

# A Quasi Curtis-Tits-Phan theorem for the symplectic group

Rieuwert J. Blok<sup>1</sup> and Corneliu Hoffman<sup>1,2</sup>

<sup>1</sup>Department of Mathematics and Statistics  
Bowling Green State University  
Bowling Green, OH 43403-1874

<sup>2</sup> School of Mathematics  
University of Birmingham  
Edgbaston, B15 2TT, United Kingdom

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Send proofs to:

Rieuwert J. Blok  
Department of Mathematics and Statistics  
Bowling Green State University  
Bowling Green, OH 43403  
Tel.: 419-372-7455  
FAX: 419-372-6092  
Email: [blokr@member.ams.org](mailto:blokr@member.ams.org)

## Abstract

We obtain the symplectic group  $\mathrm{Sp}(V)$  as the universal completion of an amalgam of low rank subgroups akin to Levi components. We let  $\mathrm{Sp}(V)$  act flag-transitively on the geometry of maximal rank subspaces of  $V$ . We show that this geometry and its rank  $\geq 3$  residues are simply connected with few exceptions. The main exceptional residue is described in some detail. The amalgamation result is then obtained by applying Tits' lemma. This provides a new way of recognizing the symplectic groups from a small collection of small subgroups.

# 1 Introduction

In the revision of the classification of finite simple groups one of the important steps requires one to prove that if a simple group  $G$  (the minimal counterexample) contains a certain amalgam of subgroups that one normally finds in a known simple group  $H$  then  $G$  is isomorphic to  $H$ . A geometric approach to recognition theorems was initiated in [BGHS03, BeSh04] and named Curtis-Phan-Tits theory. The present paper uses a natural generalization to this theory to provide a new recognition theorem for symplectic groups.

Let us outline the Curtis-Phan-Tits theory setup. For details see [BGHS03]. We consider a group  $G$  which is either semi-simple of Lie type or a Kac-Moody group. Let  $\mathcal{T} = (B_+, B_-)$  be the associated twin-building. We first define a *flip* to be an involutory automorphism  $\sigma$  of  $\mathcal{T}$  that interchanges the two halves, preserves distances and codistances and takes at least one chamber to an opposite. Given a flip  $\sigma$ , construct  $\mathcal{C}_\sigma$  as the chamber system whose chambers are the pairs of opposite chambers  $(c, c^\sigma)$  of  $\mathcal{T}$ . Let  $G_\sigma$  be the fixed subgroup under the  $\sigma$ -induced automorphism of  $G$ . Whenever the geometry  $\Gamma_\sigma$  is simply connected one obtains  $G_\sigma$  as the universal completion of the amalgam of maximal parabolics for the action of  $G_\sigma$  on  $\Gamma_\sigma$ .

We now exhibit a limitation of the Curtis-Tits-Phan setup using the building  $\mathcal{T}$  of type  $A_n$  associated to  $G = \mathrm{PSL}_{n+1}(\mathbb{F})$  for some field  $\mathbb{F}$ . In this setting a flip  $\sigma$  is induced by a polarity. The objects of the geometry  $\Gamma_\sigma$  are the non-degenerate subspaces with respect to this polarity. The requirement that  $\sigma$ -invariant pairs of opposite chambers exist enforces that at least one 1-space  $p$  does not intersect its polar hyperplane: that is,  $p$  is non-absolute. Polarities are classified by the Birkhoff-Von Neumann theorem and those with non-absolute points correspond to orthogonal or unitary forms. Thus Curtis-Phan-Tits theory applied to the  $A_n$  building only yields amalgams for orthogonal and unitary groups.

Our setup deviates from the general Curtis-Phan-Tits setup in the following way. We still start with the building of type  $A_n$ , but we relinquish the requirement that  $\Gamma_\sigma$  should consist of pairs of opposite chambers in  $\mathcal{T}$ . Instead we consider pairs of *almost opposite* chambers  $(c, c^\sigma)$  corresponding to each other under a symplectic polarity  $\sigma$ . Note that  $\sigma$  is *not a flip* since symplectic polarities have no non-absolute points. We shall construct a geometry  $\Gamma$  similar to  $\Gamma_\sigma$  and obtain a presentation of the symplectic group.

There are several reasons why this setup and our result are interesting. Recently it has become clear [DeMu, BIHo] that in order to study (simple-) connectedness of Curtis-Phan-Tits geometries related to non-spherical twin-buildings one seems forced to study complexes of pairs of almost opposite chambers corresponding to each other under some involutory automorphism. Our result involves a detailed study of just such a complex.

The geometry  $\Gamma_\sigma$ , which presented itself naturally to the authors as a Quasi Curtis-Tits-Phan geometry, is not new to the literature [Cu94, Ha88, Ha89, G04]. This indicates that this geometry is worth investigating for its own sake.

The fact that the geometry  $\Gamma_\sigma$  has higher rank than the usual  $C_n$  building geometry will cause the parabolic subgroups in our amalgam presentation to be smaller than those appearing in the amalgam of rank 2 Levi-components. This is a benefit by itself, but it will in particular yield an amalgam presentation for  $\mathrm{Sp}_4(q)$ , a group that in the Curtis-Phan-Tits approach would be of rank 2 itself and not admit an amalgam.

**The main result** We shall now briefly describe the main result. Let  $V$  be a vector space of dimension  $2n \geq 4$  over a field  $\mathbb{F}$  endowed with a symplectic form  $s$  of maximal rank. Let  $I = \{1, 2, \dots, n\}$ . Moreover consider  $\mathcal{H} = \{e_i, f_i\}_{i \in I}$  a hyperbolic basis of  $V$ . The group of linear automorphisms of  $V$  preserving the form  $s$  will be called the *symplectic group of  $V$*  and is denoted  $G = \mathrm{Sp}(V)$ . We shall define a geometry  $\Gamma$  on the subspaces of  $V$  whose radical has dimension at most 1. This geometry is transversal, residually connected, and simply connected. Moreover,  $G$  acts flag-transitively on  $\Gamma$ . This then gives a presentation of  $G$  as the universal completion of the amalgam  $\mathcal{A}$  of its maximal parabolic subgroups with respect to its action on  $\Gamma$ . Induction allows us to replace the amalgam  $\mathcal{A}$  of maximal parabolic subgroups by the amalgam  $\mathcal{A}_{\leq 2}$  of parabolic subgroups of rank at most 2. A refinement of the amalgam  $\mathcal{A}_{\leq 2}$  then leads to the following setup.

Consider the following amalgam  $\mathcal{A}^\pi = \{M_i^\pi, S_j^\pi, M_{ik}^\pi, S_{jl}^\pi, Q_{ij}^\pi\}_{j, l \in I; i, k \in I - \{n\}}$ , whose groups are characterized as follows:

- $S_i^\pi$  is the stabilizer in  $G$  of all elements of  $\mathcal{H} - \{e_i, f_i\}$  and the subspace  $\langle e_i, f_i \rangle$ ,
- $M_i^\pi$  is the stabilizer in  $G$  of all elements of  $\mathcal{H} - \{f_i, f_{i+1}\}$  and the subspace  $\langle e_i, f_i, e_{i+1}, f_{i+1} \rangle$ ,
- $M_{ij}^\pi = \langle M_i^\pi, M_j^\pi \rangle$ ,  $S_{ij}^\pi = \langle S_i^\pi, S_j^\pi \rangle$ ,  $Q_{ij}^\pi = \langle M_i^\pi, S_j^\pi \rangle$ .

We describe these groups in more detail in Section 6. In particular, in Lemma 6.8 we show that the groups in this amalgam are very small. For instance,

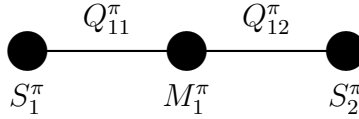
$$\begin{aligned} S_i^\pi &\cong \mathrm{Sp}_2(\mathbb{F}), \\ M_i^\pi &\cong \mathbb{F}^3. \end{aligned}$$

Our main result is the following.

**Theorem 1.1** *If  $|\mathbb{F}| > 2$  and  $V$  is a symplectic space of dimension  $2n \geq 4$  over  $\mathbb{F}$  then the symplectic group  $\mathrm{Sp}(V)$  is the universal completion of the amalgam  $\mathcal{A}^\pi$ .*

Moreover for any field  $\mathbb{F}$  the symplectic group is the universal completion of the amalgam  $\mathcal{A}$  from Section 5.2

**Corollary 1.2** *The group  $\mathrm{Sp}_4(\mathbb{F})$  is the universal completion of the following amalgam.  $\{S_1^\pi, S_2^\pi, M_1^\pi, S_{12}^\pi, Q_{11}^\pi, Q_{12}^\pi\}$ . Here,  $S_{12}^\pi \cong \mathrm{Sp}_2(\mathbb{F}) \times \mathrm{Sp}_2(\mathbb{F})$  and  $Q_{11}^\pi \cong Q_{12}^\pi \cong \mathbb{F}^3 \rtimes \mathrm{Sp}_2(\mathbb{F})$ .*



**Organization of this paper** In Section 2 we review some basic notions on geometries and some relevant facts about symplectic spaces. In Section 3 we introduce a geometry  $\Gamma$  on the almost non-degenerate subspaces of  $V$  with respect to some symplectic form of maximal rank and describe its residues. We prove that this geometry is transversal and residually connected. In Section 4 we show that the geometry and all residues of rank at least 3 are simply connected with one exception. We show that in the exceptional case the residue has a simply connected 2-cover. In Section 5 we describe the flag-transitive action of  $\mathrm{Sp}(V)$  on  $\Gamma$  and its rank 3 residues. We describe the parabolic subgroups in some detail and prove that  $\mathrm{Sp}(V)$  is the universal completion of the amalgam  $\mathcal{A}_{\leq 2}$  of parabolics of rank at most 2 for its action on  $\Gamma$ . In Section 6 we define a slim version of the amalgam  $\mathcal{A}_{\leq 2}$  by removing most of the Borel subgroup (for the action under consideration) from each of its groups. Finally, in Section 7 we define the amalgam  $\mathcal{A}^\pi$  and prove Theorem 1.1.

## 2 Preliminaries

### 2.1 Geometries

For our viewpoint on geometries we'll use the following definitions from Buekenhout [Bu95].

**Definition 2.1** A *pre-geometry* over a *type set*  $I$  is a triple  $\Gamma = (\mathcal{O}, \text{typ}, \star)$ , where  $\mathcal{O}$  is a collection of *objects* or *elements*,  $I$  is a set of *types*,  $\star$  is a binary symmetric and reflexive relation, called the *incidence relation* and  $\text{typ}: \mathcal{O} \rightarrow I$  is a *type function* such that whenever  $X \star Y$ , then either  $X = Y$  or  $\text{typ}(X) \neq \text{typ}(Y)$ .

The *rank* of the pre-geometry  $\Gamma$  is the size of  $\text{typ}(\mathcal{O})$ . A *flag*  $F$  is a (possibly empty) collection of pairwise incident objects. Its *type* (resp. *cotype*) is  $\text{typ}(F)$  (resp.  $I - \text{typ}(F)$ ). The *rank* of  $F$  is  $\text{rank}(F) = |\text{typ}(F)|$ . The *type* of  $F$  is  $\text{typ}(F) = \{\text{typ}(X) \mid X \in F\}$ . A *chamber* is a flag  $C$  of type  $I$ .

A pre-geometry  $\Gamma$  is a *geometry* if  $\text{typ}(\mathcal{O}) = I$  and if  $\Gamma$  is *transversal*, that is, if any flag is contained in a chamber.

The *incidence graph* of the pre-geometry  $\Gamma = (\mathcal{O}, \text{typ}, \star)$  over  $I$  is the graph  $(\mathcal{O}, \star)$ . This is a multipartite graph whose parts are indexed by  $I$ . We call  $\Gamma$  *connected* if its incidence graph is connected.

The *residue* of a flag  $F$  is the pre-geometry  $\text{Res}_\Gamma(F) = (\mathcal{O}_F, \text{typ}|_{\mathcal{O}_F}, \star|_{\mathcal{O}_F})$  over  $I - \text{typ}(F)$  induced on the collection  $\mathcal{O}_F$  of all objects in  $\mathcal{O} - F$  incident to  $F$ . We call  $\Gamma$  *residually connected* if for every flag of rank at least 2 the corresponding residue is connected.

For a subset  $K \subseteq I$  the *K-shadow* of a flag  $F$  is the collection of all  $K$ -flags incident to  $F$ .

We will mostly be working with connected, residually connected geometries over a set  $I$ . All our geometries will have a string diagram. This means that these geometries have a *diagram* as in Buekenhout [Bu95] and this diagram will look like a string, or a simple path. For convenience of the reader we will give the following equivalent ad-hoc definition here.

**Definition 2.2** We say that a geometry *has a string diagram* if there is a total ordering on its type set  $I$  such that for any three types  $i, j, k \in I$  with  $i < j < k$  we have the following. If  $X, Y, Z$  are objects of type  $i, j$ , and  $k$  respectively such that  $X$  and  $Z$  are incident with  $Y$ , then  $X$  is incident with  $Z$ .

Note that if a geometry has a string diagram, then so does every residue.

After choosing such a total ordering on  $I$ , we shall call the objects whose type is minimal in  $I$  *points* and those objects whose type  $i$  is minimal in  $I - \{i\}$  are called *lines*.

## 2.2 Automorphism groups and amalgams

**Definition 2.3** An *automorphism group*  $G$  of a pre-geometry  $\Gamma$  is a group of permutations of the collection of objects that preserves type and incidence. We call  $G$  *flag-transitive* if for any  $J \subseteq I$ ,  $G$  is transitive on the collection of  $J$ -flags.

Let  $G$  be a flag-transitive group of automorphisms of a geometry  $\Gamma$  over an index set  $I$ . Fix a chamber  $C$ . The *standard parabolic subgroup of type*  $J \subseteq I$  is the stabilizer in  $G$  of the residue of type  $J$  on  $C$ .

**Definition 2.4** In this paper we shall use the following definition of an amalgam of groups. Let  $(\mathcal{B}, \prec)$  be a meet-semilattice with minimal element  $\hat{0}$  in which every maximal chain has length  $s$ . An *amalgam* over  $(\mathcal{B}, \prec)$  is a collection of groups  $\mathcal{A} = \{A_\beta \mid \beta \in \mathcal{B}\}$  together with a system of homomorphisms  $\Phi = \{\varphi_{\beta,\gamma}: A_\beta \rightarrow A_\gamma \mid \beta \prec \gamma\}$  satisfying  $\varphi_{\gamma,\delta} \circ \varphi_{\beta,\gamma} = \varphi_{\beta,\delta}$  whenever  $\beta \prec \gamma \prec \delta$ . The number  $s$  is called the *rank* of  $\mathcal{A}$ .

The *universal completion* or *amalgamated sum* of  $\mathcal{A}$  is then a group  $\widehat{G}$  whose elements are words in the elements of the groups in  $\mathcal{A}$  subject to the relations between the elements of  $A_\beta$  for any  $\beta \in \mathcal{B}$  and in which for each  $\beta \prec \gamma$  each  $a \in A_\beta$  is identified with  $\varphi_{\beta,\gamma}(a) \in A_\gamma$ . We then have a homomorphism  $\widehat{\cdot}: \mathcal{A} \rightarrow \widehat{G}$ .

We note that for the appropriate choice of  $(\mathcal{B}, \prec)$  this definition of an amalgam and universal completion coincides with those given in [Se80, Ti86b].

### Note 2.5

- (i) For each  $\beta \in \mathcal{B}$  we have a homomorphism  $\widehat{\cdot}: A_\beta \rightarrow \widehat{A}_\beta \leq \widehat{G}$ , which is surjective, but not necessarily injective.
- (ii) For  $\beta, \gamma \in \mathcal{B}$  with  $\beta \prec \gamma$  we have  $\widehat{A}_\beta \leq \widehat{A}_\gamma$ .
- (iii) For  $\beta, \gamma \in \mathcal{B}$  we have  $\widehat{A}_{\beta \wedge \gamma} \leq \widehat{A}_\beta \cap \widehat{A}_\gamma$ , but we do not a priori assume equality here.

**Example 2.6** Let  $G$  be a group acting flag-transitively on a geometry  $\Gamma$  over an index set  $I$ . Let  $C$  be a chamber and, for every subset  $J \subseteq I$  with  $|J| \leq 2$  let  $P_J$  be the standard parabolic subgroup of type  $J$  in  $G$ . Then, for  $M \subseteq K$  we have the natural inclusion homomorphisms  $\varphi_{M,K}: P_M \rightarrow P_K$ . Hence  $\mathcal{A} = \{P_J \mid J \subseteq I, |J| \leq 2\}$  is an amalgam over  $\mathcal{B} = \{J \subseteq I \mid |J| \leq 2\}$  where  $M \prec K \iff M \subset K$ . For the universal completion  $\widehat{G}$  of  $\mathcal{A}$  we clearly have a surjective homomorphism  $\tau: \widehat{G} \rightarrow G$ .



### 2.3 Simple connectedness and amalgams

In order to introduce the main tool of this paper, namely Lemma 2.9 we need the notions of (closed) paths, (universal covers), simple connectedness, and the fundamental group.

In [Ti86b, Fo66, Ba80, Qu78] these notions are introduced in the context of (the face poset of) a simplicial complex in such a way that many classical results, such as can be found in [Sp81] continue to hold. In the present paper we use definitions geared towards geometries. They are equivalent to those for the (face poset of) the simplicial complex, called the *flag complex* consisting of all flags of  $\Gamma$  ordered by inclusion. For a more extensive treatment of related issues see e.g. [Ti86a, Pas94].

Let  $\Gamma$  be a connected geometry over the finite set  $I$ . A *path of length  $k$*  is a path  $x_0, \dots, x_k$  in the incidence graph. We do not allow repetitions, that is,  $x_i \neq x_{i+1}$  for all  $0 \leq i < k$ . A *cycle based at an element  $x$*  is a path  $x_0, \dots, x_k$  in which  $x_0 = x = x_k$ . Two paths  $\gamma$  and  $\delta$  are *homotopy equivalent* if one can be obtained from the other by inserting or eliminating cycles of length 2 or 3. We denote this by  $\gamma \simeq \delta$ . The homotopy classes of cycles based at an element  $x$  form a group under concatenation. This group is called the *fundamental group of  $\Gamma$  based at  $x$*  and is denoted  $\Pi_1(\Gamma, x)$ . If  $\Gamma$  is (path) connected, then the isomorphism type of this group does not depend on  $x$  and we call this group simply the *fundamental group of  $\Gamma$*  and denote it  $\Pi_1(\Gamma)$ . We call  $\Gamma$  *simply connected* if  $\Pi_1(\Gamma)$  is trivial.

Given  $k \in \mathbb{N}_{\geq 1}$ , a  *$k$ -covering* is an incidence and type preserving map  $\pi: \bar{\Gamma} \rightarrow \Gamma$ , where  $\bar{\Gamma}$  and  $\Gamma$  are geometries such that:

- (CO1) For any non-empty  $J$ -flag  $F$  in  $\Gamma$  the fiber  $\pi^{-1}(F)$  consists of exactly  $k$  distinct and disjoint  $J$ -flags.
- (CO2) Given a non-empty flag  $F$  in  $\Gamma$  and some flag  $\bar{F}$  in  $\bar{\Gamma}$  such that  $\pi(\bar{F}) = F$ , then the restriction  $\pi: \text{Res}(\bar{F}) \rightarrow \text{Res}(F)$  is an isomorphism.

We call  $\bar{\Gamma}$  a  *$k$ -cover* of  $\Gamma$ .

The key consequence of this definition in terms of proving several classical results on covers is that they possess the *unique path lifting property*:

- (L) Let  $x = x_0, \dots, x_n$  be a path in  $\Gamma$  and suppose  $\bar{x} \in \bar{\Gamma}$  satisfies  $\pi(\bar{x}) = x$ . Then, there is a unique path  $\bar{x} = \bar{x}_0, \dots, \bar{x}_n$  in  $\bar{\Gamma}$  such that  $\pi\bar{x}_i = x_i$  for all  $i = 0, 1, \dots, n$ .

We call a cover of  $\Gamma$  *universal* if it is universal in the category of coverings of  $\Gamma$  and morphisms.

**Theorem 2.7** *Given a connected geometry  $\Gamma$  and a covering  $\pi: \bar{\Gamma} \rightarrow \Gamma$ . Then, the space  $\bar{\Gamma}$  is universal among all covers of  $\Gamma$ , if and only if  $\bar{\Gamma}$  is simply connected.*

By a general categorical argument, a universal cover for  $\Gamma$  is unique up to isomorphism. Therefore  $\Gamma$  is its own universal cover if and only if  $\Pi_1(\Gamma)$  is trivial.

We can use the unique path lifting property to let  $\Pi_1(\Gamma)$  act on  $\bar{\Gamma}$  as follows. Let  $\delta = x_0, \dots, x_n$  and  $\bar{x}$  be as in (L) and denote by  $[\delta]$  the homotopy class of  $\delta$ . Then we define  $[\delta] \cdot \bar{x}_0 = \bar{x}_n$ . This action has the following property (See [Sp81, Chap.6, Lemmas 1-4] and [Pas94, Lemma 12.2]).

**Lemma 2.8** *Let  $\Gamma$  be a geometry with universal cover  $(\bar{\Gamma}, \pi)$  and let  $x$  be an object. Then the fundamental group  $\Pi_1(\Gamma)$  acts regularly on  $\pi^{-1}(x)$ .*

In Section 4 we show that for some space  $\Gamma$  we have  $\Pi_1(\Gamma) \cong \mathbb{Z}_2$ . In view of the fact that  $\Pi_1(\Gamma)$  is independent of the chosen base point, by Lemma 2.8 this is equivalent to saying that the universal cover of  $\Gamma$  is a 2-cover.

The following result, which will be referred to as Tits' Lemma, is a consequence of [Ti86b, Corollaire 1].

**Lemma 2.9** *Given a group  $G$  acting flag-transitively on a geometry  $\Gamma$ . Fix a maximal flag  $C$ . Then  $G$  is the universal completion of the amalgam consisting of the standard maximal parabolic subgroups of  $G$  with respect to  $C$  if and only if  $\Gamma$  is simply connected.*

We conclude this subsection with some methods to show that a geometry  $\Gamma$  is simply connected. This requires showing that any cycle based at a given element  $x$  is homotopy equivalent to a cycle of length 0. We call such a cycle *trivial* or *null-homotopic*.

**Lemma 2.10** *If  $\Gamma$  is a geometry, then any cycle all of whose elements are incident to a given element  $A$  is null-homotopic.*

**Proof** In this case, the cycle together with the object  $A$  forms a cone so that the cycle is null-homotopic.  $\square$

**Lemma 2.11** *Let  $\Gamma$  be a connected, residually connected geometry over a set  $I$  and let  $i, j \in I$  be distinct. Then, every cycle based at  $x$  is homotopy equivalent to a cycle consisting of objects of type  $i$  and  $j$  only.*

**Proof** Let  $x_0, x_1, \dots, x_n = x_0$  be a cycle. By transversality we may assume that  $x_0$  is an  $i$ -object. We proceed by induction on the number  $N$  of objects of type  $k \neq i, j$ . If  $N = 0$  we're done. Suppose  $N \geq 1$  and let  $l$  be minimal so that  $X = x_l$  has type different from  $i$  and  $j$ . By transversality we may assume that  $x_{l+1}$  is of

type  $i$  or  $j$ . By an easy induction argument one can show that since  $\Gamma$  is residually connected, the incidence graph induced on the collection  $X_{\{i,j\}}$  of all objects of type  $i$  and  $j$  incident to  $X$  is connected. Let  $x_{l-1} = y_0, \dots, y_m = x_{l+1}$  be a path in  $X_{\{i,j\}}$ . Then  $x_{l-1}, x_l, x_{l+1}$  and  $y_0, \dots, y_m$  are homotopy-equivalent by Lemma 2.10 since all these objects are incident to  $X$ . Thus we can replace  $x_0, x_1, \dots, x_l, \dots, x_n = x_0$  by the homotopic path  $x_0, \dots, x_{l-1} = y_0, y_1, \dots, y_m = x_{l+1}, \dots, x_n = x_0$  which contains only  $N - 1$  objects of type different from  $i, j$ . By induction we are done.  $\square$

### 3 A geometry for the symplectic group

Let  $V$  be a vector space of dimension  $n$  over a field  $\mathbb{F}$  endowed with a symplectic form  $\mathfrak{s}$  of maximal rank. We first need some notation to describe how  $\mathfrak{s}$  restricts to the various subspaces of  $V$ . Let  $\perp$  denote the orthogonality relation between subspaces of  $V$  induced by  $\mathfrak{s}$ . Thus, for  $U, W \leq V$  we have

$$U \perp W \iff \mathfrak{s}(u, w) = 0 \text{ for all } u \in U, w \in W.$$

We write

$$U^\perp = \{v \in V \mid \mathfrak{s}(u, v) = 0 \forall u \in U\} \leq V.$$

The *radical* of a subspace  $U$  is the subspace  $\text{Rad}(U) = U \cap U^\perp$ . The *rank* of  $U$  is  $\text{rank}(U) = \dim U - \dim \text{Rad}(U)$ . Note that since  $\mathfrak{s}$  is symplectic, we have  $\mathfrak{s}(v, v) = 0$  for all  $v \in V$  and so  $V$  has no anisotropic part with respect to  $\mathfrak{s}$ .

Let  $2r = \text{rank}(U)$  and  $d = \dim(U) - 2r$ . A *hyperbolic basis* for  $U$  is a basis  $\{e_i, f_j \mid 1 \leq i \leq r + d, 1 \leq j \leq r\}$  such that

(i) for all  $1 \leq i, j \leq r$ ,

$$\begin{aligned} \mathfrak{s}(e_i, e_j) = \mathfrak{s}(f_i, f_j) &= 0, \\ \mathfrak{s}(e_i, f_j) &= \delta_{ij}, \text{ and} \end{aligned}$$

(ii)  $\{e_{r+i} \mid 1 \leq i \leq d\}$  is a basis for  $\text{Rad}(U)$ .

**Lemma 3.1** *Suppose that  $W \leq U - \text{Rad}(U)$ . Then, any hyperbolic basis for  $W$  extends to a hyperbolic basis for  $U$ .*

**Proof** This is in some sense Witt's theorem. See e.g. Taylor [Ta92] for a proof.  $\square$

**Lemma 3.2** *Suppose that  $W \leq U - \text{Rad}(U)$  and  $r = \text{rank}(U)$ . If  $\dim(W) = 2r$ , then  $U = W \oplus \text{Rad}(U)$  and  $W$  is non-degenerate.*

**Proof** That  $U = W \oplus \text{Rad}(U)$  is simple linear algebra. As a consequence, and since  $\text{Rad}(U) \perp U$ , we have  $\text{Rad}(W) \leq \text{Rad}(U)$ . Thus  $\text{Rad}(W) \leq \text{Rad}(U) \cap W = \{0\}$ .  $\square$

**The Quasi Curtis-Phan-Tits geometry** The *quasi-Phan geometry*  $\Gamma = \Gamma(V)$  is a geometry over  $I = \{0, 1, \dots, n-1\}$  defined as follows. For  $i \in I$ , the  $i$ -objects, or objects of type  $i$ , are the  $i$ -spaces  $U \leq V - \text{Rad}(V)$  such that  $\dim(\text{Rad}(U)) \leq 1$ . More explicitly, since  $\mathfrak{s}$  is symplectic, this means that

$$\dim(\text{Rad}(U)) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

We say that two objects  $X$  and  $Y$  are *incident* whenever  $X \subseteq Y - \text{Rad}(Y)$  or vice versa.

**Note 3.3** The geometry  $\Gamma$  defined above can also be defined simply using points and (hyperbolic) lines. The objects of type  $i \geq 3$  are then generated by  $i$  points placed in suitable configuration (See for instance Lemma 3.2 in Blok [Bl07]). As such these geometries have been studied by Hall [Ha88, Ha89], Cuypers [Cu94], and Gramlich [G04].

It is not too difficult to see that  $\text{Sp}(V)$  is an automorphism group for  $\Gamma$ . In fact, Lemma 5.1 shows that it acts flag-transitively on  $\Gamma$ .

**Corollary 3.4** *Any hyperbolic basis for  $V$  gives rise to a unique chamber of  $\Gamma$  and, conversely, any chamber gives rise to a (not necessarily unique) hyperbolic basis for  $V$ .*

**Proof** Let  $C$  be a chamber. Then, for any two consecutive objects  $W, U \in C \cup \{V\}$  we have  $W \leq U - \text{Rad}(U)$ . Hence, using Lemma 3.1 repeatedly we find a hyperbolic basis  $\mathcal{H}$  for  $V$  such that, for any  $X \in C$ ,  $\mathcal{H} \cap X$  is a hyperbolic basis for  $X$ .

Conversely, let  $\mathcal{H}$  be a hyperbolic basis for  $V$ . Set  $d = \dim(\text{Rad}(V))$ . We have  $\mathcal{H} = \{e_i, f_j \mid 1 \leq i \leq r + d, 1 \leq j \leq r\}$ . For  $1 \leq i \leq r + d$ , let  $h_{2i-1} = e_i$  and, for  $1 \leq j \leq r$ , let  $h_{2j} = f_j$ . Then, setting  $C_l = \langle h_1, h_2, \dots, h_l \rangle_V$ , the collection  $C = \{C_l\}_{l=1}^{n-1}$  is a chamber of  $\Gamma$ .  $\square$

In the remainder of this paper we shall use the following standard setup. Let  $2r = \text{rank}(V)$  and let  $d = \dim(\text{Rad}(V)) \leq 1$ , so that  $n = 2r + d$ . We fix a hyperbolic basis

$$\mathcal{H} = \{e_i, f_j \mid 1 \leq i \leq r + d, 1 \leq j \leq r\}.$$

Alternatively we write

$$\begin{aligned} \mathcal{H} &= \{h_k\}_{k=1}^n, \text{ where} \\ h_k &= \begin{cases} e_i & \text{if } k = 2i - 1, \\ f_j & \text{if } k = 2j. \end{cases} \end{aligned}$$

We call this the *standard hyperbolic basis*. The *standard chamber* is the chamber  $C = \{C_k\}_{k=1}^{n-1}$  associated to  $\mathcal{H}$  as in Corollary 3.4. That is,  $C_k = \langle h_1, h_2, \dots, h_k \rangle_V$ , for all  $1 \leq k \leq n - 1$ .

**Lemma 3.5** *The pre-geometry  $\Gamma$  is transversal and has a string diagram.*

**Proof** Let  $F$  be a flag. Then, for any two consecutive objects  $W, U \in (F \cup \{V\})$  we have  $W \leq U - \text{Rad}(U)$ . Hence, using Lemma 3.1 repeatedly we find a hyperbolic basis  $\mathcal{H}$  for  $V$  such that, for any  $X \in C$ ,  $\mathcal{H} \cap X$  is a hyperbolic basis for  $X$ . According to Corollary 3.4,  $\mathcal{H}$  defines a unique chamber  $C$ . One verifies that  $F \subseteq C$ . The natural ordering on  $I$  provides  $\Gamma$  with a string diagram.  $\square$

**Lemma 3.6** *The pre-geometry  $\Gamma$  is connected. More precisely, the collinearity graph of the  $\{1, 2\}$ -shadow geometry has diameter at most 2 with equality if  $n \geq 3$ .*

**Proof** Let  $X, Z$  be 1-spaces in  $V - \text{Rad}(V)$ . If  $X$  and  $Z$  span a non-degenerate 2-space, then we are done. In particular, if  $n = 2$ , then this diameter is 1.

Other wise let  $W$  be a point on  $\langle X, Z \rangle$  different from  $X$  and  $Z$ . In case  $\langle X, Z \rangle \supseteq \text{Rad}(V)$ , let  $W = \text{Rad}(V)$ . Since  $V/\text{Rad}(V)$  is non-degenerate, there is a point  $Y$  in  $W^\perp$  that is not in  $\langle X, Z \rangle^\perp$ . Then clearly  $X, Z \not\subseteq Y^\perp$  and so  $X, Y, Z$  is a path in  $\Gamma$  from  $X$  to  $Z$ . Thus the collinearity graph of the  $\{1, 2\}$ -shadow geometry has diameter at most 2. Clearly equality holds.  $\square$

### 3.1 Residual geometries

Let  $C = \{C_i\}_{i \in I}$  be the standard chamber of  $\Gamma$  associated to the hyperbolic basis  $\mathcal{H}$ . For every  $J \subseteq I$  the *standard residue of type  $J$* , denoted  $R_J$ , is the residue of the  $(I - J)$ -flag  $\{C_i\}_{i \in I - J}$ . Let  $\bigsqcup_{m=1}^M J_m$  be the partition of  $J$  into maximal contiguous subsets. (We call  $K \subseteq I$  contiguous if, whenever  $i, k \in K$  and  $i < j < k$ , then  $j \in K$ .)

In this case,  $R_J = R_{J_1} \times R_{J_2} \times \cdots \times R_{J_M}$  since  $\Gamma$  has a string diagram. It now suffices to describe  $R_J$ , where  $J$  is contiguous. Let  $a = \min J$  and let  $b = \max J$ . There are two cases according as  $a$  is even or odd.

For odd  $a$ , the residue is the geometry  $\Gamma(C_{b+1}/C_{a-1}) \cong \Gamma((C_{a-1}^\perp \cap C_{b+1})/C_{a-1})$  of rank  $b - a + 2$ . We set  $C_0 = \{0\}$  and  $C_n = V$  for convenience.

For even  $a$ , we may assume that  $V = (C_{a-2}^\perp \cap C_{b+1})/C_{a-2}$  and that  $a = 2$ . Thus we need to describe the residue of  $C_1$ . We will show that  $\text{Res}_\Gamma(C_1)$  is isomorphic to a geometry  $\Pi(p, H)$  defined as follows.

**Definition 3.7** Note that  $\dim(\text{Rad}(V)) \leq 1$ . Let  $p$  be a 1-dimensional subspace of  $V - \text{Rad}(V)$  and let  $H$  be some complement of  $p$  in  $V$  containing  $\text{Rad}(V)$ . Note that  $\text{Rad}(H)$  if it is non-trivial, is not contained in  $p^\perp$ . Namely,  $\text{Rad}(H) \cap p^\perp \subseteq \text{Rad}(V)$ , which is 0 if  $\dim(V)$  is even.

Then we define  $\Pi(p, H)$  to be the geometry on the following collection of subspaces of  $H$ :

$$\{U \leq H \mid \text{Rad}(V) \not\subseteq U, \dim(\text{Rad}(U)) \leq 2 \text{ and } \text{Rad}(U) = \{0\} \text{ or } \text{Rad}(U) \not\subseteq p^\perp\}.$$

Let  $U$  and  $W$  be in  $\Pi(p, H)$  with  $\dim U < \dim W$ . We say that  $U$  is incident to  $W$  if either  $\dim(W)$  is odd and  $U \subseteq W$  or  $\dim(W)$  is even and  $U \subseteq W - \text{Rad}(W \cap p^\perp)$ .

More precisely, the objects are subspaces  $U \leq H$  not containing  $\text{Rad}(V)$  with the following properties

1. If  $U$  is odd dimensional then  $\text{Rad}(U)$  has dimension 1 and does not lie in  $p^\perp$ .

2. If  $U$  is even dimensional then  $U$  is either non-degenerate or  $\text{Rad}(U)$  has dimension 2 and is not contained in  $p^\perp$ .

**Lemma 3.8** *The map given by*

$$\begin{array}{ccc} \varphi: \text{Res}_\Gamma(p) & \rightarrow & \Pi(p, H) \\ X & \mapsto & X \cap H. \end{array}$$

*is an isomorphism.*

**Proof** We first show that  $X \in \text{Res}_\Gamma(p)$  if and only if  $X \cap H \in \Pi(p, H)$ . Note that since  $p$  is isotropic,  $p^\perp \cap H$  is a codimension 1 subspace of  $H$ . Note that if  $X \in \text{Res}_\Gamma(p)$ , then since  $X \in \Gamma$ ,  $\text{Rad}(V) \not\subseteq X$ . Moreover  $X \cap H$  is a complement of  $p$  in  $X$  so  $\text{Rad}(X \cap H)$  cannot have dimension more than two. Also  $\text{Rad}(X \cap H) \cap p^\perp \leq \text{Rad}(X)$  and so if  $X$  is even dimensional  $\text{Rad}(X \cap H) \cap p^\perp = \{0\}$  and if  $X$  is odd dimensional and  $X \cap H$  is degenerate then  $\text{Rad}(X \cap H) \not\subseteq p^\perp$ .

Conversely if  $U \in \Pi(p, H)$  it is easy to see that  $X = \langle U, p \rangle$  is in  $\text{Res}_\Gamma(p)$ . Indeed if  $U$  is odd dimensional, since  $p$  is not perpendicular to  $\text{Rad}(U)$ , the space  $X$  is non-degenerate. If  $U$  is even dimensional and non-degenerate then the space  $X$  is of maximal possible rank,  $p$  is not in  $\text{Rad}(X)$  and  $\text{Rad}(V) \not\subseteq X$ . Finally, if  $U$  is even dimensional and  $\text{Rad}(U)$  has dimension 2, then, since  $p$  is not perpendicular to the whole of  $\text{Rad}(U)$ , we have  $\text{Rad}(X) = \text{Rad}(U) \cap p^\perp \neq \text{Rad}(V)$ ,  $p$  and so  $X \in \text{Res}_\Gamma(p)$ .

Suppose that  $X$  and  $Y$  are incident elements of  $\text{Res}_\Gamma(p)$  and  $\dim(X) < \dim(Y)$ . If  $\dim(Y)$  is even then incidence is containment in both geometries. If  $\dim(Y)$  is odd then we need to prove that  $\text{Rad}(Y) \not\subseteq X$  iff  $\text{Rad}(Y^\varphi \cap p^\perp) \not\subseteq X^\varphi$ . We note that for any  $Z \in \text{Res}_\Gamma(p)$ , we have  $\text{Rad}(Z)^\varphi = \text{Rad}(Z^\varphi \cap p^\perp)$  and so the conclusion follows.  $\square$

**Lemma 3.9** *Two points  $p_1, p_2$  of  $\Pi(p, H)$  are collinear iff  $\text{Rad}(V) \not\subseteq \langle p_1, p_2 \rangle$ . In particular if  $\dim(V)$  is even then the collinearity graph of  $\Pi(p, H)$  is a complete graph and if  $\dim(V)$  is odd then the collinearity graph has diameter two.*

**Proof** If  $p_1, p_2$  are two points in  $\Pi(p, H)$ , then  $\langle p_1, p_2 \rangle$  is either totally isotropic but not contained in  $p^\perp$  or non-degenerate. So this is a line of  $\Pi(p, H)$  if and only if  $\text{Rad}(V) \not\subseteq \langle p_1, p_2 \rangle$ . Therefore the conclusion follows.  $\square$

**Lemma 3.10** *The pre-geometry  $\Gamma$  is residually connected.*

Let  $J \subseteq I$  and let  $R_J$  be the residue of the  $(I - J)$ -flag  $\{C_i\}_{i \in I - J}$ . Let  $\bigsqcup_{m=1}^M J_m$  be the partition of  $J$  into contiguous subsets. If  $M \geq 2$ , then the residue is connected since it is a direct product of geometries. Otherwise, the residue is isomorphic to  $\Gamma(V)$  for some vector space  $V$  of dimension at least 3, or to  $\Pi(C_1, H)$  inside some  $\Gamma(V)$  for some vector space  $V$  of dimension at least 4. Thus the connectedness follows from Lemmas 3.6 and 3.9.  $\square$

**Corollary 3.11** *The pre-geometry  $\Gamma$  is a geometry with a string diagram.*

**Proof** By Lemma 3.5,  $\Gamma$  is transversal and has a string diagram and by Lemma 3.10, it is residually connected.  $\square$

## 4 Simple connectedness

In this section we prove that the geometry  $\Gamma$  and all of its residues of rank at least 3 are simply connected. Note that  $\Gamma$  and all its residues are geometries with a string diagram. Therefore by Lemma 2.11 it suffices to show that all point-line cycles are null-homotopic.

**Lemma 4.1** *If  $|\mathbb{F}| \geq 3$  or  $\dim(V)$  is even, then any point-line cycle of  $\Gamma$  is null-homotopic.*

**Proof** Let  $\gamma$  be a point-line cycle based at a point  $x$ . We identify  $\gamma$  with the sequence  $x_0, \dots, x_k$  of points on  $\gamma$  (so  $\gamma$  is in fact a  $2k$ -cycle). We show by induction on  $k$  that  $\gamma$  is null-homotopic.

If  $k \leq 3$ , then  $U = \langle x_0, x_1, x_2 \rangle_V$  is non-isotropic as it contains the hyperbolic line  $\langle x_0, x_1 \rangle_V$ . If  $\text{Rad}(V) \not\subseteq U$ , then it is an object of the geometry and so by Lemma 2.10,  $\gamma$  is null-homotopic. If  $\text{Rad}(V) \subseteq U$ , then since  $|\mathbb{F}| \geq 3$  there is a point  $x$  such that  $x$  is collinear to  $x_0$ ,  $x_1$ , and  $x_2$ . Namely, consider the points  $\text{Rad}(V)x_i$  in the non-degenerate space  $V/\text{Rad}(V)$ . These are 3 distinct points on the non-degenerate 2-space  $U/\text{Rad}(V)$ . Take a fourth point  $y$  on this line. Since  $V/\text{Rad}(V)$  is non-degenerate, there is a point  $x$  orthogonal to  $y$  but not to any other point on  $U/\text{Rad}(V)$ .

Now let  $k \geq 4$ . If two non-consecutive points  $x_i$  and  $x_j$  in  $\gamma$  are collinear, then let  $\gamma_1 = \delta_1 \circ \delta_2 \circ \delta_1^{-1}$  and  $\gamma_2 = \delta_1 \circ \delta_3$ , where

$$\begin{aligned} \delta_1 &= x_0, \dots, x_i \\ \delta_2 &= x_i, x_{i+1}, \dots, x_{j-1}, x_j, x_i. \\ \delta_3 &= x_i, x_j, \dots, x_k. \end{aligned}$$

Clearly  $\gamma_1 \circ \gamma_2$  is homotopic to  $\gamma$ . Also, the cycles  $\delta_2$  and  $\gamma_2$  are both shorter than  $\gamma$ . By induction,  $\delta_2$  and  $\gamma_2$  are null-homotopic and hence so are  $\gamma_1$  and  $\gamma$  itself.

Therefore we may assume that no two non-consecutive elements in  $\gamma$  are collinear. In particular  $x_1 \perp x_{k-1}$  and  $x_{k-2} \perp x_0$ . The line  $\langle x_{k-1}, x_k \rangle_V$  has at least three points. Take any  $y \in \langle x_{k-1}, x_k \rangle - \{x_{k-1}, x_k\}$ . Then,  $y \not\perp x_1, x_{k-2}$ . Thus, since  $x_1 \not\perp x_k = x_0$  and  $x_1 \perp x_{k-1}$ , the point  $y$  is collinear to  $x_1$ . By the same reasoning  $y$  is collinear to  $x_{k-2}$ .



Now let  $\gamma_1 = \delta_1 \circ \delta_2$  and  $\gamma_2 = \delta_2^{-1} \circ \delta_3 \circ \delta_2$  and  $\gamma_3 = \delta_2^{-1} \circ \delta_4 \circ \delta_2$ , where

$$\begin{aligned}\delta_1 &= x_0, x_1, y \\ \delta_2 &= y, x_0 \\ \delta_3 &= y, x_1, x_2, \dots, x_{k-2}, y \\ \delta_4 &= y, x_{k-2}, x_{k-1}, y\end{aligned}$$

Then,

$$\gamma \simeq \gamma_1 \circ \gamma_2 \circ \gamma_3 \simeq \gamma_2 \simeq 0,$$

where the second equivalence holds since  $\gamma_1$  and  $\gamma_3$  are triangles, and the third equivalence holds since  $\gamma_2 \simeq 0$  by induction.  $\square$

**Proposition 4.2** *The geometry  $\Gamma$  is simply connected.*

**Proof** First of all,  $\Gamma$  is connected by Lemma 3.6. Thus it suffices to show that any cycle is null-homotopic. By Lemma 2.11 such a cycle is homotopic to a point-line cycle. Finally, by Lemma 4.1 point-line cycles are null-homotopic and the result follows.  $\square$

#### 4.1 Residual geometries

We shall now prove that, apart from one exception, all residues of rank at least 3 are simply connected. As we saw above any residue is either isomorphic to  $\Gamma(V)$  for some  $V$ , or to  $\Pi(p, H)$  for some point  $p$  inside  $\Gamma(V)$  for some  $V$  or it is a direct product of such geometries and possibly rank 1 residues.

We already proved that  $\Gamma(V)$  is simply connected. We shall now prove that, apart from one exception,  $\Pi(p, H)$  is a simply connected geometry. By Lemma 3.9, if  $n$  is even, then the collinearity graph of  $\Pi(p, H)$  is a complete graph and so we only have to show that all triangles are null-homotopic. If  $n$  is odd, then the diameter of the collinearity graph is 2 and so all  $k$ -cycles can be decomposed into cycles of length 3, 4, and 5.

**Lemma 4.3** *Suppose that  $\dim(V)$  is odd. Then any cycle of length 4 or 5 can be decomposed into triangles.*

**Proof** We first note that two points  $p_1$  and  $p_2$  are at distance 2 only if  $\langle p_1, p_2 \rangle \supseteq \text{Rad}(V)$ . Therefore, if  $p_1, p_2, p_3, p_4$  is a 4-cycle, then  $\text{Rad}(V) \subseteq \langle p_1, p_3 \rangle, \langle p_2, p_4 \rangle$ . Since  $|\mathbb{F}| \geq 2$  there exists a line  $L$  on  $\text{Rad}(V)$  different from these two lines and any point of  $L - p^\perp$  is collinear to all of  $p_1, p_2, p_3, p_4$ . Thus we decompose the 4-cycle into triangles. Now suppose  $p_1, p_2, p_3, p_4, p_5$  is a 5-cycle. Then  $\text{Rad}(R)$  lies on at most 1 of  $p_1p_3$  and  $p_1p_4$  and so one of these lines is in fact a line of the geometry. Thus, we can decompose the 5-cycle into shorter cycles.  $\square$

**Lemma 4.4** *Consider a hyperplane  $W$  of  $H$ . If  $\dim(V)$  is even, then  $W \in \Pi(p, H)$ . If  $\dim(V)$  is odd, then  $W \in \Pi(p, H)$  if and only if  $\text{Rad}(V) \not\subseteq W$ .*

**Proof** Let  $\dim(V)$  be even and let  $S = \text{Rad}(H)$ . Then  $\dim(S) = 1$  and  $S \not\subseteq p^\perp$ . If  $S \not\subseteq W$ , then  $W$  is non-degenerate and so it belongs to  $\Pi(p, H)$ . If  $S \subseteq W$ , then  $S \subseteq \text{Rad}(W)$ , which has dimension 2 and so  $\text{Rad}(W) \not\subseteq p^\perp$ . So again  $W \in \Pi(p, H)$ .

Now let  $\dim(V)$  be odd and let  $R = \text{Rad}(V)$ . Then  $R \subseteq \text{Rad}(H)$ , which has dimension 2 and is not included in  $p^\perp$ . Now since  $W$  is a hyperplane of  $H$  it either contains  $\text{Rad}(H)$  or it intersects it in a 1-dimensional space. In the former case,  $R \subseteq \text{Rad}(W)$  and  $W \notin \Pi(p, H)$ . In the latter case, either  $R \subseteq \text{Rad}(W)$  and  $W \notin \Pi(p, H)$ , or  $\text{Rad}(W) \not\subseteq p^\perp$  and  $W \in \Pi(p, H)$ .  $\square$

**Lemma 4.5** *If  $\dim(V)$  is odd, then any triangle of  $\Pi(p, H)$  is null-homotopic.*

**Proof** Take a triangle on the points  $p_1, p_2, p_3$ . Note that this means that  $R = \text{Rad}(V)$  does not lie on any of the lines  $p_i p_j$ . Let  $U = \langle p_1, p_2, p_3 \rangle$ . We have two cases: 1)  $R \not\subseteq U$ . Then pick a hyperplane  $W$  of  $H$  containing  $U$  but not  $R$ . Then by Lemma 4.4,  $W \in \Pi(p, H)$ . Moreover, since  $\dim(W)$  is odd it is incident to each of the lines  $p_i p_j$ .

2)  $R \subseteq U$ . Let  $S$  be a point of  $\text{Rad}(H) - R$ . Then  $S \not\subseteq p^\perp$ . Also note that for any  $i$  and  $j$ ,  $p_i, p_j, S$  is a triangle of type (1), so the triangle  $p_1, p_2, p_3$  is null-homotopic.  $\square$

**Lemma 4.6** *Suppose  $\dim(V) \geq 7$  or  $|\mathbb{F}| \geq 3$ . Then any triangle of  $\Pi(p, H)$  is null-homotopic.*

**Proof** If  $\dim(V)$  is odd, then we are done by Lemma 4.5. So now let  $\dim(V) = n$  be even. Let  $Q = \text{Rad}(H)$ . Take a triangle on the points  $p_1, p_2, p_3$  and let  $U = \langle p_1, p_2, p_3 \rangle$ . If  $U$  is a plane, then we're done by Lemma 2.10. From now on assume this is not the case. So  $U$  is either totally isotropic, or it has rank 2 and  $T = \text{Rad}(U)$  is contained in  $p^\perp \cap H$ .

Let  $L = U \cap p^\perp$ . For  $1 \leq i < j \leq 3$ , let  $L_{ij} = p_i p_j$  and  $q_{ij} = L_{ij} \cap L$ . Our aim is to find an object  $W$  of  $\Pi(p, H)$  that is incident to all lines  $L_{ij}$ . Then, by Lemma 2.10 we're done.

Suppose we can find a point  $r$  in  $p^\perp \cap H$  such that  $r^\perp \supseteq L$ . For such a point let  $W = \langle r^\perp \cap p^\perp \cap H, U \rangle$ . First note that  $\dim(r^\perp \cap p^\perp \cap H) = n - 3$ . This is because  $p^\perp \cap H$  is a non-degenerate symplectic space of even dimension  $n - 2$ . Clearly  $r^\perp \cap p^\perp \cap H$  has radical  $r$ .

Note that  $L \subseteq r^\perp \cap p^\perp \cap H$ . Also, since  $U \not\subseteq p^\perp \cap H$  and  $U \cap p^\perp = L$  we have  $U \cap (W \cap p^\perp) = L$  and  $W \cap p^\perp = r^\perp \cap p^\perp \cap H$ . Hence  $W = \langle (W \cap p^\perp), U \rangle$  and

so  $\dim(W) = \dim(W \cap p^\perp) + \dim(U) - \dim(L) = n - 2$ . Thus  $W$  is an object of  $\Pi(p, H)$  by Lemma 4.4 as required.

We now show that we can find such a point  $r$  that is different from  $q_{12}$ ,  $q_{13}$ , and  $q_{23}$ . Then, since  $r = \text{Rad}(W \cap p^\perp)$ , all lines  $L_{ij}$  are incident to  $W$  as required.

There are two cases. (1)  $n = 6$  and  $|\mathbb{F}| \geq 3$ . Since  $|\mathbb{F}| \geq 3$  we can find a point  $r \in L - \{q_{12}, q_{13}, q_{23}\}$ . Note that  $L$  is totally isotropic and so  $L \subseteq r^\perp$ , as required.

(2)  $n \geq 7$ . Note that in this case in fact  $n \geq 8$  since we assume that  $n$  is even. We find a point  $r$  in  $L^\perp \cap p^\perp \cap H - L$ . This is possible since  $p^\perp \cap H$  is non-degenerate and  $L$  is a 2-space so that  $\dim(L^\perp \cap p^\perp \cap H) = (n - 2) - 2 \geq 4 > \dim(L) = 2$ . Again we have found  $r$  as required.  $\square$

**Proposition 4.7** *If  $|\mathbb{F}| \geq 3$  then the residues of rank at least 3 are simply connected.*

**Proof** Let  $J \subseteq I$  and let  $R_{I-J}$  be the residue of the  $J$ -flag  $\{C_j\}_{j \in J}$ . Set  $r = |I - J|$ . Let  $\bigsqcup_{m=1}^M I_m$  be the partition of  $I - J$  into contiguous subsets. If  $M \geq 2$ , then the residue is a direct product of two geometries, at least one of which is a residue of rank at least 2. Such rank 2 residues are connected by Lemma 3.10. Therefore  $R$  is simply connected in this case. Otherwise, the residue is isomorphic to  $\Gamma(V)$  for some vector space  $V$  of dimension at least 3, or to  $\Pi(p, H)$  for some point  $p$  inside some  $\Gamma(V)$  for some vector space  $V$  of dimension at least 4. Therefore the simple connectedness follows from Lemmas 4.6, 4.5 and Proposition 4.2.  $\square$

## 4.2 The exceptional residue

We are now left with the intriguing case  $n = 6, |\mathbb{F}| = 2$ . Let us first describe the geometry  $\Pi(p, H)$ . The points are the 16 points of  $H - p^\perp$ . The lines are those lines of  $H$  not contained in  $p^\perp$ . Thus there are two types of lines, totally isotropic and hyperbolic ones. Each line has exactly two points and any two points are on exactly one line.

The planes of  $\Pi(p, H)$  are those non-isotropic planes of  $H$  whose radical is not contained in  $p^\perp$ . Such a plane has rank 2 and its radical is a point of  $\Pi(p, H)$ , i.e. it is in  $H - p^\perp$ . Each plane has exactly 4 points. A plane may or may not contain  $\text{Rad}(H)$ . Any point or line contained in a plane is incident to that plane.

The 4-spaces of  $\Pi(p, H)$  are those 4-spaces of  $H$  that are either non-degenerate or have a radical of dimension 2 that is not contained in  $p^\perp$ . A 4-space  $W$  is incident to all 8 points it contains and is incident to any line or plane it contains that doesn't pass through the point  $\text{Rad}(W \cap p^\perp)$ . Thus in fact  $W$  is incident to all planes  $Y$  of  $\Pi(p, H)$  contained in  $W$ . Namely, if  $\text{Rad}(W \cap p^\perp) \leq Y$ , then  $p^\perp \cap Y$  is a totally isotropic line, implying that  $\text{Rad}(Y) \leq p^\perp$ , a contradiction.

The geometry  $\Pi(p, H)$  is not simply connected. To see this, we construct a simply connected rank 4 geometry  $\bar{\Pi}(p)$  which is a degree two cover of  $\Pi(p, H)$ . Let

us now describe the geometry  $\overline{\Pi}(p)$ . A point  $q \in \Gamma$  is a point of  $\overline{\Pi}(p)$  if the line  $pq$  is non-degenerate. In particular the points of  $\Pi(p, H)$  are among the points of  $\overline{\Pi}(p)$ . We construct a map, which on the point set of  $\overline{\Pi}(p)$  is given by

$$\begin{aligned} \psi: \overline{\Pi}(p) &\rightarrow \Pi(p, H) \\ q &\mapsto \langle p, q \rangle \cap H. \end{aligned}$$

This map is two-to-one and for any  $q \in \Pi(p, H)$  we set  $q^- = q$  and denote by  $q^+$  the point of  $\overline{\Pi}(p)$  such that  $\psi^{-1}(q) = \{q^-, q^+\}$ . We extend this notation to arbitrary point sets  $S$  of  $\Pi(p, H)$  by setting  $S^\epsilon = \{q^\epsilon \mid q \in S\}$  for  $\epsilon = \pm$ .

We now note that every object of  $\Pi(p, H)$  can be identified with its point-shadow. It follows from the above description of  $\Pi(p, H)$  however that inclusion of point-shadows does not always imply incidence.

In order to describe  $\overline{\Pi}(p)$ , we shall identify objects in  $\Pi(p, H)$  and  $\overline{\Pi}(p)$  with their point-shadow. We denote point-shadows by roman capitals. If we need to make the distinction between objects and their point-shadows explicit, we'll use calligraphic capitals for the objects and the related roman capitals for their point-shadows. Thus  $\mathcal{X}$  may denote an object of  $\Pi(p, H)$  (or  $\overline{\Pi}(p)$ ) whose point-shadow is  $X$ .

For any object  $\mathcal{X}$  of  $\Pi(p, H)$  we will define exactly two objects  $\mathcal{X}_-$  and  $\mathcal{X}_+$  in  $\overline{\Pi}(p)$  such that  $\psi(X_-) = \psi(X_+) = X$  as point-sets. We then define objects  $\mathcal{X}$  and  $\mathcal{Y}$  of  $\overline{\Pi}(p)$  to be incident whenever  $X \subseteq Y$  or  $Y \subseteq X$  and  $\psi(\mathcal{X})$  and  $\psi(\mathcal{Y})$  are incident in  $\Pi(p, H)$ .

We shall obtain  $\mathcal{X}_-$  and  $\mathcal{X}_+$  by defining a partition  $X_0 \uplus X_1$  of the point-set of  $\mathcal{X}$  and setting  $X_+ = X_0^+ \uplus X_1^-$  and  $X_- = X_0^- \uplus X_1^+$ .

First let  $\mathcal{X}$  be a line. If  $\mathcal{X}$  is non-degenerate, then  $X_0 = X$  and  $X_1 = \emptyset$  so that  $X_+ = X^+$  and  $X_- = X^-$ . If  $\mathcal{X}$  is isotropic, then  $X = \{x_0\}$  and  $X_1 = \{x_1\}$  so that  $X_- = \{x_0^-, x_1^+\}$  and  $X_+ = \{x_0^+, x_1^-\}$ .

Next, let  $\mathcal{X}$  be a plane. Then  $X_0 = \{\text{Rad}(X)\}$  and  $X_1 = X - X_0$ . Note that for every line  $\mathcal{Y}$  incident to  $\mathcal{X}$  the partition  $Y_0 \uplus Y_1$  agrees with the partition  $X_0 \uplus X_1$ . Hence if  $\mathcal{Y}$  is a line and  $\mathcal{X}$  is a plane in  $\overline{\Pi}(p)$  such that  $\psi(\mathcal{Y})$  is incident to  $\psi(\mathcal{X})$ , then either  $Y \subseteq X$  or  $Y \cap X = \emptyset$ .

Finally, let  $\mathcal{X}$  be a 4-space. Let  $r = \text{Rad}(X \cap p^\perp)$ . Note that the projective lines  $L_i$  ( $i = 1, 2, 3, 4$ ) of  $\mathcal{X}$  on  $r$  meeting  $H - p^\perp$  are not incident to  $\mathcal{X}$  in  $\Pi(p, H)$ .

First let  $\mathcal{X}$  be non-degenerate. Then  $L_i$  is non-degenerate for all  $i$ . For  $i = 1, 2, 3, 4$ , let  $L_i = \{r, p_i, q_i\}$  such that  $q_i = p_1^\perp \cap L_i$  for  $i = 2, 3, 4$ . Note that this implies the fact that if  $i \neq j$ , then  $p_i \perp q_j$  but  $q_i \not\perp q_j$  and  $p_i \not\perp p_j$ . Then  $X_0 = \{p_1, \dots, p_4\}$  and  $X_1 = \{q_1, \dots, q_4\}$ . We now claim that for every line  $\mathcal{Y}$  incident to  $\mathcal{X}$  the partition  $Y_0 \uplus Y_1$  agrees with  $X_0 \uplus X_1$ . Thus we must show that if  $\mathcal{Y}$  is non-degenerate, then  $Y \subseteq X_0$  or  $Y \subseteq X_1$  and if  $\mathcal{Y}$  is isotropic, then  $Y$  intersects  $X_0$  and  $X_1$  non-trivially. First, since  $\mathcal{X}$  is non-degenerate, the lines  $q_i q_j$

are all non-degenerate for  $2 \leq i < j \leq 4$ . Moreover,  $q_1q_i$  is non-degenerate as well, for  $i = 2, 3, 4$  since otherwise  $L_i$  must be totally isotropic, a contradiction. As a consequence,  $q_1p_i$  is totally isotropic. Hence by interchanging the  $p_i$ 's for  $q_i$ 's we see that  $p_ip_j$  is non-degenerate for all  $1 \leq i < j \leq 4$ . Considering the non-degenerate lines  $L_i$  and  $L_j$  we see that  $p_iq_j$  must be isotropic for all  $1 \leq i \neq j \leq 4$ . We have exhausted all lines incident to  $\mathcal{X}$  and it is clear that the claim holds.

If  $\mathcal{X}$  is degenerate, then it has a radical  $R = \text{Rad}(X)$  of dimension 2 passing through the radical  $r = \text{Rad}(X \cap p^\perp)$ . In this case the lines  $L_i$  are all totally isotropic. Let  $R = L_1$ . Then  $X_0 = \{p_1, q_1\}$  and  $X_1 = \{p_i, q_i \mid i = 2, 3, 4\}$ . We now claim that for every line  $\mathcal{Y}$  incident to  $\mathcal{X}$  the partition  $Y_0 \uplus Y_1$  agrees with  $X_0 \uplus X_1$ . Clearly every line meeting  $X_0$  and  $X_1$  is totally isotropic. Next consider a line  $\mathcal{Y}$  meeting  $L_i$  and  $L_j$  in  $X_1$ . If  $\mathcal{Y}$  were totally isotropic, then  $X = \langle L_1, L_i, L_j \rangle$  is totally isotropic, a contradiction. We have exhausted all lines incident to  $X$  and it is clear that the claim holds.

Next we claim that for every plane  $\mathcal{Y}$  incident to  $\mathcal{X}$  the partition  $Y_0 \uplus Y_1$  agrees with  $X_0 \uplus X_1$ . First note that  $\mathcal{Y}$  doesn't contain  $\text{Rad}(X \cap p^\perp)$ , because that would make  $Y \cap p^\perp$  totally isotropic and  $\text{Rad}(Y) \leq p^\perp$  contrary to the description of planes of  $\Pi(p, H)$ . As a consequence, if  $\mathcal{L}$  is a line incident to  $\mathcal{Y}$ , then it doesn't meet  $\text{Rad}(X \cap p^\perp)$  and so is incident to  $\mathcal{X}$  as well. Hence since for every line  $\mathcal{L}$  incident to  $\mathcal{Y}$ , the partition  $L_0 \uplus L_1$  agrees with the partition  $X_0 \uplus X_1$  as well as with the partition  $Y_0 \uplus Y_1$ , also the latter two partitions agree. We can conclude that if  $\mathcal{Y}$  is a point, line or plane and  $\mathcal{X}$  is a 4-space of  $\overline{\Pi}(p)$  such that  $\psi(\mathcal{Y})$  and  $\psi(\mathcal{X})$  are incident, then either  $Y \subseteq X$  or  $Y \cap X = \emptyset$ .

Let  $\Psi$  be the extension of  $\psi$  to the entire collection of objects in  $\overline{\Pi}(p)$ .

**Lemma 4.8** (a) *The map  $\Psi: \overline{\Pi}(p) \rightarrow \Pi(p, H)$  is 2-to-1 on the objects. For any object  $\mathcal{X}$  of  $\Pi(p, H)$ , if  $\Psi^{-1}(\mathcal{X}) = \{\mathcal{X}_-, \mathcal{X}_+\}$ , then  $\psi^{-1}(X) = X_- \uplus X_+$ .*

(b) *Two objects  $\mathcal{X}$  and  $\mathcal{Y}$  of  $\overline{\Pi}(p)$  are incident if and only if  $\Psi(\mathcal{X})$  and  $\Psi(\mathcal{Y})$  are incident and  $X \cap Y \neq \emptyset$ .*

(c) *For any non-empty flag  $F_\bullet$  in  $\overline{\Pi}(p)$ , the map  $\Psi: \text{Res}(F_\bullet) \rightarrow \text{Res}(\Psi(F_\bullet))$  is an isomorphism of geometries.*

(d) *The map  $\Psi: \overline{\Pi}(p) \rightarrow \Pi(p, H)$  is a 2-cover.*

(e) *The pre-geometry  $\overline{\Pi}(p)$  is transversal.*

**Proof** (a) This clear from the construction of  $\overline{\Pi}(p)$ .

(b) By definition of incidence in  $\overline{\Pi}(p)$ ,  $\mathcal{X}$  and  $\mathcal{Y}$  can only be incident if  $\Psi(\mathcal{X})$  and  $\Psi(\mathcal{Y})$  are incident. The preceding discussion has shown that in this case either

$X \cap Y = \emptyset$  or  $X \subseteq Y$  or  $Y \subseteq X$ . Now  $\mathcal{X}$  and  $\mathcal{Y}$  are defined to be incident precisely in the latter case.

(c) Let  $\mathcal{X}_\bullet$  and  $\mathcal{Y}_\bullet$  denote objects of  $\overline{\Pi}(p)$ . Also, let  $\mathcal{X} = \Psi(\mathcal{X}_\bullet)$ ,  $\mathcal{Y} = \Psi(\mathcal{Y}_\bullet)$  and  $F = \Psi(F_\bullet)$ . By definition of the objects in  $\overline{\Pi}(p)$ ,  $\psi: X_\bullet \rightarrow X$  is a bijection of points. Therefore if  $\mathcal{Y}$  is incident with  $\mathcal{X}$ , then  $\mathcal{X}_\bullet$  is incident with exactly one object in  $\Psi^{-1}(\mathcal{Y})$ . Let  $\mathcal{Y}$  be incident with  $F$ . Then by the same token  $F_\bullet$  is incident with at most one object in  $\Psi^{-1}(\mathcal{Y})$ . We now show that there is at least one such object. Suppose that  $\mathcal{Y}_\bullet$  is incident to at least one object  $\mathcal{Z}_\bullet$  of  $F_\bullet$ . Without loss of generality assume  $Y_\bullet \subseteq Z_\bullet$ . If  $X_\bullet$  is an element of  $F_\bullet$  and  $Z_\bullet \subseteq X_\bullet$  then  $\mathcal{Y}_\bullet$  is incident to  $\mathcal{X}_\bullet$  as well. Now assume  $X_\bullet \subseteq Z_\bullet$ . Since  $\mathcal{X}$  and  $\mathcal{Y}$  are incident,  $\mathcal{Y}_\bullet$  must be either incident to  $\mathcal{X}_\bullet$  or to the object in  $\Psi^{-1}(\mathcal{X})$  different from  $\mathcal{X}_\bullet$ . In the latter case it follows that  $\mathcal{Z}_\bullet$  is incident to both objects in  $\Psi^{-1}(\mathcal{X})$ , a contradiction. A similar argument holds when  $Z_\bullet \subseteq Y_\bullet$ . We conclude that  $\Psi: \text{Res}(F_\bullet) \rightarrow \text{Res}(F)$  is a bijection. Clearly incidence is preserved by  $\Psi$ , but we must show the same holds for  $\Psi^{-1}: \text{Res}(F) \rightarrow \text{Res}(F_\bullet)$ . Let  $\mathcal{X}, \mathcal{Y} \in \text{Res}(F)$  be incident and let  $\mathcal{X}_\bullet, \mathcal{Y}_\bullet \in \text{Res}(F_\bullet)$ . Then there is a point  $q$  incident to  $\mathcal{X}, \mathcal{Y}$  and  $F$ . Suppose  $q^\epsilon \in F_\bullet$ . Then  $q^\epsilon \in X_\bullet \cap Y_\bullet$  and by (b) we find that  $\mathcal{X}_\bullet$  and  $\mathcal{Y}_\bullet$  are incident.

(d) This follows from (a) and (c).

(e) This is immediate from (c).  $\square$

**Lemma 4.9** *The pre-geometry  $\overline{\Pi}(p)$  is connected. Any two points are at distance at most 2, except the points  $\text{Rad}(H)^\pm$ , which are at distance 3 from one another.*

**Proof** Let  $\epsilon \in \{+, -\}$ . Let  $Q = \text{Rad}(H)$ . Then for any point  $q \neq Q$ , since the line  $qQ$  is totally isotropic,  $Q^\epsilon$  is collinear to  $q^{-\epsilon}$  but not to  $q^\epsilon$ . In particular any two points with the same sign are at distance at most 2. It is also clear that  $Q^+$  and  $Q^-$  have no common neighbors and are at distance at least 3.

Now consider two points  $q_1, q_2 \neq Q$ . If the line  $q_1q_2$  is totally isotropic, then  $q_1^\epsilon$  is collinear to  $q_2^{-\epsilon}$  in  $\overline{\Pi}(p)$ . If the line  $q_1q_2$  is non-degenerate, we claim that there exists  $q_3 \in \Pi(p, H)$  with  $q_1 \perp q_3 \not\perp q_2$ . Namely, we must show that  $q_1^\perp - (q_2^\perp \cup p^\perp) \neq \emptyset$ . However, this is clear since both  $p^\perp$  and  $q_2^\perp$  define proper hyperplanes of the 4-space  $q_1^\perp$ . Since no linear subspace of  $V$  is the union of two of its hyperplanes our claim follows. In  $\overline{\Pi}(p)$  we find both  $q_1^\epsilon$  and  $q_2^{-\epsilon}$  collinear to  $q_3^{-\epsilon}$  so again  $q_1$  and  $q_2$  are at distance at most 2.

Finally consider  $Q^+$  and  $Q^-$ . Let  $q_1$  be a point of  $\Pi(p, H)$  and let  $q_2$  be a point of  $\Pi(p, H)$  in  $q_1^\perp - \{Q\}$ . Then  $Q^-, q_1^+, q_2^-, Q^+$  is a path of length 3.  $\square$

**Lemma 4.10** *The pre-geometry  $\overline{\Pi}(p)$  is residually connected.*

**Proof** By Lemma 4.9,  $\overline{\Pi}(p)$  is connected so it suffices to show that every residue of rank at least 2 is connected. This follows immediately from Lemmas 4.8 and 3.10.  $\square$

**Lemma 4.11** *The pre-geometry  $\overline{\Pi}(p)$  is a geometry with a string diagram.*

**Proof** By Lemma 4.8 it is transversal and by Lemma 4.10 it is residually connected. Therefore it is a geometry. That it has a string diagram is clear since it is a cover of  $\Pi(p, H)$ , which does have a string diagram.  $\square$

**Lemma 4.12** *The geometry  $\overline{\Pi}(p)$  is simply connected.*

**Proof** By Lemma 4.11 and 2.11 it suffices to show that any point-line cycle is null-homotopic.

Let  $Q = \text{Rad}(H)$ . For any point  $q \in \Pi(p, H)$ , let  $q^*$  denote one of  $q^+, q^-$ . We claim that any  $k$ -cycle with  $k \geq 5$  can be decomposed into triangles, quadrangles, and pentagons. Namely, let  $\gamma = q_1^*, q_2^*, \dots, q_k^*, q_1^*$  be a  $k$ -cycle in  $\overline{\Pi}(p)$ . If  $q_1$  and  $q_4$  are not both  $Q$ , then they are at distance at most 2 by Lemma 4.9 and so we can decompose  $\gamma$  as  $(q_1^*, q_2^*, q_3^*, q_4^*) \circ \delta \circ \delta^{-1} \circ (q_4^*, \dots, q_k^*, q_1^*)$ , where  $\delta$  is a path from  $q_1^*$  to  $q_4^*$  of length at most 2. Thus we can decompose the  $k$ -cycle into a  $(k-1)$ -cycle and a quadrangle or pentagon. If  $q_1$  and  $q_2$  are both  $Q$ , then replacing  $q_1$  and  $q_4$  by  $q_2$  and  $q_5$ , we can again decompose the  $k$ -cycle into a  $(k-1)$ -cycle and a quadrangle or pentagon.

We shall now analyze the triangles, quadrangles, and pentagons case by case.

**Triangles** The points of a triangle  $q_1^*, q_2^*, q_3^*$  either all have the same sign or one has a sign different from the others. Note that a point that is collinear to another point with the same sign can not cover  $Q$ . In both cases  $X = \langle q_1, q_2, q_3 \rangle$  has dimension 3. Also,  $X$  is non-isotropic because at least one of the lines is. We'll show that  $r = \text{Rad}(X)$  does not lie in  $p^\perp$ . In the former case  $q_1, q_2, q_3$  form a triangle in  $\Pi(p, H)$  whose lines are non-degenerate. In particular  $r$  does not lie on any of these lines. The three remaining points on these lines are  $X \cap p^\perp$ . In the latter case this is because  $r$  is covered by the point of the triangle with the deviating sign. Thus  $X$  is a plane of  $\Pi(p, H)$  and  $q_1^*, q_2^*, q_3^*$  belong to a plane of  $\overline{\Pi}(p)$ .

**Quadrangles** Next we consider a quadrangle  $q_1^*, q_2^*, q_3^*, q_4^*$ . There are four cases. Let  $\epsilon \in \{+, -\}$ .

(1) All points have the same sign, say  $\epsilon$ . As we saw with the triangles, none of these is  $Q^\epsilon$ . Hence  $Q^{-\epsilon}$  is connected to all 4 points and so the quadrangle decomposes into triangles.

(2) All points but one, have the same sign, say  $+$ . Note that again the points on the same level do not cover  $Q$ , but are all collinear to  $Q^-$ . This decomposes the cycle into triangles and a quadrangle with two points of each sign.

(3) There are two points of each sign and these points are consecutive. Without loss of generality let  $q_1^-, q_2^-, q_3^+, q_4^+$  be the quadrangle. We first note that the points  $q_1, q_2, q_3, q_4$  are all distinct. This is because no point of  $\overline{\Pi}(p)$  is collinear to both covers of the same point in  $\Pi(p, H)$ . Also note that  $q_1q_4$  and  $q_2q_3$  are totally isotropic, but  $q_1q_2$  and  $q_3q_4$  are not. We may also assume that  $q_1q_3$  and  $q_2q_4$  are non-degenerate, for otherwise we can decompose the quadrangle into triangles. Consider the space  $Y = q_1^\perp \cap q_2^\perp \cap H$ . It is a 3-dimensional space of rank 2 whose radical is  $Q$ . Now both  $q_3^\perp \cap Y$  and  $q_4^\perp \cap Y$  are lines of  $Y$  through  $Q$ . Note that  $p^\perp \cap Y$  is a line of  $Y$  not through  $Q$ . Hence there is a point  $q \in Y - q_3^\perp - q_4^\perp - p^\perp$ . We find that  $q^+$  is collinear to all points on the quadrangle  $q_1^-, q_2^-, q_3^+, q_4^+$ , which therefore decomposes into triangles.

(4) There are two points of each sign and these points are not consecutive. Without loss of generality let  $q_1^-, q_2^+, q_3^-, q_4^+$  be the quadrangle. Let  $X = \langle q_1, q_2, q_3, q_4 \rangle$ . We first note that we can assume that the points  $q_1, q_2, q_3, q_4$  are all distinct. No two consecutive ones can be the same so if for example  $q_1 = q_3$  then the quadrangle is just a return. Hence  $\dim(X) = 3, 4$ . Note that if either  $q_1q_3$  or  $q_2q_4$  is non-degenerate, then we can decompose the quadrangle into triangles. Therefore all lines  $q_iq_j$  are totally isotropic and it follows that  $X$  is a totally isotropic 3-space. This means that  $Q = q_i$  for some  $i$ , which we may assume to be 4. Consider a totally isotropic 3-space  $Y$  on  $q_2q_4$  different from  $X$ . Then  $q_2q_4$  and  $p^\perp \cap Y$  are intersecting lines of  $Y$  and so there is a point  $q \in Y - q_2q_4 - p^\perp$ . Note that  $q \not\in q_1, q_3$  for otherwise  $\langle q, X \rangle$  is a totally isotropic 4-space. We find that  $q^-$  is collinear to all points of the quadrangle  $q_1^-, q_2^+, q_3^-, q_4^+$ , which therefore decomposes into triangles.

**Pentagons** We first note that we may assume that a pentagon has no more than 2 consecutive points of the same sign. If it contains 4 or more of sign  $\epsilon$ , then these points are all collinear to  $Q^{-\epsilon}$  which then yields a decomposition of the pentagon into triangles and quadrangles. If it contains exactly 3 consecutive points at the same level the same argument decomposes it into 2 triangles and a pentagon that contains no more than 2 consecutive points at the same level.

Therefore we may assume without loss of generality that the pentagon is  $q_1^-, q_2^-, q_3^+, q_4^-, q_5^+$ . If the point  $q_4 = Q$  then we can pick  $q'_4$  to be the fourth point of  $\langle q_3, q_4, q_5 \rangle - p^\perp$  and decompose the pentagon into the quadrangle  $q'_4, q_5^+, q_4^-, q_3^+$  and the pentagon  $q_1^-, q_2^-, q_3^+, q'_4, q_5^+$ . We therefore can assume that  $q_4 \neq Q$ . Moreover modifying this pentagon, if necessary, by the quadrangle  $q_2^-, q_3^+, q_4^-, Q^+$ , we may assume that  $q_3 = Q$ . But then  $q_3^+$  is collinear to  $q_1^-$  as well and we can decompose the pentagon into the triangle  $q_1^-, q_2^-, q_3^+$  and the quadrangle  $q_1^-, q_3^+, q_4^-, q_5^+$ .  $\square$

**Corollary 4.13** *If  $|\mathbb{F}| = 2$  and  $n = 6$  then the fundamental group of  $\Pi(p, H)$  is  $\mathbb{Z}/2\mathbb{Z}$ .*



**Proof** The geometry  $\Pi(p, H)$  has a 2-cover  $\overline{\Pi}(p)$  by Lemma 4.8. This 2-cover is simply connected by Lemma 4.12. Therefore  $\overline{\Pi}(p)$  is the universal cover of  $\Pi(p, H)$  and the fundamental group is  $\mathbb{Z}/2\mathbb{Z}$ .  $\square$

## 5 The group action

### 5.1 The classical amalgam

In this section we shall prove that  $G = \mathrm{Sp}(V)$  is the universal completion of subgroups of small rank. Our first aim is to prove Theorem 5.3 using Tits' Lemma 2.9. To this end, we proved in Section 4 that the geometry  $\Gamma$  is simply connected. The other result we shall need is the following.

**Lemma 5.1** *The symplectic group  $\mathrm{Sp}(V)$  acts flag-transitively on  $\Gamma$ .*

**Proof** Let  $F_1$  and  $F_2$  be two flags of the same type. We prove the lemma by induction on  $|F_1| = |F_2|$ . Let  $M_i$  be the object of maximal type in  $F_i$  for  $i = 1, 2$ . These objects have the same isometry type since their dimensions are equal. Since neither of them intersects  $R = \mathrm{Rad}(V)$ , by Witt's theorem (See e.g. [Ta92]) there is an isometry  $g \in \mathrm{Sp}(V)$  with  $gM_1 = M_2$ .

By induction, there is an element in  $\mathrm{Sp}(M_2)$  sending  $gF_1$  to  $F_2$ . Again by Witt's theorem, this local isometry can be extended to an isometry  $h \in \mathrm{Sp}(V)$ . Thus, the element  $hg \in \mathrm{Sp}(V)$  sends  $F_1$  to  $F_2$ , as desired.  $\square$

### Definition 5.2

Our next aim is to describe the stabilizers of flags of  $\Gamma$  in  $G = \mathrm{Sp}(V)$ . Let  $I = \{1, 2, \dots, n-1\}$ . Let  $C = \{C_i\}_{i \in I}$  be a chamber of  $\Gamma$ . For any  $J \subseteq I$ , let  $R_J$  be the  $J$ -residue on  $C$  and let  $F_J$  be the flag of cotype  $J$  on  $C$ . We set

$$\begin{aligned} P_J &= \mathrm{Stab}_G(R_J) \\ B &= \mathrm{Stab}_G(C). \end{aligned}$$

Note that  $P_J = \mathrm{Stab}_G(F_J)$  and  $B = P_\emptyset = \bigcap_{J \subseteq I} P_J$ . The group  $B$  is called the *Borel group* for the action of  $G$  on  $\Gamma$ .

**Theorem 5.3** *The group  $\mathrm{Sp}(V)$  is the universal completion of the amalgam of the maximal parabolic subgroups for the action on  $\Gamma$ .*

**Proof** The group  $G = \mathrm{Sp}(V)$  acts flag-transitively on  $\Gamma$  by Lemma 5.1. By Proposition 4.2,  $\Gamma$  is simply connected and now the result follows from Tits' Lemma 2.9.  $\square$

**Corollary 5.4** *The parabolic subgroup  $P_J$  acts flag-transitively on the residue  $R_J$ .*

**Proof** This follows immediately from Lemma 5.1 and the definition of  $P_J$  and  $R_J$ .  $\square$

**Theorem 5.5** *Assume  $|\mathbb{F}| \geq 3$ . Let  $J \subseteq I$  with  $|J| \geq 3$ . Then, the parabolic  $P_J$  of  $\mathrm{Sp}(V)$  is the universal completion of the amalgam  $\{P_{J-\{j\}} \mid j \in J\}$  of rank  $(r-1)$  parabolic subgroups contained in  $P_J$ .*

**Proof** The group  $P_J$  acts flag-transitively on the residue  $R_J$  by Corollary 5.4. By Proposition 4.7 the residue  $R_J$  is simply connected if  $|J| \geq 3$ . Again the result follows from Tits' Lemma 2.9.  $\square$

**Corollary 5.6** *Let  $|\mathbb{F}| \geq 3$  and  $\dim(V) \geq 4$ . Then,  $\mathrm{Sp}(V)$  is the universal completion of the amalgam  $\mathcal{A}_{\leq 2} = \{P_J \mid |J| \leq 2\}$  of rank  $\leq 2$  parabolic subgroups for the action on  $\Gamma$ .*

**Proof** This follows from Theorems 5.3 and 5.5 by induction on the rank.  $\square$

The remainder of this paper is devoted to replacing even the rank  $\leq 2$  parabolics in the amalgamation result above by smaller groups.

## 5.2 Parabolic subgroups

In this section we analyze the parabolic subgroups of rank  $\leq 2$  of  $\mathrm{Sp}(V)$ , where  $V$  is non-degenerate over a field  $\mathbb{F}$  with  $|\mathbb{F}| \geq 3$ . These are the groups in the amalgam  $\mathcal{A}_{\leq 2}$  of Corollary 5.6. In order to study these groups in some detail we will use the following setup. Let  $n = 2r$ . and  $\mathcal{H} = \{e_i, f_j \mid 1 \leq i \leq r, 1 \leq j \leq r\}$  be a hyperbolic basis corresponding to  $C$  as in Corollary 3.4. That is, if we relabel  $\mathcal{H}$  such that

$$\begin{aligned} h_{2i-1} &= e_i & \text{for } 1 \leq i \leq r, \\ h_{2j} &= f_j & \text{for } 1 \leq j \leq r, \end{aligned}$$

then,  $C = \{C_l\}_{l \in I}$ , where  $C_l = \langle h_1, h_2, \dots, h_l \rangle_V$ . For  $i = 1, 2, \dots, r$ , let  $H_i = \langle e_i, f_i \rangle_V$ . We will use this setup throughout the remainder of the paper.

Let

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrix defining the symplectic form with respect to the basis  $\mathcal{H}$  is

$$S = \begin{pmatrix} E & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & E \end{pmatrix}$$

Since the Borel group is contained in all parabolic subgroups, even the ones of rank  $\leq 2$ , we must know exactly what it looks like.

**Lemma 5.7** *We have*

$$B \cong (\mathbb{F} \rtimes \mathrm{GL}_1(\mathbb{F}))^{(n/2)}$$

where, for  $j = 1, 2, \dots, r$ ,  $\mathbb{F} \rtimes \mathrm{GL}_1(\mathbb{F})$  is realized on  $\langle e_j, f_j \rangle$  as

$$B_j = \left\{ \begin{pmatrix} a_j & b_j \\ 0 & a_j^{-1} \end{pmatrix} \mid a_j \in \mathbb{F}^*, b_j \in \mathbb{F} \right\}.$$

Also, the kernel of the action of  $G$  on  $\Gamma$  is  $H = \{\pm 1\}$

**Proof** First note that if  $n = 2$ , then  $G = \mathrm{SL}_2(\mathbb{F})$  and the stabilizer of  $C$  is the usual Borel group

$$B_1 = \left\{ \begin{pmatrix} a_1 & b_1 \\ 0 & a_1^{-1} \end{pmatrix} \mid a_1 \in \mathbb{F}^*, b_1 \in \mathbb{F} \right\}.$$

Clearly,  $B_1 \cong \mathbb{F} \rtimes \mathrm{GL}_1(\mathbb{F})$ .

Now let  $n \geq 3$ . We observe that, if  $g \in G$  stabilizes  $C$ , then it also stabilizes  $C_l^\perp \cap C_m$  for any  $1 \leq l \leq m \leq n-1$ . Hence,  $g$  stabilizes the subspaces spanned by  $e_i$  and it stabilizes the subspaces spanned by  $\{e_i, f_i\}$  for all  $1 \leq i \leq r$ . Thus

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_r \end{pmatrix}$$

where if  $j = 1, 2, \dots, r$  we have

$$g_j = \begin{pmatrix} a_j & b_j \\ 0 & a_j^{-1} \end{pmatrix}$$

for some  $a_j \in \mathbb{F}^*$  and  $b_j \in \mathbb{F}$ .

The kernel of the action is given by  $\bigcap_{x \in G} xBx^{-1}$ . This means that if  $g$  is described as above, then, for all  $1 \leq j \leq r$  we have  $b_j = 0$  and  $a_j = a_j^{-1} = a$  for some fixed  $a \in \mathbb{F}^*$ . This means  $a = \pm 1$ . The result follows.  $\square$

We now focus on the parabolic subgroups of rank  $\leq 2$ . Since we want to make a distinction between the various rank  $\leq 2$  parabolic subgroups, we shall give them an individual name.

**Definition 5.8** We assign the following names to the various rank  $\leq 2$  parabolic

subgroups:

$$\begin{aligned}
S_j &= P_{2j-1} && \text{for } 1 \leq j \leq r \\
M_i &= P_{2i} && \text{for } 1 \leq i \leq r-1 \\
S_{ij} &= P_{2i-1, 2j-1} && \text{for } 1 \leq i < j \leq r \\
M_{ij} &= P_{2i, 2j} && \text{for } 1 \leq i < j \leq r-1 \\
Q_{ij} &= P_{2i, 2j-1} && \text{for } 1 \leq i \leq r-1 \text{ and } 1 \leq j \leq r
\end{aligned}$$

Thus the collection of groups in the amalgam  $\mathcal{A}_{\leq 2}$  is  $\{M_i, S_j, S_{jl}, M_{ik}, Q_{ij} \mid 1 \leq i, k \leq r-1 \text{ and } 1 \leq j, l \leq r\}$ .

**Definition 5.9** In order to describe the groups in  $\mathcal{A}_{\leq 2}$  abstractly and as matrix groups we define the following matrix groups:

$$M = \left\{ \begin{pmatrix} a_1 & b_1 & 0 & w \\ 0 & a_1^{-1} & 0 & 0 \\ 0 & a_1^{-1}wa_2 & a_2 & b_2 \\ 0 & 0 & 0 & a_2^{-1} \end{pmatrix} \mid a_1, a_2 \in \mathbb{F}^*, b_1, b_2, w \in \mathbb{F} \right\},$$

$$S = \text{SL}_2(\mathbb{F}) \cong \text{Sp}_2(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F}, \text{ where } ad - bc = 1 \right\},$$

$$M_* = \left\{ \left( \begin{array}{cc|cc|cc} a_1 & b_1 & 0 & w_1 & 0 & 0 \\ 0 & a_1^{-1} & 0 & 0 & 0 & 0 \\ \hline 0 & a_1^{-1}w_1a_2 & a_2 & b_2 & 0 & w_3 \\ 0 & 0 & 0 & a_2^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & a_2^{-1}w_3a_3 & a_3 & b_3 \\ 0 & 0 & 0 & 0 & 0 & a_3^{-1} \end{array} \right) \mid \begin{array}{l} a_1, a_2, a_3 \in \mathbb{F}^* \\ b_1, b_2, b_3, w_1, w_3 \in \mathbb{F} \end{array} \right\},$$

$$Q_- = \left\{ \begin{pmatrix} a_1 & b_1 & 0 & w_1 \\ c_1 & d_1 & 0 & w_2 \\ v_2 & v_1 & a_2 & b_2 \\ 0 & 0 & 0 & a_2^{-1} \end{pmatrix} \mid \begin{array}{l} a_1, b_1, c_1, d_1, w_1, w_2, v_1, v_2, a_2, b_2 \in \mathbb{F} \text{ such that} \\ a_1d_1 - b_1c_1 = 1 \\ a_1w_2 - c_1w_1 + v_2a_2^{-1} = 0 \\ b_1w_2 - d_1w_1 + v_1a_2^{-1} = 0 \end{array} \right\},$$

and

$$Q_+ = \left\{ \begin{pmatrix} a_2 & b_2 & v_2 & v_1 \\ 0 & a_2^{-1} & 0 & 0 \\ 0 & w_1 & a_1 & b_1 \\ 0 & w_2 & c_1 & d_1 \end{pmatrix} \mid \begin{array}{l} a_1, b_1, c_1, d_1, w_1, w_2, v_1, v_2, a_2, b_2 \in \mathbb{F} \text{ such that} \\ a_1d_1 - b_1c_1 = 1 \\ a_1w_2 - c_1w_1 + v_2a_2^{-1} = 0 \\ b_1w_2 - d_1w_1 + v_1a_2^{-1} = 0 \end{array} \right\}.$$

**Lemma 5.10** (a) *For all indices that apply, we have the following isomorphisms:*

$$\begin{aligned} S_j &\cong S \times \prod_{i \neq j} B_i, \\ M_i &\cong M \times \prod_{j \neq i, i+1} B_j, \\ S_{ij} &= \langle S_i, S_j \rangle_G, \\ M_{ij} &= \langle M_i, M_j \rangle_G, \\ Q_{ij} &= \langle M_i, S_j \rangle_G. \end{aligned}$$

Moreover, we have the following isomorphisms,

(b)

$$M_{ij} \cong \begin{cases} M_* \times \prod_{k \neq i, i+1, j, j+1} B_k & \text{if } |i - j| = 1 \\ M \times M \times \prod_{k \neq i, i+1, j, j+1} B_k & \text{if } |i - j| \geq 2 \end{cases},$$

(c)

$$S_{ij} \cong (S \times S) \times \prod_{k \neq i, j} B_k,$$

(d) *If  $j \notin \{i, i+1\}$ , then*

$$Q_{ij} \cong (M \times S) \times \prod_{k \neq i, i+1, j} B_k.$$

Furthermore,

$$Q_{ii} \cong Q_- \times \prod_{k \neq i, i+1} B_k,$$

and

$$Q_{ii+1} \cong Q_+ \times \prod_{k \neq i, i+1} B_k.$$

**Proof** (a) By definition  $S_i$  is the stabilizer of the flag of type  $I - \{2i - 1\}$  on the standard chamber  $C$ . Therefore it acts on  $H_j$  as  $B_j$  for all  $j \neq i$  and it stabilizes and acts as  $S \cong \mathrm{Sp}_2(\mathbb{F})$  on  $H_i$ . Similarly, by definition,  $M_i$  acts on  $H_j$  as  $B_j$  for all  $j \neq i, i+1$  and it stabilizes  $\langle e_i \rangle$  and  $\langle e_i, f_i, e_{i+1} \rangle$  and  $H_i \oplus H_{i+1}$ . Therefore it acts as  $M$  on  $H_i \oplus H_{i+1}$ . The last three equalities follow from the fact that rank 2 residues are connected.

The remaining isomorphisms follow from similar considerations. The parabolic subgroup under consideration acts as  $B_j$  on  $H_j$  for all but 2, 3, or 4 values of  $j$ . Then the action on the subspace generated by the remaining  $H_j$  determines the non-Borel part  $(S, M, M_*, Q_-, Q_+)$  of the group.  $\square$

## 6 The slim amalgam

In this section we define a slim version  $\mathcal{A}^\pi$  of the amalgam  $\mathcal{A}_{\leq 2}$  by eliminating a large part of the Borel group from each of its groups. More precisely, the collection of groups in  $\mathcal{A}^\pi$  is  $\{X^\pi \mid X \in \mathcal{A}_{\leq 2}\}$ , where  $X^\pi$  is given in Definitions 6.1 and 6.3. Note that the groups  $X^\pi$  are subgroups of  $G$  and the inclusion maps for the amalgam  $\mathcal{A}^\pi$  are the inclusion maps induced by  $G$ . In Section 7 we shall construct an abstract version of  $\mathcal{A}^\pi$ , whose groups are not considered to be subgroups of  $G$ .

**Definition 6.1** For  $1 \leq i \leq r-1$ , the group  $M_i^\pi$  fixes every vector in  $H_k$ , for every  $k \neq i, i+1$  and a generic element acts on  $H_i \oplus H_{i+1}$  as

$$m(b_1, w, b_2) = \begin{pmatrix} 1 & b_1 & 0 & w \\ 0 & 1 & 0 & 0 \\ 0 & w & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ where } w, b_1, b_2 \in \mathbb{F}.$$

For  $1 \leq j \leq r$ , the group  $S_j^\pi$  fixes every vector in  $H_k$ , for every  $k \neq j$  and acts on  $H_j$  as  $\text{Sp}(H_j) \cong \text{Sp}_2(\mathbb{F})$ . A generic element of  $S_j^\pi$  is denoted

$$s(a, b, c, d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the matrix defines the action on  $H_j$  with respect to the basis  $\{e_j, f_j\}$ .

**Note 6.2** We note that  $S_j^\pi$  is generated by two opposite long-root groups.

**Definition 6.3** For  $1 \leq i < j \leq r$ , let

$$S_{ij}^\pi = \langle S_i^\pi, S_j^\pi \rangle_G.$$

For  $1 \leq i < j \leq r-1$ , let

$$M_{ij}^\pi = \langle M_i^\pi, M_j^\pi \rangle_G.$$

For  $1 \leq i \leq r-1$  and  $1 \leq j \leq r$ , let

$$Q_{ij}^\pi = \langle M_i^\pi, S_j^\pi \rangle_G.$$

**Definition 6.4** For any  $i = 1, 2, \dots, r$ , we set

$$U_i^\pi = \begin{cases} M_i^\pi \cap S_i^\pi & \text{if } 1 \leq i < r \\ M_{i-1}^\pi \cap S_i^\pi & \text{if } i = r \end{cases},$$

and

$$B_i^\pi = N_{S_i^\pi}(U_i^\pi).$$

Moreover,

$$B^\pi = \langle B_1^\pi, \dots, B_r^\pi \rangle \cong B_1^\pi \times \dots \times B_r^\pi.$$

**Definition 6.5** In order to describe the groups in  $\mathcal{A}^\pi$  abstractly and as matrix groups we define the following matrix groups:

$$M^\pi = \left\{ \begin{pmatrix} 1 & b_1 & 0 & w \\ 0 & 1 & 0 & 0 \\ 0 & w & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid b_1, b_2, w \in \mathbb{F} \right\},$$

$$S^\pi = S,$$

$$M_*^\pi = \left\{ \left( \begin{array}{cc|cc|cc} 1 & b_1 & 0 & w_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & w_1 & 1 & b_2 & 0 & w_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & w_3 & 1 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \mid b_1, b_2, b_3, w_1, w_3 \in \mathbb{F} \right\},$$

$$Q_-^\pi = \left\{ \left( \begin{array}{cccc} a_1 & b_1 & 0 & w_1 \\ c_1 & d_1 & 0 & w_2 \\ v_2 & v_1 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{array} \right) \mid \begin{array}{l} a_1, b_1, c_1, d_1, w_1, w_2, v_1, v_2, b_2 \in \mathbb{F} \text{ such that} \\ a_1 d_1 - b_1 c_1 = 1 \\ v_2 = -a_1 w_2 + c_1 w_1 \\ v_1 = -b_1 w_2 + d_1 w_1 \end{array} \right\},$$

and

$$Q_+^\pi = \left\{ \left( \begin{array}{cccc} 1 & b_2 & v_2 & v_1 \\ 0 & 1 & 0 & 0 \\ 0 & w_1 & a_1 & b_1 \\ 0 & w_2 & c_1 & d_1 \end{array} \right) \mid \begin{array}{l} a_1, b_1, c_1, d_1, w_1, w_2, v_1, v_2, b_2 \in \mathbb{F} \text{ such that} \\ a_1 d_1 - b_1 c_1 = 1 \\ v_2 = -a_1 w_2 + c_1 w_1 \\ v_1 = -b_1 w_2 + d_1 w_1 \end{array} \right\}.$$

**Lemma 6.6** For  $i = 1, 2, \dots, r-1$  and  $j = 1, 2, \dots, r$ , the sets  $M_i^\pi$  and  $S_j^\pi$  are subgroups of  $\text{Sp}(V)$  and

- (a)  $M_i^\pi \cong M^\pi \cong \mathbb{F}^3$ ,
- (b)  $S_j^\pi \cong S^\pi \cong \text{Sp}_2(\mathbb{F})$ .

**Proof** For  $m = m(b_1, w, b_2)$  one verifies easily that  $(me_i, mf_i)$  and  $(me_{i+1}, mf_{i+1})$  are two orthogonal hyperbolic pairs. Hence  $m$  is a symplectic matrix. Clearly  $S_j^\pi = \text{Stab}_{\text{Sp}(V)}(H_j)$  is a subgroup of  $\text{Sp}(V)$ .

- (a) It is straightforward to check that  $M_i^\pi$  is an abelian group isomorphic to  $\mathbb{F}^3$ .
- (b) This is true by definition. □

**Lemma 6.7** *For all indices  $i \neq j$  that apply, we have*

(a)

$$M_{ij}^\pi \cong \begin{cases} M_*^\pi & \text{if } |i - j| = 1 \\ M_i^\pi \times M_j^\pi & \text{if } |i - j| \geq 2 \end{cases},$$

(b)

$$S_{ij}^\pi \cong S_i^\pi \times S_j^\pi.$$

(c) *If  $j \notin \{i, i + 1\}$ , then*

$$Q_{ij}^\pi \cong M_i^\pi \times S_j^\pi.$$

*Furthermore,*

$$\begin{aligned} Q_{ii}^\pi &\cong Q_-^\pi, \\ Q_{ii+1}^\pi &\cong Q_+^\pi. \end{aligned}$$

**Proof** Part (a) and (b) are straightforward. As for part (c), if  $j \notin \{i, i + 1\}$ , then clearly  $M_i^\pi$  and  $S_j^\pi$  commute and intersect trivially.

We now turn to the cases  $Q_{ii}^\pi$  and  $Q_{ii+1}^\pi$ . First note that conjugation by the permutation matrix that switches  $(e_i, f_i)$  and  $(e_{i+1}, f_{i+1})$ , interchanges  $S_i^\pi$  and  $S_{i+1}^\pi$ , but fixes every element in  $M_i^\pi$ . Thus it suffices to prove the claim for  $Q_{ii}^\pi$ .

We consider  $Q_-^\pi$  to be represented as a matrix group with respect to the basis  $\{e_i, f_i, e_{i+1}, f_{i+1}\}$ . We shall now prove that with this identification  $Q_-^\pi = Q_{ii}^\pi$ . To this end we show that  $Q_-^\pi$  is the stabilizer of the vector  $e_{i+1}$  in  $\text{Sp}(H_i \oplus H_{i+1})$ . It is clear from the shape of the third column that  $Q_-^\pi$  stabilizes  $e_{i+1}$ . On the other hand, any matrix  $A$  in  $\text{Sp}(H_i \oplus H_{i+1})$  stabilizing  $e_{i+1}$  must have such a third column. It must also have zeroes in the last row as in  $Q_-^\pi$  since in fixing  $e_{i+1}$  it must also stabilize  $e_{i+1}^\perp$ . Any such matrix  $A$  must satisfy the conditions on the entries as indicated in the description of  $Q_-^\pi$  since  $\{Ae_i, Af_i, e_{i+1}, Af_{i+1}\}$  must be isometric to  $\{e_i, f_i, e_{i+1}, f_{i+1}\}$  in order for  $A$  to be symplectic. Therefore  $Q_-^\pi$  equals this stabilizer and hence is a group.

Clearly  $Q_-^\pi$  contains  $S_i^\pi$  and  $M_i^\pi$ . We now show that  $\langle S_i^\pi, M_i^\pi \rangle = Q_-^\pi$ . We note that if  $m = m(0, w, b_2)$  and  $s = s(a, b, c, d)$ ,  $s' = s(a_1, b_1, c_1, d_1)$  then  $sms^{-1}s'$  is the



following matrix:

$$\begin{pmatrix} a_1 & b_1 & 0 & aw \\ c_1 & d_1 & 0 & cw \\ v_2 & v_1 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $v_1, v_2$  are as in the definition of  $Q_{\pi}^{\pi}$ . Therefore all the elements of  $Q_{\pi}^{\pi}$  can be obtained that way. □

**Lemma 6.8** *We have*

(a)

$$M_{ij}^{\pi} \cong \begin{cases} \mathbb{F}^5 & \text{if } |i - j| = 1 \\ \mathbb{F}^6 & \text{if } |i - j| \geq 2 \end{cases},$$

(b)

$$S_{ij}^{\pi} \cong \mathrm{Sp}_2(\mathbb{F}) \times \mathrm{Sp}_2(\mathbb{F}),$$

(c) *If  $j \notin \{i, i + 1\}$ , then*

$$Q_{ij}^{\pi} \cong \mathbb{F}^3 \times \mathrm{Sp}_2(\mathbb{F}).$$

*Furthermore,*

$$Q_{ii}^{\pi} \cong (U \times V) \rtimes S_i^{\pi} \cong \mathbb{F}^3 \rtimes \mathrm{Sp}_2(\mathbb{F}).$$

*Here, taking the labelling as in Lemma 6.7 we have*

$$U = \left\{ \left( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid b_2 \in \mathbb{F} \right) \right\} \cong \mathbb{F},$$

$$V = \left\{ \left( \begin{pmatrix} 1 & 0 & 0 & w_1 \\ 0 & 1 & 0 & w_2 \\ v_2 & v_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \begin{array}{l} w_1, w_2, v_1, v_2 \in \mathbb{F} \text{ such that} \\ v_2 = -w_2 \\ v_1 = w_1 \end{array} \right) \right\} \cong \mathbb{F}^2,$$

$$S_i^{\pi} = \left\{ \left( \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & d_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \begin{array}{l} a_1, b_1, c_1, d_1 \in \mathbb{F} \text{ such that} \\ a_1 d_1 - b_1 c_1 = 1 \end{array} \right) \right\} \cong \mathrm{Sp}_2(\mathbb{F}).$$

Moreover, the action of  $S_i^\pi$  on  $V$  by conjugation is the natural action from the left of  $\mathrm{Sp}_2(\mathbb{F})$  on  $V$ . Also  $U = Z(Q_{ii}^\pi)$ .

Finally,

$$Q_{ii+1}^\pi \cong Q_{ii}^\pi$$

where the isomorphism is given by the labelling of the entries.

**Proof** (a) and (b): This is straightforward. As for (c), if  $j \notin \{i, i+1\}$ , then the result follows from the corresponding part in Lemma 6.7.

Since  $M_i^\pi \leq V \leq Q_{ii}^\pi$  and  $U \leq Q_{ii}^\pi$ , we have  $Q_{ii}^\pi = \langle M_i^\pi, S_i^\pi \rangle = \langle V, S_i^\pi \rangle = \langle U, V, S_i^\pi \rangle$ . Also,  $U \cap V = 1$  and  $\langle U, V \rangle$  is abelian so that  $\langle U, V \rangle \cong U \times V$ . Furthermore,  $\langle U, V \rangle \cap S_i^\pi = 1$  and  $S_i^\pi \leq N_{Q_{ii}^\pi}(\langle U, V \rangle)$ . In fact  $S_i^\pi \leq C_{Q_{ii}^\pi}(U)$  and one calculates that conjugation of  $V$  by  $S_i^\pi$  is the natural action from the left of  $\mathrm{Sp}_2(\mathbb{F})$  on  $\mathbb{F}^2 = V$ .

We now show that  $U = Z(Q_{ii}^\pi)$ . This follows from a general argument about semidirect products using the fact that  $S_i^\pi$  acts faithfully and fixed point free on  $V$ .

Finally, the isomorphism  $Q_{ii}^\pi \cong Q_{ii+1}^\pi$  is given by conjugation as in the proof of Lemma 6.7.  $\square$

## 7 The concrete amalgam

**Definition 7.1** We will define a concrete amalgam  $\mathcal{A}^\circ$ . Its set of subgroups is  $\{M_{ij}^\circ, Q_{ik}^\circ, S_{kl}^\circ, M_i^\circ, S_k^\circ \mid 1 \leq i, j \leq r-1, 1 \leq k, l \leq r\}$ , where for each  $X \in \mathcal{A}_{\leq 2}^\circ$ ,  $X^\circ$  is a copy of  $X^\pi$  and the inclusion homomorphisms are as follows:

$$\begin{aligned} \varphi_{i,\{i,j\}}^P &: M_i^\circ \rightarrow M_{ij}^\circ \\ \varphi_{j,\{i,j\}}^P &: M_j^\circ \rightarrow M_{ij}^\circ \\ \varphi_{k,\{k,l\}}^S &: S_k^\circ \rightarrow S_{kl}^\circ \\ \varphi_{l,\{k,l\}}^S &: S_l^\circ \rightarrow S_{kl}^\circ \\ \varphi_{i,\{i,k\}}^{PQ} &: M_i^\circ \rightarrow Q_{ik}^\circ \\ \varphi_{k,\{i,k\}}^{SQ} &: S_k^\circ \rightarrow Q_{ik}^\circ \end{aligned}$$

These inclusions are given by the presentations of  $X^\pi$  as matrix groups as given in Definitions 6.1 and 6.3. We denote the universal completion of  $\mathcal{A}^\circ$  by  $G^\circ$ .

**Definition 7.2** We now define the map  $\pi: \mathcal{A}^\circ \rightarrow G$ . For any  $X \in \mathcal{A}_{\leq 2}^\circ$ , it identifies  $X^\circ$  with its isomorphic copy,  $X^\pi$ , in  $G$ . Thus the image of  $\mathcal{A}^\circ$  under  $\pi$  is  $\mathcal{A}^\pi$ .

**Lemma 7.3** For any element  $X \in \mathcal{A}_{\leq 2}^\circ$ , the map  $\pi: X^\circ \rightarrow X^\pi$  is an isomorphism.

**Proof** This is true by the definition of  $\pi$ . □

It then follows that the map  $\pi$  extends to a surjective map from the universal cover  $G^\circ$  to  $G$ .

**Lemma 7.4** *If  $i \geq 2$  then  $U_i^\circ = S_i^\circ \cap M_i^\circ = S_i^\circ \cap M_{i-1}^\circ = M_i^\circ \cap M_{i-1}^\circ$ .*

**Proof** Note that the above are true for the images under  $\pi$ . By definition the group  $M_i^\pi$  stabilizes the spaces  $H_k$  if  $k \neq i, i+1$  and it also stabilizes  $e_i$  and  $e_{i+1}$ . Moreover  $S_i^\pi$  stabilizes all the  $H_k$  for  $k \neq i$ . Therefore  $S_i^\pi \cap M_i^\pi$  stabilizes all vectors in  $H_k$  for  $k \neq i$  and it also stabilizes  $e_i$ . Similarly for  $S_i^\pi \cap M_{i-1}^\pi$  and  $M_i^\pi \cap M_{i-1}^\pi$ . We notice that  $\pi$  is an isomorphism when restricted to  $\mathcal{A}^\circ$  and so the conclusion follows. □

**Definition 7.5** For  $i = 1, 2, \dots, r$ , we define the following subgroup of  $S_i^\circ \in \mathcal{A}^\circ$ :  $B_i^\circ = N_{S_i^\circ}(U_i^\circ)$ . Furthermore, we set  $B^\circ = \langle B_i^\circ \mid i = 1, 2, \dots, r \rangle$  as a subgroup of  $B^\circ$ .

**Lemma 7.6** *For  $i = 1, 2, \dots, r$ , we have*

(a)  $U_i^\pi = \pi(U_i^\circ)$  and  $B_i^\pi = \pi(B_i^\circ)$ .

(b)

$$U_i^\circ = \left\{ \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \mid b_1 \in \mathbb{F} \right\} \cong \mathbb{F}.$$

(c)

$$B_i^\circ = \left\{ \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \mid a_1, b_1, d_1 \in \mathbb{F} \text{ such that } a_1 d_1 = 1 \right\} \cong \mathbb{F} \rtimes \mathbb{F}^*.$$

**Lemma 7.7** *For  $X \in \mathcal{A}_{\leq 2}^\circ$ ,  $B^\circ$  normalizes  $X^\circ$ . The action of  $B_i^\circ$  on the rank 2 parabolics  $Q_{ii}^\circ, Q_{ji}^\circ$  ( $j \neq i$ ) as well as the action on  $S_{ij}^\circ$  is given by conjugation. Therefore,*

(a)  $[B_i^\circ, S_j^\circ] = 1$  if  $i \neq j$  and  $B_i^\circ$  acts on  $S_i^\circ$  as inner automorphisms.

(b)  $[B_i^\circ, M_j^\circ] = 1$  if  $j \neq i, i-1$ . Moreover,  $B_i^\circ$  acts on  $M_i^\circ$  as the conjugation in  $Q_{ii}^\circ$  and  $B_{i+1}^\circ$  acts on  $M_i^\circ$  as the conjugation in  $Q_{ii+1}^\circ$ .

(c) The action of  $B_i^\circ$  on the rank 2 parabolics  $M_{ij}^\circ$  is given by the action on its subgroups  $M_i^\circ$  and  $M_j^\circ$ .

**Proof** Note that the actions are as above for the  $B_i^\pi \leq S_i^\pi$  acting on the various  $X^\pi$ . Moreover the map  $\pi$  is an isomorphism when restricted to the various  $X^\circ \in \mathcal{A}^\circ$ . Therefore, the action of  $B_i^\circ$  on any (subgroup of) a rank 2-parabolic  $X^\circ$  containing  $S_i^\circ$  is the same as the action of  $B_i^\pi$  on the (corresponding subgroup of) the corresponding rank 2-parabolic  $X^\pi$ . This explains the action of  $B_i^\circ$  on the following groups: (1)  $S_{ij}^\circ$  and its subgroups  $S_i^\circ, S_j^\circ$ , (2)  $Q_{ii}^\circ$  and its subgroup  $M_i^\circ$ , (3)  $Q_{ji}^\circ$  and its subgroup  $M_j^\circ$  ( $j \neq i$ ). This settles parts (a) and (b).

Part (c) follows immediately since the groups in  $\mathcal{A}^\circ$  are all embedded in  $G^\circ$ .  $\square$

**Lemma 7.8** (a) *The group  $B^\circ$  is the internal direct product*

$$B^\circ = B_1^\circ \times B_2^\circ \times \cdots \times B_r^\circ,$$

(b)

$$\pi: B^\circ \rightarrow B^\pi \text{ is an isomorphism.}$$

**Proof** For any  $1 \leq i < j \leq r$ , we have  $B_i^\circ \leq S_i^\circ$  and since  $S_{ij}^\circ = S_i^\circ \times S_j^\circ$  we have  $\langle B_1^\circ, B_j^\circ \rangle = B_i^\circ \times B_j^\circ$ .  $\square$

**Lemma 7.9** *For  $X \in \mathcal{A}_{\leq 2}$  we have  $\langle B^\pi, X^\pi \rangle_G = X$ .*

**Proof** This is an easy calculation inside  $G$ .  $\square$

**Lemma 7.10** *For any  $X \in \mathcal{A}_{\leq 2}$ ,  $\pi(X^\circ \cap B^\circ) = X^\pi \cap B^\pi$ .*

**Proof** The inclusion  $\subseteq$  is trivial. We now prove  $\supseteq$ . We do this case by case for any  $X \in \mathcal{A}$ .

Let  $X = S_i$ . Then  $X^\pi \cap B^\pi = B_i^\pi$  and since  $B_i^\circ \leq S_i^\circ \cap B^\circ$ , we find  $\pi(X^\circ \cap B^\circ) \supseteq \pi(B_i^\circ) = B_i^\pi$ .

Let  $X = P_i$ . Then  $X^\pi \cap B^\pi = U_i^\pi \times U_{i+1}^\pi$  and since  $U_i^\circ \times U_{i+1}^\circ \leq M_i^\circ \cap B^\circ$ , we find  $\pi(X^\circ \cap B^\circ) \supseteq \pi(U_i^\circ \times U_{i+1}^\circ) = U_i^\pi \times U_{i+1}^\pi$ .

Let  $X = S_{ij}$ . Then  $X^\pi \cap B^\pi = B_i^\pi \times B_j^\pi$  and since  $B_i^\circ \times B_j^\circ \leq S_{ij}^\circ \cap B^\circ$ , we find  $\pi(X^\circ \cap B^\circ) \supseteq \pi(B_i^\circ \times B_j^\circ) = B_i^\pi \times B_j^\pi$ .

Let  $X = Q_{ij}$ . Then  $X^\pi \cap B^\pi = \langle U_i^\pi, U_{i+1}^\pi, B_j^\pi \rangle$  and since  $\langle U_i^\circ, U_{i+1}^\circ, B_j^\circ \rangle \leq Q_{ij}^\circ \cap B^\circ$ , we find  $\pi(X^\circ \cap B^\circ) \supseteq \pi(\langle U_i^\circ, U_{i+1}^\circ, B_j^\circ \rangle) = \langle U_i^\pi, U_{i+1}^\pi, B_j^\pi \rangle$ .

Let  $X = P_{ij}$ . Then  $X^\pi \cap B^\pi = \langle U_i^\pi, U_{i+1}^\pi, U_j^\pi, U_{j+1}^\pi \rangle$  and since  $\langle U_i^\circ, U_{i+1}^\circ, U_j^\circ, U_{j+1}^\circ \rangle \leq M_{ij}^\circ \cap B^\circ$ , we find  $\pi(X^\circ \cap B^\circ) \supseteq \pi(\langle U_i^\circ, U_{i+1}^\circ, U_j^\circ, U_{j+1}^\circ \rangle) = \langle U_i^\pi, U_{i+1}^\pi, U_j^\pi, U_{j+1}^\pi \rangle$ .  $\square$

Our next aim is to define a map  $\chi: \mathcal{A}_{\leq 2} \rightarrow G^\circ$  extending  $\pi^{-1}: X^\pi \rightarrow X^\circ$  for every  $X \in \mathcal{A}_{\leq 2}$ .

**Lemma 7.11**  $\chi$  is well-defined on  $X^\pi \cap B^\pi$  for all  $X \in \mathcal{A}_{\leq 2}$ .

**Proof** This follows from Lemma 7.10 □

Define  $\chi$  on  $X = B^\pi X^\pi$  for any  $X \in \mathcal{A}_{\leq 2}$  as follows:  $\chi(bx) = \chi(b)\chi(x)$ .

**Lemma 7.12**  $\chi$  is well-defined and injective on  $X$ .

**Proof** Note that if  $b, b' \in B^\pi$  and  $x, x' \in X^\pi$  then  $bx = b'x' \Rightarrow b^{-1}b' = xx'^{-1} \in B^\pi \cap X^\pi$ . Moreover by Lemma 7.11  $\chi(b^{-1}b') = \chi(xx'^{-1})$  and so  $\chi(bx) = \chi(b'x')$  and  $\chi$  is well defined.

Also if  $bx \in X$  with  $\chi(bx) = 1$  it follows that  $\chi(b) = \chi(x)^{-1} \in B^\circ \cap X^\circ$  (because  $\chi$  is the inverse of  $\pi$  when restricted to  $B^\pi$  and  $X^\pi$ ). Therefore  $b = \pi(\chi(b))$  and  $x = \pi(\chi(x))$  are both in  $\pi(X^\circ \cap B^\circ) = X^\pi \cap B^\pi$ . But  $\chi$  is a bijection when restricted to  $X^\pi \cap B^\pi$  and so  $b = x^{-1}$  and  $\chi$  is injective. □

**Lemma 7.13**  $\chi$  is an embedding of the amalgam  $\mathcal{A}_{\leq 2}$  into  $G^\circ$ .

**Proof** Let  $X \in \mathcal{A}$  then by lemma 7.9 and Lemma 7.7 we know that  $X = B^\pi X^\pi$  and so if  $bx, b'x' \in X$  then  $\chi(b'x'bx) = \chi(b'b)\chi(b^{-1}x'bx) = \chi(b'b)\chi(b^{-1}x'b)\chi(x)$ . Also  $\chi(b'x')\chi(bx) = \chi(b')\chi(x')\chi(b)\chi(x) = \chi(b')\chi(b)\chi(b^{-1})\chi(x')\chi(b)\chi(x)$ . so we only need to prove that  $\chi(b^{-1})\chi(x')\chi(b) = \chi(b^{-1}x'b)$  which is equivalent to the fact that the action of  $B^\pi$  on  $X^\pi$  is the same as the action of  $B^\circ$  on  $X^\circ$ . This follows from Lemma 7.7. □

As a consequence, we find the following result.

**Proposition 7.14** The map  $\chi$  extends to a surjective homomorphism  $G \rightarrow G^\circ$  which we also denote by  $\chi$ .

**Proof** By Lemma 7.13 the map  $\chi$  extends to a homomorphism  $G \rightarrow G^\circ$  whose image contains the subgroup of  $G^\circ$  generated by the subgroups in  $\mathcal{A}^\circ$ . Since  $\langle \mathcal{A}^\circ \rangle = G^\circ$ , the conclusion follows. □

**Proof** Of Theorem 1.1. By Corollary 5.6  $G$  is the universal completion of the amalgam  $\mathcal{A}_{\leq 2}$ .

The map  $\pi: \mathcal{A}^\circ \rightarrow G$  given by  $\pi: X^{c}irc \rightarrow X^\pi$  for all  $X \in \mathcal{A}_{\leq 2}$  extends to surjection  $\pi: G^\circ \rightarrow H^\pi$  where  $H^\pi$  is the subgroup of  $G$  generated by the subgroups in  $\mathcal{A}^\pi$ . By Lemma 7.9  $H^\pi = G$ , and so  $\pi: G^\circ \rightarrow G$  is surjective.

By Proposition 7.14 there is a surjective map  $\chi: G \rightarrow G^\circ$ , which by Lemma 7.12 is injective on the subgroups of  $\mathcal{A}_{\leq 2}$ .

The composition  $\chi \circ \pi: G^\circ \rightarrow G^\circ$  is a surjective homomorphism which is the identity on every subgroup in the amalgam  $\mathcal{A}^\circ$ . Since  $G^\circ$  is the universal completion of  $\mathcal{A}^\circ$ , the only such map is the identity.

Hence  $\pi: G^\circ \rightarrow G$  is an isomorphism. □

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