On the nucleus of the Grassmann embedding of the symplectic dual polar space $\mathrm{DSp}(2n, \mathbb{F})$, $\mathrm{char}(\mathbb{F}) = 2$

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Abstract

Let $n \geq 3$ and let $\mathbb{F}$ be a field of characteristic 2. Let $DSp(2n, \mathbb{F})$ denote the dual polar space associated with the building of Type C_n over $\mathbb{F}$ and let $\mathcal{G}_{n-2}$ denote the $(n-2)$-Grassmannian of type C_n. Using the bijective correspondence between the points of $\mathcal{G}_{n-2}$ and the quads of $DSp(2n, \mathbb{F})$, we construct a full projective embedding of $\mathcal{G}_{n-2}$ into the nucleus of the Grassmann embedding of $DSp(2n, \mathbb{F})$. This generalizes a result of the paper [9] which contains an alternative proof of this fact in the case when $n = 3$ and $\mathbb{F}$ is finite.

1 Introduction and preliminaries

We assume the reader is familiar with the concept of a *partial linear rank two incidence geometry* $\Gamma = (\mathcal{P}, \mathcal{L})$, also called a *point-line geometry* (See e.g. [5, 19]). The distance $\mathrm{dist}(x, y)$ between two points $x, y \in \mathcal{P}$ of Γ will be measured in the *collinearity graph* of Γ, that is the graph $(\mathcal{P}, E)$ whose set of edges E consists of all unordered pairs of points belonging to a line of Γ. By a *subspace* of Γ we mean a subset S of $\mathcal{P}$ such that if $l \in \mathcal{L}$ and $|l \cap S| \geq 2$, then $l \subseteq S$. If S is a subspace of Γ, then we denote by $\widetilde{S}$ the point-line geometry $(S, \mathcal{L}_S)$ where $\mathcal{L}_S := \{l \in \mathcal{L} \,|\, l \subseteq S\}$. A subspace S is called *convex* if for any three points $x, y, z \in \mathcal{P}$, $\mathrm{dist}(x, y) + \mathrm{dist}(y, z) = \mathrm{dist}(x, z)$ and $x, z \in S$ imply that also $y \in S$. The maximal distance between two points of a convex subspace S is called the *diameter* of S.

Let $\Gamma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ and $\Gamma_2 = (\mathcal{P}_2, \mathcal{L}_2)$ be two point-line geometries with respective distance functions $\mathrm{dist}_1(\cdot, \cdot)$ and $\mathrm{dist}_2(\cdot, \cdot)$. A *full embedding* of Γ_1 into Γ_2 is an injective mapping e from $\mathcal{P}_1$ to $\mathcal{P}_2$ such that $e(L) := \{e(x) \,|\, x \in L\}$ is a line of Γ_2 for every line L of Γ_1. A full embedding is called *isometric* if $\mathrm{dist}_2(e(x), e(y)) = \mathrm{dist}_1(x, y)$ for all $x, y \in \mathcal{P}_1$. If Γ_2 is a projective space and if $e(\mathcal{P}_1)$ generates the whole of Γ_2, then e is called a *full projective embedding.* In this case, the dimension of the projective space Γ_2 is called the *projective dimension of* e. Isomorphisms between full (projective) embeddings, which we will denote by the symbol $\cong$, are defined in the usual way.

Suppose $e : \Gamma \to \Sigma$ is a full embedding of the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ into the projective space Σ, and α is a subspace of Σ satisfying:

(C1) $e(x) \notin \alpha$ for every point $x \in \mathcal{P}$;

(C2) $\langle \alpha, e(x) \rangle \neq \langle \alpha, e(y) \rangle$ for every two distinct points x and y of Γ.

Then the mapping $e/\alpha : \Gamma \to \Sigma/\alpha; x \mapsto \langle \alpha, e(x) \rangle$ is a full embedding of Γ into the quotient projective space Σ/α. If $e_1 : \Gamma \to \Sigma_1$ and $e_2 : \Gamma \to \Sigma_2$ are two full projective embeddings of Γ, then we say that $e_1 \geq e_2$ if there exists a subspace α of Σ_1 satisfying (C1), (C2) such that e_1/α is isomorphic to e_2.

An important class of point-line geometries are the dual polar spaces. With every non-degenerate polar space Π of rank n, there is associated a *dual polar space* Δ *of rank* n. The points of Δ are the maximal singular subspaces of Π (i.e. the singular subspaces of projective dimension $n-1$) and the lines of Δ are the sets L_α of maximal singular subspaces containing a given singular subspace α of projective dimension $n-2$. There is a bijective correspondence between the possibly empty singular subspaces of Π and the convex subspaces of Δ. If β is an $(n-1-k)$-dimensional, $k \in \{0, \ldots, n\}$, singular subspace of Π, then the set of all singular subspaces containing β is a convex subspace of Δ of diameter k. These convex subspaces are called *quads* if $k = 2$, *hexes* if $k = 3$ and *maxes* if $k = n-1$. If F is a convex subspace of Δ of diameter $k \geq 2$, then $\widetilde{F}$ is a dual polar space of rank k.

Suppose $\Delta = (\mathcal{P}, \mathcal{L})$ is a thick dual polar space of rank n. For every point x of Δ, let H_x denote the set of points at non-maximal distance (i.e. distance at most $n-1$) from x. If $e : \Delta \to \Sigma$ is a full projective embedding of Δ and if F is a convex subspace of Δ, then e induces a full embedding e_F of $\widetilde{F}$ into a subspace Σ_F of Σ. A full embedding e of Δ into a projective space Σ is called *polarized* if $\langle e(H_x) \rangle$ is a hyperplane of Σ for every point x of Δ. If e is a full polarized embedding of Δ, then $\mathcal{N}_e := \bigcap_{x \in \mathcal{P}} \langle e(H_x) \rangle$ is called the *nucleus* of the embedding e. The nucleus $\mathcal{N}_e$ satisfies the properties (C1) and (C2) above and hence there exists a full embedding $\overline{e} := e/\mathcal{N}_e$ of Δ into the projective space $\Sigma/\mathcal{N}_e$. If e_1 is an arbitrary full polarized embedding of Δ, then by Cardinali, De Bruyn and Pasini [8], $e_1 \geq \overline{e}$ and $\overline{e}_1 \cong \overline{e}$. The embedding $\overline{e}$ is called the *minimal full polarized embedding* of Δ. The following is also proved in [8].

Lemma 1.1 *If F is a convex subspace of diameter at least 2 of Δ, then $(\overline{e})_F$ is isomorphic to the minimal full polarized embedding of $\widetilde{F}$.*

Now, let $n \geq 3$ and let $\mathbb{F}$ be a field of characteristic 2. Consider the dual polar space $DSp(2n, \mathbb{F})$ associated with the building of type C_n over $\mathbb{F}$ (see Section 2). This dual polar space admits a full embedding e_{gr} into a projective space of dimension $\binom{2n}{n} - \binom{2n}{n-2} - 1$, called the *Grassmann embedding* of $DSp(2n, \mathbb{F})$. If Q is a quad of $DSp(2n, \mathbb{F})$, then $e_Q := (e_{gr})_Q$ is isomorphic to the Grassmann embedding of $DSp(4, \mathbb{F}) \cong O(5, \mathbb{F})$ into $\mathrm{PG}(4, \mathbb{F})$ and hence $\mathcal{N}_{e_Q}$ is a singleton, see e.g. Section 2. Let $\mathcal{G}_{n-2}$ denote the following point-line geometry: the points of $\mathcal{G}_{n-2}$ are the quads of $DSp(2n, \mathbb{F})$ and the lines of $\mathcal{G}_{n-2}$ are the sets of quads of $DSp(2n, \mathbb{F})$ which contain a given line of $DSp(2n, \mathbb{F})$ and which are contained in a given hex of $DSp(2n, \mathbb{F})$. The following is the main result of this paper:

Main Theorem (1) *The dimension of $\mathcal{N}_{e_{gr}}$ is equal to $\binom{2n}{n} - \binom{2n}{n-2} - 2^n - 1$.*
(2) *For every quad Q of $DSp(2n, \mathbb{F})$, the singleton $\mathcal{N}_{e_Q}$ is contained in $\mathcal{N}_{e_{gr}}$.*
(3) *The map $Q \mapsto \mathcal{N}_{e_Q}$ defines a full projective embedding of $\mathcal{G}_{n-2}$ into $\mathcal{N}_{e_{gr}}$.*

The geometry $\mathcal{G}_{n-2}$ is isomorphic to the $(n-2)$-*Grassmannian* of type C_n, that is the point-line geometry with points the objects of rank $n-2$ of C_n (i.e. the spaces of vector dimension $n-2$) and with lines the sets $l_{[A,B]} := \{x \mid \mathrm{rank}(x) = n-2, A \subset x \subset B\}$, where A and B are objects of rank $n-3$ and $n-1$, respectively.
Grassmannians of polar spaces have attracted some attention recently. Embeddings, generating ranks, special subspaces, and hyperplane complements have recently been under investigation in the literature, see e.g. [1, 14, 12, 2, 13, 3, 4].

We will prove the main theorem in Section 4. This main theorem generalizes Theorem 1.3 of the paper [9] which contains an alternative proof of Main Theorem for the case when $n = 3$ and $\mathbb{F}$ is a finite field of even characteristic.

2 Notation and the dimension of $\mathcal{N}_{e_{gr}}$

Let $n \in \mathbb{N} \setminus \{0, 1\}$ and let $\mathbb{F}$ be a field of characteristic 2. Let V be a $2n$-dimensional vector space over $\mathbb{F}$ equipped with a non-degenerate alternating form $(\cdot, \cdot)$. An ordered basis $(\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_n, \overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n)$ of V is called a *hyperbolic basis* of V (with respect to $(\cdot, \cdot)$) if $(\overline{e}_i, \overline{e}_j) = (\overline{f}_i, \overline{f}_j) = 0$ and $(\overline{e}_i, \overline{f}_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$. Let $\bigwedge^n V$ denote the n-th exterior power of V and let W denote the subspace of $\bigwedge^n V$ generated by all vectors of the form $\overline{v}_1 \wedge \overline{v}_2 \wedge \cdots \wedge \overline{v}_n$, where $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$ are vectors of V satisfying $(\overline{v}_i, \overline{v}_j) = 0$ for all $i, j \in \{1, \ldots, n\}$. The dimension of W is equal to $\binom{2n}{n} - \binom{2n}{n-2}$, see e.g. Cooperstein[11] or De Bruyn [15].

The subspaces of V which are totally isotropic with respect to $(\cdot, \cdot)$ define a building of type C_n. We denote the associated dual polar space by $DSp(2n, \mathbb{F})$. We denote by $DO(2n+1, \mathbb{F})$ the dual polar space associated with the building of type B_n which arises from a vector space of dimension $2n+1$ over $\mathbb{F}$ equipped with a non-degenerate quadratic form of Witt-index n. The dual polar spaces $DSp(2n, \mathbb{F})$ and $DO(2n+1, \mathbb{F})$ are isomorphic if and only if the field $\mathbb{F}$ is perfect (see e.g. De Bruyn and Pasini [18]). In [18] it is also shown that for any field $\mathbb{F}$ (of characteristic 2), there exists an isometric full embedding of $DSp(2n, \mathbb{F})$ into $DO(2n+1, \mathbb{F})$.

For every point $p = \langle \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n \rangle$ of $DSp(2n, \mathbb{F})$, let $e_{gr}(p)$ denote the point $\langle \overline{v}_1 \wedge \overline{v}_2 \wedge \cdots \wedge \overline{v}_n \rangle$ of $\mathrm{PG}(W)$. Notice that $e_{gr}(p)$ is independent of the generating set $\{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n\}$ of p. The map e_{gr} defines a full embedding of $DSp(2n, \mathbb{F})$ into $\mathrm{PG}(W)$, called the *Grassmann embedding* of $DSp(2n, \mathbb{F})$. Let $\mathcal{N} := \mathcal{N}_{e_{gr}}$ denote the nucleus of the embedding e_{gr}. The dual polar space $DO(2n+1, \mathbb{F})$ admits a full polarized embedding into the projective space $\mathrm{PG}(2^n - 1, \mathbb{F})$ called the *spin embedding* of $DO(2n+1, \mathbb{F})$ (Chevalley [10], Buekenhout and Cameron [6], Cameron [7]). In view of the existence of an isometric full embedding of $DSp(2n, \mathbb{F})$ into $DO(2n+1, \mathbb{F})$, the dual polar space $DSp(2n, \mathbb{F})$ admits a full polarized embedding e_{sp} into a subspace Σ of $\mathrm{PG}(2^n - 1, \mathbb{F})$. Now, by De Bruyn and Pasini [17, Corollary 1.5], any full polarized embedding of a thick dual polar space of rank n has projective dimension at least $2^n - 1$. Hence $\Sigma = \mathrm{PG}(2^n - 1, \mathbb{F})$ and e_{sp} is the minimal full polarized embedding of $DSp(2n, \mathbb{F})$. This implies that the projective dimension of $\mathcal{N}$ equals $\binom{2n}{n} - \binom{2n}{n-2} - 2^n - 1$.

3 The nucleus in the case $n = 2$

In this section, we suppose that $n = 2$. Then $\mathcal{N}$ is a singleton and the vector space W has dimension 5. Since $p_1 = \langle \overline{e}_1, \overline{e}_2 \rangle$, $p_2 = \langle \overline{f}_1, \overline{f}_2 \rangle$, $p_3 = \langle \overline{e}_1, \overline{f}_2 \rangle$, $p_4 = \langle \overline{e}_2, \overline{f}_1 \rangle$

and $p_5 = \langle \overline{e}_1 + \overline{e}_2, \overline{f}_1 + \overline{f}_2 \rangle$ are points of $DSp(4, \mathbb{F})$ and $\overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_2, \overline{e}_2 \wedge \overline{f}_1, (\overline{e}_1 + \overline{e}_2) \wedge (\overline{f}_1 + \overline{f}_2)$ are linearly independent vectors of W, we have

$$\begin{aligned} W &= \langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_2, \overline{e}_2 \wedge \overline{f}_1, (\overline{e}_1 + \overline{e}_2) \wedge (\overline{f}_1 + \overline{f}_2) \rangle \\ &= \langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_2, \overline{e}_2 \wedge \overline{f}_1, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle. \end{aligned}$$

Now, the image of e_{gr} is a quadric $Q \cong O(5, \mathbb{F})$ of $\mathrm{PG}(W)$. The tangent hyperplane $T(p_1)$ at the point $e_{gr}(p_1)$ of Q is equal to $\langle \overline{e}_1 \wedge \overline{e}_2, \overline{e}_1 \wedge \overline{f}_2, \overline{e}_2 \wedge \overline{f}_1, (\overline{e}_1 + \overline{e}_2) \wedge (\overline{f}_1 + \overline{f}_2) \rangle = \langle \overline{e}_1 \wedge \overline{e}_2, \overline{e}_1 \wedge \overline{f}_2, \overline{e}_2 \wedge \overline{f}_1, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle$. Similarly, the tangent hyperplanes $T(p_2)$, $T(p_3)$ and $T(p_4)$ at the points $e_{gr}(p_2)$, $e_{gr}(p_3)$ and $e_{gr}(p_4)$ of Q are respectively equal to $\langle \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_2, \overline{e}_2 \wedge \overline{f}_1, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle$, $\langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_2, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle$ and $\langle \overline{e}_1 \wedge \overline{e}_2, \overline{f}_1 \wedge \overline{f}_2, \overline{e}_2 \wedge \overline{f}_1, \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle$. Since $\langle \overline{e}_1 \wedge \overline{f}_1 + \overline{e}_2 \wedge \overline{f}_2 \rangle$ is the unique point in the intersection $T(p_1) \cap T(p_2) \cap T(p_3) \cap T(p_4)$, it necessarily is the unique point of the singleton $\mathcal{N}$.

4 Proof of Main Theorem

If F is a convex subspace of diameter at least 2 of $DSp(2n, \mathbb{F})$, then the Grassmann embedding e_{gr} of $DSp(2n, \mathbb{F})$ induces a full embedding $e_F := (e_{gr})_F$ of $\widetilde{F}$ into a subspace Σ_F of $\mathrm{PG}(W)$. This embedding e_F is isomorphic to the Grassmann embedding of $\widetilde{F}$, see e.g. Cardinali, De Bruyn and Pasini [8, Proposition 4.10]. So, for every quad Q of $DSp(2n, \mathbb{F})$, the nucleus of e_Q consists of a single point. We will denote this point by $e_{\mathcal{N}}(Q)$. By the following lemma, $e_{\mathcal{N}}$ can be regarded as a map between the set of points of $\mathcal{G}_{n-2}$ and the set of points of $\mathcal{N}$.

Lemma 4.1 *(i) For every quad Q of $DSp(2n, \mathbb{F})$, $e_{\mathcal{N}}(Q) \in \mathcal{N}$.*

(ii) $\mathcal{N}$ coincides with the subspace of $\mathrm{PG}(W)$ *generated by the points $e_{\mathcal{N}}(Q)$, where Q is a quad of $DSp(2n, \mathbb{F})$.*

Proof : Suppose $\mathcal{N}'$ is a subspace satisfying properties (C1) and (C2) of Section 1 with respect to the embedding e_{gr}. Then for every quad Q of $DSp(2n, \mathbb{F})$, $\mathcal{N}' \cap \Sigma_Q$ satisfies properties (C1) and (C2) with respect to the embedding e_Q. Moreover,

$$e_Q / (\mathcal{N}' \cap \Sigma_Q) \cong (e_{gr} / \mathcal{N}')_Q. \tag{4.1}$$

(i) Since $e_{gr}/\mathcal{N}$ is the minimal full polarized embedding of $DSp(2n, \mathbb{F})$, $(e_{gr}/\mathcal{N})_Q$ is isomorphic to the minimal full polarized embedding of $\widetilde{Q}$ for every quad Q of $DSp(2n, \mathbb{F})$ (see Lemma 1.1). From (4.1), it then follows that $\mathcal{N} \cap \Sigma_Q = \mathcal{N}_{e_Q} = \{e_{\mathcal{N}}(Q)\}$. Hence, $e_{\mathcal{N}}(Q) \in \mathcal{N}$.

(ii) Suppose $\mathcal{N}'$ is the subspace of $\mathcal{N}$ generated by all points $e_{\mathcal{N}}(Q)$ where Q is a quad of $DSp(2n, \mathbb{F})$. Then for every quad Q of $DSp(2n, \mathbb{F})$, $\{e_{\mathcal{N}}(Q)\} \subseteq \mathcal{N}' \cap \Sigma_Q \subseteq$

$\mathcal{N}\cap\Sigma_Q = \{e_{\mathcal{N}}(Q)\}$. Hence, $\mathcal{N}'\cap\Sigma_Q = \{e_{\mathcal{N}}(Q)\}$. By (4.1), the embedding $(e_{gr}/\mathcal{N}')_Q$ has projective dimension 3. Now, by De Bruyn [16, Theorem 1.6], if e' is a full polarized embedding of a dual polar space of rank n such that every induced quad embedding has projective dimension 3, then e' has projective dimension $2^n - 1$. Applying this here, we see that the full polarized embedding $e_{gr}/\mathcal{N}'$ has projective dimension $2^n - 1$. This implies that $\mathcal{N}' = \mathcal{N}$. $\square$

Lemma 4.2 *$e_{\mathcal{N}}$ maps in a bijective way any line of $\mathcal{G}_{n-2}$ to some line of $\mathcal{N}$.*

Proof : If H is a hex of $DSp(2n,\mathbb{F})$, then the full embedding e_H of $\widetilde{H}$ induced by e_{gr} is isomorphic to the Grassmann embedding of $\widetilde{H}$. So, it suffices to prove the lemma in the case $n = 3$. Consider the line L^* of $\mathcal{G}_{n-2}$ which consists of all quads of $DSp(6,\mathbb{F})$ which contain a given line L of $DSp(6,\mathbb{F})$. We can choose a hyperbolic basis $(\bar{e}_1,\bar{e}_2,\bar{e}_3,\bar{f}_1,\bar{f}_2,\bar{f}_3)$ of V in such a way that $L = \langle\bar{e}_1,\bar{e}_2\rangle$. Let Q denote the quad of $DSp(6,\mathbb{F})$ corresponding to $\langle\bar{e}_1\rangle$ and for every $t \in \mathbb{F}$, let Q_t denote the quad of $DSp(6,\mathbb{F})$ corresponding to $\langle\bar{e}_2 + t\bar{e}_1\rangle$. Then by Section 3, $e_{\mathcal{N}}(Q) = \langle\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_2+\bar{e}_1\wedge\bar{e}_3\wedge\bar{f}_3\rangle$. Since $(\bar{e}_2+t\bar{e}_1,\bar{e}_1,\bar{e}_3,\bar{f}_2,\bar{f}_1+t\bar{f}_2,\bar{f}_3)$ is a hyperbolic basis of V, we have $e_{\mathcal{N}}(Q_t) = \langle(\bar{e}_2 + t\bar{e}_1)\wedge\bar{e}_1\wedge(\bar{f}_1 + t\bar{f}_2) + (\bar{e}_2 + t\bar{e}_1)\wedge\bar{e}_3\wedge\bar{f}_3\rangle = \langle(\bar{e}_2\wedge\bar{e}_1\wedge\bar{f}_1 + \bar{e}_2\wedge\bar{e}_3\wedge\bar{f}_3) + t(\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_2 + \bar{e}_1\wedge\bar{e}_3\wedge\bar{f}_3)\rangle$. Hence, $e_{\mathcal{N}}$ defines a bijection between the line L^* of $\mathcal{G}_{n-2}$ and a line of $\mathcal{N}$. $\square$

Lemma 4.3 *The map $e_{\mathcal{N}}$ is injective.*

Proof : Let Q_1 and Q_2 be two distinct quads of $DSp(2n,\mathbb{F})$. We need to show that $e_{\mathcal{N}}(Q_1) \neq e_{\mathcal{N}}(Q_2)$.

(i) If $Q_1 \cap Q_2$ is a line, then Lemma 4.2 implies that $e_{\mathcal{N}}(Q_1) \neq e_{\mathcal{N}}(Q_2)$.

(ii) Suppose that $Q_1 \cap Q_2$ is a singleton $\{x\}$. Let F denote the convex subspace of diameter 4 containing Q_1 and Q_2. Since the embedding e_F of $\widetilde{F}$ induced by e is isomorphic to the Grassmann embedding of $\widetilde{F}$, we may suppose that $n = 4$. We can choose a hyperbolic basis $(\bar{e}_1,\bar{e}_2,\bar{e}_3,\bar{e}_4,\bar{f}_1,\bar{f}_2,\bar{f}_3,\bar{f}_4)$ of V in such a way that the point x corresponds to $\langle\bar{e}_1,\bar{e}_2,\bar{e}_3,\bar{e}_4\rangle$ and that the quads Q_1 and Q_2 correspond to $\langle\bar{e}_1,\bar{e}_2\rangle$ and $\langle\bar{e}_3,\bar{e}_4\rangle$, respectively. Then by Section 3, $e_{\mathcal{N}}(Q_1) = \langle\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3\wedge\bar{f}_3 + \bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_4\wedge\bar{f}_4\rangle \neq \langle\bar{e}_3\wedge\bar{e}_4\wedge\bar{e}_1\wedge\bar{f}_1 + \bar{e}_3\wedge\bar{e}_4\wedge\bar{e}_2\wedge\bar{f}_2\rangle = e_{\mathcal{N}}(Q_2)$.

(iii) Suppose Q_1 and Q_2 are disjoint. Then there exist maxes M_1 and M_2 such that $Q_1 \subseteq M_1$, $Q_2 \subseteq M_2$ and $M_1 \cap M_2 = \emptyset$. We can choose a hyperbolic basis $(\bar{e}_1,\bar{e}_2,\ldots,\bar{e}_n,\bar{f}_1,\bar{f}_2,\ldots,\bar{f}_n)$ of V in such a way that M_1 corresponds to $\langle\bar{e}_1\rangle$ and M_2 corresponds to $\langle\bar{f}_1\rangle$. Then $\langle e_{gr}(M_1)\rangle$ is the subspace of PG(W) generated by all points of the form $\langle\bar{e}_1\wedge\bar{g}_2\wedge\bar{g}_3\wedge\cdots\wedge\bar{g}_n\rangle$, where $\bar{g}_2,\bar{g}_3,\ldots,\bar{g}_n$ are linearly independent vectors of $\langle\bar{e}_2,\ldots,\bar{e}_n,\bar{f}_2,\ldots,\bar{f}_n\rangle$ satisfying $(\bar{g}_i,\bar{g}_j) = 0$ for all $i,j \in \{2,\ldots,n\}$. Similarly, $\langle e_{gr}(M_2)\rangle$ is the subspace of PG(W) generated by all points of the form $\langle \overline{f}_1 \wedge \overline{g}_2 \wedge \overline{g}_3 \wedge \cdots \wedge \overline{g}_n \rangle$, where $\overline{g}_2, \overline{g}_3, \ldots, \overline{g}_n$ are linearly independent vectors of $\langle \overline{e}_2, \ldots, \overline{e}_n, \overline{f}_2, \ldots, \overline{f}_n \rangle$ satisfying $(\overline{g}_i, \overline{g}_j) = 0$ for all $i, j \in \{2, \ldots, n\}$. Clearly, $\langle e_{gr}(M_1) \rangle$ and $\langle e_{gr}(M_2) \rangle$ are disjoint. This implies that $e_{\mathcal{N}}(Q_1) \neq e_{\mathcal{N}}(Q_2)$. $\square$

References

[1] R. J. Blok. The generating rank of the symplectic line-Grassmannian. *Beiträge Algebra Geom.* 44 (2003), 575–580.

[2] R. J. Blok. The generating rank of the symplectic Grassmannians: hyperbolic and isotropic geometry. *European J. Combin.* 28 (2007), 1368–1394.

[3] R. J. Blok and B. N. Cooperstein. Projective subgrassmannians of polar Grassmannians. Submitted.

[4] R. J. Blok and B. N. Cooperstein. The generating ranks of the Symplectic and Unitary polar Grassmannians. Preprint.

[5] F. Buekenhout (editor). *Handbook of incidence geometry.* North-Holland, Amsterdam, 1995.

[6] F. Buekenhout and P. J. Cameron. Projective and affine geometry over division rings. In *Handbook of incidence geometry*, pages 27–62. North-Holland, Amsterdam, 1995.

[7] P. J. Cameron. Flat embeddings of near $2n$-gons. In *Finite geometries and designs (Proc. Conf., Chelwood Gate, 1980)*, volume 49 of *London Math. Soc. Lecture Note Ser.*, pages 61–71. Cambridge Univ. Press, Cambridge - New York, 1981.

[8] I. Cardinali, B. De Bruyn and A. Pasini. Minimal full polarized embeddings of dual polar spaces. *J. Algebraic Combin.* 25 (2007), 7–23.

[9] I. Cardinali and G. Lunardon. A geometric description of the spin-embedding of symplectic dual polar spaces of rank 3. *J. Combin. Theory Ser. A*, to appear.

[10] C. C. Chevalley. *The algebraic theory of spinors.* Columbia University Press, New York, 1954.

[11] B. N. Cooperstein. On the generation of dual polar spaces of unitary type over finite fields. *J. Combin. Theory Ser. A* 83 (1998), 221–232.

[12] B. N. Cooperstein. Classical subspaces of symplectic Grassmannians. *Bull. Belg. Math. Soc. Simon Stevin* 12 (2005), 719–725.

[13] B. N. Cooperstein. Symplectic subspaces of symplectic Grassmannians. *European J. Combin.* 28 (2007), 1442–1454.

[14] B. N. Cooperstein, A. Kasikova, and E. E. Shult. Witt-type theorems for Grassmannians and Lie incidence geometries. *Adv. Geom.* 5 (2005), 15–36.

[15] B. De Bruyn. A decomposition of the natural embedding spaces for the symplectic dual polar spaces. *Linear Algebra Appl.* 426 (2007), 462–477.

[16] B. De Bruyn. The structure of the spin-embeddings of dual polar spaces and related geometries. *European J. Combin.*, to appear.

[17] B. De Bruyn and A. Pasini. Minimal scattered sets and polarized embeddings of dual polar spaces. *European J. Combin.* 28 (2007), 1890–1909.

[18] B. De Bruyn and A. Pasini. On symplectic polar spaces over non-perfect fields of characteristic 2. *Linear Multilinear Algebra*, to appear.

[19] A. Pasini. *Diagram geometries.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.