SPANNING POINT-LINE GEOMETRIES IN BUILDINGS OF SPHERICAL TYPE

Rieuwert J. Blok   Andries E. Brouwer

We consider the point-line geometries that arise as a shadow space in a spherical building with a diagram of type A_n, B_n, C_n, D_n or E_n, and determine in which cases the geometry is spanned by the set of points on an apartment. It turns out that this happens precisely in the cases corresponding to a minimal weight.

1 INTRODUCTION

In this paper, M denotes a Dynkin diagram defined over an index set I and $(W, \{r_i\}_{i\in I})$ will be the associated Coxeter system. The diagram M is called simply-laced if it has only single bonds. For any subset $J \subset I$, M_J denotes the subdiagram of M defined over J and W_J will be the subgroup of W having generator set $\{r_i\}_{i\in J}$. The pair $(W_J, \{r_i\}_{i\in J})$ then is a Coxeter system with diagram M_J.

Let X be a building of type M, i.e. X is a chamber system over I such that each rank-one residue contains at least two chambers, and having a W-valued distance-function

$$\delta \,:\, X \times X \longrightarrow W \;,$$

such that $\delta(x, y) = r_f$ if and only if there exists a gallery of reduced type f with endpoints x and y respectively. We mention that the set W together with the distance function defined by $\delta(x, y) = x^{-1}y$ for all $x, y \in W$ is a building. Note that, in this case, if $\delta(x, y) = r_{i_0} \cdots r_{i_k}$ $(i_j \in I\,,\, 0 \le j \le k)$, then $(x, xr_{i_0}, \ldots, xr_{i_0} \cdots r_{i_k} = y)$ is a gallery from x to y.

An apartment of X is a subset of X which is δ-isometric to W. The collection of all apartments of X is denoted by $\mathcal{A}$.

If we identify an apartment with W, then left multiplication by an element from W induces a δ-isometry of the apartment onto itself. Let $r \in W$ be a reflection, i.e. a conjugate ${r_i}^w$ ($w \in W$) of some generator r_i. Then, there is a unique partition $\alpha \sqcup \beta$ of W such that $r\alpha = \beta$, and for any two chambers $x \in \alpha$, $y \in W$ and for any expression $\delta(x, y) = r_{i_0} \cdots r_{i_k}$, the number of elements j ($1 \leq j \leq k$) such that $\delta(x, rx) = {r_{i_j}}^{r_{i_{j-1}} \cdots r_{i_0}}$ is odd if and only if $y \in \beta$ (In particular, $\delta(x, rx) \in {W_{\{i\}}}^W$ for some $i \in I$). The elements of such a partition are called roots and we denote the collection of all roots in X by $\mathcal{R}$.

Two subsets of X will be called incident if their intersection is non-empty.

The residue of type J, or J-residue, on a chamber $x \in X$ is the inverse image of W_J under the mapping $\delta(x, \cdot) : X \to W$. This is a building of type M_J. The rank of a J-residue is the number $|J|$. We will call a residue of type $I - \{j\}$ ($j \in I$) an object of type j. The set of objects incident with a subset $Y \subset X$ is denoted by $\mathrm{obj}(Y)$.

A point-line geometry is a triple $(P, L, \star)$ in which P and L are sets whose elements are called 'points' and 'lines' respectively and in which $\star$ is a symmetric relation between the elements of P and L, called the incidence relation. A point p and a line l for which $p \star l$ are called incident. For a point set C, we let $C^\perp$ be set of points collinear with each of its elements.

A subspace of a point-line geometry $(P, L, \star)$ is a triple $(P', L', \star')$ such that $P' \subset P$, $L' \subset L$ and $\star'$ is the restriction of $\star$ to P' and L', satisfying the following requirements: If $p \in P$ is incident with a line in L', then $p \in P'$ and if $l \in L$ is incident with at least two points in P', then $l \in L'$. If $\mathcal{G} = (P, L, \star)$ is a point-line geometry and S is a subset of P, then the smallest subspace of $\mathcal{G}$ containing S will be denoted by $\langle S \rangle_{\mathcal{G}}$. If $\langle S \rangle_{\mathcal{G}} = \mathcal{G}$, then S is said to span $\mathcal{G}$.

Fix $i_0 \in I$ and let $I^\circ = \{j \ : \ m_{i_0 j} \geq 3\}$. Then, $X(i_0)$ is the point-line geometry $(\mathcal{P}, \mathcal{L}, \star)$, where $\mathcal{P}$ and $\mathcal{L}$ are the sets of residues in X of type $I - \{i_0\}$ and $I - I^\circ$ respectively, and $\star$ is induced by the incidence relation for subsets of X.

If Y is a subset of X, then the set of points that are incident with Y is denoted by $\mathcal{P}(Y)$. For a subset $T \subset \mathcal{A}$ moreover, we let $\mathcal{P}(T) := \bigcup_{A \in T} \mathcal{P}(A)$. Given a diagram M and a node with label i, let M_i denote M with designated node i, and let $M_i^\bullet$ denote the connected component of M_i containing i.

The aim of this paper is, to characterize by their diagram and the label i_0 those buildings X for which $X(i_0)$ is spanned by the set of points on an apartment among all thick spherical buildings with a diagram of type A_n, B_n, C_n, D_n or E_n. In Section 2 we will give conditions on the diagram and the label i_0 such that $X(i_0)$ is spanned by the points on an apartment, using the connectedness of a certain graph on $\mathcal{A}$ and a special decomposition of the conjugacy classes of the generators of the Coxeter group. This part does not, except possibly for the cases B_n and C_n, assume thickness. In Section 3 we will prove, using the embedding of $X(i_0)$ into a Lie algebra module, that the conditions given in Section 2 are in fact sufficient. This is summarized in the following theorem.

Theorem 1 *Suppose X is a (spherical) building with a diagram M of type A_n, D_n, E_n, B_n or C_n. If M is of type B_n or C_n, assume moreover that X is obtained from the group $O_{2n+1}(\mathbb{F})$ or $Sp_{2n}(\mathbb{F})$ respectively, with $Char(\mathbb{F}) \neq 2$. Then $\langle \mathcal{P}(A) \rangle_{X(i_0)} = X(i_0)$ for any apartment A in X if and only if $M^{\bullet}_{i_0}$ is of one of the following types* [1]*:*

1) A_{n,i_0} *with* $i_0 \in \{1, 2, \ldots, n\}$, *and* $n \in \mathbb{N}_{\geq 1}$,

2) $D_{n,1}$, $D_{n,n-1}$, $D_{n,n}$ *with* $n \in \mathbb{N}_{\geq 3}$,

3) $E_{6,1}$, $E_{6,6}$, $E_{7,7}$,

4) $B_{n,n}$ *with* $n \in \mathbb{N}_{\geq 2}$,

5) $C_{n,1}$ *with* $n \in \mathbb{N}_{\geq 2}$.

Finally, in Section 4, we will show in a geometric way why rank-two geometries of type D_{n,i_0} with $n \geq 4$ and $2 \leq i_0 \leq n-2$ are not spanned by the points on an apartment.

Remark:
The main result of Cooperstein and Shult in [4] is rather similar to THEOREM 1 and THEOREM 2 (Section 2), but was shown using a somewhat lengthy case-by-case analysis. In their paper, they define a *frame* of a geometry to be a subset of the point set that spans the geometry and is independent for some projective embedding. They show that apartments are frames in precisely the cases discussed here.

2 SPANNING APARTMENTS

First, we state the main result of this section.

Theorem 2 *Suppose X is a spherical building with a diagram M. Let i_0 be a node of M. Then, $\langle \mathcal{P}(A) \rangle_{X(i_0)} = X(i_0)$ for any apartment A in X if $M^{\bullet}_{i_0}$ is of one of the following types:*

1) A_{n,i_0} *with* $i_0 \in \{1, 2, \ldots, n\}$ *and* $n \in \mathbb{N}_{\geq 1}$,

[1] The nodes in the diagrams are labeled as in [3] Ch. 6 §4.

2) $D_{n,1}$, $D_{n,n-1}$, $D_{n,n}$ with $n \in \mathbb{N}_{\geq 3}$,

3) $E_{6,1}$, $E_{6,6}$, $E_{7,7}$,

4) $B_{n,n}$ with $n \in \mathbb{N}_{\geq 2}$, provided that its rank-two residues of type $B_{2,2}$ as a building are spanned by the points on an apartment [2].

5) $C_{n,1}$, with $n \in \mathbb{N}_{\geq 2}$, provided that its rank-two residues of type $C_{2,1}$ as a building are spanned by the points on an apartment [3].

We have

Lemma 2.1 *Suppose X is a spherical building with a diagram M. Let i_0 be a node of M. Then, for any apartment B in X and any pair α, β of roots with $B = \alpha \sqcup \beta$ we have $\mathcal{P}(B) \subseteq \bigcup_{a \in obj(\alpha)} \mathcal{P}(a)$ if $M^{\bullet}_{i_0}$ is of one of the following types:*

1) A_{n,i_0} with $i_0 \in \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}_{\geq 1}$,

2) $D_{n,1}$, $D_{n,n-1}$, $D_{n,n}$ with $n \in \mathbb{N}_{\geq 3}$,

3) $E_{6,1}$, $E_{6,6}$, $E_{7,7}$,

4) $B_{n,n}$ with $n \in \mathbb{N}_{\geq 3}$

and $\mathcal{P}(B) \subseteq \bigcup_{a \in obj(\alpha)} \mathcal{P}(a) \cup \bigcup_{p \in \mathcal{P}(\alpha) \cap \mathcal{P}(\beta)} p^{\perp}$ if $M^{\bullet}_{i_0}$ is of type

5) $C_{n,1}$ with $n \geq 2$.

Proof:
Let r be the reflection interchanging α and β. Clearly, $\mathcal{P}(B) = \mathcal{P}(\alpha) \cup \mathcal{P}(\beta)$ and $\mathcal{P}(\alpha) \subseteq \bigcup_{a \in \mathrm{obj}(\alpha)} \mathcal{P}(a)$. Now choose any $p \in \mathcal{P}(\beta)$ and assume that $x \in \beta$ is a chamber on p. If $\delta(x, rx) \in {W_J}^{W_{I-\{i_0\}}}$ $(J \subset I)$, then p contains a chamber y such that $\delta(y, ry) \in W_J$. For any $i \in I - J$, ry and y are in a common object of type i. Clearly, this object is incident with α and p, and so $p \in \bigcup_{a \in \mathrm{obj}(\alpha)} \mathcal{P}(a)$. If $\delta(x, rx) \in {W_J}^{W_{I-\{i_0\}} r_{i_0}}$ $(J \subset I)$, then the chamber xr_1 lies on a point collinear with p and as $\delta(xr_{i_0}, rxr_{i_0}) \in W_{I-\{i_0\}}$, this point lies both on α and on β.

[2]This holds, for instance, for the buildings obtained from $O_{2n+1}(\mathbb{F})$ with $\mathrm{Char}(\mathbb{F}) \neq 2$.
[3]This holds, for instance, for the buildings obtained from $Sp_{2n}(\mathbb{F})$ with $\mathrm{Char}(\mathbb{F}) \neq 2$.

We note that $\{\delta(x,rx)|x\in\beta\}=\bigcup_{i\in I}{W_{\{i\}}}^W$ and that if $S\subseteq I$ is the index set of a connected simply-laced subdiagram of M then ${W_{\{i\}}}^W={W_{\{j\}}}^W$ for all $i,j\in S$. The lemma will follow from the fact that in the cases 1), 2) and 3), we have

$$ {W_{\{i_0\}}}^W=\bigcup_{j\in I}{W_{\{j\}}}^{W_{I-\{i_0\}}}, $$

and in cases 4) and 5), we have

$$ \begin{aligned} W_{\{n-1\}}^W &= \left({W_{\{n-1\}}}^{W_{\{n\}}}\right)^{W_{I-\{n\}}}, \\ {W_{\{n\}}}^W &= \{{r_n}^{r_{n-1}\cdot r_{n-2}\cdots r_{k+1}\cdot r_k}|1\le k\le n-1\}\cup\{1,r_n\} \quad \text{and} \\ {W_{\{1\}}}^W &= {W_{\{1\}}}^{W_{I-\{1\}}}. \end{aligned} $$

Note that case 4) indeed follows from the equation above because $W_{\{n-1\}}^{W_{\{n\}}}\subset W_J$ with $J=\{n-1,n\}$ and as $n\ge 3$ we have $I\setminus J\ne\emptyset$. The computations that establish the equations for cases 4) and 5) are slightly longer but otherwise completely similar to the one needed to obtain the equation for cases 1), 2) and 3) and are therefore omitted.

We will prove the first equation using induction on $|I|$. For $|I|=1$ the equality evidently holds. Now let $|I|\ge 2$. We will show that $\bigcup_{j\in I}{W_{\{j\}}}^{W_{I-\{i_0\}}}$, which is invariant under conjugation by $W_{I-\{i_0\}}$, is also invariant under conjugation by r_{i_0} and is therefore equal to ${W_{\{i_0\}}}^W$. Since M is a forest, each $j\in I^\circ$ is in the index set of a different component of $M_{I-\{i_0\}}$. Call this index set I_j. Using induction together with the fact that M is a forest, we find for any $l\in I_j$

$$ \left({W_{\{l\}}}^{W_{I-\{i_0\}}}\right)^{r_{i_0}}=\left({W_{\{j\}}}^{W_{I_j}}\right)^{r_{i_0}}=\left(\bigcup_{i\in I_j}{W_{\{i\}}}^{W_{I_j-\{j\}}}\right)^{r_{i_0}}=\bigcup_{i\in I_j}{W_{\{i\}}}^{r_{i_0}W_{I_j-\{j\}}} $$

and this is a subset of $\bigcup_{j\in I}{W_{\{j\}}}^{W_{I-\{i_0\}}}$ since ${r_i}^{r_{i_0}}=r_i$ for $i\in I_j-\{j\}$ and ${r_j}^{r_{i_0}}={r_{i_0}}^{r_j}$. It remains to consider the case $l=i_0$. Since the Coxeter graphs of type $\mathrm{M}_{I-\{i_0\},\{j,k\}}$ (with $j,k\in I^\circ$ not necessarily distinct) have diameter less than or equal to 2 (cf. [1] §10.4.3), every element of $W_{I-\{i_0\}}$ has a product representation containing at most two generators r_j with $j\in I^\circ$. As the group W_J with $J=I-(I^\circ\cup\{i_0\})$ centralizes r_{i_0} and ${r_{i_0}}^{r_j}={r_j}^{r_{i_0}}$ for $j\in I^\circ$, we find

$$ \left({W_{\{i_0\}}}^{W_{I-\{i_0\}}}\right)^{r_{i_0}}\subseteq\{1\}\cup\{r_{i_0}\}\cup\bigcup_{j\in I^\circ}{r_j}^{W_J}\cup\bigcup_{j,k\in I^\circ}{r_k}^{W_J r_{i_0} r_j r_{i_0} W_J} $$

and since ${r_k}^{W_J r_{i_0} r_j r_{i_0} W_J} = {r_k}^{W_J r_j r_{i_0} r_j W_J} = {r_k}^{W_{I_k-\{k\}} r_j r_{i_0} r_j W_J}$ we can use the fact that r_j centralizes W_{I_k} in case $j \neq k$ and induction otherwise to conclude that ${r_k}^{W_{I_k-\{k\}} r_j} \subset \bigcup_{i \in I_k} {r_i}^{W_{I_k-\{k\}}}$. It follows that $\left(W_{\{i_0\}}^{W_{I-\{i_0\}}}\right)^{r_{i_0}}$ is contained in $\bigcup_{j \in I} {W_{\{j\}}}^{W_{I-\{i_0\}}}$. □

Proof of Theorem 2:
We will use induction on $|I|$. If $|I| \leq 2$, then the statement can easily be verified for cases 1), 2) and 3), whereas for cases 4) and 5), this is precisely the assumption. The fact that the buildings obtained from $O_{2n+1}(\mathbb{F})$ and $Sp_{2n}(\mathbb{F})$ with $\mathrm{Char}(\mathbb{F}) \neq 2$, satisfy the conditions of case 4) and 5) respectively, follows from Proposition 2 in [8].

Now let $|i| \geq 3$. Let p be any point in $X(i_0)$. Then, p is incident with some apartment B of X. At this point, we note that the graph whose vertices are the elements of $\mathcal{A}$ and in which two vertices are joined by an edge if and only if the corresponding apartments contain a common root, is connected. This follows from the fact that the automorphism group of X, generated by the root groups U_α ($\alpha \in \mathcal{R}$), acts transitively on $\mathcal{A}$. It therefore suffices, by induction on the distance between A and B in the graph $\mathcal{A}$, to show that, if there exists an apartment $C \in \mathcal{A}$ with $\mathcal{P}(C) \subset \langle \mathcal{P}(A) \rangle_{X(i_0)}$, that has a root α in common with B, then also $p \in \langle \mathcal{P}(A) \rangle_{X(i_0)}$. Suppose there is such an apartment. Suppose that p is incident with some object Y on α. If $p = Y$, we are done. If $p \neq Y$, then Y is a building with a diagram N for which $N_{i_0}^{\bullet}$ is again of one of the types in the list above and $C \cap Y$ is an apartment of Y. Since each point (line) of $Y(i_0)$ is contained in a unique point (line) of $X(i_0)$, the geometry $Y(i_0)$ is isomorphic to a subspace of $X(i_0)$ and we can apply the induction hypothesis to finish the proof. Suppose that $M_{i_0}^{\bullet}$ has type $C_{n,1}$ and p is collinear with a point q on $\mathcal{P}(C) \cap \mathcal{P}(B)$. The point q is a building Y of type C_{n-1} over $I - \{1\}$ and $Y(2)$ is the point-line geometry of lines and planes on q. Now use the $A_{2,1}$-case to see that we may use the induction hypothesis applied to $Y(2)$ to show that $p \in \langle \mathcal{P}(A) \rangle$. □

Remark:
A result very similar to LEMMA 2.1 can be found as Lemma 4.5 in Ronan & Smith [7]. Instead of reducing the proof to a computation in the Weyl group, these authors use a case-by-case analysis.

3 POINT-LINE GEOMETRIES THAT ARE SPANNED BY THE POINTS ON AN APARTMENT

In this section, except in the proof of THEOREM 1, we will assume that $|I| \geq 3$.

By an embedding of a point-line geometry Γ into a projective space $\mathbb{P}(V)$, we will mean an injective mapping $\Gamma \longrightarrow \mathbb{P}(V)$, sending points to points and lines to lines preserving incidence.

Let $\mathfrak{L}$ be the semisimple Lie algebra over $\mathbb{C}$ with root system Φ of type M. Let λ_{i_0} be the minimal fundamental dominant weight corresponding to the node of M with label i_0 and suppose $V = V(\lambda_{i_0})$ is the unique irreducible $\mathfrak{L}$-module of highest weight λ_{i_0} with maximal vector v^+.

We construct a Chevalley group G over a field $\mathbb{F}$ together with a G-module V' from this representation (see [5] §§25/27). Choose a minimal admissible lattice L in V and put $V' = L \otimes_{\mathbb{Z}} \mathbb{F}$. Suppose that $(\{\mathfrak{h}_i\}_{i \in I}, \{\mathfrak{x}_\alpha\}_{\alpha \in \Phi})$ is a Chevalley basis for $\mathfrak{L}$ and that ϕ is the representation defining the action of $\mathfrak{L}$ on V. The Chevalley group $G = G_{V'}$ is the subgroup of $\mathrm{GL}(V')$ generated by all elements of the form $x_\alpha(t) = \exp t\phi(\mathfrak{x}_\alpha)$ ($t \in \mathbb{F}$, $\alpha \in \Phi$). An element $\exp t\phi(\mathfrak{x}_\alpha) = \sum_{k=0}^{\infty} t^k \phi \left[\frac{\mathfrak{x}_\alpha^k}{k!}\right]$ inherits its action on V' from the action of $\mathfrak{U}_{\mathbb{Z}}$ on L. (Here, the $\left[\frac{\mathfrak{x}_\alpha^k}{k!}\right]$ are elements of a $\mathbb{Z}$-form $\mathfrak{U}_{\mathbb{Z}}$ of the universal enveloping algebra of $\mathfrak{L}$.)

Since G has a (B, N)-pair (see e.g. [9] §3.), there exists a building Δ_G with diagram M whose chamber system can be identified with the collection G/B of left cosets of B. If M is simply-laced, then this is the unique ($|I| \geq 3$) spherical building over $\mathbb{F}$ with diagram M, and if $M = B_n$ or C_n, it is the building obtained from $O_{2n+1}(\mathbb{F})$ or $Sp_{2n}(\mathbb{F})$ respectively. In the sequel, we will identify X with Δ_G. The sets of points and lines of $X(i_0)$ will then be identified with the set G/P of left cosets of the standard parabolic subgroup $P = P_{I-\{i_0\}}$ and the collection of subsets $gP_{\{i_0\}}P$ ($g \in G$) respectively.

Lemma 3.1 *The mapping $\zeta : G/P \longrightarrow \mathbb{P}(V')$ sending gP to the projective point $[gv^+]$ is an embedding such that $\langle \zeta(G/P) \rangle = \mathbb{P}(V')$.*

Proof:

We will first prove that ζ is well-defined and that it sends points to points and lines to lines, preserving incidence. As G is a subgroup of $\mathrm{GL}(V')$, it suffices to consider the image of P and $P_{\{i_0\}}P$ only. In order to determine the action of the standard parabolic subgroups P and $P_{\{i_0\}}$ on $[v^+]$, we will first examine the action of an arbitrary minimal standard parabolic subgroup $P_{\{i\}}$ $(i \in I)$ on $[v^+]$. Using the Levi decomposition of a standard parabolic subgroup, we write $P_{\{i\}} = L_{\{i\}} \ltimes U_{\{i\}}$, where $L_{\{i\}} = \langle H, U_{\alpha_i}, U_{-\alpha_i} \rangle$ and $U_{\{i\}} = \langle U_\alpha | \alpha \in \Phi^+ - \{\alpha_i\} \rangle$. Here, U_α denotes the root group of a root $\alpha \in \Phi$ and $H = N \cap B$. Furthermore, as H normalizes each U_α, we have $L_{\{i\}} = \langle U_{\alpha_i}, U_{-\alpha_i} \rangle H$. By definition, $[v^+]$ is stabilized by B and hence in particular by H and $U_{\{i\}}$ which implies that in fact $P_{\{i\}}[v^+] = \langle U_{\alpha_i}, U_{-\alpha_i} \rangle [v^+]$.

We now turn to the group $\langle U_{\alpha_i}, U_{-\alpha_i} \rangle$. Since $U_\alpha = \{x_\alpha(t) | t \in \mathbb{F}\}$ $(\alpha \in \Phi)$, this is in fact the Chevalley group of the Lie subalgebra $\mathfrak{l}$ with basis $\{\mathfrak{h}_i, \mathfrak{x}_{\alpha_i}, \mathfrak{x}_{-\alpha_i}\}$, acting on $\mathbb{P}(V')$ by the restriction of ϕ to $\mathfrak{l}$. Let S be the irreducible $\mathfrak{l}$-submodule of V' containing v^+. This module has dimension $\lambda_{i_0}(\mathfrak{h}_i) + 1$ because $\mathfrak{l} \cong \mathfrak{sl}_2$. Since λ_{i_0} is a fundamental weight, $\lambda_{i_0}(\mathfrak{h}_i) \leq 1$, implying that $S = \mathbb{F}\langle U_{\alpha_i}, U_{-\alpha_i} \rangle v^+$. Looking again at the dimension of S we find that P stabilizes $[v^+]$ and that $P_{\{i_0\}}P[v^+]$ is the set of all points on a line. Thus, ζ is well-defined and sends points to points and lines to lines. As a line in G/P is a set of points, ζ preserves incidence.

Injectivity of ζ follows from the fact that the stabilizer $\mathrm{Stab}_G([v^+])$ of $[v^+]$, by containing the maximal standard parabolic subgroup P, is itself standard parabolic and hence must be equal to P. Finally, by the minimal choice of the admissible lattice L, the $\mathbb{F}G$-module V' is cyclic (see [5] §27.5), whence the equality $\langle \zeta(G/P) \rangle = \mathbb{P}(V')$ follows. $\square$

In the sequel, we will think of the Weyl group of Δ_G as the quotient $W \cong N/H$. We note that for $w \in W$, the expressions wB, wP and $\zeta(wP)$ are well-defined since $H \subset B \subset P$. Let A be the apartment of Δ_G with chamber set $\{wB | w \in W\}$, then, the set of points on this apartment equals WP.

Lemma 3.2 *If W is the Weyl group of Δ_G, then $\langle \zeta(WP) \rangle = \mathbb{P}(V')$ if and only if λ_{i_0} is a minimal weight.*

Proof:

A point in $\zeta(WP)$ looks like $[wv^+]$ $(w \in W)$. Recall that the action of G, hence also of W, on $\mathbb{P}(V')$ is induced by the action of G on V', which in turn was induced by the action of $\mathfrak{U}_{\mathbb{Z}}$ on L. Suppose $\mathcal{H}$ is the set of weights of V and for $\mu \in \mathcal{H}$, we denote by V_μ its weight space. Then $L = \bigoplus_{\mu\in\mathcal{H}}(L \cap V_\mu)$ and therefore $V' = \bigoplus_{\mu\in\mathcal{H}} V'_\mu$, where $V'_\mu = (L \cap V_\mu) \otimes_{\mathbb{Z}} \mathbb{F}$. If $\mathbb{F} = \mathbb{C}$, then $V'_\mu = V_\mu$ and for $w \in W$ and $\mu \in \mathcal{H}$, we have $wV_\mu = V_{w\mu}$ and in fact, $w(V_\mu \cap L) = V_{w\mu} \cap L$ $(w \in W)$ (see [5] §21.2) . From the preceding, it follows that then also $wV'_\mu = V'_{w\mu}$ $(w \in W)$. By Proposition 6 in Ch. 8 §7 of [2], $\mathcal{H} = W\lambda_{i_0}$ if and only if λ_{i_0} is a minimal weight. Since $V_{\lambda_{i_0}}$ is spanned by v^+ only, $V'_{\lambda_{i_0}} = \langle v^+\rangle$. This proves the lemma. □

We note here that $\zeta(WP)$ is a set of $[W : W_{I-\{i_0\}}]$ independent points in $\mathbb{P}(V')$. We are ready to prove THEOREM 1.

Proof of Theorem 1:
The 'if' part follows directly from THEOREM 2. We now prove the 'only if' part. For $|I| \leq 2$, we only need to show that if $M^\bullet_{i_0}$ is of type $B_{2,1}$ or $C_{2,2}$, then $X(i_0)$ is not spanned by the points on an apartment but the natural embedding of $X(i_0)$, in both cases being the $O_5(\mathbb{F})$ geometry, is 5-dimensional and consequently $X(i_0)$ cannot be spanned by the 4 points that an apartment has.

For $|I| \geq 3$, we may identify X with Δ_G. We may assume that A is the apartment with chamber set $\{wB|w \in W\}$ because G is transitive on the collection of apartments. Above, all diagrams of type A_n, D_n, E_n, B_n and C_n that have a minimal weight, together with the labels i_0 for which the fundamental dominant weight λ_{i_0} is a minimal weight, are listed (see [2] Ch.8 §7 [4]). Hence, if $M^\bullet_{i_0}$ is not of one of the types listed in the theorem, then, by the previous lemma, $\langle\zeta(\mathcal{P}(A))\rangle$ is a proper subspace of $\mathbb{P}(V')$, which by LEMMA 3.1 equals $\langle\zeta(X(i_0))\rangle$. Using LEMMA 3.1, we conclude that $\langle\mathcal{P}(A)\rangle$ is a proper subspace of $X(i_0)$. □

4 AN EXAMPLE

In Section 2, we have seen that the geometry $X(i_0)$ is spanned by the set of points on an apartment if $M^\bullet_{i_0}$ is in a certain list. In this section, for a

[4]The notation here refers to the Dynkin diagrams of the dual of the root system. For simply-laced diagrams, these diagrams are identical. Note however, that the Dynkin diagram for the dual of the root system of type B_n is of type C_n.

building X with diagram of type D_n with $n \geq 4$, we will give a geometric argument showing why the set of points on an apartment in the geometry $X(i_0)$ $(2 \leq i_0 \leq n-2)$ does not span the entire geometry.

Let V be a vector space of dimension $2n$ over a field $\mathbb{F}$ provided with a non-degenerate quadratic form $\mathcal{Q}$ of maximal Witt index. For any subspace U of V, let $U^\perp$ denote the subspace that is orthogonal to U with respect to the symmetric bilinear form associated to $\mathcal{Q}$. We will call two subspaces U, W of V of the same dimension *partially orthogonal* if $U \cap W^\perp \neq 0$, or, equivalently, $U^\perp \cap W \neq 0$.

The building X will be the incidence geometry whose objects are the totally singular (abbreviated: t.s.) subspaces of V of dimension different from $n-1$. There are two types of t.s. n-spaces, two t.s. n-spaces having the same type if their intersection has even codimension in both. Incidence is defined by inclusion with the exception that two t.s. n-spaces of different type are called incident if their intersection has codimension one in both. The building X has diagram D_n and all buildings of type D_n arise in this way (see [10] and [6]).

Let $I = \{1, \hat{1}, \ldots, n, \hat{n}\}$ and let $\hat{}: I \longrightarrow I$ be the involution interchanging i and $\hat{\imath}$. We can choose a basis $B = \{e_i, e_{\hat{\imath}} | 1 \leq i \leq n\}$ of V such that $\mathcal{Q}(x) = x_1 x_{\hat{1}} + \cdots + x_n x_{\hat{n}}$ with respect to B. For any k-set $S \subseteq I$ let E_S denote the k-space $\langle e_i | i \in S\rangle$. The collection of t.s. subspaces of dimension different from $n-1$ that are spanned by subsets of B is an apartment of X. Call this apartment A. The chambers on A can be identified with the ordered sequences $(i_1, \ldots, i_{n-1})$ for which $E_{\{i_1,\ldots,i_{n-1}\}}$ is a t.s. $(n-1)$-space.

Fix i_0 with $2 \leq i_0 \leq n-2$ so that i_0 does not correspond to an end node of the diagram of X. The point-line geometry $X(i_0) = (\mathcal{P}, \mathcal{L}, \star)$ is obtained by taking for $\mathcal{P}$ the set of t.s. i_0-spaces of V, for $\mathcal{L}$ the set of pairs (P, M) where M is a t.s. (i_0+1)-space in V and P is a t.s. (i_0-1)-space in M, and stipulating that for $L \in \mathcal{P}$, $L \star (P, M)$ whenever $P \subseteq L \subseteq M$.

Proposition 4.1 *i) Let H be any i_0-space in V. Then the set $\mathcal{P}_H$ of t.s. i_0-spaces that are partially orthogonal to H is a proper hyperplane of $X(i_0)$.*

ii) Let $\mathcal{H}$ be the collection of i_0-sets in I that contain at least one of the subsets $\{i, \hat{\imath}\}$ $(1 \leq i \leq n)$. Then, $\langle \mathcal{P}(A)\rangle_{X(i_0)} \subseteq \bigcap_{S \in \mathcal{H}} \mathcal{P}_{E_S}$.

Proof:

i) Let $(P, M) \in \mathcal{L}$. We show that $\mathcal{P}_H$ contains either one or all t.s. i_0-spaces incident with (P, M). If $P \cap H^\perp \neq 0$ we are done. If $P \cap H^\perp = 0$, then, since $\mathrm{codim}_M(P) = 2$ and $\dim(M \cap H^\perp) \geq 1$, either one or all lines incident with (P, M) are in $P + (M \cap H^\perp)$.

We will now prove that $\mathcal{P}_H$ is a proper hyperplane. Since the automorphism group of X is transitive on the collection $\mathcal{A}$ of apartments in X, we may assume that $H = E_S$ for some i_0-set $S \subset I$. We have $V = E_{S\cap\hat{S}} + E_{\hat{S}-S} + E_{S-\hat{S}} + E_{I-(S\cup\hat{S})}$. Let $J \subset \{1, \ldots, n\}$ be such that $\{\{j, \hat{\jmath}\} | j \in J\}$ is a partition of $S \cap \hat{S}$. Since $i_0 \leq n$ we can find distinct k_j $(j \in J)$ such that $\{k_j, \hat{k_j}\} \subset I - (S \cup \hat{S})$. For $j \in J$, choose $u_j \in \langle e_j, e_{k_j}\rangle - \{e_j, e_{k_j}\}$ and $u_{\hat{\jmath}} \in \langle e_{\hat{\jmath}}, e_{\hat{k_j}}\rangle - \{e_{\hat{\jmath}}, e_{\hat{k_j}}\}$ such that $U_j = \langle u_j, u_{\hat{\jmath}}\rangle$ is a t.s. 2-space. As the k_j $(j \in J)$ are distinct, also $U' = \sum_{j\in J} U_j$ is a t.s. subspace. We note that $E_{S\cap\hat{S}} + E_{I-(S\cup\hat{S})} = U' + E_{I-(S\cup\hat{S})}$. Let U be the t.s. i_0-space $U' + E_{\hat{S}-S}$. Then $U + H^\perp = U' + E_{\hat{S}-S} + E_{S-\hat{S}} + E_{I-(S\cup\hat{S})} = E_{S\cap\hat{S}} + E_{\hat{S}-S} + E_{S-\hat{S}} + E_{I-(S\cup\hat{S})} = V$ and since $\dim(U) = \mathrm{codim}_V(H^\perp)$ we have $U \cap H^\perp = 0$. This shows that U is a t.s. i_0-space which is not in $\mathcal{P}_H$.

ii) Let $E_T \in \mathcal{P}(A)$ be any t.s. i_0-space. Then for any $S \in \mathcal{H}$ there is at least one $j \in T$ such that $j \in \{i, \hat{\imath}\}$ and $S \cap \{i, \hat{\imath}\} = \emptyset$. It follows that $E_{\{j\}} \subseteq E_T \cap E_S^\perp$. This shows that $\mathcal{P}(A)$ is contained in the subspace $\bigcap_{S\in\mathcal{H}} \mathcal{P}_{E_S}$. $\square$

By exhibiting a proper subspace of $X(i_0)$ that contains $\mathcal{P}(A)$, we have shown explicitly why $\mathcal{P}(A)$ cannot span the geometry $X(i_0)$. We expect that equality holds in *ii)* and hope to come back to this later.

Acknowledgement

We would like to thank A. M. Cohen for useful suggestions regarding Section 3.

References

[1] Brouwer, A.E., Cohen, A.M. and Neumaier, A.: *Distance-regular graphs*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3, **18**, Springer, Berlin, 1989.

[2] Bourbaki, N.: *Groupes et algèbres de Lie, Chapitres 7 et 8*, Diffusion C.C.L.S., Paris, 1975.

[3] Bourbaki, N.: *Groupes et algèbres de Lie, Chapitres 4, 5 et 6*, Masson, Paris, 1981.

[4] Cooperstein, B.N. and Shult, E.E.: 'Frames of Lie incidence geometries', preprint.

[5] Humphreys, J.E.: *Introduction to Lie algebras and representation theory*, Graduate Texts in Math. **9**, Springer, New York, 1972.

[6] Ronan, M.A.: *Lectures on Buildings*, Perspectives in Math. **7**, Academic Press, Boston, New York, 1989.

[7] Ronan, M.A. and Smith, S.D.: 'Sheaves on buildings and modular representations of Chevalley groups', *J. Algebra* **96**(1985), 319-346.

[8] Shult, E.E. and Thas, J.A.: 'Hyperplanes of dual polar spaces and the spin module', *Arch. Math.* **59**(1992), 610-23.

[9] Steinberg, R.: *Lectures on Chevalley groups*, Yale Lecture Notes, Yale University Press, 1967.

[10] Tits, J.: *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Math. **386**, Springer, Berlin, 1986.

R. J. Blok
Dept. of Pure Mathematics
Delft University of Technology
P.O. Box 5031
2600 GA Delft
The Netherlands
e-mail to: blok@twi.tudelft.nl

A. E. Brouwer
Dept. of Mathematics
Techn. Univ. Eindhoven
P.O. Box 512
5600 MB Eindhoven
The Netherlands
e-mail to: aeb@cwi.nl