On the nucleus of the Grassmann embedding of the symplectic dual polar space $\text{DSp}(2n, \mathbb{F})$, $\text{char}(\mathbb{F}) = 2$

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Abstract

Let $n \geq 3$ and let $\mathbb{F}$ be a field of characteristic 2. Let $DSp(2n, \mathbb{F})$ denote the dual polar space associated with the building of Type $C_n$ over $\mathbb{F}$ and let $G_{n-2}$ denote the $(n-2)$-Grassmannian of type $C_n$. Using the bijective correspondence between the points of $G_{n-2}$ and the quads of $DSp(2n, \mathbb{F})$, we construct a full projective embedding of $G_{n-2}$ into the nucleus of the Grassmann embedding of $DSp(2n, \mathbb{F})$. This generalizes a result of the paper [9] which contains an alternative proof of this fact in the case when $n = 3$ and $\mathbb{F}$ is finite.
1 Introduction and preliminaries

We assume the reader is familiar with the concept of a partial linear rank two incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, also called a point-line geometry (See e.g. [5, 19]). The distance $\text{dist}(x, y)$ between two points $x, y \in \mathcal{P}$ of $\Gamma$ will be measured in the collinearity graph of $\Gamma$, that is the graph $(\mathcal{P}, E)$ whose set of edges $E$ consists of all unordered pairs of points belonging to a line of $\Gamma$. By a subspace of $\Gamma$ we mean a subset $S$ of $\mathcal{P}$ such that if $l \in \mathcal{L}$ and $|l \cap S| \geq 2$, then $l \subseteq S$. If $S$ is a subspace of $\Gamma$, then we denote by $\tilde{S}$ the point-line geometry $(S, \mathcal{L}_S)$ where $\mathcal{L}_S := \{l \in \mathcal{L} | l \subseteq S\}$. A subspace $S$ is called convex if for any three points $x, y, z \in \mathcal{P}$, $\text{dist}(x, y) + \text{dist}(y, z) = \text{dist}(x, z)$ and $x, z \in S$ imply that also $y \in S$. The maximal distance between two points of a convex subspace $S$ is called the diameter of $S$.

Let $\Gamma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\Gamma_2 = (\mathcal{P}_2, \mathcal{L}_2)$ be two point-line geometries with respective distance functions $\text{dist}_1(\cdot, \cdot)$ and $\text{dist}_2(\cdot, \cdot)$. A full embedding of $\Gamma_1$ into $\Gamma_2$ is an injective mapping $e$ from $\mathcal{P}_1$ to $\mathcal{P}_2$ such that $e(L) := \{e(x) | x \in L\}$ is a line of $\Gamma_2$ for every line $L$ of $\Gamma_1$. A full embedding is called isometric if $\text{dist}_2(e(x), e(y)) = \text{dist}_1(x, y)$ for all $x, y \in \mathcal{P}_1$. If $\Gamma_2$ is a projective space and if $e(\mathcal{P}_1)$ generates the whole of $\Gamma_2$, then $e$ is called a full projective embedding. In this case, the dimension of the projective space $\Gamma_2$ is called the projective dimension of $e$. Isomorphisms between full (projective) embeddings, which we will denote by the symbol $\cong$, are defined in the usual way.

Suppose $e : \Gamma \rightarrow \Sigma$ is a full embedding of the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ into the projective space $\Sigma$, and $\alpha$ is a subspace of $\Sigma$ satisfying:

(C1) $e(x) \not\in \alpha$ for every point $x \in \mathcal{P}$;
(C2) $\langle \alpha, e(x) \rangle \neq e(\alpha)$ for every two distinct points $x$ and $y$ of $\Gamma$.

Then the mapping $e/\alpha : \Gamma \rightarrow \Sigma/\alpha; x \mapsto \langle \alpha, e(x) \rangle$ is a full embedding of $\Gamma$ into the quotient projective space $\Sigma/\alpha$. If $e_1 : \Gamma \rightarrow \Sigma_1$ and $e_2 : \Gamma \rightarrow \Sigma_2$ are two full projective embeddings of $\Gamma$, then we say that $e_1 \geq e_2$ if there exists a subspace $\alpha$ of $\Sigma_1$ satisfying (C1), (C2) such that $e_1/\alpha$ is isomorphic to $e_2$.

An important class of point-line geometries are the dual polar spaces. With every non-degenerate polar space $\Pi$ of rank $n$, there is associated a dual polar space $\Delta$ of rank $n$. The points of $\Delta$ are the maximal singular subspaces of $\Pi$ (i.e. the singular subspaces of projective dimension $n - 1$) and the lines of $\Delta$ are the sets $L_\alpha$ of maximal singular subspaces containing a given singular subspace $\alpha$ of projective dimension $n - 2$. There is a bijective correspondence between the possibly empty singular subspaces of $\Pi$ and the convex subspaces of $\Delta$. If $\beta$ is an $(n - 1 - k)$-dimensional, $k \in \{0, \ldots, n\}$, singular subspace of $\Pi$, then the set of all singular subspaces containing $\beta$ is a convex subspace of $\Delta$ of diameter $k$. These convex subspaces are called quads if $k = 2$, hexes if $k = 3$ and maxes if $k = n - 1$. If $F$ is a
convex subspace of $\Delta$ of diameter $k \geq 2$, then $\tilde{F}$ is a dual polar space of rank $k$.

Suppose $\Delta = (\mathcal{P}, \mathcal{L})$ is a thick dual polar space of rank $n$. For every point $x$ of $\Delta$, let $H_x$ denote the set of points at non-maximal distance (i.e., distance at most $n - 1$) from $x$. If $e : \Delta \to \Sigma$ is a full projective embedding of $\Delta$ and if $F$ is a convex subspace of $\Delta$, then $e$ induces a full embedding $e_F$ of $F$ into a subspace $\Sigma_F$ of $\Sigma$. A full embedding $e$ of $\Delta$ into a projective space $\Sigma$ is called polarized if $\langle e(H_x) \rangle$ is a hyperplane of $\Sigma$ for every point $x$ of $\Delta$. If $e$ is a full polarized embedding of $\Delta$, then $N_e := \bigcap_{x \in \mathcal{P}} \langle e(H_x) \rangle$ is called the nucleus of the embedding $e$. The nucleus $N_e$ satisfies the properties (C1) and (C2) above and hence there exists a full embedding $\tilde{e} := e/N_e$ of $\Delta$ into the projective space $\Sigma/N_e$. If $e_1$ is an arbitrary full polarized embedding of $\Delta$, then by Cardinali, De Bruyn and Pasini [8], $e_1 \geq \tilde{e}$ and $\tilde{e}_1 \cong \tilde{e}$. The embedding $\tilde{e}$ is called the minimal full polarized embedding of $\Delta$. The following is also proved in [8].

**Lemma 1.1** If $F$ is a convex subspace of diameter at least 2 of $\Delta$, then $(\tilde{e})_F$ is isomorphic to the minimal full polarized embedding of $\tilde{F}$.

Now, let $n \geq 3$ and let $\mathbb{F}$ be a field of characteristic 2. Consider the dual polar space $DSp(2n, \mathbb{F})$ associated with the building of type $C_n$ over $\mathbb{F}$ (see Section 2). This dual polar space admits a full embedding $e_{gr}$ into a projective space of dimension $\left(\binom{2n}{n} - \binom{2n}{n-2}\right) - 1$, called the Grassmann embedding of $DSp(2n, \mathbb{F})$. If $Q$ is a quad of $DSp(2n, \mathbb{F})$, then $e_Q := (e_{gr})_Q$ is isomorphic to the Grassmann embedding of $DSp(4, \mathbb{F}) \cong O(5, \mathbb{F})$ into $PG(4, \mathbb{F})$ and hence $N_{e_Q}$ is a singleton, see e.g. Section 2. Let $G_{n-2}$ denote the following point-line geometry: the points of $G_{n-2}$ are the quads of $DSp(2n, \mathbb{F})$ and the lines of $G_{n-2}$ are the sets of quads of $DSp(2n, \mathbb{F})$ which contain a given line of $DSp(2n, \mathbb{F})$ and which are contained in a given hex of $DSp(2n, \mathbb{F})$. The following is the main result of this paper:

**Main Theorem** 1. The dimension of $N_{e_{gr}}$ is equal to $\left(\binom{2n}{n} - \binom{2n}{n-2}\right) - 2^n - 1$.
2. For every quad $Q$ of $DSp(2n, \mathbb{F})$, the singleton $N_{e_Q}$ is contained in $N_{e_{gr}}$.
3. The map $Q \mapsto N_{e_Q}$ defines a full projective embedding of $G_{n-2}$ into $N_{e_{gr}}$.

The geometry $G_{n-2}$ is isomorphic to the $(n-2)$-Grassmannian of type $C_n$, that is the point-line geometry with points the objects of rank $n-2$ of $C_n$ (i.e. the spaces of vector dimension $n-2$) and with lines the sets $l_{[A,B]} := \{x \mid \text{rank}(x) = n-2, A \subset x \subset B\}$, where $A$ and $B$ are objects of rank $n-3$ and $n-1$, respectively. Grassmannians of polar spaces have attracted some attention recently. Embeddings, generating ranks, special subspaces, and hyperplane complements have recently been under investigation in the literature, see e.g. [1, 14, 12, 2, 13, 3, 4].


We will prove the main theorem in Section 4. This main theorem generalizes Theorem 1.3 of the paper [9] which contains an alternative proof of Main Theorem for the case when \( n = 3 \) and \( \mathbb{F} \) is a finite field of even characteristic.

2 Notation and the dimension of \( \mathcal{N}_{e_{gr}} \)

Let \( n \in \mathbb{N} \setminus \{0, 1\} \) and let \( \mathbb{F} \) be a field of characteristic 2. Let \( V \) be a 2\( n \)-dimensional vector space over \( \mathbb{F} \) equipped with a non-degenerate alternating form \((\cdot, \cdot)\). An ordered basis \((\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n, \overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n)\) of \( V \) is called a hyperbolic basis of \( V \) (with respect to \((\cdot, \cdot)\)) if \((\overline{v}_i, \overline{v}_j) = (\overline{f}_i, \overline{f}_j) = 0\) and \((\overline{v}_i, \overline{f}_j) = \delta_{ij}\) for all \( i, j \in \{1, \ldots, n\} \).

Let \( \wedge^n V \) denote the \( n \)-th exterior power of \( V \) and let \( W \) denote the subspace of \( \wedge^n V \) generated by all vectors of the form \( \overline{v}_1 \wedge \overline{v}_2 \wedge \cdots \wedge \overline{v}_n \), where \( \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n \) are vectors of \( V \) satisfying \((\overline{v}_i, \overline{v}_j) = 0\) for all \( i, j \in \{1, \ldots, n\} \). The dimension of \( W \) is equal to \( \binom{2n}{n} - \binom{2n}{n-2} \), see e.g. Cooperstein[11] or De Bruyn [15].

The subspaces of \( V \) which are totally isotropic with respect to \((\cdot, \cdot)\) define a building of type \( \mathbb{B}_n \). We denote the associated dual polar space by \( \text{DSp}(2n, \mathbb{F}) \). We denote by \( \text{DO}(2n+1, \mathbb{F}) \) the dual polar space associated with the building of type \( \mathbb{B}_n \) which arises from a vector space of dimension \( 2n + 1 \) over \( \mathbb{F} \) equipped with a non-degenerate quadratic form of Witt-index \( n \). The dual polar spaces \( \text{DSp}(2n, \mathbb{F}) \) and \( \text{DO}(2n+1, \mathbb{F}) \) are isomorphic if and only if the field \( \mathbb{F} \) is perfect (see e.g. De Bruyn and Pasini [18]). In [18] it is also shown that for any field \( \mathbb{F} \) (of characteristic 2), there exists an isometric full embedding of \( \text{DSp}(2n, \mathbb{F}) \) into \( \text{DO}(2n+1, \mathbb{F}) \).

For every point \( p = \langle \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n \rangle \) of \( \text{DSp}(2n, \mathbb{F}) \), let \( e_{gr}(p) \) denote the point \( \langle \overline{v}_1 \wedge \overline{v}_2 \wedge \cdots \wedge \overline{v}_n \rangle \) of \( \text{PG}(W) \). Notice that \( e_{gr}(p) \) is independent of the generating set \( \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n\} \) of \( p \). The map \( e_{gr} \) defines a full embedding of \( \text{DSp}(2n, \mathbb{F}) \) into \( \text{PG}(W) \), called the Grassmann embedding of \( \text{DSp}(2n, \mathbb{F}) \). Let \( \mathcal{N} := \mathcal{N}_{e_{gr}} \) denote the nucleus of the embedding \( e_{gr} \). The dual polar space \( \text{DO}(2n+1, \mathbb{F}) \) admits a full polarized embedding into the projective space \( \text{PG}(2^n - 1, \mathbb{F}) \) called the spin embedding of \( \text{DO}(2n+1, \mathbb{F}) \) (Chevalley [10], Buekenhout and Cameron [6], Cameron [7]). In view of the existence of an isometric full embedding of \( \text{DSp}(2n, \mathbb{F}) \) into \( \text{DO}(2n+1, \mathbb{F}) \), the dual polar space \( \text{DSp}(2n, \mathbb{F}) \) admits a full polarized embedding \( e_{sp} \) into a subspace \( \Sigma \) of \( \text{PG}(2^n - 1, \mathbb{F}) \). Now, by De Bruyn and Pasini [17, Corollary 1.5], any full polarized embedding of a thick dual polar space of rank \( n \) has projective dimension at least \( 2^n - 1 \). Hence \( \Sigma = \text{PG}(2^n - 1, \mathbb{F}) \) and \( e_{sp} \) is the minimal full polarized embedding of \( \text{DSp}(2n, \mathbb{F}) \). This implies that the projective dimension of \( \mathcal{N} \) equals \( \binom{2n}{n} - \binom{2n}{n-2} - 2^n - 1 \).

3 The nucleus in the case \( n = 2 \)

In this section, we suppose that \( n = 2 \). Then \( \mathcal{N} \) is a singleton and the vector space \( W \) has dimension 5. Since \( p_1 = \langle \overline{v}_1, \overline{v}_2 \rangle \), \( p_2 = \langle \overline{f}_1, \overline{f}_2 \rangle \), \( p_3 = \langle \overline{v}_1, \overline{f}_2 \rangle \), \( p_4 = \langle \overline{v}_2, \overline{f}_1 \rangle \)
and \( p_5 = \langle \overline{e}_1 + \overline{e}_2, \overline{f}_1 + \overline{f}_2 \rangle \) are points of \( DS\Sigma(4, \mathbb{F}) \) and \( \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_1, \overline{e}_2 \wedge \overline{f}_1, (\overline{e}_1 + \overline{e}_2) \wedge (\overline{f}_1 + \overline{f}_2) \) are linearly independent vectors of \( W \), we have
\[
W = \langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, (\overline{e}_1 + \overline{e}_2) \wedge (\overline{f}_1 + \overline{f}_2) \rangle = \langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_1, \overline{e}_2 \wedge \overline{f}_1, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle.
\]

Now, the image of \( e_{gr} \) is a quadric \( Q \cong O(5, \mathbb{F}) \) of \( PG(W) \). The tangent hyperplane \( T(p_1) \) at the point \( e_{gr}(p_1) \) of \( Q \) is equal to \( \langle \overline{e}_1 \wedge \overline{e}_2, \overline{e}_1 \wedge \overline{f}_1, (\overline{e}_1 + \overline{e}_2) \wedge (\overline{f}_1 + \overline{f}_2) \rangle = \langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle \). Similarly, the tangent hyperplanes \( T(p_2) \), \( T(p_3) \) and \( T(p_4) \) at the points \( e_{gr}(p_2), e_{gr}(p_3) \) and \( e_{gr}(p_4) \) of \( Q \) are respectively equal to \( \langle \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle \), \( \langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle \), \( \langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle \) and \( \langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle \). Since \( \langle \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle \) is the unique point in the intersection \( T(p_1) \cap T(p_2) \cap T(p_3) \cap T(p_4) \), it necessarily is the unique point of the singleton \( N \).

## 4 Proof of Main Theorem

If \( F \) is a convex subspace of diameter at least 2 of \( DS\Sigma(2n, \mathbb{F}) \), then the Grassmann embedding \( e_{gr} \) of \( DS\Sigma(2n, \mathbb{F}) \) induces a full embedding \( e_F := (e_{gr})_F \) of \( \tilde{F} \) into a subspace \( \Sigma_F \) of \( PG(W) \). This embedding \( e_F \) is isomorphic to the Grassmann embedding of \( \tilde{F} \), see e.g. Cardinali, De Bruyn and Pasini [8, Proposition 4.10]. So, for every quad \( Q \) of \( DS\Sigma(2n, \mathbb{F}) \), the nucleus of \( e_Q \) consists of a single point. We will denote this point by \( e_N(Q) \). By the following lemma, \( e_N \) can be regarded as a map between the set of points of \( G_{n-2} \) and the set of points of \( N \).

**Lemma 4.1**

(i) For every quad \( Q \) of \( DS\Sigma(2n, \mathbb{F}) \), \( e_N(Q) \in N \).

(ii) \( N \) coincides with the subspace of \( PG(W) \) generated by the points \( e_N(Q) \), where \( Q \) is a quad of \( DS\Sigma(2n, \mathbb{F}) \).

**Proof:** Suppose \( N' \) is a subspace satisfying properties (C1) and (C2) of Section 1 with respect to the embedding \( e_{gr} \). Then for every quad \( Q \) of \( DS\Sigma(2n, \mathbb{F}) \), \( N' \cap \Sigma_Q \) satisfies properties (C1) and (C2) with respect to the embedding \( e_Q \). Moreover,
\[
e_Q/(N' \cap \Sigma_Q) \cong (e_{gr}/N')_Q. \tag{4.1}
\]

(i) Since \( e_{gr}/N \) is the minimal full polarized embedding of \( DS\Sigma(2n, \mathbb{F}) \), \( (e_{gr}/N)_Q \) is isomorphic to the minimal full polarized embedding of \( \tilde{Q} \) for every quad \( Q \) of \( DS\Sigma(2n, \mathbb{F}) \) (see Lemma 1.1). From (4.1), it then follows that \( N \cap \Sigma_Q = N_{e_Q} = \{e_N(Q)\} \). Hence, \( e_N(Q) \in N \).

(ii) Suppose \( N' \) is the subspace of \( N \) generated by all points \( e_N(Q) \) where \( Q \) is a quad of \( DS\Sigma(2n, \mathbb{F}) \). Then for every quad \( Q \) of \( DS\Sigma(2n, \mathbb{F}) \), \( \{e_N(Q)\} \subseteq N' \cap \Sigma_Q \subseteq
$\mathcal{N} \cap \Sigma_Q = \{ e_{\mathcal{N}}(Q) \}$. Hence, $\mathcal{N}' \cap \Sigma_Q = \{ e_{\mathcal{N}'}(Q) \}$. By (4.1), the embedding $(e_{gr}/\mathcal{N}')_Q$ has projective dimension 3. Now, by De Bruyn [16, Theorem 1.6], if $e'$ is a full polarized embedding of a dual polar space of rank $n$ such that every induced quad embedding has projective dimension 3, then $e'$ has projective dimension $2^n - 1$. Applying this here, we see that the full polarized embedding $e_{gr}/\mathcal{N}'$ has projective dimension $2^n - 1$. This implies that $\mathcal{N}' = \mathcal{N}$.

**Lemma 4.2** $e_{\mathcal{N}}$ maps in a bijective way any line of $\mathcal{G}_{n-2}$ to some line of $\mathcal{N}$.

**Proof:** If $H$ is a hex of $DSp(2n, \mathbb{F})$, then the full embedding $e_H$ of $\widetilde{H}$ induced by $e_{gr}$ is isomorphic to the Grassmann embedding of $\widetilde{H}$. So, it suffices to prove the lemma in the case $n = 3$. Consider the line $L^*$ of $\mathcal{G}_{n-2}$ which consists of all quads of $DSp(6, \mathbb{F})$ which contain a given line $L$ of $DSp(6, \mathbb{F})$. We can choose a hyperbolic basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ of $V$ in such a way that $L = \langle \tilde{e}_1, \tilde{e}_2 \rangle$. Let $Q$ denote the quad of $DSp(6, \mathbb{F})$ corresponding to $\langle \tau_1 \rangle$ and for every $t \in \mathbb{F}$, let $Q_t$ denote the quad of $DSp(6, \mathbb{F})$ corresponding to $\langle \tilde{e}_1 + t\tilde{e}_2 \rangle$. Then by Section 3, $e_{\mathcal{N}}(Q) = \langle \tilde{e}_1 \land \tilde{e}_2 \land \tilde{f}_2 + \tilde{e}_1 \land \tilde{e}_3 \land \tilde{f}_3 \rangle$. Since $(\tilde{e}_2 + t\tilde{e}_1, \tilde{e}_1, \tilde{e}_3, \tilde{f}_2, \tilde{f}_1 + t\tilde{f}_2, \tilde{f}_3)$ is a hyperbolic basis of $V$, we have $e_{\mathcal{N}}(Q_t) = \langle (\tilde{e}_2 + t\tilde{e}_1) \land \tilde{e}_1 \land (\tilde{f}_1 + t\tilde{f}_2) \rangle = \langle (\tilde{e}_2 \land \tilde{e}_1 \land \tilde{f}_1 + \tilde{e}_2 \land \tilde{e}_3 \land \tilde{f}_3) + t(\tilde{e}_1 \land \tilde{e}_2 \land \tilde{f}_2 + \tilde{e}_1 \land \tilde{e}_3 \land \tilde{f}_3) \rangle$. Hence, $e_{\mathcal{N}}$ defines a bijection between the line $L^*$ of $\mathcal{G}_{n-2}$ and a line of $\mathcal{N}$.

**Lemma 4.3** The map $e_{\mathcal{N}}$ is injective.

**Proof:** Let $Q_1$ and $Q_2$ be two distinct quads of $DSp(2n, \mathbb{F})$. We need to show that $e_{\mathcal{N}}(Q_1) \neq e_{\mathcal{N}}(Q_2)$.

(i) If $Q_1 \cap Q_2$ is a line, then Lemma 4.2 implies that $e_{\mathcal{N}}(Q_1) \neq e_{\mathcal{N}}(Q_2)$.

(ii) Suppose that $Q_1 \cap Q_2$ is a singleton $\{x\}$. Let $F$ denote the convex subspace of diameter 4 containing $Q_1$ and $Q_2$. Since the embedding $e_F$ of $\widetilde{F}$ induced by $e$ is isomorphic to the Grassmann embedding of $\widetilde{F}$, we may suppose that $n = 4$. We can choose a hyperbolic basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4)$ of $V$ in such a way that the point $x$ corresponds to $\langle \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \rangle$ and that the quads $Q_1$ and $Q_2$ correspond to $\langle \tilde{e}_1, \tilde{e}_2 \rangle$ and $\langle \tilde{e}_3, \tilde{e}_4 \rangle$, respectively. Then by Section 3, $e_{\mathcal{N}}(Q_1) = \langle \tilde{e}_1 \land \tilde{e}_2 \land \tilde{e}_3 \land \tilde{f}_3 + \tilde{e}_1 \land \tilde{e}_2 \land \tilde{e}_4 \land \tilde{f}_4 \rangle = e_{\mathcal{N}}(Q_2)$.

(iii) Suppose $Q_1$ and $Q_2$ are disjoint. Then there exist maxes $M_1$ and $M_2$ such that $Q_1 \subseteq M_1$ and $Q_2 \subseteq M_2$ and $M_1 \cap M_2 = \emptyset$. We can choose a hyperbolic basis $(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n, \tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$ of $V$ in such a way that $M_1$ corresponds to $\langle \tilde{e}_1 \rangle$ and $M_2$ corresponds to $\langle \tilde{f}_1 \rangle$. Then $e_{gr}(M_1)$ is the subspace of $PG(W)$ generated by all points of the form $\langle \tilde{e}_1 \land \tilde{g}_2 \land \tilde{g}_3 \land \cdots \land \tilde{g}_n \rangle$, where $\tilde{g}_2, \tilde{g}_3, \ldots, \tilde{g}_n$ are linearly independent vectors of $\langle \tilde{e}_2, \ldots, \tilde{e}_n, \tilde{f}_2, \ldots, \tilde{f}_n \rangle$ satisfying $\langle \tilde{g}_i, \tilde{g}_j \rangle = 0$ for all $i, j \in \{2, \ldots, n\}$. Similarly, $e_{gr}(M_2)$ is the subspace of $PG(W)$ generated by all points.
of the form \( \langle f_1 \land g_2 \land g_3 \land \cdots \land g_n \rangle \), where \( g_2, g_3, \ldots, g_n \) are linearly independent vectors of \( \langle e_2, \ldots, e_n, f_2, \ldots, f_n \rangle \) satisfying \( \langle g_i, g_j \rangle = 0 \) for all \( i, j \in \{2, \ldots, n\} \). Clearly, \( \langle e_{gr}(M_1) \rangle \) and \( \langle e_{gr}(M_2) \rangle \) are disjoint. This implies that \( e_N(Q_1) \neq e_N(Q_2) \).

\[ \square \]

References


