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A thin near hexagon with 50 points

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Abstract

We show the existence and uniqueness of a thin near hexagon which has 50 points and an affine plane of order 3 as a local space.

1. Definitions and motivation

In this paper we solve the open problem mentioned at the end of [BDB]:

Does there exist a near hexagon which satisfies conditions (C1), (C2) and (C3) below?

($C1$) Every line is incident with exactly two points.
($C2$) Every two points at distance 2 are contained in a good quad.
($C3$) At least one local space is an affine plane of order 3.

We will prove that such a near hexagon exists and that it is unique. This completes the classification of all near hexagons with good quads and with at least one local space isomorphic to an affine plane, see [BDB]. Since the paper [BDB] is published in this issue of JCT-A, we will use it as a reference for some basic notions like "near hexagon", "quad", and so on. A quad Q is called *good* if every point of Q is incident with the same number of lines of Q; this number then is denoted by $t_Q + 1$. From now on we assume that $\mathcal{S}$ is a near hexagon which satisfies (C1)–(C3). Since $\mathcal{S}$ has only lines of size 2, it can be regarded as a bipartite graph of diameter 3 (see [BDB, Lemma 1]). The edges then play the role of the lines. For a set A of vertices, let $A^{\perp}$

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denote the set of all vertices adjacent to every element of A. If x and y are two points at distance 2 then $Q(x,y) := \{x,y\}^{\perp} \cup (\{x,y\}^{\perp})^{\perp}$ is the unique quad through x and y. The fact that this quad is good means that $|\{x,y\}^{\perp}| = |\{x,y\}^{\perp\perp}|$. For every point x, we can define the following linear space $\mathscr{L}(x)$: the points of $\mathscr{L}(x)$ are the elements of $x^{\perp}$ and the lines are all the sets $\{a,b\}^{\perp\perp}$ where a and b are two different vertices of $x^{\perp}$. The linear space $\mathscr{L}(x)$ is called the *local space at* x. In [BDB] another definition was given, but in this special case both definitions are easily seen to be equivalent. By Theorem 5 of [BDB], we know that $\mathscr{S}$ has precisely $v = 50$ points, that every point is incident with exactly $t+1 = 9$ lines and that every quad Q has order $(1, t_Q)$ with $t_Q \in \{1,2,3\}$. Moreover, there exists a quad Q with $t_Q = 3$. The point set $\mathscr{P}$ of $\mathscr{S}$ can be partitioned into two subsets $\mathscr{P}_+$ and $\mathscr{P}_-$, each of size 25, such that two points of the same subset are never collinear. We sometimes call a point of $\mathscr{P}_\varepsilon$ a *point of type* ε. The set $\mathscr{P}_\varepsilon$, $\varepsilon \in \{+,-\}$, can be given the structure of a linear space $\mathscr{S}_\varepsilon$ by taking all sets $\{x,y\}^{\perp\perp}$, $x,y \in \mathscr{P}_\varepsilon$ and $x \neq y$, as lines. Clearly $\mathscr{L}(a)$ is a subgeometry of $\mathscr{S}_\varepsilon$ for every $a \in \mathscr{P}_{-\varepsilon}$.

2. The structure of $\mathscr{S}$

Let n_+ denote a point whose local space is an affine plane of order 3. We may suppose that $n_+ \in \mathscr{P}_+$. For every point x and every $i \in \{1,2,3\}$, let $N_i(x)$ or shortly N_i denote the number of quads of order $(1,i)$ through x, or equivalently, the number of lines of size $i+1$ in $\mathscr{L}(x)$. Since $t+1 = 9$, $N_3 \in \{0,1,2,3\}$. Every two lines through x are contained in a unique quad; hence $72 = (t+1)t = 12N_3 + 6N_2 + 2N_1$. Since there are $|\mathscr{P}_+| - 1 = |\mathscr{P}_-| - 1 = 24$ points at distance 2 from x, $24 = 3N_3 + 2N_2 + N_1$. These equations give

Lemma 2.1. *For any point x one of the following holds*:

(n) $N_3 = 0$, $N_2 = 12$, $N_1 = 0$: *x is called a* nice *point*;
(e) $N_3 = 1$, $N_2 = 9$, $N_1 = 3$; *x is called an* elusive *point*;
(g) $N_3 = 2$, $N_2 = 6$, $N_1 = 6$: *x is called a* good *point*;
(b) $N_3 = 3$, $N_2 = 3$, $N_1 = 9$: *x is called a* bad *point.*

The local space of a nice point is a Steiner system $S(2,3,9)$ and hence an affine plane of order 3.

Lemma 2.2. *If n is a nice point, then $d(n,R) \leqslant 1$ for every quad R of order $(1,2)$ or $(1,3)$.*

Proof. Suppose that $\mathrm{d}(n,R) = 2$. If R has order $(1,3)$, then the four quads through n meeting R contain 5 points of $\Gamma_2(R)$, contradicting $|\Gamma_2(R)| = |\mathscr{P}| - |R| - |\Gamma_1(R)| = 50 - 8 - 40 = 2$. Suppose therefore that R has order $(1,2)$. Let S_i, $i \in \{1,2,3\}$, denote the three quads through n meeting R, put $\{v_i\} := S_i \cap R$, and let w_i denote the unique point of S_i at distance 2 from n and v_i. Let K_1, K_2 and K_3 denote the three lines through v_1 not contained in $S_1 \cup R$. Let T_i, $i \in \{2,3\}$, denote the unique quad through v_1 and w_i. All the lines of T_i through v_1 are contained in $\{K_1, K_2, K_3\}$. Hence, we may suppose that T_i, $i \in \{2,3\}$, contains K_1 and K_i. Every quad of order $(1,3)$ through v_1 necessarily contains the lines K_2 and K_3. Hence $N_3(v_1) \leqslant 1$ and v_1 is either a nice or an elusive point. Suppose that v_1 is elusive and let T denote the unique quad of order $(1,3)$ through v_1. The quads T_2 and T_3 then have order $(1,1)$. Since each of the lines $T \cap S_1$ and $T \cap R$ is contained in a quad of order $(1,1)$, $N_1(v_1) \geqslant 4$. This however contradicts our assumption that v_1 is elusive. Hence v_1 is nice and $T_2 = T_3$ has order $(1,2)$. The points v_1, w_2 and w_3 are all contained in the quad T_2. Similarly, v_2 and v_3 are nice and each of the sets $\{v_2, w_1, w_3\}$ and $\{v_3, w_1, w_2\}$ is contained in a quad of order $(1,2)$. The quad R' through $\{v_2, w_1, w_3\}$ has order $(1,2)$ and satisfies $\mathrm{d}(n, R') = 2$. By the same argument as above, w_1 and w_3 are nice. Clearly also w_2 is nice. Notice that

$$A := \{n, v_1, v_2, v_3, w_1, w_2, w_3\}$$

together with the collection of quads of order $(1,2)$ intersecting A in at least two points has the structure of a Fano plane. Recall that there exists a quad Q of order $(1,3)$. Let $z \in A$. By the first paragraph of the proof, $d(z, Q) = 1$. Since Q is a quad, z is collinear with a unique point $\tilde{z}$ of Q. If $\tilde{z}_1 = \tilde{z}_2$ for two points $z_1, z_2 \in A$, then also the third point of A in the quad through z_1 and z_2 is collinear with $\tilde{z}_1$. Let $\tilde{A} = \{\tilde{z} \mid z \in A\}$. Since $t_Q = 3$, $|\tilde{A}| \leqslant 4$ and hence every point of A is collinear with the same point q of Q. Since q is collinear with v_1, v_2 and v_3, q belongs to R. Now $\mathrm{d}(n, q) = 1$ contradicts $\mathrm{d}(n, R) = 2$. $\square$

Lemma 2.3. *There are no elusive points. Exactly one point n_- in $\Gamma_1(n_+)$ is nice while the other* 8 *are good. The* 8 *points of $\Gamma_2(n_+)$ collinear with n_- are good, while the other* 16 *points are bad. Finally, all* 16 *points in $\Gamma_3(n_+)$ are bad.*

Proof. We first show

(I) Every point $z \in \Gamma_1(n_+)$ is either nice or good.

The line n_+z is contained in 4 quads of order $(1,2)$, proving that z is not bad. Suppose z is elusive and let Q denote the unique quad of order $(1,3)$ through z. Since $N_1 = 3$, there exists a line K of Q through z not contained in a quad of order $(1,1)$. The line K then is contained in 1 quad of order $(1,3)$ and $\frac{5}{2}$ quads of order $(1,2)$, a contradiction.

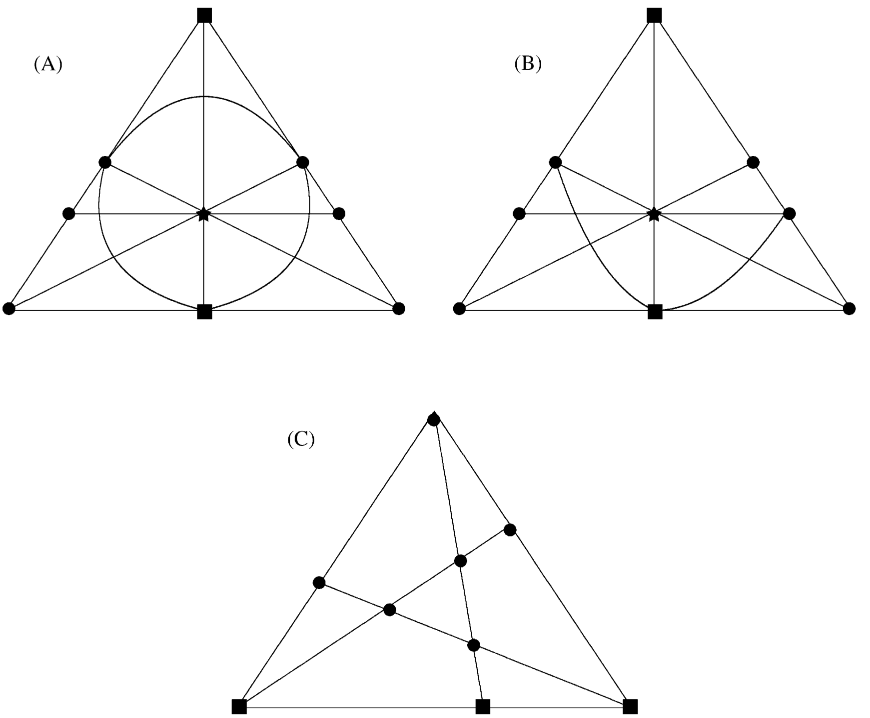
(II) All 16 points in $\Gamma_3(n_+)$ are bad, $\Gamma_1(n_+)$ has one nice and eight good points.

Let $z \in \Gamma_3(n_+)$. By Lemma 2.2 every quad of order $(1,2)$ or $(1,3)$ through z has a unique point in common with $\Gamma_1(n_+)$. Thus $N_2 + N_3 \leqslant 9$ and z is either of good or bad. Moreover, there are exactly $(9 - N_2 - N_3)$ quads of order $(1,1)$ through z which meet $\Gamma_1(n_+)$. Let α_i and β_i, respectively, denote the number of points of type (i), $i \in \{n, e, g, b\}$, in $\Gamma_1(n_+)$ and $\Gamma_3(n_+)$. Counting the number of quads of order $(1,1)$ connecting a point of $\Gamma_1(n_+)$ with a point of $\Gamma_3(n_+)$, we find $6\alpha_g = \beta_g + 3\beta_b$. Counting the number of quads of order $(1,3)$ connecting a point of $\Gamma_1(n_+)$

with a point of $\Gamma_3(n_+)$, we find $2\alpha_g = \frac{1}{3}(2\beta_g + 3\beta_b)$. Hence $\beta_g = 0$, $\beta_b = 16$, $\alpha_g = 8$ and $\alpha_n = 1$.

Applying (I) and (II) to the unique nice point n_- in $\Gamma_1(n_+)$, we find that $\Gamma_2(n_+)$ contains 8 good points and 16 bad points. $\square$

Lemma 2.4. *The lines of size at least three in the local space of a good point determine a configuration as in* (A) *or* (B). *The lines of size at least three in the local space of a bad point determine a configuration as in* (C). (*In these pictures nice and good points are indicated with stars and squares, respectively.*)



Proof. Any two collinear good points are contained in a unique quad of order $(1,2)$ containing both nice points. Thus the local space of a good point g has one nice point n, two good points, and six bad points, and the nice and good points form a line.

Since quads are good, the four lines of length 3 on g in $\mathscr{L}(n)$ correspond to four lines of length 3 on n in $\mathscr{L}(g)$. Ignoring lines of size 2 the local space of g is a partial linear space having two lines of size 4 and six lines of size 3. This yields configurations (A) and (B).

Let $b \in \mathscr{P}_\varepsilon$ be a bad point. The unique quad on b and n_ε has order $(1,2)$ and contains three good points. Conversely, if g, h are any two good points collinear to b, then the unique quad on g and h contains b and n_ε. Thus the local space of a bad point b contains three good points, all on one line, and six bad points.

Ignoring the lines of size 2 the local space of the bad point b is a partial linear space having three lines of size 4 and three lines of size 3. This yields configuration (C). Notice that the three good points are precisely the common neighbors of b and n_ε.

It remains to show that the nice and good points are as indicated by the stars and squares. For (C) this is obvious, as there are no nice points in $\mathscr{L}(b)$ and the good points form line of size three. Since all lines through the nice point in $\mathscr{L}(g)$ have size three, the nice point in (A) and (B) is uniquely determined. Let x be the unique point in $\mathscr{L}(g)$ such that $\{x, g\}$ lies in two quads of order $(1, 3)$. Then in $\mathscr{L}(x)$ there exist two lines of size four meeting in the good point g. So by (C), x cannot be bad. Thus x is good. $\square$

We say that a line in some local space has type L_{ngg} if it consists of one nice point and two good points. Similarly, we define L_{ggg}, L_{nbb}, L_{gbb}, and L_{gbbb}. By $L_{ggg}^{\perp} = L_{nbb}$ we mean that if a line l has type L_{ggg}, then $l^{\perp}$ has type L_{nbb}.

Lemma 2.5. (a) $L_{ngg}^{\perp} = L_{ngg}$,
(b) $L_{ggg}^{\perp} = L_{nbb}$, $L_{nbb}^{\perp} = L_{ggg}$,
(c) $L_{gbb}^{\perp} = L_{gbb}$, *and*
(d) $L_{gbbb}^{\perp} = L_{gbbb}$.

Proof. First, note that if l is a line in some local space, then $|l| = |l^{\perp}|$ since the quad $l \cup l^{\perp}$ is good.

(a) Suppose that n_ε is the nice point on the line l of type L_{ngg}. Then $n_{-\varepsilon}$ is a point in $l^{\perp}$ and by looking at the affine plane $\mathscr{L}(n_\varepsilon)$ we find that $l^{\perp}$ has type L_{ngg}.
(b) If the line l of type L_{ggg} belongs to $\mathscr{P}_\varepsilon$ then clearly $n_{-\varepsilon} \in l^{\perp}$. By (a) $l^{\perp}$ cannot contain any good points. Thus $L_{ggg}^{\perp} = L_{nbb}$. Conversely, all common neighbors of a nice and a bad point are good. Hence $L_{nbb}^{\perp} = L_{ggg}$.
(c) Let l be a line of type L_{gbb}. Then $l^{\perp}$ is a line of the local space of both the good and the bad points of l. By Lemma 2.4 $l^{\perp}$ must have type L_{gbb}.
(d) By Lemma 2.4 all lines of length 4 are of type L_{gbbb}. $\square$

A line of $\mathscr{S}$ is said to have *valency* $3^k 4^l$ if it is contained in exactly k quads of order $(1, 2)$ and l quads of order $(1, 3)$. A point b of the local space $\mathscr{L}(a)$ is said to have *valency* $v_a(b) = 3^k 4^l$ if it is contained in exactly k lines of size 3 and l lines of size 4. Clearly $v_a(b)$ is equal to the valency of the line ab. Hence $v_a(b) = v_b(a)$. For every good point a of $\mathscr{S}$, let $\bar{a}$ denote the unique good point of valency $3^1 4^2$ in $\mathscr{L}(a)$. This point exists; see (A) and (B) of Lemma 2.4. Similarly, for a bad point x, let $\bar{x}$ denote the unique bad point of valency $3^2 4^1$ in $\mathscr{L}(x)$. Also define $\overline{n_+} = n_-$ and $\overline{n_-} = n_+$. Clearly $\bar{\bar{a}} = a$ for every point a of $\mathscr{S}$. For every good or bad point a of $\mathscr{S}$ there exists a unique nice point n at distance 2 from a. The unique point of $\{n,a\}^{\perp\perp}$ different from n and a will be denoted by $-a$. Note that n, a, and $-a$ all have the same type. Also define $-n_+ = n_+$ and $-n_- = n_-$. Clearly $-(-a) = a$ for every point a of $\mathcal{S}$. If a and b are distinct points in the affine plane $\mathcal{L}(n_\varepsilon)$, $\varepsilon \in \{+,-\}$, then the unique third point on the line ab will be denoted by $a * b$. Also define $a * a = a$. Clearly $-a = n_{-\varepsilon} * a$ for every point a of $\mathcal{L}(n_\varepsilon)$.

We record:

Lemma 2.6. *Let $a, b, x \in \mathcal{P}_\varepsilon$ with a, b good and x bad. Then*

(a) *$\bar{a}$ is the unique good point of valency $3^1 4^2$ in $\mathcal{L}(a)$.*
(b) *$a = \bar{\bar{a}}$, that is a is the unique point of valency $3^1 4^2$ in $\mathcal{L}(\bar{a})$.*
(c) *$\{a,b\}^{\perp\perp} = \{a, b, a*b\}$ is the affine line in $\mathcal{L}(n_{-\varepsilon})$ determined by a and b.*
(d) *$\{n_\varepsilon, a\}^{\perp\perp} = \{n_\varepsilon, a, -a\}$ is the affine line in $\mathcal{L}(n_{-\varepsilon})$ determined by n_ε and a.*
(e) *$-(-a) = a$ and $Q(a, n_\varepsilon) = \{n_\varepsilon, a, -a; n_{-\varepsilon}, \bar{a}, -\bar{a}\}$.*
(f) *$\bar{x}$ is the unique bad point of valency $3^2 4^1$ in $\mathcal{L}(x)$.*

Since $\mathcal{L}(n_\varepsilon)$ is an affine plane with a distinguished point $n_{-\varepsilon}$, it carries the structure of a two-dimensional vector space $V_{-\varepsilon}$ over $\mathbb{F}_3$. The vector addition and scalar multiplication are as follows.

(SM) $0a := n_{-\varepsilon}$, $1a := a$ and $(-1)a := -a$ for every point a of $\mathcal{L}(n_\varepsilon)$.
(VA) $a + b := -(a * b)$ for all points a and b of $\mathcal{L}(n_\varepsilon)$.

Lemma 2.7. *We have $\overline{-a} = -\bar{a}$ for every good point a of $\mathcal{S}$.*

Proof. Suppose that $a \in \mathcal{P}_\varepsilon$ with $\varepsilon \in \{+,-\}$ and let $Q = Q(a, n_\varepsilon)$. The point $\bar{a}$ is a common neighbor of a and n_ε and hence belongs to Q. Since $-a \in Q$, also $\overline{-a} \in Q$ and hence $\overline{-a} \in \{\bar{a}, -\bar{a}\}$. If $\overline{-a} = \bar{a}$, then a and $-a$ are two points of valency $3^1 4^2$ in $\mathcal{L}(\bar{a})$, a contradiction to Lemma 2.4. Hence $\overline{-a} = -\bar{a}$. □

Lemma 2.8. *Let x be a bad point and let a and c be the two good points of valency $3^2 4^1$ in $\mathcal{L}(x)$; see* (C) *of* Lemma 2.4. *Then $\overline{a*c} = \bar{a} + \bar{c} = -\overline{a+c}$.*

Proof. Suppose that $x \in \mathcal{P}_\varepsilon$. Then $Q(a, \bar{x})$ and $Q(c, \bar{x})$ are quads of order $(1,2)$ and $Q(a*c, \bar{x})$ has order $(1,3)$. By Lemma 2.2 each of the quads $Q(a, \bar{x})$, $Q(c, \bar{x})$ and $Q(a*c, \bar{x})$ contains a unique point collinear with $n_{-\varepsilon}$. The three good points so obtained are precisely the three common neighbors of $n_{-\varepsilon}$ and $\bar{x}$. The point $Q(a, \bar{x}) \cap n_{-\varepsilon}^\perp$ is a common neighbor of a, $\bar{x}$, and $n_{-\varepsilon}$ and must therefore be equal to either $\bar{a}$ or $-\bar{a}$. If $\bar{a} \in Q(a, \bar{x})$, then Lemma 2.4 applied to $\mathcal{L}(\bar{a})$ implies that $Q(a, \bar{x})$ has order $(1,3)$, a contradiction. Hence $Q(a, \bar{x}) \cap n_{-\varepsilon}^\perp = \{-\bar{a}\}$. Similarly $Q(c, \bar{x}) \cap n_{-\varepsilon}^\perp = \{-\bar{c}\}$. Now, $Q(a*c, \bar{x})$ is a quad of order $(1,3)$ through $a*c$ and so by definition of $\bar{\cdot}$ contains $\overline{a*c}$. Since $-\bar{a}$, $-\bar{c}$ and $\overline{a*c}$ are the three common neighbors of $\bar{x}$ and $n_{-\varepsilon}$, we have $\overline{a*c} = (-\bar{a}) * (-\bar{c}) = \bar{a} + \bar{c}$. □

For a bad point x let a, c be as in the previous lemma. Define

$$\mathscr{F}(x) = \{\bar{a}, \bar{c}\} \text{ and } \hat{x} = \bar{a} + \bar{c}.$$

For a set of points A, let $\bar{A} = \{\bar{a} \mid a \in A\}$ and $-A = \{-a \mid a \in A\}$.

From the proof of Lemma 2.8 we obtain:

Corollary 2.9. *Let x be a bad point and $\mathscr{F}(x) = \{a, c\}$. Then*

(a) *$\bar{a}$ and $\bar{c}$ are the two good points of valency $3^2 4^1$ in $\mathscr{L}(x)$.*
(b) *$\bar{\hat{x}}$ is the unique good point of valency $3^1 4^1$ in $\mathscr{L}(x)$.*
(c) $\hat{x} = a + c = \overline{\bar{a} * \bar{c}}$.
(d) $\bar{\hat{x}} = \bar{a} * \bar{c} = -\bar{a} - \bar{c} = -\overline{a * c}$.
(e) $\{n_\varepsilon, x, -x\}^\perp = \{\bar{a}, \bar{c}, \bar{a} * \bar{c}\} = \overline{\mathscr{F}(x) \cup \{\hat{x}\}}$.
(f) $\{n_{-\varepsilon}, \bar{x}, -\bar{x}\}^\perp = \{-a, -c, a + c\} = -\mathscr{F}(x) \cup \{\hat{x}\}$.

Lemma 2.10. (a) *If x is a bad point, then $\mathscr{F}(x) = \mathscr{F}(-x)$ and $\hat{x} = \widehat{-x}$.*

(b) *Let x and y be bad points. Then $\hat{x} = \hat{y}$ if and only if $x = \pm y$.*

(c) *Every local space of a good point is isomorphic to the linear space* (A) *of Lemma* 2.4.

Proof. Let x be a bad point in $\mathscr{P}_\varepsilon$. By Corollary 2.9(e) applied to x and $-x$, $\mathscr{F}(x) \cup \{\hat{x}\} = \mathscr{F}(-x) \cup \{\widehat{-x}\}$. With $\mathscr{F}(x) = \{u, v\}$ and $\mathscr{F}(-x) = \{u', v'\}$, we have $\hat{x} = -2\,\hat{x} = -u - v - \hat{x} = -u' - v' - (\widehat{-x}) = \widehat{-x}$ and hence also $\mathscr{F}(x) = \mathscr{F}(-x)$. Now fix a good point a. Then by Corollary 2.9(b) $a = \hat{x}$ if and only if $\bar{a}$ is the unique good point of valency $v_x(\bar{a}) = 3^1 4^1$ and this happens if and only if $v_{\bar{a}}(x) = 3^1 4^1$. For each of the two possibilities for $\mathscr{L}(\bar{a})$, see Lemma 2.4, there are exactly two solutions for x. This combined with (a) proves (b). Since by (b) both solutions must lie on the same line through the nice point we see that $\mathscr{L}(\bar{a})$ cannot be isomorphic to the linear space (B) of Lemma 2.4. This proves (c). □

Lemma 2.11. *There exists an $\mathbb{F}_3$-linear isomorphism $f_+ : V_+ \to V_-$ with $f_+(a) = \pm\bar{a}$ for all $a \in V_+$.*

Proof. Let $\mathscr{N}_\varepsilon$ be the set of lines of V_ε containing n_ε. By Lemma 2.7, $\bar{\cdot}$ induces a bijection from $\mathscr{N}_+$ to $\mathscr{N}_-$. Let g be any $\mathbb{F}_3$-linear isomorphism from V_+ to V_-. Since $\mathrm{GL}(V_-)$ induces the full symmetric group on $\mathscr{N}_-$, there exists an $h \in \mathrm{GL}(V_-)$ such that hg and $\bar{\cdot}$ induce the same map from $\mathscr{N}_+$ to $\mathscr{N}_-$. Then $f_+ = hg$ works. □

The function f_+ from the previous lemma is essentially unique. Only $-f_+$ has the same property. Now, let $f_- = f_+^{-1}$ and define $f : V_+ \cup V_- \to V_+ \cup V_-, a \to f_\varepsilon(a)$, if $a \in V_\varepsilon$. Then f^2 is the identity and for all good points a there exists a unique $|a| \in \mathbb{F}_3$

with $f(a) = |a|\bar{a}$. Since $f(-a) = -f(a)$ and $\overline{-a} = -\bar{a}$ we have $|-a| = |a|$. The element $|a|$ is called the *norm* of a. Define $|n| = 0$ if n is a nice point. For all points $a, b \in V_\varepsilon$, we define $\langle a, b \rangle = \frac{1}{2}(|a+b| - |a| - |b|) = |a| + |b| - |a+b|$.

Lemma 2.12. (a) *If a is a good point, then $|a| = |\bar{a}| = |f(a)|$.*

(b) *For all good points a and b of the same type, $\langle a, b \rangle = \langle f(a), f(b) \rangle$.*

Proof. From $a = f(f(a)) = f(|a|\bar{a}) = |a| f(\bar{a}) = |a|\,|\bar{a}|\,\bar{\bar{a}} = |a|\,|\bar{a}|a$, it follows that $|\bar{a}| = |a|$. Since $f(a) = \pm\bar{a}$ we have $|f(a)| = |\bar{a}| = |a|$. If a and b are good points of the same type $\varepsilon \in \{+, -\}$, then $\langle a, b \rangle = |a| + |b| - |a+b| = |f(a)| + |f(b)| - |f(a+b)| = \langle f(a), f(b) \rangle$. □

Lemma 2.13. *Let $V = V_\varepsilon$ for some ε.*

(a) *$|\cdot|$ is a non-degenerate quadratic form in V and $\langle \cdot, \cdot \rangle$ is its associated symmetric bilinear form.*
(b) *Let $a, c \in V^\# := V \backslash \{0\}$. Then $\{a, c\} = \mathscr{F}(x)$ for a bad point x if and only if $\langle a, c \rangle = 0$.*
(c) *Let $b \in V^\#$. There exists a uniquely determined set $\mathscr{F}(b) := \{a, c\}$ with $a, c \in V^\#$, $a + c = b$ and $\langle a, c \rangle = 0$.*
(d) *$\mathscr{F}(x) = \mathscr{F}(\hat{x})$.*
(e) *$\mathscr{F}(\bar{x}) = -\overline{\mathscr{F}(x)}$ and $\hat{\bar{x}} = \bar{\hat{x}}$.*

Proof. (a) Let l be a line of $\mathscr{L}(n_{-\varepsilon})$ with $n_\varepsilon \notin l$ and let x denote a bad point in $l^\perp$. Putting $\mathscr{F}(x) = \{a, c\}$, we have $l = \{\bar{\hat{x}}, \bar{a}, \bar{c}\}$ and $\hat{x} = a + c$. Thus by definition of $|\cdot|$

$$|\bar{a}| f(\bar{a}) + |\bar{c}| f(\bar{c}) = a + c = \hat{x} = |\bar{\hat{x}}| f(\bar{\hat{x}}) = |\bar{\hat{x}}| f(\bar{a} * \bar{c}) = -|\bar{\hat{x}}| f(\bar{a}) - |\bar{\hat{x}}| f(\bar{c}).$$

Since $\bar{a} \neq \pm\bar{c}$ also $f(\bar{a}) \neq \pm f(\bar{c})$ and hence $|\bar{a}| = |\bar{c}| = -|\bar{\hat{x}}|$. As a consequence every line l of $\mathscr{L}(n_{-\varepsilon})$, $n_\varepsilon \notin l$, contains vectors of norm 1 and -1. Since $\mathscr{L}(n_{-\varepsilon}) \simeq \mathrm{AG}(2,3)$ and $|a| = |-a|$ for every $a \in V^\#$ it is easy to verify that exactly four points in $V^\#$ have norm 1, while the other four have norm -1. This proves (a).

(b) If $a = \pm c$ then a and c are neither orthogonal nor is $\{a, c\}$ of the form $\mathscr{F}(x)$. So we may assume that $n_\varepsilon \notin \{a, c\}^{\perp\perp}$. Suppose $\{a, c\} = \mathscr{F}(x)$ for a bad point x, then by the proof of (a), $|\bar{a}| = |\bar{c}|$. Hence $|a| = |c| = -|a+c|$ and $\langle a, c \rangle = |a| + |c| - |a+c| = 0$. Conversely, suppose that $\langle a, c \rangle = 0$ and let x denote a bad point of $\{\bar{a}, \bar{c}\}^\perp$. Then $\mathscr{F}(x)$ is a set of two orthogonal points. Since $\mathscr{F}(x) \subseteq \{a, c, *\bar{\bar{c}}\}$, we necessarily have $\mathscr{F}(x) = \{a, c\}$.

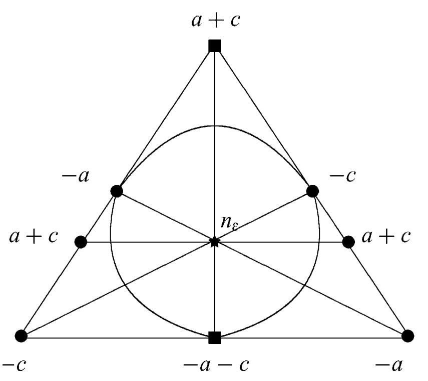
(c) Readily verified.

(d) $\mathscr{F}(x) = \{a, c\} = \mathscr{F}(a + c) = \mathscr{F}(\hat{x})$.

(e) Applying Corollary 2.9(e) to x and Corollary 2.9(f) to $\bar{x}$ we have $\overline{\mathscr{F}(x) \cup \{\hat{x}\}} = -\mathscr{F}(\bar{x}) \cup \hat{\bar{x}}$. Any affine line in $V_{-\varepsilon}$ which is not incident with $n_{-\varepsilon}$ contains a unique pair of orthogonal vectors. Thus $\overline{\mathscr{F}(x)} = -\mathscr{F}(\bar{x})$ and $\bar{\hat{x}} = \hat{\bar{x}}$. □

For a good or nice point a we define $\hat{a} = a$. For any point a of $\mathscr{S}$ we call $\hat{a}$ the *label* of a. Note that $\hat{a}$ is a nice or good point of the same type as a.

Lemma 2.14. *Let $V = V_\varepsilon$ for some ε. Let $b \in V^\#$ be a good point and $\mathscr{F}(b) = \{a, c\}$. Then the labels of the points in $\mathscr{L}(\bar{b})$ are as follows*:



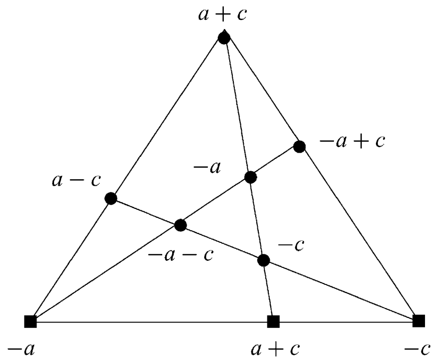
Proof. Since a and c are orthogonal, there are a bad points x_1 and $-x_1$ satisfying $\mathscr{F}(x_1) = \mathscr{F}(-x_1) = \{a, c\}$ (see Lemmas 2.13(b) and 2.10(a)). Then $\widehat{x_1} = \widehat{-x_1} = a + c = b$ and $x_1, -x_1 \in \overline{\widehat{x_1}}^\perp = \bar{b}^\perp$. Since $a + c$ and $a - c$ are orthogonal, there are bad points x_2 and $-x_2$ satisfying $\mathscr{F}(x_2) = \mathscr{F}(-x_2) = \{a + c, a - c\}$ and $\widehat{x_2} = \widehat{-x_2} = (a + c) + (a - c) = -a$. Clearly $x_2, -x_2 \in \overline{a + c}^\perp = \bar{b}^\perp$. Similarly, since $a + c$ and $c - a$ are orthogonal, there are bad points x_3 and $-x_3$ in $\bar{b}^\perp$ satisfying $\widehat{x_3} = \widehat{-x_3} = (a + c) + (c - a) = -c$. By Lemma 2.6(b), b is the unique good point of valency $3^1 4^2$ in $\mathscr{L}(\bar{b})$. If y is one of the two bad points of valency $3^1 4^1$ in $\mathscr{L}(\bar{b})$, then $\bar{b}$ has valency $3^1 4^1$ in $\mathscr{L}(y)$. So by Corollary 2.9(b), $\bar{\hat{y}} = \bar{b}$ and $\hat{y} = b$. Hence the labels are as claimed. □

Corollary 2.15. (a) *Let l be a line of size four in $\mathscr{S}_\varepsilon$, $\varepsilon \in \{+, -\}$, and let a denote the unique good point on l. Then there exists a unique bad point b on l with $\hat{b} = a$. If c and d are the other two bad points on l, then* $\langle \hat{c}, \hat{d} \rangle = 0$ and $\hat{c} + \hat{d} = -a$.

(b) *Let l be a line of size three in $\mathscr{S}_\varepsilon$, $\varepsilon \in \{+, -\}$, with one good point a and two bad points b and c. Then $\{\hat{b}, \hat{c}\} = \mathscr{F}(a)$.*

Proof. Let f denote the unique good point in $l^\perp$. Applying the previous lemma to $f^\perp$, we immediately see that (a) and (b) hold. □

Lemma 2.16. *Let x be a bad point and $\{a, c\} = \mathscr{F}(\bar{x})$. Then the labels of the points in $\mathscr{L}(x)$ are as follows*:



Proof. By Corollary 2.9(e), $\overline{\mathscr{F}(x) \cup \{\hat{x}\}}$ is the set of good points in $\mathscr{L}(x)$. By parts (d) and (e) of Lemma 2.13 these points are $-a, -c$ and $\hat{\bar{x}} = \hat{\hat{x}} = a+c$. By Corollary 2.9(b), $\bar{\hat{x}}$ has valency $3^1 4^1$ in $\mathscr{L}(x)$. So all the good points are as claimed.

Also $\bar{x}$ is labeled $a+c$ and by Lemma 2.6(f) has valency $3^2 4^1$ in $\mathscr{L}(x)$. Let f be the third point on $\{-a, \bar{x}\}^{\perp\perp}$. By Corollary 2.15(b) $\hat{f} = a - c$. Similarly the third point g on $\{-c, \bar{x}\}^{\perp\perp}$ has label $-a+c$.

The two missing points on the line $\{\bar{x}, a+c\}^{\perp\perp}$ need labels $-a$ and $-c$. The missing points on the line $\{-a, g\}^{\perp\perp}$ need labels $-a$ and $-a-c$. So the intersection point needs to be labelled $-a$ and all the labels are as claimed. $\square$

Let x and y be bad points of opposite type. The preceding lemma combined with Lemma 2.10 tells us whether one of $\pm y$ is perpendicular to x. But if one of $\pm y$ is perpendicular we do not know yet which one it is. For this we introduce an equivalence relation on the set of bad points which helps us to distinguish y and $-y$.

For two bad points x and y of the same type we define $x \sim y$ if $|\{x, y\}^{\perp\perp}| = 4$ and $\langle \hat{x}, \hat{y} \rangle = 0$. Then $\sim$ is a symmetric relation. Let $\approx$ be the transitive extension of $\sim$. Let Δ_ε be the set of equivalence classes of $\approx$ on the bad points of type ε and put $\Delta = \Delta_+ \cup \Delta_-$.

Lemma 2.17. *Let x be a bad point of $\mathscr{P}_\varepsilon$.*

(a) *Let $y \notin \{x, -x\}$ be a bad point of $\mathscr{P}_\varepsilon$. Then $x \sim y$ if and only if $|\{x, -y\}^{\perp\perp}| = 3$.*
(b) *There exist exactly two bad points y, z in $\sim$-relation to x.*
(c) *There exists a bad point $u \neq x$ with $y \sim u \sim z$.*
(d) *$[x] := \{x, y, z, u\}$ is the equivalence class of $\approx$ containing x.*
(e) *The labels of $[x]$ consists of all the four good points of same norm and type as $\hat{x}$. More precisely, $\hat{x} = -\hat{u}$, $\hat{y} = -\hat{z}$ and $\hat{x}$ and $\hat{y}$ are orthogonal.*
(f) $[-x] = -[x]$.

Proof. (a) Assume that $x \sim y$ or that $|\{x, -y\}^{\perp\perp}| = 3$. Let a be the good point in $\{x, y\}^\perp$ in the first case and in $\{x, -y\}^\perp$ in the second case. Then x, y and $-y$ are in

$a^\perp$. A glance at Lemma 2.14 shows that $|\{x,-y\}^{\perp\perp}|=3$ if and only if $\hat{x}\perp\hat{y}$ and $|\{x,y\}^{\perp\perp}|=4$.

(b) By (a) the number of bad points y with $x\sim y$ equals the number of quads of order $(1,2)$ through x different from $Q(x,n_\varepsilon)$. By Lemma 2.1 and since x is bad, this number equals 2.

(c) Both of the lines $\{x,y\}^\perp$ and $\{x,z\}^\perp$ of $\mathscr{L}(x)$ have size 4 and hence intersect in a bad point d; see Lemma 2.4. Let $\mathscr{F}(\bar{\hat{x}})=\{a,c\}$. Now x, y and z are points of $\mathscr{L}(d)$ such that $|\{x,y\}^\perp|=|\{x,z\}^\perp|=4$ and $\langle\hat{x},\hat{y}\rangle=\langle\hat{x},\hat{z}\rangle=0$. By Lemma 2.16, it follows that $\hat{x}=-a-c$ and $\{\hat{y},\hat{z}\}=\{a-c,-a+c\}$. If v is the unique bad point of valency 3^24^1 in $\mathscr{L}(d)$, then yv and zv are lines of size 3. Hence by (a) $u:=-v$ satisfies the required properties.

(d) For each $t\in\{x,y,z,u\}$, $\{x,y,u,z\}$ contains two and thus all points in $\sim$-relation with t.

(e) By the proof of (c), $\hat{y}=-\hat{z}$ and $\hat{u}=\hat{v}=-\hat{x}$. Since $x\sim y$, $\hat{x}$ and $\hat{y}$ are orthogonal.

(f) Both $[-x]$ and $-[x]$ consists of those four bad points outside $[x]$ whose labels have the same norm and type as x. □

Lemma 2.18. *Let x be a bad point.*

(a) $\{x,\hat{x}\}^\perp=\{\bar{x},\bar{\hat{x}}\}^{\perp\perp}$ *and the quad* $Q(x,\hat{x})$ *has order* $(1,3)$.

(b) *Let* t_1, t_2 *and s be bad points different from x such that* $t_1,t_2\in\{x,\hat{x}\}^{\perp\perp}$, $t_1\neq t_2$, *and* $x\in\{s,\hat{s}\}^{\perp\perp}$. *Then* $[t_1]=[t_2]=-[s]$.

(c) *Define* $i\cdot[x]:=[t]$ *where t is a bad point on* $\{x,\hat{x}\}^{\perp\perp}$ *different from x. Then* $i\cdot[x]$ *is well defined, that is does not depend on the choice of x in* $[x]$.

(d) $|\hat{t}|=-|\hat{x}|$.

(e) $i\cdot(i\cdot[x])=-[x]$.

(f) *Let* $r\neq\bar{x}$ *be a bad point on* $\{x,\hat{x}\}^\perp$. *Then* $[\bar{t}]=-[r]$.

(g) $[\bar{x}]=\overline{[x]}(:=\{\bar{y}\mid y\in[x]\})$.

(h) $\overline{i\cdot[x]}=-i\cdot\overline{[x]}$.

Proof. (a) By definition, $\bar{\hat{x}}$ (resp. $\bar{x}$) is the unique good (resp. bad) point of valency 3^14^1 (resp. valency 3^24^1) in $\mathscr{L}(x)$. By Lemma 2.16, $|\{\bar{\hat{x}},\bar{x}\}^{\perp\perp}|=4$. By definition of $\bar{\cdot}$ we have $\bar{x}\in x^\perp$ and $\bar{\hat{x}}\in\hat{x}^\perp$. By Corollary 2.9(c), $\bar{\hat{x}}\in x^\perp$. By Lemma 2.13(e) $\hat{\bar{x}}=\bar{\hat{x}}$ and so $\bar{x}\in\hat{x}^\perp$. This proves (a).

(b) By Corollary 2.15, $\widehat{t_1}$ and $\widehat{t_2}$ are orthogonal. Hence $t_1\sim t_2$ and $[t_1]=[t_2]$. By (a) and Lemma 2.4, the two lines $\{x,\hat{x}\}^\perp$ and $\{s,\hat{s}\}^\perp$ of $\mathscr{L}(x)$ intersect in a bad point x'. The points x, $\hat{x}$, t_1, t_2, s and $\hat{s}$ are all contained in $\mathscr{L}(x')$. By Lemma 2.16 and taking innerproducts into account, either $|\{t_1,s\}^{\perp\perp}|=3$ or $|\{t_2,s\}^{\perp\perp}|=3$. Thus either $-s\sim t_1$ or $-s\sim t_2$ and so $-[s]=[-s]=[t_1]=[t_2]$.

(c) Note that by (b) the definition does not depend on the choice of t. Let y be a bad point with $y\sim x$, so $\hat{x}$ and $\hat{y}$ are orthogonal. By Corollary 2.15 $\{x,y\}^{\perp\perp}=\{x,y,s,\hat{s}\}$

for some bad point s. If $\tilde{t}$ is a bad point of $\{y, \hat{y}\}^{\perp\perp}$ different from y, then by (b) applied twice, $[t] = -[s] = [\tilde{t}]$. Since $\approx$ is the transitive extension of $\sim$, (c) holds.

(d) Clear, for example by Corollary 2.15 and the fact that non-orthogonal vectors of V_ε have different norm.

(e) By (b) applied to t in place of x, $i \cdot [t] = -[x]$.

(f) Considering the line $\{x, \hat{x}\}^{\perp\perp}$ in $\mathscr{L}(\bar{x})$ and using Lemma 2.16 we find $\hat{t} \in \bar{x}^{\perp}$. Hence $\bar{x} \in \{t, \hat{t}\}^{\perp}$ and so using (a), $\bar{x} \in \{\bar{t}, \hat{\bar{t}}\}^{\perp\perp}$. Now by (a) r is a bad point on $\{\bar{x}, \hat{\bar{x}}\}^{\perp\perp}$ different from $\bar{x}$ and hence $[\bar{t}] = -[r]$ by (b).

(g) Let $x \sim y$ and put $\{x, y\}^{\perp\perp} = \{x, y, s, \hat{s}\}$. Let u be a bad point on $\{x, y\}^{\perp}$ other than $\bar{s}$. Applying (f) twice with s in place of x, $[\bar{x}] = -[u] = [\bar{y}]$. Since $\approx$ is the transitive extension of $\sim$, (g) holds.

(h) We have $\overline{i \cdot [x]} = \overline{[t]} = [\bar{t}] = -[r] = -i \cdot [\bar{x}] = -i \cdot \overline{[x]}$. □

By part (g) we obtain a well-defined map $\bar{\cdot} : \Delta \to \Delta$, $[x] \to [\bar{x}]$. For every $\delta \in \Delta$, we define $|\delta| := |\hat{x}|$ for any $x \in \delta$. Note that this is well-defined by (e) of Lemma 2.17. Put

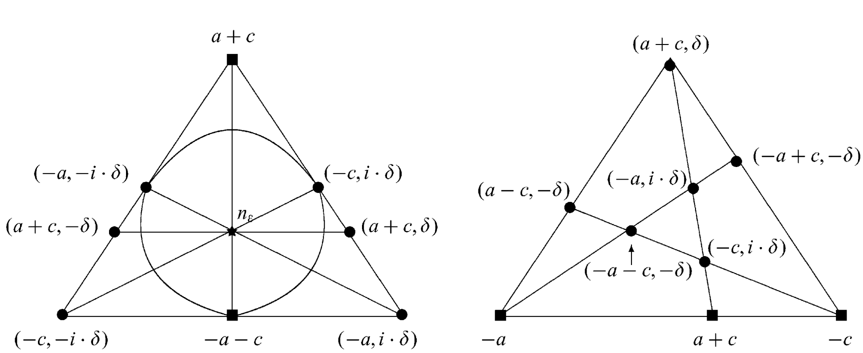
$$\mathscr{B}_\varepsilon = \{(a, \delta) \mid a \in V_\varepsilon^{\#}, \delta \in \Delta_\varepsilon, |\delta| = |a|\}.$$

Lemma 2.19. *The map $\Phi_\varepsilon : x \to (\hat{x}, [x])$ from the set of bad points of type ε to $\mathscr{B}_\varepsilon$ is a bijection.*

Proof. By (e) of Lemma 2.17, $\hat{x} \neq \hat{y}$ for any two distinct bad points with $[x] = [y]$. Thus Φ_ε is one to one. Since there are sixteen bad points of a given type and since also $|\mathscr{B}_\varepsilon| = 16$, Φ_ε is a bijection. □

Let $d \in P_\varepsilon$. If d is bad, define $\Phi(d) = \Phi_\varepsilon(d) = (\hat{d}, [d])$. If d is nice or good, let $\Phi(d) = d$. We call $\Phi(d)$ the *coordinates* of d. For (a, δ) in $\mathscr{B}_\varepsilon$ we define $\overline{(a, \delta)} = (\bar{a}, \bar{\delta})$. By (a) and (g) of Lemma 2.18, $\Phi(\bar{d}) = \overline{\Phi(d)}$.

Lemma 2.20. *Let a and c be orthogonal good ε-points and $\delta \in \Delta_\varepsilon$ with $|\delta| = |a + c|$. Let d and x be the points with coordinates $\overline{a + c}$ and $\overline{(a + c, \delta)}$, respectively. Then the coordinates of the points in $\mathscr{L}(d)$ and $\mathscr{L}(x)$ are as follows:*



Proof. This easily follows from Lemmas 2.14, 2.16 and 2.18. □

Remark. Let $i \in \mathbb{F}_9$ be satisfying $i^2 = -1$ and let $\bar{\cdot}$ denote the unique non-trivial automorphism of $\mathbb{F}_9$. Comparing the norms of $\pm[x]$, $\pm i[x]$, $\pm\overline{[x]}$, and $\pm i\overline{[x]}$ using Lemma 2.18, one can define a map $k : \Delta \to \{1, -1, i, -i\}$ such that the following holds for every $\delta \in \Delta$:

(i) $|\delta| = [k(\delta)]^2$;
(ii) $k(i \cdot \delta) = i\, k(\delta)$;
(iii) $k(\bar{\delta}) = \overline{k(\delta)} = [k(\delta)]^{-1}$.

3. Construction and uniqueness of the near hexagon

Based on the properties given in the previous section, we are now able to give a construction for $\mathscr{S}$. Choosing a suitable base in V_+, we may suppose that V_+ is the vector space $\mathbb{F}_3^2$ and that $\langle \cdot, \cdot \rangle$ is the standard symmetric bilinear form on $\mathbb{F}_3^2$, i.e. $\langle (a_1, b_1), (a_2, b_2) \rangle = a_1 a_2 + b_1 b_2$. We choose for V_- the dual space of V_+ and for $f_+ = f_-^{-1}$ the $\mathbb{F}_3$-linear transformation $V_+ \to V_- : a \to \langle a, \cdot \rangle$. Let 0_ε denote the 0-vector in V_ε and put $V_\varepsilon^\# = V_\varepsilon \setminus \{0_\varepsilon\}$. For all $a, b \in V_-$ define $\langle a, b \rangle = \langle f_-(a), f_-(b) \rangle$. Notice that this makes f_+ and f_- isometries. For every $a \in V_\varepsilon$, put $|a| = \langle a, a \rangle$ and define $\bar{a} = |a| f_\varepsilon(a)$. Let $\mathscr{B}_\varepsilon = \{(a, \delta) \mid a \in V_\varepsilon^\#, \delta \in \mathbb{F}_9, \delta^4 = 1 \text{ and } \delta^2 = |a|\}$, and let $\mathscr{P}_\varepsilon = V_\varepsilon \cup \mathscr{B}_\varepsilon$. For every $(a, \delta) \in \mathscr{B}_\varepsilon$ define $\overline{(a, \delta)} = (\bar{a}, \bar{\delta}) = (\bar{a}, \delta^{-1}) \in \mathscr{B}_{-\varepsilon}$. Let Γ denote the bipartite graph whose vertices are the elements of $\mathscr{P}_+ \cup \mathscr{P}_-$, with two vertices d and e adjacent if and only if one of the following possibilities occurs for a certain $\varepsilon \in \{+, -\}$:

(1) $d = 0_{-\varepsilon}$ and $e \in V_\varepsilon$;
(2) $d = \overline{a + b}$ for some $a, b \in V_\varepsilon^\#$ with $\langle a, b \rangle = 0$ and $e \in \{0_\varepsilon, a + b, -(a + b), (a + b, \delta), (a + b, -\delta), (-a, i\delta), (-a, -i\delta), (-b, i\delta), (-b, -i\delta)\}$ for any $\delta \in \{\pm 1, \pm i\}$ satisfying $\delta^2 = |a + b|$;
(3) $d = \overline{(a + b, \delta)}$ and $e \in \{-a, -b, a + b, (a + b, \delta), (-a - b, -\delta), (a - b, -\delta), (-a + b, -\delta), (-a, i\delta), (-b, i\delta)\}$ for some $a, b \in V_\varepsilon^\#$ and $\delta \in \{\pm 1, \pm i\}$ satisfying $\langle a, b \rangle = 0$ and $\delta^2 = |a + b|$.

By Lemma 2.20, any near hexagon satisfying (C1)–(C3) is isomorphic to the geometry $\mathscr{G}(\Gamma)$ whose points and lines are the vertices and edges, respectively, of the graph Γ. So, it remains to show that $\mathscr{G}(\Gamma)$ is a near hexagon in which conditions (C2) and (C3) hold.

Claim. *If* $(*)$ $|\{x, y\}^\perp| = |\{x, y\}^{\perp\perp}|$ *holds for all vertices* x *and* y *of the same type, then* $\mathscr{G}(\Gamma)$ *is a near hexagon in which* (C2) *and* (C3) *hold.*

Proof. Let x and y be two vertices of the same type. Since $\{x, y\} \subseteq \{x, y\}^{\perp\perp}$, we have $(\{x, y\}^{\perp\perp})^\perp \subseteq \{x, y\}^\perp$. Clearly, also $\{x, y\}^\perp \subseteq (\{x, y\}^{\perp\perp})^\perp$ and hence

$(\{x,y\}^{\perp\perp})^{\perp} = \{x,y\}^{\perp}$. Now suppose that condition $(*)$ holds. Since $|\{x,y\}^{\perp}| = |\{x,y\}^{\perp\perp}| \geqslant 2$, Γ is a graph of diameter 3. As it is bipartite it is a near hexagon with lines of size 2. If $\{a,b\} \subseteq \{x,y\}^{\perp}$, then $\{x,y\}^{\perp\perp} \subseteq \{a,b\}^{\perp}$ and $\{a,b\}^{\perp\perp} \subseteq \{x,y\}^{\perp}$. Applying condition $(*)$ to the pairs (x,y) and (a,b), we easily see that $\{x,y\}^{\perp} \cup \{x,y\}^{\perp\perp} = \{a,b\}^{\perp} \cup \{a,b\}^{\perp\perp}$. Similarly, if $\{a,b\} \subseteq \{x,y\}^{\perp\perp}$, then $\{x,y\}^{\perp} \cup \{x,y\}^{\perp\perp} = \{a,b\}^{\perp} \cup \{a,b\}^{\perp\perp}$. As a consequence, every two points of the same type are contained in a unique set of the form $\{x,y\}^{\perp} \cup \{x,y\}^{\perp\perp}$. This proves (C2). By construction, $\mathscr{L}(0_+)$ and $\mathscr{L}(0_-)$ are affine spaces of order 3 and hence also (C3) holds. $\square$

The verification of condition $(*)$ is straightforward but tedious and we leave it to the dedicated reader. The results of our verification are listed in the Table below. In this table a, b and c are elements of $V_{\varepsilon}^{\#}$ satisfying $b = a + c$ and $\langle a, c \rangle = 0$. The elements δ, μ satisfy $|\delta| = |b|$ and $|\mu| = |a|$. The notation $L_{gbbb}(\cdots)$ means that this line contains one good point and three bad points.

It might be helpful to observe that our object has an automorphism group of order 64. Indeed, let $\alpha \in GL(V_+)$ and $\sigma \in F_3^{\#}$ with $\langle \alpha(a), \alpha(b) \rangle = \sigma \langle a, b \rangle$ for all $a, b \in V_+$. Let $\lambda \in \mathbb{F}_9$ with $\lambda^2 = \sigma$. Define a bijection of $\mathscr{P}$ as follows:

$$a \to \alpha(a), \text{ for all } a \in V_+,$$

$$(a,\delta) \to (\alpha(a), \lambda\delta), \text{ for all } (a,\delta) \in \mathscr{B}_+,$$

$$a \to a \circ \alpha^{-1}, \text{ for all } a \in V_-,$$

$$(a,\delta) \to (a \circ \alpha^{-1}, \bar{\lambda}\delta), \text{ for all } (a,\delta) \in \mathscr{B}_-.$$

It is straightforward from the definitions that this map preserves the collinearity of points. This gives us 32 automorphisms. Since our description is symmetric in V_+ and V_- there also exists an automorphism interchanging the types of points.

Line l of $\mathscr{S}_\varepsilon$	the points on l	Line $l^\perp$ of $\mathscr{S}_{-\varepsilon}$	the points on $l^\perp$
$L_{gbbb}(b,\delta)$	$b, (b,\delta), (-a, i\delta), (-c, i\delta)$	$L_{gbbb}(\bar{b},\bar{\delta})$	$\bar{b}, (\bar{b},\bar{\delta}), (\bar{a}, i\bar{\delta})(\bar{c}, i\bar{\delta})$
$L_{ngg}(\pm a)$	$0_\varepsilon, a, -a$	$L_{ngg}(\pm\bar{a})$	$0_{-\varepsilon}, \bar{a}, -\bar{a}$
$L_{ggg}(b)$	$b, -a, -c$	$L_{nbb}(\bar{b})$	$0_{-\varepsilon}, (\bar{b},\bar{\delta}), (\bar{b}, -\bar{\delta})$
$L_{nbb}(b)$	$0_\varepsilon, (b,\delta), (b,-\delta)$	$L_{ggg}(\bar{b})$	$\bar{b}, \bar{a}, \bar{c}$
$L_{gbb}(a,\mu,c)$	$b, (a,\mu), (c,-\mu)$	$L_{gbb}(\bar{a},\bar{\mu},\bar{c})$	$-\bar{b}, (\bar{a},\bar{\mu}), (\bar{c},-\bar{\mu})$
$L_{gb}(a,\mu)$	$-a, (a,\mu)$	$L_{gb}(-\bar{a},-\bar{\mu})$	$\bar{a}, (-\bar{a},-\bar{\mu})$
$L_{gb}(a,c,\mu)$	$a, (c,\mu)$	$L_{bb}(\bar{a}+\bar{c}, i\bar{\mu}, -\bar{a}+\bar{c})$	$(\bar{a}+\bar{c}, i\bar{\mu}), (\bar{a},-\bar{\mu})$
$L_{bb}(a,\mu,c)$	$(a,\mu), (-a+c, i\mu)$	$L_{gb}(\bar{a}-\bar{c}, \bar{a}+\bar{c}, i\bar{\mu})$	$\bar{a}-\bar{c}, (\bar{a}+\bar{c}, i\bar{\mu})$
$L_{bb}(\pm a,\mu)$	$(a,\mu), (-a,\mu)$	$L_{bb}(\pm\bar{c},-\bar{\mu})$	$(\bar{c},-\bar{\mu}), (-\bar{c},-\bar{\mu})$
$L_{bb}(\pm(a,\mu))$	$(a,\mu), (-a,-\mu)$	$L_{bb}(\pm(\bar{a},\bar{\mu}))$	$(\bar{a},\bar{\mu}), (-\bar{a},-\bar{\mu})$

The construction we gave for the near hexagon is one which is the result of a coordinatization process. There exists another construction using a so-called *two-intersection set*. This is a set of points in a projective space which intersects each hyperplane in either n_1 or n_2 points, $n_1 \neq n_2$.

Suppose that X is a set of 25 points in PG(3,4) satisfying the following properties:

(I) every plane intersects X in either 5 or 9 points,
(II) there exists a point $x \in X$ such that $|l \cap X| \in \{1, 3\}$ for every line l through x.

Call a plane which intersects X in 9 points a *thick plane*. Consider the following incidence structure $\mathcal{S}_X$: the points of $\mathcal{S}_X$ are the elements of X together with the thick planes of PG(3,4); the lines of $\mathcal{S}_X$ are the pairs $\{y, \pi\}$ with $y \in X$ and π a thick plane through y; incidence is containment. If λ_l denotes the number of thick planes through a line l of PG(3,4), then $25 = |X| = |l \cap X| + \lambda_l(9 - |l \cap X|) + (5 - \lambda_l)(5 - |l \cap X|)$ or $\lambda_l = |l \cap X|$. Hence, every two points of the same type have at least two common neighbors and the maximal distance between two points of $\mathcal{S}_X$ is equal to 3. As a consequence $\mathcal{S}_X$ is a thin near hexagon. For every line l of PG(3,4) with $|l \cap X| \geqslant 2$ we put $Q_l := (l \cap X) \cup \{\pi | \pi \text{ is a thick plane through } l\}$. Clearly Q_l is a good quad of $\mathcal{S}_X$. Moreover, every two points p_1 and p_2 of the same type are contained in a unique quad Q_l: if p_1 and p_2 are points of PG(3,4), then $l = p_1p_2$; if p_1 and p_2 are thick planes, then $l = p_1 \cap p_2$. This proves (C2). By (II) the local space at x only contains lines of size three and hence is an affine plane of order 3. This proves (C3).

We will now give a set of 25 points satisfying (I) and (II). The set is related to a partial spread of PG(3,4); that is, a set of pairwise disjoint lines. If we have a partial spread of size 12 in PG(3,4), then the set X of points not contained in one of the lines of the partial spread satisfies (I): if a plane π contains a line of the partial spread, then $|\pi \cap X| = 5$; otherwise $|\pi \cap X| = 9$. By Soicher [LS] there are three nonequivalent partial spreads of size 12 which are not contained in a partial spread of size 13 (a so-called *maximal partial spread*). We verified that the sets X arising from these maximal partial spreads satisfy (II). We will give the set X corresponding to the first example of [LS]. If $x = (1, 0, 1, 1)$ and

$$\begin{array}{llll} a_1 = (1,0,\alpha,\alpha) & a_2 = (1,\alpha,0,1) & a_3 = (1,\alpha^2,0,\alpha) & a_4 = (1,\alpha^2,0,\alpha^2) \\ b_1 = (1,0,\alpha^2,\alpha^2) & b_2 = (0,1,\alpha^2,0) & b_3 = (1,1,\alpha^2,0) & b_4 = (0,1,\alpha,\alpha^2) \end{array}$$

$$\begin{array}{llll} a_5 = (1,1,1,0) & a_6 = (1,\alpha^2,1,0) & a_7 = (1,1,\alpha,\alpha^2) & a_8 = (1,1,\alpha^2,\alpha^2) \\ b_5 = (1,\alpha,1,\alpha^2) & b_6 = (1,1,1,\alpha^2) & b_7 = (1,\alpha,0,\alpha) & b_8 = (0,1,\alpha,\alpha) \end{array}$$

$$\begin{array}{llll} a_9 = (1,\alpha,\alpha,1) & a_{10} = (1,\alpha,\alpha^2,\alpha) & a_{11} = (1,\alpha^2,\alpha,\alpha^2) & a_{12} = (1,0,\alpha^2,0) \\ b_9 = (1,\alpha^2,0,1) & b_{10} = (0,1,1,\alpha) & b_{11} = (0,1,1,\alpha^2) & b_{12} = (1,0,\alpha,\alpha^2) \end{array}$$

with $\alpha \in \mathbb{F}_4 \setminus \{0, 1\}$, then $X := \{x, a_1, \ldots, a_{12}, b_1, \ldots, b_{12}\}$ satisfies (I) and $xa_i \cap X = \{x, a_i, b_i\}$ for every $i \in \{1, \ldots, 12\}$.

References

[BDB] B. De Bruyn, Characterizations of near polygons by means of one local space, J. Combin. Theory Ser. A 102 (2003) 282–291.

[LS] L.H. Soicher, http://www.maths.qmw.ac.uk/~leonard/partialspreads/