout [Bu95].

Definition 2.1 A *pre-geometry* over a *type set* I is a triple $\Gamma = (\mathcal{O}, \mathrm{typ}, \star)$, where $\mathcal{O}$ is a collection of *objects* or *elements*, I is a set of *types*, $\star$ is a binary symmetric and reflexive relation, called the *incidence relation* and $\mathrm{typ}\colon \mathcal{O} \to I$ is a *type function* such that whenever $X \star Y$, then either $X = Y$ or $\mathrm{typ}(X) \neq \mathrm{typ}(Y)$.

The *rank* of the pre-geometry Γ is the size of $\mathrm{typ}(\mathcal{O})$. A *flag* F is a (possibly empty) collection of pairwise incident objects. Its *type* (resp. *cotype*) is $\mathrm{typ}(F)$ (resp. $I - \mathrm{typ}(F)$). The *rank* of F is $\mathrm{rank}(F) = |\,\mathrm{typ}(F)|$. The *type* of F is $\mathrm{typ}(F) = \{\mathrm{typ}(X) \mid X \in F\}$. A *chamber* is a flag C of type I. A J-*flag* is a flag of type J.

A pre-geometry Γ is a *geometry* if $\mathrm{typ}(\mathcal{O}) = I$ and if Γ is *transversal*, that is, if any flag is contained in a chamber.

The *incidence graph* of the pre-geometry $\Gamma = (\mathcal{O}, \mathrm{typ}, \star)$ over I is the graph $(\mathcal{O}, \star)$. This is a multipartite graph whose parts are indexed by I. We call Γ *connected* if its incidence graph is connected.

The *residue* R of a flag F is the pre-geometry $\mathrm{Res}_\Gamma(F) = (\mathcal{O}_F, \mathrm{typ}\,|_{\mathcal{O}_F}, \star|_{\mathcal{O}_F})$ over $I - \mathrm{typ}(F)$ induced on the collection $\mathcal{O}_F$ of all objects in $\mathcal{O} - F$ incident to all elements of F. The type of the residue R is the cotype of F. We call Γ *residually connected* if for every flag of corank at least 2 the corresponding residue is connected.

We will mostly be working with connected, residually connected geometries over a set I.

2.2 Automorphism groups and amalgams

Definition 2.2 An *automorphism group* G of a pre-geometry Γ is a group of permutations of the collection of objects that preserve type and incidence. We call G *flag-transitive* if for any $J \subseteq I$, G is transitive on the collection of J-flags.

Let G be a flag-transitive group of automorphisms of a geometry Γ over an index set I. Fix a chamber C. The *standard parabolic subgroup of type* $J \subseteq I$ is the stabilizer in G of the residue of type J on C.

Definition 2.3 In this paper we shall use the following definition of an amalgam of groups. Let $(\mathcal{B}, \prec)$ be a meet-semilattice with minimal element $\widehat{0}$ in which every maximal chain has length s. An *amalgam* over $(\mathcal{B}, \prec)$ is a collection of groups $\mathcal{A} = \{A_\beta \mid \beta \in \mathcal{B}\}$ together with a system of homomorphisms $\Phi = \{\varphi_{\beta,\gamma}\colon A_\beta \to A_\gamma \mid \beta \prec \gamma\}$ satisfying $\varphi_{\gamma,\delta} \circ \varphi_{\beta,\gamma} = \varphi_{\beta,\delta}$ whenever $\beta \prec \gamma \prec \delta$. The number s is called the *rank* of $\mathcal{A}$.

A *completion* of $\mathcal{A}$ is a group G with the property that, for each $\beta \in \mathcal{B}$, there exists a homomorphism $f_\beta\colon A_\beta \to G$ such that for any $\alpha \prec \beta$ we have $f_\alpha = f_\beta \circ \phi_{\alpha,\beta}$ and $G = \langle f_\beta(A_\beta) \mid \beta \in \mathcal{B}\rangle$. The *universal completion* or *amalgamated sum* of $\mathcal{A}$ is then a group $\widehat{G}$ whose elements are words in the elements of the groups in $\mathcal{A}$ subject to the relations between the elements of A_β for any $\beta \in \mathcal{B}$ and in which for each $\beta \prec \gamma$ each $a \in A_\beta$ is identified with $\varphi_{\beta,\gamma}(a) \in A_\gamma$. We then have a homomorphism $\widehat{\cdot} : \mathcal{A} \to \widehat{G}$.

We note that for the appropriate choice of $(\mathcal{B}, \prec)$ this definition of an amalgam and universal completion coincides with those given in [Se80, Ti86b].

Note 2.4

(i) For each $\beta \in \mathcal{B}$ we have a homomorphism $\widehat{\cdot} : A_\beta \to \widehat{A_\beta} \leq \widehat{G}$, which is surjective, but not necessarily injective.

(ii) For $\beta, \gamma \in \mathcal{B}$ with $\beta \prec \gamma$ we have $\widehat{A_\beta} \leq \widehat{A_\gamma}$.

(iii) For $\beta, \gamma \in \mathcal{B}$ we have $\widehat{A_{\beta \wedge \gamma}} \leq \widehat{A_\beta} \cap \widehat{A_\gamma}$, but we do not a priori assume equality here.

Example 2.5 Let G be a group acting flag-transitively on a geometry Γ over an index set I. Let C be a chamber and, for every subset $J \subseteq I$ with $|J| \leq 2$ let P_J be the standard parabolic subgroup of type J in G. Then, for $M \subseteq K \subseteq I$ we have the natural inclusion homomorphisms $\varphi_{M,K} \colon P_M \to P_K$. Hence $\mathcal{A} = \{P_J \mid J \subseteq I,\ |J| \leq 2\}$ is an amalgam over $\mathcal{B} = \{J \subseteq I \mid |J| \leq 2\}$ where $M \prec K \iff M \subset K$. For the universal completion $\widehat{G}$ of $\mathcal{A}$ we clearly have a surjective homomorpism $\tau \colon \widehat{G} \to G$.

2.3 Simple connectedness and amalgams

In order to introduce the main tool of this paper, namely Lemma 2.6 we need the notions of (closed) paths, (universal covers), simple connectedness, and the fundamental group.

In [Ti86b, Fo66, Ba80, Qu78] these notions are introduced in the context of (the face poset of) a simplicial complex in such a way that many classical results, such as can be found in [Sp81] continue to hold. In the present paper we use definitions geared towards geometries. They are equivalent to those for the (face poset of) the simplicial complex, called the *flag complex* consisting of all flags of Γ ordered by inclusion. For a more extensive treatment of related issues see e.g. [Ti86a, Pas94].

Let Γ be a connected geometry over the finite set I. A *path of length* k is a path $x_0, \ldots, x_k$ in the incidence graph. We do not allow repetitions, that is, $x_i \neq x_{i+1}$ for all $0 \leq i < k$. A *cycle based at an element* x is a path $x_0, \ldots, x_k$ in which $x_0 = x = x_k$. Two paths γ and δ are *homotopy equivalent* if one can be obtained from the other by inserting or eliminating cycles of length 2 or 3. We denote this by $\gamma \simeq \delta$. The homotopy classes of cycles based at an element x form a group under concatenation. This group is called the *fundamental group of* Γ *based at* x and is denoted $\Pi_1(\Gamma, x)$. If Γ is (path) connected, then the isomorphism type of this group does not depend on x and we call this group simply the *fundamental group of* Γ and denote it $\Pi_1(\Gamma)$. We call Γ *simply connected* if $\Pi_1(\Gamma)$ is trivial.

The following result, which will be referred to as Tits' Lemma, is a consequence of [Ti86b, Corollaire 1].

Lemma 2.6 *Given a group G acting flag-transitively on a geometry Γ. Fix a maximal flag C. Then G is the universal completion of the amalgam consisting of the standard maximal parabolic subgroups of G with respect to C if and only if Γ is simply connected.*

2.4 Buildings and twin-buildings

In this subsection, we give the basic definitions and a few facts on buildings and twin-buildings. We make use of the following sources: [Ab96, AbRo98, AbVa01, Ro89, Ro03, Ti86a, Ti92]. Let $I = \{0, 1, \ldots, n-1\}$ and let $M = (m_{ij})$ be a Coxeter diagram over I. Let (W, S) be the Coxeter system of type M, where $S = \{s_i \mid i \in I\}$. A *building of type M* is a pair (Δ, δ) where Δ is a set, whose elements are called *chambers*, and $\delta : \Delta \times \Delta \longrightarrow W$ is a distance function satisfying the following axioms. Let $x, y \in \mathcal{C}$ and $w = \delta(x, y)$. Then

(B1) $w = 1$ if and only if $x = y$;

(B2) if $z \in \Delta$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws; furthermore if $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$; and

(B3) if $s \in S$, there exists $z \in \Delta$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

For any subset $J \subset I$, M_J denotes the subdiagram of M defined over J and W_J will be the subgroup of W having generator set $S_J = \{s_i\}_{i \in J}$. The pair (W_J, S_J) then is a Coxeter system with diagram M_J.

The *residue of type* J, or J-residue, on a chamber $c \in \Delta$ is the inverse image of W_J under the mapping $\delta(c, \cdot) : \Delta \to W$. This is a building of type M_J. The rank of a J-residue is the number $|J|$. We will call a residue of type $I - \{i\}$ $(i \in I)$ an *object of type* i and a residue of type $\{i\}$ $(i \in I)$ an *i-panel*.

Example 2.7 Taking W and defining $\delta(x, y) = x^{-1}y$ we obtain a building of type M in which each panel has exactly two chambers. This is called the *Coxeter building* of type M. A reflection in W is a conjugate of one of the generators. Each reflection $r \in W$ fixes a subcomplex M_r of codimension 1 of the Coxeter complex. This subcomplex (the wall of r) divides the chambers of W in two disjoint convex sets called *opposite roots* (they will be denoted by α and $-\alpha$.

An *apartment* of the building Δ is a subset $\Sigma \subseteq \Delta$ that is isometric to the Coxeter building of type M. One can define roots in the building using this identification.

Since apartments are convex in Δ with respect to δ, and any two chambers are contained in some apartment Σ, the distance between any two chambers is given by their distance inside an apartment. As a consequence, a building is uniquely determined by the collection of chambers together with a system of apartments.

Given two buildings $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$, $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ of the same type M, a *codistance* (*twinning*) is a map $\delta_* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \longrightarrow W$ such that the following axioms hold where $\epsilon = \pm$, $x \in \mathcal{C}_\epsilon, y \in \mathcal{C}_{-\epsilon}$ and $w = \delta_*(x, y)$:

(T1) $\delta_*(y,x) = w^{-1}$;

(T2) if $z \in \mathcal{C}_{-\epsilon}$ such that $\delta_{-\epsilon}(y,z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x,z) = ws$; and

(T3) if $s \in S$, there exists $z \in \mathcal{C}_{-\epsilon}$ such that $\delta_{-\epsilon}(y,z) = s$ and $\delta_*(x,z) = ws$.

A *twin building* of type M is a triple $\Delta = ((\Delta_+, \delta_+), (\Delta_-, \delta_-), \delta_*)$, where (Δ_+, δ_+) and (Δ_-, δ_-) are buildings of type M and δ_* is a twinning between Δ_+ and Δ_-.

We call $c_+ \in \Delta_+$ and $c_- \in \Delta_-$ *opposite* if $\delta_*(c_+, c_-) = 1$. The opposition relation is denoted opp.

By the results of Abramenko and Van Maldeghem in [AbVa01], given two isomorphic buildings in order to make this into a twin-building, it suffices to define a sufficiently nice opposition relation (a 1-twinning).

If in (T3) above $l(ws) = l(w) + 1$, then the chamber z satisfying $\delta_*(x,z) = ws$ is unique in the s-panel π on y. In this case we call z the *projection* of x onto π and denote it $\mathrm{proj}_\pi(x)$. More generally, if R is a J-residue on y, where J is spherical, there is a unique chamber $z \in R$ such that $\delta_*(x,z)$ has maximal length among all codistances between x and a chamber on R. In this case we call z the projection of x onto R and write $z = \mathrm{proj}_R(x)$.

A *twin apartment* of Δ is a pair $\Sigma = (\Sigma_+, \Sigma_-)$ in which, for each $\varepsilon = \pm$, the set Σ_ε is an apartment of Δ_ε such that $(\Sigma_+, \Sigma_-, \delta_*)$ forms a twin-building. Apartments of Δ_ε that belong to a twin-apartment are called *interior* (see e.g. [Ro03]).

A pair $\alpha = (\alpha_+, \alpha_-)$ is called a *twin root* if there exists a twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$ such that α_ε is a root in Σ_ε and $\alpha_- = -\,\mathrm{opp}_\Sigma(\alpha_+)$. By [AbRo98, Lemma 3], twin roots as well as twin-apartments are coconvex. Following loc. cit. we define a set of chambers X to be *coconvex* if, given a chamber $x \in X$ and a panel π with $\pi \cap X \neq \emptyset$ belonging to different halves of Δ, the chamber $\mathrm{proj}_\pi(x)$ belongs to X. Following loc. cit. we define the *coconvex hull* of chambers $x_- \in \Delta_-$ and $x_+ \in \Delta_+$ to be $C(x_+, x_-) = \bigcap_C C$, where C runs through all coconvex subsets containing both x_+ and x_-.

Lemma 2.8 *Let $x_\varepsilon \in \Delta_\varepsilon$ and let R_ε be a spherical residue of Δ_ε for $\varepsilon = \pm$. Then,*

(a) *$C(x_+, x_-) = \bigcap_{(\alpha_+,\alpha_-)} \alpha_+ \cup \alpha_-$, where (α_+, α_-) runs through all twin-roots of any twin-apartment containing both x_+ and x_-.*

(b) *In particular, if $C(x_+, x_-) = (C_+, C_-)$ for some subsets $C_\varepsilon \subseteq \Delta_\varepsilon$, then C_ε is convex in Δ_ε.*

(c) *$\mathrm{proj}_{R_\varepsilon}(x_{-\varepsilon}) = R_\varepsilon \cap \bigcap_{u_\varepsilon \in R_\varepsilon} C(u_\varepsilon, x_{-\varepsilon})$.*

Proof (a) Is in [AbRo98] and (b) follows from (a) since roots of buildings are convex (see above).

(c) Let $z_\varepsilon = \mathrm{proj}_{R_\varepsilon}(x_{-\varepsilon})$. Then using the definition of coconvexity and induction on the length of a minimal gallery from x_ε to z_ε we find that $z_\varepsilon \in C(u_\varepsilon, x_{-\varepsilon})$ for all $u_\varepsilon \in R_\varepsilon$. On the other hand, $R_\varepsilon \cap \bigcap_{u_\varepsilon \in R_\varepsilon} C(u_\varepsilon, x_{-\varepsilon}) \subseteq R_\varepsilon \cap C(z_\varepsilon, x_{-\varepsilon})$ and by definition of projection, the latter set is the singleton $\{z_\varepsilon\}$. □

The root group of α_ε is the following

$$U_{\alpha_\varepsilon} = \{g \in \mathrm{Aut}(\Delta_\varepsilon) \mid g \text{ fixes every chamber having a panel } \pi \text{ with } |\pi \cap \alpha_\varepsilon| = 2\}. \quad (2.1)$$

The twin-buildings we will be dealing with are of Moufang type, which means that for each root α_ε the root group U_{α_ε} acts regularly on the apartments of Δ_ε containing α_ε. Using the twinning, one then finds that if $\alpha = (\alpha_-, \alpha_+)$ is a twin-root, we have $U_{\alpha_\varepsilon} = U_{\alpha_{-\varepsilon}}$.

Given two opposite chambers $c_+ \in \Delta_+$ and $c_- \in \Delta_-$, there is a unique twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$ such that $c_+ \in \Sigma_+$ and $c_- \in \Sigma_-$. We denote this apartment by $\Sigma(c_+, c_-)$. We have $\Sigma_\varepsilon = \{d_\varepsilon \in \Delta_\varepsilon \mid \delta^*(c_{-\varepsilon}, d_\varepsilon) = \delta_\varepsilon(c_\varepsilon, d_\varepsilon)\}$. See Tits [Ti92].

2.5 Geometric flips of twin-buildings

A *flip* φ of the twin-building Δ is an automorphism $\varphi\colon \Delta \to \Delta$ of order 2 interchanging Δ_+ and Δ_- such that

- for any $c_\varepsilon, d_\varepsilon \in \Delta_\varepsilon$, $c_{-\varepsilon}, d_{-\varepsilon} \in \Delta_{-\varepsilon}$, we have

$$\begin{aligned} \delta_{-\varepsilon}(c_\varepsilon^\varphi, d_\varepsilon^\varphi) &= \delta_\varepsilon(c_\varepsilon, d_\varepsilon) \\ \delta_*(c_{-\varepsilon}^\varphi, d_\varepsilon^\varphi) &= \delta_*(c_{-\varepsilon}, d_\varepsilon), \end{aligned}$$

- there is a pair of chambers $(c_+, c_-) \in \mathrm{opp}$ such that $c_+^\varphi = c_-$.

Note that $\varphi\colon \Delta_+ \to \Delta_-$ and $\varphi\colon \Delta_- \to \Delta_+$ are type-preserving isomorphisms.

We'll denote $\Delta_\varphi = \{(c_+, c_+^\varphi) \in \Delta_+ \times \Delta_- \mid c_+ \text{ opp } c_+^\varphi\}$. We'll define Γ_φ to be the incidence structure whose objects of type $i \in I$ are those $I - \{i\}$-residues of Δ_+ that contain a chamber c_+ where $(c_+, c_+^\varphi) \in \Delta_\varphi$. Two such objects are *incident* whenever they intersect in a chamber c_+ with $(c_+, c_+^\varphi) \in \Delta_\varphi$.

A *geometric flip* φ of the twin-building Δ is an automorphism $\varphi\colon \Delta \to \Delta$ of order 2 interchanging Δ_+ and Δ_- such that

- for any $c_\varepsilon, d_\varepsilon \in \Delta_\varepsilon$, $c_{-\varepsilon}, d_{-\varepsilon} \in \Delta_{-\varepsilon}$, we have

$$\begin{aligned} \delta_{-\varepsilon}(c_\varepsilon^\varphi, d_\varepsilon^\varphi) &= \delta_\varepsilon(c_\varepsilon, d_\varepsilon) \\ \delta_*(c_{-\varepsilon}^\varphi, d_\varepsilon^\varphi) &= \delta_*(c_{-\varepsilon}, d_\varepsilon), \end{aligned}$$

- given a panel $\pi \in \Delta_\varepsilon$ the map $\mathrm{proj}_\pi \circ \varphi\colon \pi \to \pi$ is not the identity.

In Gramlich et al. [GHMS] a flip is defined to be what we call a geometric flip here. It follows immediately that a geometric flip is a flip as defined above, but the converse is unknown.

Lemma 2.9 *Let φ be a flip. Let $c \in \Delta_\varepsilon$ be a chamber and let $\emptyset \neq J \subseteq I$ such that for each $j \in J$, the j-object on c contains a chamber that is opposite to its φ-image. Then, $\delta_*(c, c^\varphi) \in W_{I-J}$. Moreover, if φ is geometric, then the $I - J$-residue on c contains a chamber that is opposite to its φ-image.*

Proof Let $j \in J$ and let R be the j-object on c and let $d \in R$ be opposite to d^φ. Now let $\gamma = (c = c_0, \ldots, c_k = d)$ be a minimal gallery in R from c to d of type $i_1 \cdots i_k$. By (T2) and (T3) if $\delta_*(x, y) = w$ and $\delta_{-\varepsilon}(y, z) = s$, then $\delta_*(x, z) \in \{w, ws\}$. Using this repeatedly and the fact that $\delta_*(d, d^\varphi) = 1$ we find that $\delta_*(c, c^\varphi) \in W_{\{i_1,\ldots,i_k\}} \leq W_{I-\{j\}}$. Repeating this for all $j \in J$ we find that $\delta_*(c, c^\varphi) \in \bigcap_{j\in J} W_{I-\{j\}} = W_{I-J}$.

Now assume that φ is geometric and let $\delta_*(c, c^\varphi) = w$ a reduced word of type $i_l \cdots i_1$ in W_{I-J}. Note that $l(ws_{i_1}) = l(w) - 1$ and let π be the i_1-panel on c. Since φ is geometric, there exists a chamber $c_1 \in \pi$ such that $\mathrm{proj}_\pi(c_1^\varphi) \neq c_1$. More precisely, one of the following occurs. If $\mathrm{proj}_\pi(\pi^\varphi) = c$, then $\delta_*(c_1, c_1^\varphi) = s_{i_1} w s_{i_1}$ of length $l(w) - 2$. Otherwise $\delta_*(c_1, c_1^\varphi) = ws_{i_1}$ of length $l(w) - 1$. In either case $\delta_*(c_1, c_1^\varphi) \in W_{I-J}$ so that c_1 belongs to the $I - J$-residue on c. Repeating this argument we eventually find a chamber d in the $I - J$-residue on c opposite to d^φ. □

Let $\mathcal{F}_\varphi = \{R \subseteq \Delta_+ \mid R \text{ a residue with } c \,\mathrm{opp}\, c^\varphi \text{ for some } c \in R\}$.

Corollary 2.10 *Let φ be a geometric flip. Then $\mathcal{F}_\varphi$ is the flag complex of Γ_φ. In particular, Γ_φ is transversal, that is, every flag is contained in a chamber.*

Proof Clearly all objects of Γ_φ belong to $\mathcal{F}_\varphi$. Therefore it suffices to show that for any two residues $R_J, R_K \in \mathcal{F}_\varphi$ of type $I - J$ and $I - K$ respectively such that $R_J \cap R_K \neq \emptyset$ we have $R_J \cap R_K \in \mathcal{F}_\varphi$. Let $c \in R_J \cap R_K$ and for any $i \in J \cup K$ let R_i be the i-object on c. Then $R_J \cap R_K = \bigcap_{i\in J\cup K} R_i$, which is a residue of type $I - (J \cup K)$. Moreover, since $R_J, R_K \in \mathcal{F}_\varphi$ also $R_i \in \mathcal{F}_\varphi$ for all $i \in J \cup K$. It now follows from Lemma 2.9 with $R = R_J \cap R_K$ and c that $R \in \mathcal{F}_\varphi$. □

3 The affine twin building of type $\widetilde{A}_{n-1}$ over $\mathrm{k}(t)$

In this section we introduce the necessary material on the twin-building $\Delta = (\Delta_+, \Delta_-)$ of type $\widetilde{A}_{n-1}$ over the field $\mathrm{k}(t)$ of rational functions over some suitable field k.

We shall view Δ_ε mostly through a lattice-chain model as described in [AbNe02], although we also draw upon [AbVa01, Ga97, Ro89, Ti92]. We shall now describe this model in some detail.

Let V be a vector space of dimension n over $\mathrm{k}(t)$. For $\varepsilon \in \{+,-\}$, let v_ε be a discrete valuation on $\mathrm{k}(t)$ such that $v_\varepsilon(t^{\varepsilon 1}) = 1$, and let $\mathcal{O}_\varepsilon$ be the valuation ring with respect to v_ε.

Definition 3.1 An $\mathcal{O}_\varepsilon$*-lattice* is a free $\mathcal{O}_\varepsilon$-submodule Λ of V of rank n so that $V = \mathrm{k}(t)\Lambda$. Such lattices are of the form

$$\Lambda = \bigoplus_{i=1}^{n} \mathcal{O}_\varepsilon b_i,$$

where $\{b_1, b_2, \ldots, b_n\}$ is a $\mathrm{k}(t)$-basis for V. In this case we call $\{b_1, b_2, \ldots, b_n\}$ a *lattice basis* for Λ.

A chain $\cdots \subset \Lambda_i \subset \Lambda_{i+1} \subset \cdots$ of $\mathcal{O}_\varepsilon$-lattices is called *admissible* if the set $\{\Lambda_i\}$ is invariant under multiplication by integral powers of t. The admissible chain generated by the lattice Λ will be denoted $[\Lambda]$.

For each ε, we now describe a geometry Γ_ε. The collection of all admissible chains of $\mathcal{O}_\varepsilon$-lattices partially ordered by inclusion is the collection of flags of Γ_ε. More precisely, the *objects* of this geometry are the minimal admissible chains of $\mathcal{O}_\varepsilon$-lattices. In the case of an $\mathcal{O}_+$-lattice Λ, these are of the form $\cdots \subset t^i\Lambda \subset t^{i+1}\Lambda \subset \cdots$ and in the case of an $\mathcal{O}_-$-lattice Λ of the form $\cdots \subset t^{-i}\Lambda \subset t^{-(i+1)}\Lambda \subset \cdots$. Two objects are called *incident* whenever their union is again an admissible $\mathcal{O}_\varepsilon$-lattice.

We now describe the *chambers* of Γ_ε. These are the maximal admissible chains of $\mathcal{O}_\varepsilon$-lattices. They are of the form

$$\begin{aligned} c_+(a_1, a_2, \ldots, a_n) &= \cdots \supset \Lambda_0^+ \supset \Lambda_1^+ \supset \cdots \supset \Lambda_{n-1}^+ \supset t\Lambda_0^+ \supset \cdots \\ c_-(a_1, a_2, \ldots, a_n) &= \cdots \subset \Lambda_0^- \subset \Lambda_1^- \subset \cdots \subset \Lambda_{n-1}^- \subset t\Lambda_0^- \subset \cdots \end{aligned}$$

where $\{a_1, a_2, \ldots, a_n\}$ is a $\mathrm{k}(t)$-basis for V and

$$\begin{aligned} \Lambda_0^\varepsilon &= \langle a_1, \ldots, a_{n-1}, a_n \rangle_{\mathcal{O}_\varepsilon}, \\ \Lambda_1^\varepsilon &= \langle ta_1, \ldots, a_{n-1}, a_n \rangle_{\mathcal{O}_\varepsilon}, \\ &\vdots \\ \Lambda_{n-1}^\varepsilon &= \langle ta_1, ta_2, \ldots, ta_{n-1}, a_n \rangle_{\mathcal{O}_\varepsilon}. \end{aligned}$$

We call $(a_1, \ldots, a_n)$ an *ordered chain basis* for this chamber. Two chambers are *i-adjacent* if their objects of type $j \neq i$ are equal.

The set of chambers of Γ_ε is in fact the collection of chambers of a building Δ_ε. We now describe the system of *apartments* $\mathcal{A}(\mathrm{k}(t))$ of Δ_ε. Given any $\mathrm{k}(t)$-basis $\{a_1, \ldots, a_n\}$ for V, the collection of admissible chains of $\mathcal{O}_\varepsilon$-lattices

$$\Sigma_\varepsilon\{a_1, \ldots, a_n\} = \{c_\varepsilon(t^{k_1}a_{\rho(1)}, \ldots, t^{k_n}a_{\rho(n)}) \mid k_1, \cdots, k_n \in \mathbb{Z}, \rho \in S_n\}$$

is an apartment for Δ_ε. Note that each chamber may have more than one such description. The Weyl group W can now be identified with $(\mathbb{Z}^n \rtimes S_n)/Z$, where $Z = \langle(1,\ldots,1)\rangle$. The set of designated generators is $S = \{r_i\}_{i=0}^{n-1}$, where $\{r_i\}_{i=1}^{n-1}$ is the standard Coxeter system for S_n and where r_0 interchanges a_0 with $t^{-1}a_n$ and a_n with ta_0.

Note 3.2 The collection

$$\mathcal{A}(\mathrm{k}(t)) = \{\Sigma_\varepsilon\{a_1,\ldots,a_n\} \mid a_1,\ldots,a_n \text{ a k}(t)\text{-basis for } V\}$$

is a system of apartments for Δ_ε. However, the system of *all* apartments is obtained in the same manner after passing to the completion of $\mathrm{k}(t)$ with respect to v_ε as described in [Ro89].

In order to define the type of an object (and also in order to define the opposition relation) we shall need the following setup. Let $\mathsf{A} = \mathrm{k}[t,t^{-1}]$. Fix a *basis of reference* $\mathsf{B} = \{b_1,b_2,\ldots,b_n\}$ for V and let M be the A-module spanned by this basis.

The *type* of an $\mathcal{O}_\varepsilon$-lattice Λ (with respect to this basis of reference) is

$$\mathrm{typ}(\Lambda) = \varepsilon v_\varepsilon(\det(g)) \bmod n,$$

where $g \in \mathrm{GL}(V)$ is such that $g(\bigoplus_{i=1}^n \mathcal{O}_\varepsilon b_i) = \Lambda$. Clearly $\mathrm{typ}(t^k\Lambda) = \mathrm{typ}(\Lambda) + kn \bmod n$ for all $k \in \mathbb{Z}$ so that the type is well defined for minimal admissible chains. Given an arbitrary admissible chain F of lattices, we set $\mathrm{typ}(F) = \{\mathrm{typ}(\Lambda) \mid \Lambda \in F\}$.

After having defined the two buildings Δ_+ and Δ_-, we now define a twinning by describing the opposition relation. We call two chambers c_+ and c_- *opposite* if they are of the form $c_+(a_1,a_2,\ldots,a_n)$ and $c_-(a_1,a_2,\ldots,a_n)$ for some A-basis $\{a_1,a_2,\ldots,a_n\}$ of M.

Twin apartments are of the form $\Sigma\{a_1,\ldots,a_n\} = (\Sigma_+,\Sigma_-)$, where, for each $\varepsilon = \pm$, $\Sigma_\varepsilon = \Sigma_\varepsilon\{a_1,\ldots,a_n\}$, for some A-basis $\{a_1,\ldots,a_n\}$ for M. Twin roots are determined by a reflection in W. For example, consider the twin-roots (α_+,α_-) and $(-\alpha_+,-\alpha_-)$ associated with the reflection interchanging the 1-adjacent chambers $c_+(a_1,\ldots,a_n)$ and $c_+(a_2,a_1,a_3,\ldots,a_n)$. The chambers of α_+ are those of the form $c_+(t^{k_1}a_{\rho(1)},\ldots,t^{k_n}a_{\rho(n)})$, where $k_{\rho^{-1}(1)} > k_{\rho^{-1}(2)}$ or $k_{\rho^{-1}(1)} = k_{\rho^{-1}(2)}$ and $\rho^{-1}(1) < \rho^{-1}(2)$. Moreover, the chambers of α_+ having a panel on the wall $\delta\alpha_+$ are of type $c_+(t^{k_1}a_{\rho(1)},\ldots,t^{k_n}a_{\rho(n)})$, where $k_{\rho^{-1}(1)} = k_{\rho^{-1}(2)}$ and $\rho^{-1}(1)+1 = \rho^{-1}(2)$. The root $-\alpha_+$ is defined similarly, but with the roles of 1 and 2 interchanged. Then $\alpha_- = -\operatorname{opp}_\Sigma(\alpha_+)$ and $-\alpha_- = -\operatorname{opp}_\Sigma(-\alpha_+)$.

4 A flip for the twin-building of type $\widetilde{A}_{n-1}$ over $\mathrm{k}(t)$

In this section we first describe a flip φ for the twin-building of type $\widetilde{A}_{n-1}$ over $\mathrm{k}(t)$ and then show that φ is in fact geometric.

4.1 The σ-hermitian form β

In this paper, σ denotes an automorphism of $\mathrm{k}(t)$ of order 2 such that

(a) σ preserves k,

(b) σ induces an isomorphism $\mathcal{O}_\varepsilon \to \mathcal{O}_{-\varepsilon}$, for $\varepsilon = \pm$.

Lemma 4.1 *Let σ be as above. Then σ is given by*

$$\begin{aligned} b^{\sigma^2} &= b \qquad \forall b \in \mathrm{k} \\ t^{\sigma} &= \lambda t^{-1} \quad \text{for some } \lambda \in \mathrm{k}_{\sigma} \end{aligned}$$

where k_σ is the fixed field of σ. In particular, if the norm $N_\sigma\colon a \mapsto aa^\sigma$ is surjective, then we may take $\lambda = 1$.

Proof Since $\sigma^2 = \mathrm{id}$ and $\sigma(\mathrm{k}) = \mathrm{k}$, σ induces an automorphism of k of order 2. Now $v_-(t^\sigma) = 1$ so there exists some $n \in \mathbb{Z}_{\geq 0}$, $f(t) = \sum_{i=0}^n a_i t^i$ and $g(t) = \sum_{i=0}^n b_i t^i$ with $a_n \neq 0 \neq b_n$, and $\gcd(f(t), g(t)) = 1$ such that

$$t^\sigma = t^{-1}\frac{f(t)}{g(t)}.$$

Then,

$$t = t^{\sigma^2} = (t^{-1})^\sigma \left(\frac{f(t)}{g(t)}\right)^\sigma = (t^{-1})^\sigma \frac{f^\sigma(t^\sigma)}{g^\sigma(t^\sigma)} = t\frac{g(t)}{f(t)} \cdot \frac{f^\sigma(t^{-1}\frac{f(t)}{g(t)})}{g^\sigma(t^{-1}\frac{f(t)}{g(t)})}$$

Here we define $f^\sigma(t) = \sum_{i=0}^n a_i^\sigma t^i$ and likewise for g. That is,

$$f(t) \cdot g^\sigma\left(t^{-1}\frac{f(t)}{g(t)}\right) = g(t) \cdot f^\sigma\left(t^{-1}\frac{f(t)}{g(t)}\right)$$

Then, multiplying by $(tg(t))^n$ left and right, we get

$$f(t) \cdot \sum_{i=0}^n b_i^\sigma f(t)^i t^{n-i} g(t)^{n-i} = g(t) \cdot \sum_{i=0}^n a_i^\sigma f(t)^i t^{n-i} g(t)^{n-i}$$

Since $g(t)$ occurs in all terms on the left except $b_n^\sigma f(t)^{n+1} \neq 0$, and $\gcd(f(t), g(t)) = 1$, we find that $\deg(g(t)) = 0$. Hence also $\deg(f(t)) = 0$ and we find $t^\sigma = t^{-1}\lambda$ for some $\lambda \in \mathrm{k}$. Moreover, again since $t = t^{\sigma^2} = t\lambda^{-1}\lambda^\sigma$, we have $\lambda^\sigma = \lambda$ so that $\lambda \in \mathrm{k}_\sigma$. If $\lambda = aa^\sigma$, then $(a^{-1}t)^\sigma = t^{-1}a^{-\sigma}aa^\sigma = (a^{-1}t)^{-1}$. Thus, after replacing t by $a^{-1}t$ we have $t^\sigma = t^{-1}$. □

Note 4.2 From now on we shall assume that σ has the following properties.

(S) $N_\sigma\colon \mathrm{k} \to \mathrm{k}_\sigma$ is surjective and $t^\sigma = t^{-1}$.

(H) Moreover, in order to prove Lemma 4.11, we shall also assume that $\sigma|_{\mathrm{k}} \neq \mathrm{id}$.

Let $\beta\colon V \times V \to \mathrm{k}(t)$ be a σ-hermitian form. That is, it satisfies

$$\begin{aligned} \beta(\lambda u_1 + u_2, v) &= \lambda\beta(u_1, v) + \beta(u_2, v) \\ \beta(v, u) &= \beta(u, v)^\sigma \end{aligned}$$

for all $\lambda \in \mathrm{k}(t)$ and $u_1, u_2, v \in V$. Moreover, it is *non-degenerate* in the sense that if $\beta(u, v) = 0$ for all $u \in V$, then $v = 0$.

Lemma 4.3 *If σ satisfies (S) and (H), then there exists a $\mathrm{k}(t)$-basis for V that is orthonormal with respect to β.*

4.2 The flip φ induced by β

Define a non-degenerate σ-hermitian form β where σ satisfies (S) and (H), such that the basis of reference for M is orthonormal with respect to β.

Define $\varphi\colon \Delta \to \Delta$ by setting

$$\Lambda^{\varphi} = \{v \in V \mid \beta(u,v) \in \mathcal{O}_{\varepsilon}\ \forall u \in \Lambda\} \tag{4.1}$$

for each $\mathcal{O}_{\varepsilon}$-lattice Λ.

Given an $\mathrm{k}(t)$-basis $C = \{c_1, \ldots, c_n\}$ for V, then, since β is non-degenerate there exists a basis $C^* = \{c_1^*, \ldots, c_n^*\}$ such that

$$\beta(c_i, c_j^*) = \delta_{ij} \text{ for } 0 \leq i,j \leq n.$$

We call C^* the basis *dual to C with respect to β*.

Lemma 4.4 *Let β be a σ-hermitian form and let φ be the associated map. Let C be a basis for V and let C^* be its dual basis with respect to β. Then,*

(a) $\langle C\rangle^{\varphi}_{\mathcal{O}_{\varepsilon}} = \langle C^*\rangle_{\mathcal{O}_{-\varepsilon}}$.

(b) *In particular, φ sends the apartment $\Sigma_{\epsilon}(C)$ isometrically to $\Sigma_{-\epsilon}(C^*)$.*

Proof Without loss of generality assume $\varepsilon = +$. Let Λ have $\mathcal{O}_+$-basis $C = \{c_1, \ldots, c_n\}$ and let Λ^* have $\mathcal{O}_-$-basis $C^* = \{c_1^*, \ldots, c_n^*\}$. Then, C^* is an $\mathrm{k}(t)$-basis for V. So let $v = \sum_{i=1}^n a_i c_i^*$. Then $v \in \Lambda^{\varphi}$ if and only if $\beta(c_i, v) = a_i^{\sigma} \in \mathcal{O}_+$ if and only if $a_i \in \mathcal{O}_-$. This shows that $\Lambda^{\varphi} = \Lambda^*$, and (a) is proved.

Since for any $k \in \mathbb{Z}$ and $u, v \in V$ we have $\beta(t^k u, t^k v) = \beta(u,v)$, it follows that

$$\{t^{k_1}c_1, \ldots, t^{k_n}c_n\}^* = \{t^{k_1}c_1^*, \ldots, t^{k_n}c_n^*\}.$$

Hence, φ sends admissible chains to admissible chains, it preserves types, and since it reverses containment, it also preserves i-adjacency. This proves (b). □

Corollary 4.5 *φ induces a flip on Δ.*

Proof By Lemma 4.4 φ is a type-preserving map between Δ_{ε} and $\Delta_{-\varepsilon}$. Also, it sends apartments of Δ_{ε} to apartments of $\Delta_{-\varepsilon}$ so that φ induces a distance preserving map $\Delta_{\varepsilon} \to \Delta_{-\varepsilon}$. We now show that φ also preserves twin-apartments. Let C be an A-basis for M and let C^* be its dual basis with respect to β. Then C is obtained from B by some A-linear isomorphism of M and C^* is obtained from B by the adjoint of this isomorphism. Since σ preserves A and in particular its units, the adjoint operator preserves $\mathrm{GL}(\mathsf{M})$. Therefore the twin-apartment $\Sigma\{C\}$ is sent to the twin-apartment $\Sigma\{C^*\}$. Since φ preserves the opposition relation, it also preserves the codistance function δ_*.

Finally, since the basis $\mathsf{B} = \{b_1, \ldots, b_n\}$ of reference for M is self-dual with respect to β, φ interchanges the opposite chambers $c_+(b_1, \ldots, b_n)$ and $c_-(b_1, \ldots, b_n)$. □

4.3 The flip φ is geometric

The last part of this section is devoted to proving that the flip defined above is in fact geometric in the sense of Lemma 2.9. Recall that this means that for any panel $\pi_\varepsilon \in \Delta_\varepsilon$ the maps $\varphi\colon \pi_\varepsilon \to \pi_\varepsilon^\varphi$ and $\mathrm{proj}_{\pi_\varepsilon}\colon \pi_\varepsilon^\varphi \to \pi_\varepsilon$ are not each other's inverse. Clearly, if the latter is not bijective, we are done. Therefore we consider the following slightly more general situation. Here $\varepsilon \in \{+,-\}$.

(A1) π_ε is a panel on a chamber c_ε of Δ_ε,

(A2) $\Sigma = (\Sigma_+, \Sigma_-)$ is a twin-apartment with $c_\varepsilon \in \Sigma_\varepsilon$,

(A3) $\pi_\varepsilon \cap \Sigma_\varepsilon = \{c_\varepsilon, d_\varepsilon\}$, for some chamber d_ε,

(PI) The projections $\mathrm{proj}_{\pi_\varepsilon}\colon \pi_{-\varepsilon} \to \pi_\varepsilon$ are each other's inverse. We assume that $c_\varepsilon = \mathrm{proj}_{\pi_\varepsilon}(c_{-\varepsilon})$ and $d_\varepsilon = \mathrm{proj}_{\pi_\varepsilon}(d_{-\varepsilon})$.

Note that (A2) can be realized by [AbRo98, Lemma 1] due to Tits. Once we assume the first part of (PI), the second part follows since twin-apartments are coconvex (Lemma 3 of loc. cit.).

Lemma 4.6 *Let $\pm(\alpha_\varepsilon, \alpha_{-\varepsilon})$ be the twin-roots of Σ determined by $\pi_{-\varepsilon}$ so that $\Sigma_\varepsilon = \alpha_\varepsilon \uplus -\alpha_\varepsilon$ and $\Sigma_{-\varepsilon} = \alpha_{-\varepsilon} \uplus -\alpha_{-\varepsilon}$. Suppose $c_\varepsilon \in \alpha_\varepsilon$. Then, $c_{-\varepsilon} = \pi_{-\varepsilon} \cap \alpha_{-\varepsilon}$.*

Proof By [AbRo98, Lemma 3], twin-roots are coconvex. Therefore if (α_+, α_-) is the twin-root containing c_ε and meeting $\pi_{-\varepsilon}$ in a chamber $c_{-\varepsilon}$, then by coconvexity the chamber $\mathrm{proj}_{\pi_{-\varepsilon}}(c_\varepsilon)$ belongs to $\alpha_{-\varepsilon} \cap \pi_{-\varepsilon}$. This chamber is $c_{-\varepsilon}$. □

Lemma 4.7 *The panels π_ε and $\pi_{-\varepsilon}$ determine the same twin-roots of Σ.*

Proof Let $\pm(\alpha_+, \alpha_-)$ be the twin-roots of Σ determined by π_+ such that $c_+ \in \alpha_+$. Now $c_- = \pi_- \cap \alpha_-$ and $d_- = \pi_- \cap -\alpha_-$ by Lemma 4.6. Therefore the panel π_- is on the wall of the twin-root (α_+, α_-). □

Lemma 4.8 *Let $\alpha = (\alpha_+, \alpha_-)$ be a twin-root of Δ and let π_ε be a panel intersecting the root in a chamber c_ε. Then the projection map $\mathrm{proj}_{\pi_\varepsilon}\colon \pi_{-\varepsilon} \to \pi_\varepsilon$ commutes with the action of U_α, the root group of α.*

Proof Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin-apartment on α. Let $-\alpha = (-\alpha_+, -\alpha_-)$ be the opposite twin-root (so that $\alpha \uplus -\alpha = \Sigma$). Then by Lemma 4.6, $c_- = \mathrm{proj}_{\pi_-}(c_+) \in \alpha_- \subseteq \Sigma_-$.

Let U_α be the root group of α and let the isomorphism $(\mathrm{k}, +) \to U_\alpha$ be given by $\lambda \mapsto U_\alpha(\lambda)$. Since U_α is regular on the collection of twin-apartments containing α we can index these apartments as $\Sigma(\lambda) = \alpha \uplus -\alpha^{U_\alpha(\lambda)}$. For each $\lambda \in \mathrm{k}$, let $d_\varepsilon(\lambda) \in \Delta_\varepsilon$ be such that $\{c_\varepsilon, d_\varepsilon(\lambda)\} = \pi_\varepsilon \cap \Sigma_\varepsilon(\lambda)$. Then, by Lemma 4.6 we have $\mathrm{proj}_{\pi_\varepsilon}(d_{-\varepsilon}(\lambda)) = d_\varepsilon(\lambda) \in -\alpha_-(\lambda) \subseteq \Sigma_-(\lambda)$. We summarize this by saying that the maps $\mathrm{proj}_{\pi_+} \circ U_\alpha(\lambda)$ and $U_\alpha(\lambda) \circ \mathrm{proj}_{\pi_+}$ agree on $\pi_- - \{c_-\} \to \pi_+ - \{c_+\}$. □

We shall now describe a way to parametrize the chambers in a panel by the 1-spaces of a 2-dimensional vector space over k. Suppose $A = \{a_1, \dots, a_n\}$ is an A-basis for M and let $\Sigma = \Sigma\{a_1, \dots, a_n\}$. Let c_+ be a chamber of Σ. We may use an element from W to justify the assumption that $c_+ = c_+(a_1, \dots, a_n)$. We consider the various panels on c_+. If π_+ is an i-panel with $1 \leq i \leq n-1$, then the chambers of $\pi_+ - \{c_+\}$ are given by

$$c_+(a_1, \dots, a_{i-1}, a_{i+1}, a_i + \lambda a_{i+1}, a_{i+2}, \dots, a_n), \quad \lambda \in \mathrm{k}.$$

Using this description and the definition of root groups one verifies that $U_\alpha \leq \mathrm{GL}(V)$ acts as

$$\begin{aligned} U_\alpha(\lambda)\colon V &\to V \\ a_j &\mapsto a_j \qquad\qquad \text{if } j \neq i \\ a_i &\mapsto a_i + \lambda a_{i+1} \end{aligned}$$

If π_+ is the 0-panel on c_+, then the chambers of $\pi_+ - \{c_+\}$ are given by

$$c_+(t^{-1}a_n + \lambda a_1, a_2, \dots, a_{n-1}, ta_1), \quad \lambda \in \mathrm{k}.$$

In this case U_α acts as

$$\begin{aligned} U_\alpha(\lambda)\colon V &\to V \\ a_j &\mapsto a_j \qquad\qquad \text{if } j \neq n \\ a_n &\mapsto a_n + \lambda t a_1 \end{aligned}$$

A reasoning similar to the one for c_+ yields a description for the panel π_- on c_-. We point out though that in determining the chambers of the panel π_- one should take the reversal map into account (see Ronan [Ro03]).

From now, in addition to assumptions (A1-A3, PI), we add the following assumption:

(A4) π_+ has type 1,

(F1) $\pi_- = \pi_+^\varphi$.

Let

$$\begin{aligned} c_+ &= c_+(a_1, \dots, a_n) \\ c_- &= c_-(t^{k_1}a_{i_1}, \dots, t^{k_n}a_{i_n}). \end{aligned}$$

In order to ensure that $\langle t^{k_1}a_{i_1}, \dots, t^{k_n}a_{i_n}\rangle_{\mathcal{O}_-}$ has type 0, we suppose that $k_1, \dots, k_n \in \mathbb{Z}$ are such that $\sum_{i=1}^n k_i = 0$. Also assume $\{i_1, \dots, i_n\} = \{1, \dots, n\}$. Since π_- is on the wall determined by the panel π_+, the description of these panels at the end of Section 3 tells us that $i_1 = 2$ and $i_2 = 1$ and $k_1 = k_2$.

Let P be an k-vector space with basis p_1 and p_2. We define a map $\psi_+\colon \pi_+ \to \mathrm{PG}(P)$ as follows.

$$\begin{aligned} \psi_+\colon \pi_+ &\to \mathrm{PG}(P) \\ c_+(a_2, a_1 + \lambda a_2, a_3, \dots, a_n) &\mapsto \langle p_1 + \lambda p_2\rangle \text{ for all } \lambda \in \mathrm{k} \\ c_+(a_1, \dots, a_n) &\mapsto \langle p_2\rangle. \end{aligned}$$

We similarly define a map $\psi_-\colon \pi_- \to \mathrm{PG}(P)$ as follows.

$$\begin{aligned} \psi_-\colon \pi_- &\to \mathrm{PG}(P) \\ c_-(t^{k_1}(a_1+\lambda a_2), t^{k_1}a_2, t^{k_3}a_3, \ldots, t^{k_n}a_n) &\mapsto \langle p_1 + \lambda p_2\rangle \text{ for all } \lambda \in \mathrm{k} \\ c_-(t^{k_1}a_2, t^{k_1}a_1 \ldots, t^{k_n}a_n) &\mapsto \langle p_2\rangle. \end{aligned}$$

We call ψ_+ a *projective* identification for π_+ with respect to the ordered basis $\{a_1, \ldots, a_n\}$.

We now have the following description of the projection map.

Lemma 4.9 *With respect to the projective identifications ψ_+ and ψ_-, the projection map $\mathrm{proj}_{\pi_\varepsilon}\colon \pi_{-\varepsilon} \to \pi_\varepsilon$ is induced by the identity map on P.*

Proof This follows immediately from Lemma 4.8. □

Lemma 4.10 *Suppose π is a panel in Δ_+ containing chambers c and d. Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be ordered bases such that*

$$\begin{aligned} c &= c_+(x_1, \ldots, x_n) &&= c_+(y_1, \ldots, y_n) \\ d &= c_+(x_2, x_1, x_3, \ldots, x_n) &&= c_+(y_2, y_1, y_3, \ldots, y_n). \end{aligned}$$

Then, under the projective identifications ψ_X and ψ_Y for π with respect to X and Y, the identity map on π is induced by a k*-linear map on P.*

Proof Without loss of generality we may assume that π has type 1. Since π has type 1, the type i-elements on c and d are the same for all $i \neq 1$. Therefore we can view π as a panel in the residue of the 0-object $\Lambda_0 = \langle x_1, \ldots, x_n\rangle_+ = \langle y_1, \ldots, y_n\rangle_+$ that is, in the k-vector space $\Lambda_0/t\Lambda_0$. Since all the subspaces spanned by subsets of $\{x_1, x_2\}$ are equal to the subspaces spanned by the corresponding subsets of $\{y_1, y_2\}$ we find that modulo $t\Lambda_0$ we have $y_1 = \alpha_1 x_1$ and $y_2 = \alpha_2 x_2$ for some non-zero $\alpha_1, \alpha_2 \in \mathrm{k}$. Moreover, the identity map on the remaining chambers of π is now given by sending $y_1 + \lambda y_2 \mapsto \alpha_1 x_1 + \lambda\alpha_2 x_2$. So, if ψ_Y and ψ_X are the identification maps with respect to Y and X, then

$$\begin{aligned} \psi_X \circ \mathrm{id}_\pi \circ \psi_Y^{-1}\colon\ \mathrm{PG}(P) &\to \mathrm{PG}(P) \\ \langle p_1 + \lambda p_2\rangle &\mapsto \langle p_1 + \lambda \tfrac{\alpha_2}{\alpha_1} p_2\rangle \\ \langle p_2\rangle &\mapsto \langle \tfrac{\alpha_2}{\alpha_1} p_2\rangle \end{aligned}$$

This proves the claim. □

Lemma 4.11 *The flip φ is geometric.*

Proof If $\mathrm{proj}_\pi(\pi^\varphi)$ is a single chamber of π there is nothing to prove.

Let $\pi_+ \subseteq \Delta_+$ be a panel such that $\mathrm{proj}_{\pi_+} \circ \varphi\colon \pi_+ \to \pi_+$ is the identity. Without loss of generality we may assume that π_ε has type 1. Let $\pi_- = \pi_+^\varphi$ and let $c_\varepsilon, d_\varepsilon \in \pi_\varepsilon$ be such that $c_\varepsilon^\varphi = c_{-\varepsilon}$ and $d_\varepsilon^\varphi = d_{-\varepsilon}$. Then, by the discussion preceding Lemma 4.9, we have

$$\begin{aligned} c_+ &= c_+(a_1, \ldots, a_n) \\ d_+ &= c_+(a_2, a_1, a_3, \ldots, a_n) \\ c_- &= c_-(t^k a_2, t^k a_1, t^{k_3} a_{i_3}, \ldots, t^{k_n} a_{i_n}) \\ d_- &= c_-(t^k a_1, t^k a_2, t^{k_3} a_{i_3}, \ldots, t^{k_n} a_{i_n}) \end{aligned}$$

Let $A^* = \{a_1^*, \dots, a_n^*\}$ be the dual basis of $A = \{a_1, \dots, a_n\}$ so that for each $\lambda \in \mathrm{k}$ the dual basis for $\{a_2, a_1 + \lambda a_2, a_3, \dots, a_n\}$ is $\{a_2^* - \lambda^\sigma a_1^*, a_1^*, a_3^*, \dots, a_n^*\}$. Moreover, for each $\lambda \in \mathrm{k}$ we have

$$c_+(a_2, a_1 + \lambda a_2, a_3, \dots, a_n)^\varphi = c_-(a_2^* - \lambda^\sigma a_1^*, a_1^*, a_3^*, \dots, a_n^*).$$

At the same time, since $\mathrm{proj}_{\pi_+} \circ \varphi$ is the identity, we have

$$c_-(a_2^* - \lambda^\sigma a_1^*, a_1^*, a_3^*, \dots, a_n^*) = c_-(t^k(a_1 + \lambda a_2), t^k a_2, t^{k_3} a_{i_3}, \dots, t^{k_n} a_{i_n}).$$

Let ψ_ε be the projective identification for π_ε with respect to A and ψ_* be the projective identification for π_- with respect to A^*. Then by Lemmas 4.10 and 4.9, the composition $\psi_* \circ \mathrm{proj}_{\pi_-} \circ \psi_+^{-1} = (\psi_* \circ \psi_-^{-1}) \circ (\psi_- \circ \mathrm{proj}_{\pi_-} \circ \psi_+^{-1})$ is linear. On the other hand, by assumption, this composition is equal to $\psi_* \circ \varphi \circ \psi_+^{-1}$. By examining the correspondence

$$c_+(a_2, a_1 + \lambda a_2, a_3, \dots, a_n)^\varphi = c_-(a_2^* - \lambda^\sigma a_1^*, a_1^*, a_3^*, \dots, a_n^*).$$

we find that on P this map is given by

$$\begin{aligned} \psi_* \circ \varphi \circ \psi_+^{-1} \colon \quad & P && \to P \\ & p_1 + \lambda p_2 && \mapsto p_1 - \lambda^\sigma p_2 \\ & p_2 && \mapsto p_2. \end{aligned}$$

Since $\sigma \neq \mathrm{id}|_{\mathrm{k}}$, this map is not linear, and hence is different from $\psi_* \circ \mathrm{proj}_{\pi_-} \circ \psi_+^{-1}$. This contradicts the fact that $\mathrm{proj}_{\pi_+} \circ \varphi$ is the identity. □

5 The flip-flop geometry Γ_φ and β-orthogonal A-bases

Let σ and β be as in Section 4. As proved in Corollary 2.10 the set $\mathcal{F}_\varphi$ is the flag complex of Γ_φ. In this section we shall describe a way to recognize elements of $\mathcal{F}_\varphi$ among all residues in Δ_+.

Proposition 5.1 *Let $a_1, \dots, a_n$ be an A-basis for M. If φ interchanges the opposite chambers $c_+(a_1, \dots, a_n)$ and $c_-(a_1, \dots, a_n)$, then $\{a_1, \dots, a_n\}$ is orthogonal with respect to β.*

Proof Let $A^* = \{a_1^*, \dots, a_n^*\}$ be the dual basis of $A = \{a_1, \dots, a_n\}$ with respect to β. Then, $c_{-\varepsilon} = c_\varepsilon^\varphi = c_{-\varepsilon}(a_1^*, \dots, a_n^*)$ for $\varepsilon = \pm$. Now let Σ be the unique twin-apartment containing c_- and c_+. Then, $\Sigma = \Sigma\{a_1, \dots, a_n\}$. We now claim that $\Sigma = \Sigma\{a_1^*, \dots, a_n^*\}$. Let $T \in \mathrm{GL}_n(\mathsf{A})$ be the transition matrix from B to A. Then, the transition matrix from B to A^* is the adjoint T^* of T with respect to β, that is $T^* = T^{t\sigma} \in \mathrm{GL}_n(\mathsf{A})$. Therefore $\Sigma\{a_1^*, \dots, a_n^*\}$ is a twin-apartment on c_+ and c_-. Since such apartments are unique, our claim is proved. The transition from A to A^* is given by the matrix $D = T^* T^{-1}$. Since D preserves Σ, it must be monomial (D is in the group N of the twin BN-pair with respect to Σ, see e.g. [Ab96]). We now show that D is in fact diagonal. Let ρ be the permutation such that $a_i^* = \lambda_i t^{k_i} a_{\rho(i)}$ with $\lambda_i \in \mathrm{k}^*$ and $k_i \in \mathbb{Z}$ for all $i = 1, 2, \dots, n$. Then apparently we have

$$c_-(a_1, \dots, a_n) = c_+(a_1, \dots, a_n)^\varphi = c_-(t^{k_1} a_{\rho(1)}, \dots, t^{k_n} a_{\rho(n)})$$

and since W acts regularly on the apartment, it follows that $\rho = \mathrm{id}$ and $k_1 = \dots = k_n$. □

Lemma 5.2 *Any orthogonal* A*-basis for* M *with respect to* β *has a diagonal Gram matrix with coefficients in* k_σ*.*

Proof Let $A = \{a_1, \ldots, a_n\}$ be an A-basis for M that is orthogonal with respect to β and let $T = (t_{ij} = \beta(a_i, a_j))$ be its Gram matrix. Clearly T is diagonal and $t_{ii} \in \mathsf{A}_\sigma$ for all i.

Recall that the A-basis of reference B for M is orthonormal with respect to β. Let $g \in \mathrm{GL}_n(\mathsf{A})$ take B to A. Then, $T = (t_{ij}) = g^t g^\sigma$. Since the determinant of g is a unit $\lambda t^k \in \mathsf{A}^*$ for some $\lambda \in \mathrm{k}$ and $k \in \mathbb{Z}$, we find that $\det(T) = \lambda\lambda^\sigma \in \mathrm{k}_\sigma$.

On the other hand, we know that T is diagonal with coefficients in A_σ, which means that each t_{ii} is of the form $\sum_{j=0}^m \alpha_j t^j + \alpha_j^\sigma t^{-j}$. Therefore, $v_\varepsilon(t_{ii}) \leq 0$. Moreover, since, $v_\varepsilon(\det(T)) = \sum_{i=1}^n v_\varepsilon(t_{ii}) = 0$ we find that $v_\varepsilon(t_{ii}) = 0$ for all i. It now follows that $t_{ii} \in \mathrm{k}_\sigma$ for all i. □

Corollary 5.3 *Let* φ *be given by the* σ*-sesquilinear form* β*. Then* Δ_φ *consists of precisely those pairs of chambers* $c_\varepsilon(a_1, \ldots, a_n)$ *such that* $\{a_1, \ldots, a_n\}$ *is an* A*-basis for* M *that is orthonormal with respect to* β*.*

Proof Let $c \in \Delta_\varepsilon$ and $c^\varphi \in \Delta_{-\varepsilon}$ be opposite. By Lemma 5.1 there is an orthogonal basis $\{a_1, \ldots, a_n\}$ such that $c = c_\varepsilon(a_1, \ldots, a_n)$. By Lemma 5.2 we know that $\beta(a_i, a_i) \in \mathrm{k}_\sigma$. By our assumption on σ the norm $N_\sigma\colon \mathrm{k} \to \mathrm{k}_\sigma$ is surjective. So replacing a_i by $\lambda_i a_i$, where $\lambda_i \lambda_i^\sigma = t_{ii}^{-1}$ we find an orthonormal basis inducing the same chambers. As we saw in Corollary 4.5, if $\{a_1, \ldots, a_n\}$ is an A-basis for M that is orthonormal with respect to β, then the pair of opposite chambers $c_\varepsilon(a_1, \ldots, a_n)$ are interchanged by φ. □

Corollary 5.4 *Let* φ *be given by the* σ*-sesquilinear form* β*. Then a pair of opposite objects* $([\Lambda_+], [\Lambda_-])$ *of* Γ *belongs to* Γ_φ *precisely if any* A*-basis* $\{c_1, \ldots, c_n\}$ *for* M *such that* $\Lambda_\varepsilon = \langle c_1, \ldots, c_n \rangle_{\mathcal{O}_\varepsilon}$ *for* $\varepsilon = +, -$*, has a Gram matrix with coefficients in* k*.*

Proof By definition, $([\Lambda_+], [\Lambda_-])$ belongs to Γ_φ if and only if there are chambers (c_+, c_-) interchanged by φ with $c_\varepsilon \in [\Lambda_\varepsilon]$. Therefore by Corollary 5.3 this happens precisely if there is an A-basis $A = \{a_1, \ldots, a_n\}$ for M that is orthonormal with respect to β such that $c_\varepsilon = c_\varepsilon(a_1, \ldots, a_n)$. In particular, Λ_ε is spanned by an orthonormal basis, which we may assume is $\{a_1, \ldots, a_n\}$, after multiplying some of the a_i by a power of t.

Suppose that $C = \{c_1, \ldots, c_n\}$ is an A-basis for M such that for $\varepsilon = +, -$, $\Lambda_\varepsilon = \langle c_1, \ldots, c_n \rangle_{\mathcal{O}_\varepsilon}$. Then the transition matrix T from A to C has coefficients in $\mathrm{k} = \mathsf{A} \cap \mathcal{O}^+ \cap \mathcal{O}_-$. Hence, the Gram matrix for C is $T^{t\sigma} T$, which has coefficients in k.

Conversely, suppose $\Lambda_\varepsilon = \langle c_1, \ldots, c_n \rangle_{\mathcal{O}_\varepsilon}$ for $\varepsilon = +, -$, such that $C = \{c_1, \ldots, c_n\}$ is an A-basis for M that has a Gram matrix with coefficients in k. Then consider the k-vector space spanned by this basis. Since the norm of $N_\sigma\colon \mathrm{k} \to \mathrm{k}^\sigma$ is surjective we can apply Gram-Schmidt to find an orthonormal basis $A = \{a_1, \ldots, a_n\}$. Thus the transition matrix from C to A belongs to $\mathrm{GL}_n(\mathrm{k})$ and therefore A is an $\mathcal{O}_\varepsilon$-basis for Λ_ε for $\varepsilon = +, -$. By Corollary 5.3 this means that $([\Lambda_+], [\Lambda_-]) \in \Gamma_\varphi$. □

6 Residual geometries of Γ_φ and the geometry $\mathcal{N}$

In this section we shall describe the residues of Γ_φ in terms of β.

Lemma 6.1 *Let Δ be a twin-building and let φ be a flip. Suppose φ interchanges the pair (c_+, c_-) of opposite chambers. Let R^+ be a spherical residue on c_+ and let $R_- = R_+^\varphi$. Then,*

(a) *$R = (R_+, R_-)$ together with the induced (co-) distances forms a spherical twin-building,*

(b) *$\varphi|_R$ is a flip of R.*

(c) *Identifying R_+ and R_- via the isomorphism induced by projection, φ is a flip of the spherical building R_+.*

Proof (a) Since R_+ and R_- are opposite, they have the same type and are therefore isomorphic (spherical) buildings. Moreover, the distance and codistance functions restricted to R satisfy the axioms of a twinning. [Ro00] (b) The restriction $\varphi|_R \colon R_+ \to R_+^\varphi = R_-$ clearly preserves distances and codistances and interchanges the halves R_+ and R_- of the twin-building R. Also, R contains a chamber that is interchanged with its opposite by φ.

(c) The map $\mathrm{proj}\colon R_- \to R_+$ given by $x \mapsto \mathrm{proj}_{R_+}(x)$, where $\mathrm{proj}_{R_+}(x)$ is the unique chamber at maximal codistance from x in R_+ is an isomorphism ([Ro00, Prop 4.3]). Moreover, $y \in R_-$ is opposite $x \in R_+$ in Δ if and only if $\mathrm{proj}_{R_+}(y)$ is opposite x in R_+. Thus $\mathrm{proj}_{R_+} \circ \varphi$ induces a flip of the spherical building R_+. Note that a flip of a spherical building sends types to opposite types, whereas a flip of a twin-building preserves types. □

Returning to the flip φ induced by a σ-hermitian form β, we describe the flip induced on each residue.

Note 6.2 Note that the residue field $\mathcal{O}_\varepsilon/(t^\varepsilon)$ is k.

Lemma 6.3 *Let the flip φ of Δ be induced by the σ-hermitian form β on V and let φ interchange the pair $R = (R_+, R_-)$ of opposite objects of Δ. If $R_+ = [\Lambda_+] \in \Delta_+$ and $R_- = [\Lambda_-]$, where $\Lambda_- = \Lambda_+^\varphi$, then*

(a) $\begin{array}{ll} i_+\colon R_+ & \to A_{n-1}(\Lambda_+/t\Lambda_+) \\ \Lambda' & \mapsto \Lambda'/t\Lambda_+ \end{array}$ *and* $\begin{array}{ll} i_-\colon R_- & \to A_{n-1}(t\Lambda_-/\Lambda_-) \\ \Lambda'' & \mapsto \Lambda''/\Lambda_- \end{array}$
where $t\Lambda_+ \leq \Lambda' \leq \Lambda_+$ and $\Lambda_- \leq \Lambda'' \leq t\Lambda_-$, are isomorphisms. Note that these are A_{n-1}-buildings over k.

(b) *Let $\{c_1, \ldots, c_n\}$ be an* A*-basis for* M *so that $\Lambda_\varepsilon = \langle c_1, \ldots, c_n\rangle_{\mathcal{O}_\varepsilon}$. Then, the projection map $\mathrm{proj}_{R_-}\colon R_+ \to R_-$ is given by the* k*-linear map*

$$\begin{array}{rccc} \mathrm{proj}_{R_-}\colon & R_+ & \to & R_- \\ & \sum_{i=1}^n \gamma_i(t) c_i & \mapsto & \sum_{i=1}^n \gamma_i(t^{-1}) t c_i \end{array}$$

(c)

$$\begin{array}{rcl}\overline{\beta}\ :\Lambda_+/t\Lambda_+\times\Lambda_+/t\Lambda_+ & \to & \mathrm{k}\\ (u,v) & \mapsto & \beta(u,t^{-1}\operatorname{proj}_{R_-}(v))+(t)\end{array}$$

is a reflexive $\overline{\sigma}$*-sesquilinear form, where* $\overline{\sigma}=\sigma|_{\mathrm{k}}$.

(d) *The flip of the spherical building* R_+ *induced by* φ *is given by the polarity induced by* $\overline{\beta}$ *on* $\Lambda_+/t\Lambda_+$.

Proof (a) Set $W=\Lambda_+/t\Lambda_+$. By the lattice-isomorphism theorem for the $\mathcal{O}_+$-module chains $t\Lambda_+\le\Lambda'\le\Lambda_+$ the map i_+ is injective and preserves containment/incidence. It remains to show that under this correspondence the lattice Λ' is free and of maximal rank if and only if $\Lambda'/t\Lambda_+$ is a k-linear subspace of W. Let $U\le W$ be a linear subspace and let $c_1,\dots,c_n\in\Lambda$ be such that their images in W form a basis for W and $c_1,\dots,c_i$ forms a basis for U. By Nakayama's lemma, the c_i's are an $\mathcal{O}_+$-basis for Λ_+. Set $\Lambda'=\langle c_1,\dots,c_i,tc_{i+1},\dots,tc_n\rangle_{\mathcal{O}_+}$ so that $i_+(\Lambda')=U$.

(b) Let $e_1,\dots,e_n$ be an arbitrary A-basis for M such that $\Lambda_\varepsilon=\langle e_1,\dots,e_n\rangle_{\mathcal{O}_\varepsilon}$. Consider the apartment $\Sigma\{e_1,\dots,e_n\}$. Then we see the projection of $c_+(e_1,\dots,e_n)\in R_+$ as the unique chamber of R_- opposite to $c_-(e_1,\dots,e_n)$ inside $R_-\cap\Sigma_-$. This is the chamber $c_-(e_n,\dots,e_1)$.

Since both $\{c_1,\dots,c_n\}$ and $\{e_1,\dots,e_n\}$ are A-bases for M contained in $\Lambda_+\cap\Lambda_-$, the transition matrix $A=(\alpha_{ij})$ with $e_j=\sum_{i=1}^n\alpha_{ij}c_i$ has $\alpha_{ij}\in\mathrm{k}$. Let π be the map given by $\pi(\sum_{i=1}^n\gamma_i(t)c_i)=\sum_{i=1}^n\gamma_i(t^{-1})tc_i$. Then $\pi(e_j)=\pi(\sum_{i=1}^n\alpha_{ij}c_i)=\sum_{i=1}^n\alpha_{ij}tc_i=te_j$. The conclusion follows once we prove that π is well-defined.

Suppose $\sum_{i=1}^n t\gamma_i(t)c_i\in t\Lambda_+$. Then $\pi(\sum_{i=1}^n t\gamma_i(t)c_i)=\sum_{i=1}^n t^{-1}\gamma_i(t^{-1})tc_i\in\Lambda_-$. This shows that π is well-defined between the quotients $\Lambda_+/t\Lambda_+$ and $t\Lambda_-/\Lambda_-$.

(c) First note that since $\Lambda_-=\Lambda_+^\varphi$, we have $\beta(u,v')\in\mathcal{O}_+$ for all $u\in\Lambda_+$ and $v'\in\Lambda_-$. Also, if either $u\in t\Lambda_+$ or $v'\in t^{-1}\Lambda_-$, then $\beta(u,v')\in(t)$. Since $\operatorname{proj}_{R_-}(v)\in t\Lambda_-$, $\beta(u,t^{-1}\operatorname{proj}_{R_-}(v))\in\mathcal{O}_+$ for any $u,v\in\Lambda_+$. Since $\operatorname{proj}_{R_-}$ is k-linear and sends $t\Lambda_+$ to Λ_-, $\overline{\beta}$ is well-defined. Clearly, $\overline{\beta}$ is $\overline{\sigma}$-sesquilinear.

(d)

Let Λ' be such that $t\Lambda_+\le\Lambda'\le\Lambda_+$. Now

$$\begin{array}{rl}v\in\Lambda'^\varphi & \iff \beta(u,v)\in\mathcal{O}_+\ \forall u\in\Lambda'\\ & \iff \beta(u,\operatorname{proj}(\operatorname{proj}(v)))\in\mathcal{O}_+\ \forall u\in\Lambda'\\ & \iff \beta(u,t^{-1}\operatorname{proj}(\operatorname{proj}(v)))\in t\mathcal{O}_+\ \forall u\in\Lambda'\\ & \iff \overline{\beta}(u,\operatorname{proj}(v))=0\ \forall u\in\Lambda'\\ & \iff \overline{\beta}(u,\operatorname{proj}(v))=0\ \forall u\in\Lambda'/t\Lambda_+\\ & \iff \operatorname{proj}(v)\in(\Lambda'/t\Lambda_+)^\perp.\end{array}$$

Hence, $\operatorname{proj}_{R_+}(\Lambda'^\varphi)=(\Lambda'/t\Lambda_+)^\perp$, where $\perp$ is taken with respect to $\overline{\beta}$. □

Corollary 6.4 *Let* $\overline{\varphi}$ *be the flip induced on the residue of an object* $\Lambda_+\in\Gamma_\varphi$. *Then the residue of* Λ_+ *in* Γ_φ *is the flip-flop geometry of* $\overline{\varphi}$ *induced on* $\Lambda_+/t\Lambda_+$.

The pre-geometry in Corollary 6.4 is isomorphic to the pre-geometry $\mathcal{N}_n$ of k-subspaces of an n-dimensional k-vector space V that are non-degenerate with respect to the non-degenerate $\overline{\sigma}$-hermitian form $\overline{\beta}$, where incidence is given by inclusion.

The next result follows from Bennett and Shpectorov [BeSh04] and Lemma 8.3.

Lemma 6.5 *Let R_φ be an object of the pre-geometry Γ_φ. First assume that $\mathrm{k} = \mathbb{F}_{q^2}$ for some prime power q. Then, R_φ is*

(a) *connected if $(n,q) \neq (3,2)$, and*

(b) *residually connected if $q \neq 2$.*

Moreover, if $|\mathrm{k}| \geq 4$ (in particular if it is infinite), then R_φ is connected and simply connected.

7 (Simple) connectedness of Γ_φ

In this section we prove that the geometry Γ_φ and all of its residues of rank at least 3 are usually (simply) connected.

Example 7.1 For $n = 2$, the geometry Γ_φ is not always connected. Note that Γ is a tree, so that Γ_φ is a forrest. We show that there exists an apartment Σ_+ of Δ_+ and three objects $[\Lambda_0]$, $[\Lambda_1]$, and $[\Lambda_2]$ on Σ_+ such that $[\Lambda_0], [\Lambda_2] \in \Gamma_\varphi$ are both incident to $[\Lambda_1] \notin \Gamma_\varphi$.

Let V be a vector space of dimension 2 over $\mathrm{k}(t)$ and let β be the σ-hermitian form inducing the flip φ. We assume that σ satisfies (S) and (H) and in addition we require that $\sqrt{2} \in \mathrm{k}_\sigma$ (this is for instance the case if $\mathrm{k} = \mathbb{F}_{81}$). Let $\{e, f\}$ be a hyperbolic pair with respect to β. Then, $\{\frac{e-f}{\sqrt{2}}, \frac{e+f}{\sqrt{2}}\}$ is an orthonormal basis for β, so we can set $\mathsf{M} = \langle \frac{e-f}{\sqrt{2}}, \frac{e+f}{\sqrt{2}} \rangle_{\mathsf{A}}$. The transition matrix between these bases is

$$H_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathrm{SL}_2(\mathrm{k}_\sigma).$$

We consider the apartment $\Sigma_+ = \Sigma_+\{-e + tf, f\}$ and let $\Lambda_i = \langle -e + tf, t^i f \rangle_{\mathcal{O}_+}$, for $i = 0, 1, 2$. Then if $\Lambda_0 = \langle e, f \rangle_{\mathcal{O}_+}$, it follows that $[\Lambda_0] \in \Gamma_\varphi$ since $\{\frac{e-f}{\sqrt{2}}, \frac{e+f}{\sqrt{2}}\}$ is an orthonormal $\mathcal{O}_+$-basis for Λ_0.

In the residue $\Lambda_0/t\Lambda_0$, we see that Λ_1 is represented by the singular 1-space $\langle e \rangle + t\Lambda_0$ and so by Corollary 6.4, $[\Lambda_1] \notin \Gamma_\varphi$. Finally, we claim that $\Lambda_2 = \langle te, (t^2-1)e + tf \rangle_{\mathcal{O}_+}$. This is so because the transition matrix from $\{-e + tf, t^2 f\}$ to $\{te, (t^2-1)e + tf\}$ is

$$H_2 = \begin{pmatrix} -t & 1-t^2 \\ 1 & t \end{pmatrix} \in \mathrm{GL}_2(\mathrm{k}[t]).$$

Hence, we can also represent the object $[\Lambda_2]$ as $\langle e, (t-t^{-1})e + f \rangle_{\mathcal{O}_+}$. This basis is hyperbolic with respect to β. The transition matrix from $\{e, f\}$ to $\{e, (t-t^{-1})e + f\}$ is

$$U = \begin{pmatrix} 1 & t-t^{-1} \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathsf{A}).$$

That is, Λ_2 is spanned by an A-basis for M that has a Gram matrix with entries in k. It follows that $[\Lambda_2] \in \Gamma_\varphi$. Thus Γ_φ is disconnected.

7.1 A criterion for simple connectedness of Γ_φ

In order to prove that Γ_φ is connected and simply connected, we shall use the following criterion for simple connectedness of Phan chamber systems given by Devillers and Mühlherr [DeMu07].

Given a residue R, we define

$$\begin{aligned} l_*(\varphi, R) &= \min\{l(\delta_*(c, c^\varphi)) \mid c \in R\} \\ A_\varphi(R) &= \{c \in R \subseteq \Delta_+ \mid \delta_*(c, c^\varphi) = l_*(\varphi, R)\} \end{aligned}$$

Let φ be a geometric flip of the twin-building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ over the set I. Suppose that the following conditions are met:

(DMi) If $J \subsetneq I$ has cardinality at most 3, then each J-residue is spherical.

(DMii) If $J \subsetneq I$ has cardinality 2, and R is a J-residue of Δ_+, then the chamber system $(A_\varphi(R), (\sim_j)_{j\in J})$ is connected.

(DMiii) If $J \subsetneq I$ has cardinality 3, and R is a J-residue of Δ_+, then the chamber system $(A_\varphi(R), (\sim_j)_{j\in J})$ is simply 2-connected.

Then, the chamber system Δ_φ is simply 2-connected.

7.2 The geometry $A_\varphi(R)$

We shall now consider the geometry $A_\varphi(R)$. In order to eliminate the cases where the diagram of R is disconnected we have the following observation.

Lemma 7.2 *Let R be a residue that is isomorphic to a direct product of residues $R_1 \times R_2$. Then $A_\varphi(R) \cong A_\varphi(R_1) \times A_\varphi(R_2)$.*

Proof We have $\mathrm{proj} \circ\varphi|_R = \mathrm{proj} \circ\varphi|_{R_1} \times \mathrm{proj} \circ\varphi|_{R_2}$. Therefore a chamber $c = (c_1, c_2)$ belongs to $A_\varphi(R)$ if and only if $c_i \in A_\varphi(R_i)$ if and only if c_i opp $\mathrm{proj} \circ\varphi(c_i)$. □

In order to identify the geometries $A_\varphi(R)$, where R is a residue of rank 2 or 3 with a connected diagram, we present the following two results.

Lemma 7.3 *For $\varepsilon = \pm$, let $S_\varepsilon \subsetneq R_\varepsilon$ be residues of Δ_ε such that $S_\varepsilon = \mathrm{proj}_{R_\varepsilon}(R_{-\varepsilon})$ and let $x_\varepsilon \in R_\varepsilon$ be an arbitrary chamber. Then,*

(a) *$C(x_+, x_-)$ contains $z_\varepsilon = \mathrm{proj}_{R_\varepsilon}(x_{-\varepsilon})$, and*

(b) *$C(x_+, x_-)$ contains $y_\varepsilon = \mathrm{proj}_{S_\varepsilon}(x_\varepsilon)$. Moreover,*

(c) *$\mathrm{proj}_{S_\varepsilon}(y_{-\varepsilon}) = \mathrm{proj}_{R_\varepsilon}(y_{-\varepsilon}) = \mathrm{proj}_{R_\varepsilon}(x_{-\varepsilon}) = \mathrm{proj}_{S_\varepsilon}(x_{-\varepsilon}) = z_\varepsilon$.*

Proof (a) Since $C(x_+, x_-)$ is coconvex and contains $x_{-\varepsilon}$ and some chamber (namely x_ε) of R_ε, it contains $z_\varepsilon = \mathrm{proj}_{R_\varepsilon}(x_{-\varepsilon})$ as well (cf. Lemma 2.8).

(b) Since $C(x_+, x_-) \cap R_\varepsilon$ is convex by Lemma 2.8, and contains x_ε and some chamber (namely z_ε) of S_ε, it also contains $y_\varepsilon = \mathrm{proj}_{S_\varepsilon}(x_\varepsilon)$.

(c) We show that $\mathrm{proj}_{R_\varepsilon}(x_{-\varepsilon}) = \mathrm{proj}_{S_\varepsilon}(x_{-\varepsilon})$. By Lemma 2.8, $z_\varepsilon = \mathrm{proj}_{R_\varepsilon}(x_{-\varepsilon}) = R_\varepsilon \cap \bigcap_{u_\varepsilon \in R_\varepsilon} C(u_\varepsilon, x_{-\varepsilon})$. Hence, in particular $z_\varepsilon \in R_\varepsilon \cap \bigcap_{u_\varepsilon \in S_\varepsilon} C(u_\varepsilon, x_{-\varepsilon})$. Since we know that the latter is a singleton, we are done. Now since $x_{-\varepsilon}$ is arbitrary, taking $x_{-\varepsilon} = y_{-\varepsilon}$, we also find $\mathrm{proj}_{R_\varepsilon}(y_{-\varepsilon}) = \mathrm{proj}_{S_\varepsilon}(y_{-\varepsilon})$.

From (b) with $x_\varepsilon = u_\varepsilon \in R_\varepsilon$ we find that $y_{-\varepsilon} \in C(u_\varepsilon, x_{-\varepsilon})$. Hence, $C(u_\varepsilon, y_{-\varepsilon}) \subseteq C(u_\varepsilon, x_{-\varepsilon})$ for all $u_\varepsilon \in R_\varepsilon$. It follows that $\bigcap_{u_\varepsilon \in R_\varepsilon} C(u_\varepsilon, y_{-\varepsilon}) \subseteq \bigcap_{u_\varepsilon \in R_\varepsilon} C(u_\varepsilon, x_{-\varepsilon})$. Intersecting this with R_ε we find that both sets are singletons equal to $\mathrm{proj}_{R_\varepsilon}(y_{-\varepsilon})$ and $\mathrm{proj}_{R_\varepsilon}(x_{-\varepsilon})$ respectively. □

We now examine the special situation where $R_{-\varepsilon} = R_\varepsilon^\varphi$.

Corollary 7.4 *Let the situation be as in Lemma 7.3 and assume in addition that $R_{-\varepsilon} = R_\varepsilon^\varphi$ and $x_{-\varepsilon} = x_\varepsilon^\varphi$, for $\varepsilon = \pm$. Then,*

(a) $y_{-\varepsilon} = y_\varepsilon^\varphi$ *and* $z_{-\varepsilon} = z_\varepsilon^\varphi$.

(b) $x_\varepsilon \in A_\varphi(R_\varepsilon)$ *if and only if*

  (i) *x_ε belongs to a residue opposite to S_ε in R_ε whose type is also opposite to the type of S_ε in R_ε and*

  (ii) $y_\varepsilon \in A_\varphi(S_\varepsilon)$.

(c) $\mathrm{proj}_{S_\varepsilon}$, $\mathrm{proj}_{S_{-\varepsilon}}$ *and φ all define bijections between $S_{-\varepsilon}$ and S_ε.*

Proof (a) This is because φ preserves distances and codistances.

(b) Note that $x_\varepsilon \in A_\varphi(R_\varepsilon)$ if and only if x_ε is opposite to $z_\varepsilon = \mathrm{proj}_{R_\varepsilon}(x_\varepsilon^\varphi)$. This happens precisely if we have (i) and in addition, $y_\varepsilon = \mathrm{proj}_{S_\varepsilon}(x_\varepsilon)$ is opposite to z_ε. Note however, that by Lemma 7.3 and (a), we have $z_\varepsilon = \mathrm{proj}_{S_\varepsilon}(y_\varepsilon^\varphi)$. Thus, y_ε is opposite to z_ε precisely if $y_\varepsilon \in A_\varphi(S_\varepsilon)$.

(c) It follows from Lemma 7.3 (c) that $\mathrm{proj}_{R_\varepsilon}(R_{-\varepsilon}) = \mathrm{proj}_{R_\varepsilon}(S_{-\varepsilon}) = \mathrm{proj}_{S_\varepsilon}(S_{-\varepsilon})$, so $\mathrm{proj}_{S_\varepsilon} \colon S_{-\varepsilon} \to S_\varepsilon$ is surjective. Moreover, $\mathrm{proj}_{S_\varepsilon}$ and $\mathrm{proj}_{S_{-\varepsilon}}$ are each other's inverse. As for φ, since φ defines a bijection between R_ε and $R_{-\varepsilon}$, it also sends S_ε to $S_{-\varepsilon}$. □

Note 7.5 We would like to thank the referee who pointed us to the present simplified version of part (c).

Lemma 7.6 *For $\varepsilon = \pm$, let R_ε be a spherical residue of Δ_ε such that $R_{-\varepsilon} = R_\varepsilon^\varphi$ and let $x_\varepsilon \in R_\varepsilon$. Assume in addition that $\mathrm{proj}_{R_\varepsilon} \colon R_{-\varepsilon} \to R_\varepsilon$ is bijective for $\varepsilon = \pm$. Define the relation $\mathrm{opp}_R \subseteq R_+ \times R_- \cup R_- \times R_+$ by $x_\varepsilon \,\mathrm{opp}_R\, x_{-\varepsilon}$ if and only if x_ε and $\mathrm{proj}_{R_\varepsilon}(x_{-\varepsilon})$ are opposite in R_ε. Then,*

(a) opp_R *defines a twinning on* $R = (R_+, R_-)$,

(b) φ *restricts to a geometric flip on* R.

Proof In order to show that opp_R defines a twinning on R, we show that it defines a 1-twinning as in [AbVa01]. Consider two chambers $c_\varepsilon \in R_\varepsilon$ with $c_- \mathrm{opp}_R\, c_+$ and let $\pi_\varepsilon \subseteq R_\varepsilon$ be a panel of some given type i on c_ε. Let $x_\varepsilon \in \pi_\varepsilon$. We must show that there exists a unique $y_{-\varepsilon} \in \pi_{-\varepsilon}$ such that x_ε is not opposite to $Ry_{-\varepsilon}$. Let $\pi' = \mathrm{proj}_{R_{-\varepsilon}}(\pi_\varepsilon)$ and $x' = \mathrm{proj}_{R_{-\varepsilon}}(x_\varepsilon) \in \pi'$. Since $\mathrm{proj}_{R_{-\varepsilon}}: R_\varepsilon \to R_{-\varepsilon}$ is bijective, π' is again a panel. Since $c_\varepsilon \mathrm{opp}_R\, c_{-\varepsilon}$, the panels π' and $\pi_{-\varepsilon}$ are opposite in $R_{-\varepsilon}$. Therefore the chamber $y_{-\varepsilon} = \mathrm{proj}_{\pi_{-\varepsilon}}(x')$ is the unique chamber not opposite to x' in $\pi_{-\varepsilon}$ so that $y_{-\varepsilon}$ is also the unique chamber not opposite to x_ε in $\pi_{-\varepsilon}$. This shows that opp_R is a 1-twinning.

In order to use the main result from loc. cit. we show that opp_R also satisfies condition (TA) of that result. This means that we must find some $x_\varepsilon \in R_\varepsilon$ and an apartment Σ_{R_ε} of R_ε such that $\{x_\varepsilon\} = \{y_\varepsilon \in R_\varepsilon \mid y_\varepsilon \in \Sigma_{R_\varepsilon} \text{ and } c_{-\varepsilon} \mathrm{opp}_R\, y_\varepsilon\}$. To do this, simply take a twin-apartment $\Sigma = (\Sigma_+, \Sigma_-)$ of Δ and let $\Sigma_{R_\varepsilon} = \Sigma_\varepsilon \cap R_\varepsilon$. Then, there is a unique chamber x_ε in Σ_ε such that $x_\varepsilon \mathrm{opp}_{R_\varepsilon} \mathrm{proj}_{R_\varepsilon}(c_{-\varepsilon})$. This is the desired chamber.

(b) Clearly φ is an involutory isometry of R preserving distances and the twinning opp_R. Moreover, φ is geometric on R if for each panel $\pi \subseteq R_\varepsilon \subseteq \Delta_\varepsilon$ the map $\mathrm{proj}_\pi \circ\varphi: \pi \to \pi$ is not the identity. Since the projection map of R is the restriction of the projection map of Δ, geometricity immediately carries over from Δ to R. □

Corollary 7.7 *Let the situation be as in Lemma 7.6, where we now specify to* Δ *and* φ *as in Section 4. If* $R_+ \cong R_1 \times \cdots \times R_r$*, where* R_i *has type* A_{m_i}*, then* $A_\varphi(R_+) \cong A_\varphi(R_1) \times \cdots \times A_\varphi(R_r)$*, where* $A_\varphi(R_i) \cong \mathcal{N}_{m_i+1}$.

Proof The fact that the phan geometry of a product is the product of the Phan geometries follow from Lemma 7.2. Therefore assume R_+ is of type A_m. By Lemma 7.6 φ is a flip on the spherical building $R = (R_+, R_-)$. Identifying R_- with R_+ via proj_{R_+}, the flip acts as $\mathrm{proj}_{R_+} \circ\varphi$ on R_+. Since proj_{R_+} is bijective, this flip is non-degenerate on R_+. Note that flip defines a non-degenerate polarity on the A_m building R_+. Hence there exist panels π_+ for which $\mathrm{proj}_{\pi_+} \circ\varphi: \pi_+ \to \pi_+$ is bijective. Now in Section 4 we showed that proj_{π_+} is linear and φ is σ-semi-linear with respect to some projective identification. This implies that $\mathrm{proj}_{R_+} \circ\varphi$ must be a unitary flip on R_+. The Phan geometry R_φ is now isomorphic to $\mathcal{N}_{m+1}$. □

Geometric description of $A_\varphi(R)$ In order to use the criteria (DMi), (DMii), and (DMiii), we shall now explicitly describe $A_\varphi(R)$ when R has rank 2 or 3. Note that $A_\varphi(R)$ is completely characterized by Corollary 7.4 part (c) and Corollary 7.7. We shall now describe R in case R has rank 1, 2 or 3. By Corollary 7.7, we may assume that R has type A_m with $m = 2, 3$ over index set $I = \{1, \ldots, m\}$. We then distinguish cases by considering the type of S as defined in Corollary 7.4.

Extremal cases and rank 1 We first distinguish two extremal cases: $S = R$ and S is a chamber. Clearly this covers the case when R has rank 1. If $S = R$, then by Corollary 7.7 the geometry $A_\varphi(R)$ is isomorphic to $\mathcal{N}_{\mathrm{rank}(R)+1}$. If S is a chamber, then $A_\varphi(R)$ is the collection of chambers of R opposite to S in R by Corollary 7.4.

Rank 2 **and the geometry** $\mathcal{AN}$ By duality we may assume $\mathrm{typ}(S) = \{2\}$. By Corollary 7.7, $A_\varphi(S)$ is isomorphic to the non-degenerate points on a line endowed with a non-degenerate unitary form; in fact it is the dual of that geometry i.e. it is a special collection of lines lying on a given point p. We can model this as follows. Let V be a vector space of dimension 3 containing the 1-space p. Also, let m be a 2-space disjoint from p and let $\mathcal{H}$ be a hyper surface of m_σ (e.g. the set of vectors that are singular with respect to a non-degenerate unitary form). For $t \in \mathcal{H}$ let $l_t = \langle p, t\rangle_V$. Let $\mathcal{AN}$ be the geometry whose points are the 1-spaces in $V - \bigcup_{t\in\mathcal{H}} l_t$ and whose lines are the 2-spaces not containing p. Note that the chambers of $\mathcal{AN}$ are those point-line pairs (q, l), where l is opposite to (i.e. not lying on) p and where the projection $\langle q, p\rangle$ of q onto p is different from any of the l_t. Using Corollary 7.4 one verifies that in case $\mathcal{H}$ is given by a unitary form that is non-degenerate on m, $\mathcal{AN} = A_\varphi(R)$.

Rank 3 When R has rank 3, by duality $\mathrm{typ}(S) \in \{\{1, 3\}, \{1\}, \{2\}, \{1, 2\}\}$. We now describe the resulting geometries $A_\varphi(R)$ in each case.

Case $\mathrm{typ}(S) = \{1, 3\}$ **and the geometry** Γ_m Let m be a line of the building geometry of type $A_3(\mathrm{k})$, i.e., the geometry on the non-trivial subspaces of some four-dimensional k-vector space V. Let β be a non-degenerate σ-hermitian form on V such that m is non-degenerate with respect to β. As in the Case rank 2 above, let $\mathcal{H}$ be the set of singular points of V with respect to β. Let $\mathcal{H}_m = \mathcal{H} \cap m$ and let $\mathcal{H}_{m^\perp} = \mathcal{H} \cap m^\perp$. For $x \in \mathcal{H}_{m^\perp}$, let π_x be the plane $\langle m, x\rangle$.

Let Γ_m be the pre-geometry of elements opposite this set, i.e., the points of Γ_m are those points that are not incident with any of the π_x, the lines of Γ_m are those lines that do not intersect m, and the planes of Γ_m are those planes that are not incident with any of the $p \in \mathcal{H}_m$. The elements of this geometry are called *good*.

The geometry Γ_m models $A_\varphi(R)$ in case $\mathrm{typ}(S) = \{1, 3\}$: The residue S itself is the line m. Using Corollary 7.7 we have $A_\varphi(S) \cong \mathcal{N}_2 \times \mathcal{N}_2$. The chambers of $A_\varphi(S)$ are those triples (p, m, π), where p is non-degenerate with respect to $\beta|_m$, that is not on $\mathcal{H}_m$, and π corresponds to the point $\pi \cap m^\perp$, which must be non-degenerate with respect to $\beta|_{m^\perp}$. By Corollary 7.4 the chambers of $A_\varphi(R)$ are those triples $x = (p, l, \pi)$, where l is opposite to (i.e. disjoint from) m, and $\mathrm{proj}_S(x) = (\pi \cap m, m, \langle m, p\rangle)$ belongs to $A_\varphi(S)$. Clearly this is exactly the geometry Γ_m.

Cases $\mathrm{typ}(S) = \{1\}$ This geometry can be constructed in the same way as the geometry Γ_m in Case $\mathrm{typ}(S) = \{1, 3\}$. By making β have a 1-dimensional radical on either m or $m^\perp$ we obtain the geometry $A_\varphi(R)$ in case $\mathrm{typ}(S)$ is $\{3\}$ and $\{1\}$ respectively.

Case typ$(S) = \{2\}$ **and the geometry** $\Gamma_{p,\pi}$ Let p be a point incident with a plane π of the building geometry of type $A_3(\mathrm{k})$, i.e., the geometry on the non-trivial subspaces of some four-dimensional k-vector space V. Let β be a non-degenerate σ-hermitian form on V such that p and π are non-degenerate with respect to β. Let $m = p^\perp \cap \pi$, and, for each 1-space x on $\mathcal{H}_m$, let $l_x = \langle p, x\rangle$.

Let $\Gamma_{p,\pi}$ be the pre-geometry of elements opposite this set, i.e., the points of $\Gamma_{p,\pi}$ are those points that are not incident with π, the lines of $\Gamma_{p,\pi}$ are those lines that do not intersect any of the l_x, and the planes of $\Gamma_{p,\pi}$ are those planes that are not incident with p. The elements of $\Gamma_{p,\pi}$ are called *good*.

The geometry $\Gamma_{p,\pi}$ models A_φ in case typ$(S) = \{2\}$. The residue S is the flag (p, π). Using Corollary 7.7 we have $A_\varphi(S) \cong \mathcal{N}_2$. The chambers of $A_\varphi(S)$ are those triples (p, m, π), where $m = \langle p, x\rangle$ and x is non-degenerate with respect to $\beta|_{p^\perp\cap\pi}$. By Corollary 7.4 the chambers of $A_\varphi(R)$ are those triples $x = (q, m, \pi')$, where q and π' are opposite to (i.e. not lying on) π and p respectively, and where $\mathrm{proj}_S(x) = (p, \langle p, m\cap\pi\rangle, \pi)$ belongs to $A_\varphi(S)$. This is exactly the geometry $\Gamma_{(p,\pi)}$.

Case typ$(S) = \{1, 2\}$ **and the geometry** Γ_π Let π be a plane of the building geometry of type $A_3(k)$, i.e., the geometry on the non-trivial subspaces of some four-dimensional k-vector space V. Let β be a non-degenerate σ-hermitian form on V such that π is non-degenerate. Let $\mathcal{H}_\pi = \mathcal{H} \cap \pi$, and for each $t \in \mathcal{H}_\pi$, let $l_t = t^\perp \cap \pi$. We study the pre-geometry Γ_π opposite of this residue, i.e., the points of Γ_π are those points that are not contained in π, the lines of Γ_π are those lines that intersect π in a point not in S_π, and the planes of Γ_π are those planes that do not contain any l_t. The elements of Γ_π are called *good*.

The geometry Γ_π models A_φ in case typ$(S) = \{1, 2\}$. The residue S is the plane π. Using Corollary 7.7 we have $A_\varphi(S) \cong \mathcal{N}_3$. The chambers of $A_\varphi(S)$ are those triples (p, l, π), where p and l are non-degenerate with respect to $\beta|_\pi$. By Corollary 7.4 the chambers of $A_\varphi(R)$ are those triples $x = (q, m, \pi')$, where q is opposite to (i.e. not lying on) π, and where $\mathrm{proj}_S(x) = (m \cap \pi, \pi' \cap \pi, \pi)$ belongs to $A_\varphi(S)$. This is exactly the geometry Γ_π.

Theorem 7.8 *The geometry* Γ_φ *is simply connected, if* $n \geq 4$ *and* $|\mathrm{k}_\sigma| \geq 4$.

Proof Since by Lemma 4.11 the flip φ is geometric, we need to verify criteria (DMi), (DMii) and (DMiii).

(DMi) Since Δ has diagram $\widetilde{A}_{n-1}$, condition (DMi) is satisfied.

(DMii) and (DMiii): By a result of Gramlich et al. [GHMS] conditions (DMii) and (DMiii) are satisfied when the diagram is $\widetilde{A}_{n-1}$ for $n \geq 4$, when $\mathrm{k} = \mathbb{F}_{q^2}$ and $q \geq 4$. In Corollaries 8.6 and 8.10 we also prove these results for infinite fields. □

8 (Simple) connectedness of $A_\varphi(R)$

In this section we prove Lemma 8.3, which is analogous to a result from Bennett and Shpectorov [BeSh04], but now for infinite fields, needed in Lemma 10.2.

We also prove Corollaries 8.6 and 8.10, which are analogous to results from Gramlich et al. [GHMS], but now for infinite fields, needed in Theorem 7.8.

More precisely, we show that the small rank geometries $A_\varphi(R)$ will satisfy the conditions (DMii) and (DMiii) needed in the proof of Theorem 7.8 in the case of an infinite field. These geometries were described explicitly in Section 7. We shall use those descriptions to show (simple) connectedness.

We learned that generalized versions of these geometries are considered in [DeGrMu] as well. There they are called *generalized Phan geometries*. In fact the end results in this section are implied by some of the results there but, as the methods are different and independent from those in [DeGrMu], we decided to keep the results included here in order for the paper to be as self-contained as possible.

8.1 Criteria for simple connectedness over infinite fields

In the following results the geometry Γ will have a string diagram. In that context, points will refer to one fixed end node and lines and planes will then refer to the following two nodes.

Lemma 8.1 *Let Γ be a residually connected geometry with a string diagram, in which the following holds:*

(NI) *Given a set Y of three lines, there exists a line l not intersecting any of the lines in Y.*

(PL) *Given a set X of at most three points and a set Y of at most three lines, and a line l not intersecting any of the lines in Y, there exists a point p on l such that p is collinear to each point in X and forms a plane with each line in Y.*

Then, Γ is connected and simply connected.

Proof We first show that Γ is connected. Since Γ is transversal, it suffices to show that, given two points x_1 and x_2, there is a point p that is collinear to both. This follows from (PL).

Since Γ has a string diagram and is residually connected, any cycle is homotopic to a cycle of points and lines. Therefore it suffices that any such cycle is 0-homotopic. Because the diameter of the collinearity graph of Γ is 2, any cycle is homotopic to a cycle of length at most 5. We first show that any cycle of length 3 is 0-homotopic. Let $x_1, y_1, x_2, y_2, x_3, y_3, x_1$ be a 3-cycle, where x_i is a point and y_i is a line for each $i = 1, 2, 3$. Then by (NI) there exists a line l not intersecting the lines y_1, y_2, and y_3. By (PL) there is a point p on l that forms a plane π_i with y_i, for $i = 1, 2, 3$. Hence, the subcomplex induced on $\{x_i, y_i, \pi_i, p \mid i = 1, 2, 3\}$ forms a cone, which is 0-homotopic.

Next, suppose that $x_1, y_1, \ldots, x_4, y_4, x_1$ is a 4-cycle. Then, by (PL), there is a point on the line y_4 that is collinear to x_2 and x_3. This decomposes the 4-cycle into three 3-cycles.

Next, suppose that $x_1, y_1, \ldots, x_5, y_5, x_1$ is a 5-cycle. Then, by (PL), there is a point on the line y_5 that is collinear to x_2 and x_4. This decomposes the 5-cycle into two 3-cycles and a 4-cycle. □

Now let V be an k-vector space of dimension $n \geq 2$ with a non-degenerate σ-hermitian form β. Here σ is a non-trivial automorphism of order 2 of the infinite field k.

We shall reprove the required results from [GHMS] by replacing some of the counting arguments used there with algebraic geometric arguments. In order to do this, we will use some considerations over the fixed field k_σ. If $U \leq V$ is an m-dimensional k-subspace of V, then we denote by U_σ the vector space U, viewed as a $2m$-dimensional vector space over k_σ.

Let $\mathcal{H}$ be the set of singular points of V with respect to β.

Lemma 8.2 (a) *The set $\mathcal{H}$ consists of the 1-spaces of V corresponding to 2-spaces of the affine space V_σ lying on the algebraic hyper surface given by β.*

(b) *Let l be a 2-space of V such that β is non-trivial on l. View l_σ as an affine space over k_σ. The β-non-degenerate 1-spaces on l are those 2-dimensional k_σ-spaces of l_σ lying outside the algebraic hyper-surface $\mathcal{H}_l = \mathcal{H} \cap l$ over k_σ.*

Proof (a) We note that, by definition of σ, the extension k/k_σ is Galois of degree 2. Therefore $k = k_\sigma(\alpha)$, where the minimal polynomial m_α is quadratic. If $a = x + y\alpha \in k$, then $N_\sigma(a) = aa^\sigma = x^2 + \mathrm{tr}(\alpha)xy + N_\sigma(\alpha)y^2 = P(x,y)$. As a consequence, given an orthonormal k-basis $\{e_i\}_{i=1}^n$ for V, then for $v = \sum_{i=1}^n (x_i + y_i\alpha)e_i$, we have $\beta(v,v) = \sum_{i=1}^n P(x_i, y_i)$. Therefore $\mathcal{H}$ is the zero-set of a polynomial with coefficients in k_σ.

(b) This is immediate from (a). □

8.2 The geometries $\mathcal{N}$ and $\mathcal{AN}$ and condition (DMii)

Let $\mathcal{N} = \mathcal{N}_n$ be the pre-geometry of subspaces of V that are non-degenerate with respect to β, where incidence is given by inclusion.

The following lemma was proved by Bennett and Shpectorov in [BeSh04] in the case that $k = \mathbb{F}_{q^2}$.

Lemma 8.3 *If* k *is infinite and $n \geq 4$, then $\mathcal{N}_n$ is a connected, residually connected, and simply connected geometry.*

Proof We first show $\mathcal{N}$ is transversal. Let F be a flag of $\mathcal{N}$ that is not maximal and suppose that $(A, C) \subseteq F$ is a subflag with $\dim(C) - \dim(A) \neq 1$. Since there exists a non-degenerate 1-space in $A^\perp \cap C$, we can extend F so as to include an object of type $\dim(A) + 1$. The same argument can be applied to flags F of rank 1, by setting $A = 0$ or $C = V$.

We show that $\mathcal{N}$ satisfies the criteria (NI) and (PL) as in Lemma 8.1.

Let y_i be a line for $i = 1, 2, 3$. Pick a point p not on any of these lines. Then the planes $\langle p, y_i \rangle$ are represented as three lines in $p^\perp$ (which does not contain p itself). Note that this requires that $n \geq 4$. Since $p^\perp$ is non-degenerate, it contains at least one non-degenerate point q not on any of the lines. Then $\langle p, q \rangle$ is the required line l. Thus $\mathcal{N}$ satisfies (NI).

Let l be a line and X be the set of points and Y the set of lines as in property (PL). For $x \in X$, let $m_x = x^\perp \cap \langle x, l\rangle$ and define the isomorphism of k_σ-varieties $\varphi_x: l \to m_x$ by $t \mapsto \langle x, t\rangle \cap m_x$. Then the 1-space t does not form a line of $\mathcal{N}$ with x if and only if $\varphi_x(t) \in S_{m_x}$.

For $y \in Y$, let $m_y = y^\perp \cap \langle l, y\rangle$ and define the isomorphism of k_σ-varieties $\psi_y: l \to m_y$ by $t \mapsto \langle y, t\rangle \cap m_y$. Then the 1-space t does not form a plane of $\mathcal{N}$ with y if and only if $\psi_y(t) \in S_{m_y}$.

It follows that all the points on l that fail one of the conditions in (PL) lie on the union $S_l \cup \cup \bigcup_{x\in X} \varphi_x^{-1}(S_{m_x}) \cup \bigcup_{y \in Y} \psi_y^{-1}(S_{m_y})$. By Lemma 8.2 this is a union of hypersurfaces. Since the field k_σ is infinite, the affine plane l_σ cannot be covered by finitely many hyper surfaces. Therefore $\mathcal{N}$ satisfies (NI) and (PL) from Lemma 8.1.

Next, we prove that any residue R of type $J \subseteq I = \{1, 2 \ldots, n-1\}$, where $|J| \geq 2$, is connected. If J is disconnected, then R is the direct product of two geometries, and is therefore connected. If J is connected, then R is isomorphic to the geometry $\mathcal{N}$ in rank $|J|$. Since $\mathcal{N}$ has a string diagram and is transversal, it suffices to show that any two points are connected by a path of points and lines only. Let x_1 and x_2 be two points of R and let l be a line through x_1. Let $m_2 = x_2^\perp \cap \langle x_2, l\rangle$. Define the isomorphism of k_σ-varieties $\phi_2: l \to m_2$ by $t \mapsto \langle t, x_2\rangle \cap m_2$. Then the 1-space t does not form a line of $\mathcal{N}$ with x_2 if and only if $\phi_2(t) \in S_{m_2}$.

Now a 1-space p on l is a point that is collinear to x_2 unless it lies on the union of the two hyper surfaces S_l and $\phi_2^{-1}(m_2)$ of l_σ. Here we again make use of Lemma 8.2. Since the field k_σ is infinite, the affine plane l_σ cannot be covered by finitely many hyper surfaces. Therefore, some point p is collinear to both x_1 and x_2. It follows that R is connected.

We have proved that $\mathcal{N}$ is transversal, has a string diagram, and is residually connected. Since it also satisfies (IN) and (PL) from Lemma 8.1 $\mathcal{N}$ is connected and simply connected. □

Lemma 8.4 *$\mathcal{AN}$ is transversal and connected.*

Proof Note that two points x_1 and x_2 both different from p are collinear if and only if $\langle p, x_1\rangle$ and $\langle p, x_2\rangle$ intersect m in different points. If this is not the case, then any third point x_3 such that $\langle x_3, p\rangle$ intersects m elsewhere is collinear to both. Since $m - \mathcal{H}$ contains more than one 1-space, such a third point x_3 always exists. Therefore $\mathcal{AN}$ is connected.

Next we show that $\mathcal{AN}$ is transversal. Any point is contained in some 2-space not passing through p, so any point lies on some line. Since k is infinite, by Lemma 8.2 the 2-space m contains a point q. If y is a line, then y does not contain p and so the point $\langle p, q\rangle \cap y$ is a point on y. □

Lemma 8.5 *$\mathcal{N}_3$ is transversal and connected.*

Proof Let x_1 and x_2 be two points. Let $m_i = x_i^\perp$ for $i = 1, 2$. If $x_1 \in m_2$, then x_1 is collinear to x_2 and we're done. Suppose this is not the case. Let $\varphi: m_1 \to m_2$ be the the isomorphism of algebraic k_σ-varieties given by $t \mapsto \langle x_2, t\rangle \cap m_2$. Then t is collinear to both x_1 and x_2 provided it does not belong to the union of hyper surfaces $S_{m_1} \cup \varphi^{-1}(S_{m_2})$. Since k is infinite, m_1 is not covered by this union and so such a point t exists.

We now show that $\mathcal{N}_3$ is transversal. By the preceding, every points lies on some line. Each line y is $x^\perp$ for some point x. By the preceding, y has a point. □

Corollary 8.6 *The chamber system Δ_φ satisfies condition (DMii) in the proof of Theorem 7.8 if* k *is infinite.*

Proof Let R be a rank-2 residue. Consider $A_\varphi(R)$. First note that if R is of type $A_1 \times A_1$, then by Lemma 7.2 we have $A_\varphi(R) = A_\varphi(\pi_1) \times A_\varphi(\pi_2)$. Since φ is geometric, both factors are non-emtpy so that $A_\varphi(R)$ is connected.

Next assume that R is of type A_2. There are three cases for $\mathrm{proj}_R(R^\varphi)$: it is either a chamber c, a panel π, or all of R. In the first case, $A_\varphi(R)$ consists of all chambers of R opposite c. By [Ab96, Br93], this is connected. In the second case, $A_\varphi(R)$ is isomorphic to $\mathcal{AN}$ or its dual. In the third case, $A_\varphi(R)$ is isomorphic to $\mathcal{N}_3$ or its dual. Therefore connectedness follows from Lemma 8.4 and 8.5.

We note here that condition (DMii) refers to connectedness of chamber systems, whereas we have proved connectedness for the geometry. Since the residue R we consider has rank 2, these notions are equivalent. □

8.3 (Simple) connectedness of $A_\varphi(R)$ in rank 3 and condition (DMiii)

We show that the geometries $A_\varphi(R)$ described in Section 7 satisfy condition (DMiii). We note again that the condition (DMiii) refers to simple connectedness of chamber systems, whereas we shall work with geometries. Since the residues R we consider are of rank 3 or less, these notions are equivalent.

We first consider the case where R is of type $A_3(\mathrm{k})$ for some infinite field k.

Lemma 8.7 *The pre-geometry Γ_m is a simply connected geometry.*

Proof We need to show that Γ_m is transversal, residually connected and satisfies (NI) and (PL). To show (NI) we proceed as the proof of Lemma 8.3 adding the line m to the set $\{y_1, y_2, y_3\}$.

To prove (PL) we first fix a line l and the sets X and Y as in the definition. Define the map $\phi_{m^\perp}: l \to m^\perp$ by $t \mapsto \langle m, t\rangle \cap m^\perp$. Now t is a point of the geometry if and only if $t \notin \phi_{m^\perp}^{-1}(S_{m^\perp})$.

If $x \in X$ then the only 1-dimensional subspace of l not collinear with x is $p_x = \langle m, x\rangle \cap l$. Moreover if $y \in Y$ then the map $\varphi_y: t \mapsto \langle t, y\rangle \cap m$ gives an isomorphism between the points of l that do not form a good plane with y and the points on the hyper surface S_m. Note that this means that $\varphi_y^{-1}(S_m)$ is a hyper surface of l_σ. It follows that all the points on l that fail one of the conditions in (PL) lie on the union $\phi_{m^\perp}^{-1}(S_{m^\perp}) \cup \bigcup_{y\in Y} \varphi_y^{-1}(S_m) \cup \{p_x \mid x \in X\}$. As before this union of subvarieties does not cover l_σ so there is a point t on l that is collinear to all of X and coplanar to all of Y.

We now show that Γ_m is transversal and residually connected. Note that above we proved that each line contains a point. Moreover, both the residues of a point and of a plane are isomorphic to $\mathcal{AN}$. Therefore by Lemma 8.4 we are done. □

Lemma 8.8 *The pre-geometry $\Gamma_{p,\pi}$ is a simply connected geometry.*

Proof We first show that $\Gamma_{p,\pi}$ satisfies properties (NI) and (PL). Let y_i, with $i = 1, 2, 3$ be lines of $\Gamma_{p,\pi}$. Let x be a 1-space in $m - S_m - \bigcup_{i=1}^{3} y_i$ so that any 2-space on x not in π is a line. Repeating the argument in the proof of Lemma 8.3 we find the required line l.

We now prove that $\Gamma_{p,\pi}$ has property (PL). Let X, Y and l be as in (PL). For each $x \in X$, define the isomorphism $\varphi_x : l \to m$ by setting $z \mapsto \langle x, z, p\rangle \cap m$. Then, z does not form a good line with x if and only if $\varphi_x(z) \in S_m$.

For $y \in Y$, the only point on l that does not form a good plane with y is the point $p_y = l \cap \langle p, y\rangle$.

It follows that all the points on l that fail one of the conditions in (PL) lie on the union $S_l \cup \{p_y, y \in Y\} \cup \bigcup_{x\in X} \varphi_x^{-1}(S_m)$. As before this union of subvarieties does not cover l_σ.

We now show that $\Gamma_{p,\pi}$ is transversal and residually connected. Clearly the residue of a line is connected since its diagram is disconnected. Moreover, the residue of a point and the dual of the residue of a plane are isomorphic to $\mathcal{AN}$. Therefore by Lemma 8.4 we are done. □

Lemma 8.9 *The pre-geometry Γ_π is a simply connected geometry.*

Proof We first show that Γ_π satisfies properties (NI) and (PL). Let y_i, with $i = 1, 2, 3$ be lines of Γ_π. Let x be a 1-space on $\pi - S_\pi - \bigcup_{i=1}^{3} y_i$ so that any 2-space on x not in π is a good line. Repeating the argument in the proof of Lemma 8.3 we find the required line l.

We now prove that Γ_π has property (PL). Let X, Y and l be as in (PL).

Let $x \in X$. Define the isomorphism $\varphi_x : l \to m_x$, where $m_x = \langle x, l\rangle \cap \pi$, by $t \mapsto \langle x, t\rangle \cap \pi$. Then $\langle x, t\rangle$ is not a good line if and only if $\varphi_x(t) \in S_{m_x}$.

Let $y \in Y$ and let $m_y = (y \cap \pi)^\perp \cap \pi$. Define $\psi_y : l \to m_y$ to be the isomorphism given by $t \mapsto \langle t, y\rangle \cap m_y$. Then, the 1-space t does not form a good plane with y if and only if $t \in \psi^{-1}(S_{m_y})$. It follows that all the points on l that fail one of the conditions in (PL) lie on the union $S_l \cup \cup \bigcup_{x\in X} \varphi_x^{-1}(S_{m_x}) \cup \bigcup_{y\in Y} \psi_y^{-1}(S_{m_y})$. As before this union of subvarieties does not cover l_σ.

We now show that Γ_π is transversal and residually connected. Clearly the residue of a line is connected since its diagram is disconnected. Moreover, the residue of a point is isomorphic to $\mathcal{N}_3$ and the dual of the residue of a plane is isomorphic to $\mathcal{AN}$. Therefore by Lemmas 8.4 and 8.5 we are done. □

Corollary 8.10 *The chamber system Δ_φ satisfies condition (DMiii) in the proof of Theorem 7.8 if* k *is infinite.*

Proof Let R be a J-residue for some $J \subseteq I$ with $|J| = 3$. If the diagram underlying J is disconnected, then $A_\varphi(R)$ is a direct product by Lemma 7.2. Moreover, each rank-2 component is connected by Corollary 8.6. Simple 2-connectedness then follows from [BeSh04, Lemma 3.8].

Therefore we may assume that R is of type A_3 and $J = \{1, 2, 3\}$. Let $X = \mathrm{proj}_R \circ \varphi(R)$ have type K. Then X is a residue in R, which can be of rank 0, 1, 2, or 3. If X is a chamber, then $A_\varphi(R)$ is the collection of chambers in R opposite to X.

The result follows from Lemma 8.7 if X has type $K = \{1, 3\}$, $\{1\}$, $\{3\}$; in case $K = \emptyset$ (i.e. X is a chamber) this is a trivial calculation, and follows from [Ab96]; if follows from Lemma 8.8 if $K = \{2\}$; and it follows from Lemma 8.9 if $K = \{1, 2\}$ or $K = \{2, 3\}$. Finally in case $K = J$, the result follows from Lemma 8.3. □

9 The group G_φ

Although the full group of type preserving automorphisms of Δ is larger, we shall only consider the following subgroup. Here we identify $\mathrm{GL}(\mathsf{M})$ with $\mathrm{GL}_n(\mathsf{A})$ via the action on the basis of reference B.

$$\mathrm{GL}_n^0(\mathsf{A}) = \{g \in \mathrm{GL}_n(\mathsf{A}) \mid v_\varepsilon(\det(g)) = 0\}.$$

This group acts strongly transitively on Δ ([Ro89, Ab96, Br89, Ga97]).

Definition 9.1 Given a non-degenerate σ-hermitian form β with an A-basis for M that is orthonormal for β, where σ satisfies (S) and (H). An automorphism $f \in \mathrm{GL}_n^0(\mathsf{A})$ is called an *isometry* if

$$\beta(f(u), f(v)) = \beta(u, v) \quad \forall u, v \in V.$$

The group of all such isometries is denoted $\mathrm{GU}_n(\mathsf{A})$.

From now on we shall work with the following subgroup

$$\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta) = \mathrm{SU}_n(\mathsf{A}) = \mathrm{GU}_n(\mathsf{A}) \cap \mathrm{SL}_n(\mathsf{A})$$

Lemma 9.2 *The group* $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ *acts flag-transitively on* Γ_φ*.*

Proof Since Γ_φ is transversal and $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ preserves types, it suffices to show that $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ is transitive on chambers. By Proposition 5.1 chambers of Γ_φ are given by β-orthogonal A-bases for M. Let c be a chamber given by the orthonormal basis A. Then, there exists and element $g \in \mathrm{GU}_n(\mathsf{A})$ that takes the basis of reference B to A. By applying, if necessary, a scalar multiplication by $\det(g) \in \mathrm{k}$ to one of the basis elements of A, we can ensure that g is transformed into an element of $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ (Note that since $\det(g) \in \mathsf{A}^*$ and $\varepsilon(\det(g)) = 0$, in fact $\det(g) \in \mathrm{k}^*$). Therefore A and the modified basis determine the same chamber c. □

10 The amalgam

In this section we shall prove that $G = \mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ is the universal completion of an amalgam $\mathcal{A}_{(2)}$ (Defined in 10.1) of finite subgroups of small rank.

10.1 The amalgam $\mathcal{M}$ of maximal parabolics

Let c be the chamber associated to the ordered basis of reference $\mathsf{B} = \{b_1, \ldots, b_n\}$. For $i = 0, 1, \ldots, n-1$ let M_i be the stabilizer of the i-object on c. Then, we have $M_0 = \mathrm{SU}_n(\mathrm{k})$.

Let the "shift" operator be given by:

$$s = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ t & 0 & 0 & 0 \end{pmatrix}.$$

Then, $M_i = M_0^{s^i} = s^{-i} M_0 s^i$ for $i = 1, \ldots, n-1$. Define the amalgam

$$\mathcal{M} = \{\bigcap_{j \in J} M_j \mid \emptyset \neq J \subseteq I\},$$

where the connecting homomorphisms are given by inclusion.

Proposition 10.1 *Let $n \geq 4$ and let* k *be infinite, or* $\mathrm{k} = \mathbb{F}_{q^2}$ *with $q \geq 4$. Then, the group* $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ *is the universal completion of the amalgam $\mathcal{M} = \{M_i \mid i = 0, \ldots, n-1\}$.*

Proof By Lemma 4.11, Corollary 2.10, and Lemma 6.5, Γ_φ is a residually connected geometry. By the results in Section 7 this geometry is connected and simply connected under the conditions given in the Proposition. By Lemma 9.2, $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ is flag-transitive on Γ_φ. Therefore, the result follows from Tits' Lemma. □

10.2 The amalgams of all (slim) parabolics

We shall now describe the amalgam $\mathcal{A}$ of slim parabolics for $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$. Let

$$A_1 = \left(\begin{array}{c|c} \mathrm{SU}_2(\mathrm{k}) & 0 \\ \hline 0 & I_{n-2} \end{array}\right),$$

Moreover, for any $i \in I = \{0, 1, \ldots, n-1\}$, and taking subscripts modulo n, we define $A_{i+1} = A_1^{s^i}$. For any subset $\emptyset \neq J \subsetneq I$, let $A_J = \langle A_j \mid j \in J\rangle_{\mathrm{SU}_n(\mathrm{k}[t,t^{-1}],\beta)}$. Note that for any $i \notin J$, we have $A_J \leq M_i$.

We now set

$$\mathcal{A} = \{A_J \mid \emptyset \neq J \subsetneq I\},$$

where the connecting homomorphisms are given by inclusion. Note that the amalgam $\mathcal{A}_k = \{A_J \mid \emptyset \neq J \subsetneq I - \{k\}\}$, is an amalgam for M_k. This follows from [BeSh04, Ph77]. In the case where $\mathrm{k} = \mathbb{F}_4$, this is not the case, but this will not affect our results, since we will assume that $|\mathrm{k}| \geq 16$.

However, these groups are not the full parabolic subgroups for the residual geometries. Next, we describe the amalgam of all parabolics of $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$. Let

$$B_1 = \left(\begin{array}{c|c} \mathrm{GU}_2(\mathrm{k}) & 0 \\ \hline 0 & D_{n-2} \end{array}\right) \cap \mathrm{SU}_n(\mathrm{k}),$$

where D_l is the diagonal subgroup of $\mathrm{SU}_l(\mathrm{k})$. Moreover, for any $i \in I = \{0, 1, \ldots, n-1\}$, and taking subscripts modulo n, we define $B_{i+1} = B_1^{s^i}$. For any subset $\emptyset \neq J \subsetneq I$, let $B_J = \langle B_j \mid j \in J \rangle_{\mathrm{SU}_n(\mathrm{k}[t,t^{-1}],\beta)}$. Note that for any $i \notin J$, we have $B_J \leq M_i$. We now set

$$\mathcal{B} = \{B_J \mid \emptyset \neq J \subsetneq I\},$$

where the connecting homomorphisms are given by inclusion. Note that the amalgam $\mathcal{B}_k = \{B_J \mid \emptyset \neq J \subsetneq I - \{k\}\}$, is the amalgam of maximal parabolics for the residual geometry of M_k.

Lemma 10.2 *Let $n \geq 4$ and let* k *be infinite, or* $\mathrm{k} = \mathbb{F}_{q^2}$ *with $q \geq 4$. For each $k \in I$, the maximal parabolic M_k is the universal completion of the amalgam $\mathcal{B}_k$.*

Proof This follows by Tits' Lemma and the fact that the residue of the k-object is connected, simply connected and that M_k acts flag-transitively on it. The connectedness and simple connectedness follows from Lemma 6.5. Flag-transitivity follows from the flag-transitive action of $\mathrm{SU}_n(\mathrm{k}[t,t^{-1}],\beta)$ on Γ_φ, which was proved in Lemma 9.2 □

In order to prove Theorem 1, we need to show that the universal completion of $\mathcal{A}_k$ is M_k and use induction on the rank. In the case of finite k, this is proved by Bennett and Shpectorov [BeSh04, Theorem 1.1] by first showing that the universal completion of $\mathcal{A}_k$ is equal to the universal completion of $\mathcal{B}_k$. Unfortunately, their proof makes use of the fact that k is finite. Therefore we shall reprove some of their results in order to obtain the result for infinite k.

Lemma 10.3 *Let $n \geq 4$. Then, the universal completion of $\mathcal{A}_k$ is also the universal completion of $\mathcal{B}_k$.*

Proof Let $\tilde{G}_k$ be the universal completion of $\mathcal{A}_k$. Since M_k is a completion of $\mathcal{A}_k$, the image of $\mathcal{A}_k$ in $\tilde{G}_k$ is isomorphic to $\mathcal{A}_k$ and there is a surjective homomorphism $\pi\colon \tilde{G}_k \to M_k$. Hence it suffices to show that the image of $\mathcal{A}_k$ in $\tilde{G}_k$ can be extended to a copy of $\mathcal{B}_k$. Let D_i be the intersection of A_i with the diagonal subgroup D of M_k. Notice that D (and hence every D_i) normalizes every A_J. We adopt the tilde convention, so that for any element x or subgroup H from $\mathcal{A}_k$, $\tilde{x}$ and $\tilde{H}$ denote their images in $\tilde{G}_k$. Notice that D is not a subgroup of the amalgam $\mathcal{A}_k$, and consequently we cannot use this convention to define $\tilde{D}$. Therefore we define it indirectly as follows. Let $\tilde{D}$ be equal to the product of all the subgroups $\tilde{D}_i$ (Notice that $D_i \subseteq A_i \in \mathcal{A}_k$ and hence $\tilde{D}_i$ is defined.) We claim that there exists an isomorphism $D \to \tilde{D}$ extending the isomorphisms $D_i \to \tilde{D}_i$. Indeed D_i and D_j are both contained in $A_{i,j}$ and they commute element-wise. Therefore $\tilde{D}_i$ and $\tilde{D}_j$ also commute elementwise. This proves that there exists a surjective homomorphism $\phi\colon \Pi_{i\neq k}\tilde{D}_i \to \tilde{D}$. However, we also have a canonical isomorphism of abstract groups $\gamma\colon \Pi_{i\neq k}D_i \to \Pi_{i\neq k}\tilde{D}_i$. Here the former group is D. Now the composition $\phi \circ \gamma$ is a surjective homomorphism, which restricts to an isomorphism $D_i \to \tilde{D}_i$ for each i. The restriction $\pi\colon \tilde{D} \to D$ is surjective and has $\phi \circ \gamma$ as its inverse. This proves our claim.

In a similar spirit, for $J \subseteq I - \{k\}$, define $\tilde{B}_J$ to be the product of the subgroups $\tilde{A}_J$ with $\tilde{D}$. For this definition to make sense, we must show that every $\tilde{D}_i$ normalizes $\tilde{A}_J$. Note that A_J is generated by the subgroups A_j with $j \in J$. Inside $A_{i,j}$ we see that D_i normalizes A_i and A_j. Hence $\tilde{D}_i$ normalizes every $\tilde{A}_j$, implying that $\tilde{D}_i$ normalizes $\tilde{A}_J$. Thus the subgroups $\tilde{B}_J$ are well defined.

We claim that with respect to the natural homomorphism $\pi\colon \tilde{G}_k \to M_k$, $\tilde{B}_J$ maps isomorphically onto the group B_J. This map is clearly surjective. Since $\tilde{D}$ normalizes $\tilde{B}_J$, there is a surjective homomorphism $\tilde{A}_J \rtimes \tilde{D} \to \tilde{B}_J$, whose kernel is $\tilde{K} = \{(a, a^{-1}) \in \tilde{A}_J \rtimes \tilde{D} \mid a \in \tilde{A}_J \cap \tilde{D}\}$. Similarly, there is a surjective homomorphism $A_J \rtimes D \to B_J$, whose kernel is $K = \{(a, a^{-1}) \in A_J \rtimes D \mid a \in A_J \cap D\}$. The natural homomorphism π now induces a surjective homomorphism $(\tilde{A}_J \rtimes \tilde{D})/\tilde{K} \to (A_J \rtimes D)/K$. Now note that $A_J \cap D = \langle D_i \mid i \in J\rangle$. Moreover, $\tilde{A}_J \cap \tilde{D} \geq \langle \tilde{D}_i \mid i \in J\rangle$. This shows that the natural homomorphism is in fact injective.

Now it is clear that the natural homomorphism $\pi\colon \tilde{G}_k \to M_k$ induces an isomorphism from the amalgam $\{\tilde{B}_J \mid \emptyset \neq J \subsetneq I - \{k\}\}$ onto the amalgam $\mathcal{B}_k$. Thus we have extended the image of the amalgam $\mathcal{A}_k$ to a copy of the $\mathcal{B}_k$. It follows that these amalgams have the same completions. □

Corollary 10.4 *Let $n \geq 4$ and let* k *be infinite, or* $\mathrm{k} = \mathbb{F}_{q^2}$ *with* $q \geq 4$. *Then, the universal completion of* $\mathcal{A}_k$ *is* M_k.

Proof This follows by combining Lemmas 10.2 and 10.3. □

Corollary 10.5 *Let $n \geq 4$ and let* k *be infinite, or* $\mathrm{k} = \mathbb{F}_{q^2}$ *with* $q \geq 4$. *Then, for each subset* $\emptyset \neq J \subsetneq I$ *of size at least* 3*, the group* A_J *is the universal completion of* $\mathcal{A}_J = \{A_{J'} \mid \emptyset \neq J' \subsetneq J\}$. *In particular,* M_k *is the universal completion of the amalgam* $\mathcal{A}_{k,(2)} = \{A_J \mid \emptyset \neq J \subsetneq I - \{k\}$ *and* $|J| \leq 2\}$.

Proof This follows by induction on $|J|$ from Corollary 10.4. □

10.3 The slim rank-2 amalgam $\mathcal{A}_{(2)}$ and the proof of Theorem 1

Let

$$\mathcal{A}_{(2)} = \{A_J \mid \emptyset \neq J \subsetneq I \text{ and } |J| \leq 2\}. \tag{10.1}$$

Proof (Of Theorem 1.) By Proposition 10.1, $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ is the universal completion of $\mathcal{M}$.

Let $\widehat{G}$ be the universal completion of the amalgam $\mathcal{A}_{(2)}$ and let $\widehat{M_k}$ be the subgroup of $\widehat{G}$ generated by the subgroups corresponding to $\mathcal{A}_{k,(2)}$.

Since $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ is a completion of $\mathcal{A}_{(2)}$, there is a surjective homomorphism $\tau\colon \widehat{G} \to \mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$. We now show that this map has an inverse. By Corollary 10.4, each $\widehat{M_k}$ is a completion of $\mathcal{A}_{k,(2)}$. Therefore, there are surjective homomorphisms $\chi_k\colon M_k \to \widehat{M_k}$, which are isomorphisms on the elements of $\mathcal{A}_{(2)}$ and so they extend a map between $\mathcal{M}$ and $\widehat{G}$. Thus, $\widehat{G}$ is realized as a completion of $\mathcal{M}$. Since $\mathrm{SU}_n(\mathrm{k}[t, t^{-1}], \beta)$ is the universal completion of $\mathcal{M}$, we have a surjective homomorphism χ between $\mathrm{SU}_n(\mathrm{k}[t,t^{-1}],\beta)$ and $\widehat{G}$, which is the identity on $\mathcal{A}_{(2)}$. The composition of χ and τ is a surjective homomorphism which is the identity on $\mathcal{A}_{(2)}$. By the universal property of $\widehat{G}$ it has to be the identity. $\square$

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