Extensions of isomorphisms for affine dual polar spaces and strong parapolar spaces

Rieuwert J. Blok
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027 USA
blokr@member.ams.org

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Send proofs to:

Rieuwert J. Blok
Dipartimento di matematica
Università di Roma "Tor Vergata"
00133 Roma
Italy

Tel.: 39-06-7259-4709
FAX: 39-06-7259-4699
Email: blokr@member.ams.org

Abstract

Let B be a class of point-line geometries. Given $\Gamma_i \in \mathsf{B}$ with subspace $\mathcal{S}_i$ for $i = 1, 2$, does any isomorphism $\Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ extend to a unique isomorphism $\Gamma_1 \longrightarrow \Gamma_2$? It is known to be true if B is the class of almost all projective spaces or the class of almost all non-degenerate polar spaces. We show that this is true for the class of almost all strong parapolar spaces, including dual polar spaces.

A special case occurs when $\Gamma_1 = \Gamma_2 = \Gamma$ has an embedding into a projective space $\mathbb{P}(V)$ that is natural in the sense that $\mathrm{Aut}(\Gamma) \leq \mathrm{P\Gamma L}(V)$. Then the question becomes whether $\mathbb{P}(V)$ is also the natural embedding for $\Gamma - \mathcal{S}$. Our result shows that in most cases the stabilizer $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(\Gamma - \mathcal{S})$ is faithful on $\Gamma - \mathcal{S}$ and equals $\mathrm{Aut}(\Gamma - \mathcal{S})$ and so the answer is affirmative. We know that there exist some interesting exceptions. These will be covered in a subsequent paper.

1 Introduction

A *point-line geometry* is a pair $\Gamma = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a set whose elements are called *points* and $\mathcal{L}$ is a set whose elements are subsets of $\mathcal{P}$ called *lines*. A point-line geometry Γ is a *partial linear space*, if any two points are contained in at most one line. We call Γ *thick*, if every line has at least three points. Note that this means that a grid, although not thick as a building, is thick as a point-line geometry in the sense defined here. Throughout the paper we will assume that point-line geometries are partial linear and thick, unless specified otherwise.

Given a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, let X be any subset of $\mathcal{P}$. If $|l \cap X| \geq 2$ for some line $l \in \mathcal{L}$, then we call this intersection a *line of* X. The collection of all lines of X is denoted $\mathcal{L}(X)$. We call X a *subspace* if all lines of X are in fact lines of Γ. The subspace X is *proper* if $\emptyset \neq X \neq \mathcal{P}$. A (geometric) *hyperplane* of Γ is a proper subspace H with the property that $l \cap H \neq \emptyset$ for all $l \in \mathcal{L}$. Note that, although hyperplanes are "large", it is not true that they are necessarily the subspaces that are maximal with respect to containment.

Given a subspace $\mathcal{S}$ of Γ, let $\mathcal{A} = \mathcal{P} - \mathcal{S}$ and denote the geometry induced on it by $A = (\mathcal{A}, \mathcal{L}(\mathcal{A}))$. The point-line geometry A is called the *(generalized) affine geometry* or *subspace complement* associated to $\mathcal{S}$ in Γ. We often loosely denote this affine geometry by $\Gamma - \mathcal{S}$.

Definition 1.1 A class B of point-line geometries is called *affinely rigid* (AR) if and only if

(AR) given $\Gamma_i \in \mathsf{B}$ with a subspace $\mathcal{S}_i$ $(i = 1, 2)$, then any isomorphism $\Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ extends uniquely to an isomorphism $\Gamma_1 \longrightarrow \Gamma_2$.

We consider the following question:

(Q) Under what conditions is a class B affinely rigid?

Taking $\Gamma_1 = \Gamma_2 = \Gamma$, this problem can be phrased as follows: is the group induced by $\mathrm{Stab}_{Aut(\Gamma)}(\Gamma - \mathcal{S})$ faithful on $\Gamma - \mathcal{S}$ and does it coincide with $\mathrm{Aut}(\Gamma - \mathcal{S})$? Consider also the following, more refined, question: Suppose that Γ is embedded into the projective space $\mathbb{P}(V)$ for some vector space V. Then one may think of V as a "natural" embedding if every automorphism of Γ is induced by some (semi-) linear automorphism of V. This is for instance true of the projective geometry $\mathbb{P}(V)$ itself by the fundamental theorem of projective geometry. Thus the question of whether $\mathrm{Aut}(\Gamma - \mathcal{S})$ is contained in $\mathrm{Aut}(\Gamma)$ is strictly linked to the question of whether the embedding of $\Gamma - \mathcal{S}$ into $\mathbb{P}(V)$ is natural also.

Example 1.2 Let $\Gamma = \mathbb{P}(V)$, where $V = \mathbb{F}_2^{n+1}$ for some $n \geq 3$. Suppose $\mathcal{S} = \mathbb{P}(H)$ for some subspace $H \subseteq V$ of codimension 1. Then $\Gamma - \mathcal{S}$ as a geometry is the complete graph on 2^n points. Now $\mathrm{Aut}(\Gamma) = \mathrm{SL}_{n+1}(\mathbb{F}_2)$ and $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(\Gamma - \mathcal{S}) = 2^n.\,\mathrm{SL}_n(\mathbb{F}_2)$. This group is faithful on $\Gamma - \mathcal{S}$, but is smaller than $\mathrm{Aut}(\Gamma - \mathcal{S}) = \mathrm{Sym}(2^n)$. Note that $\mathrm{Aut}(\Gamma - \mathcal{S}) \leq \mathrm{SL}_{2^n}(\mathbb{F}_2)$, which is the group of linear automorphisms of the 2^n-dimensional embedding for $\Gamma - \mathcal{S}$.

Let us consider the lines of a generalized affine geometry $\Gamma - \mathcal{S}$. Given a line L of Γ and some subset of points $l \subseteq L$, if $|l| \geq 2$ then, since Γ is a partial linear space, l determines the line L uniquely. We say that l *supports* L and write $\overline{l} = L$. We call a line l of $\Gamma - \mathcal{S}$ *short* if $\overline{l} \neq l$. Note that in this case $\overline{l} \cap \mathcal{S}$ consists of exactly one point. Lines of $\Gamma - \mathcal{S}$ that are not short are called *long*.

Given a short line l of $\Gamma - \mathcal{S}$, the unique point on $\overline{l} - l \in \mathcal{S}$ is a *non-deep* point of $\mathcal{S}$. Points in $\mathcal{S}$ that are not collinear to a point in $\Gamma - \mathcal{S}$ are called *deep*. We denote the non-deep and deep points for $\mathcal{S}$ by $\mathcal{N}(\mathcal{S})$ and $\mathcal{D}(\mathcal{S})$ respectively. The following refinement of this notion will be crucial in this paper.

Definition 1.3 Following Shult [14] we define a sequence of subsets $D_i(\mathcal{S})$ as follows: Set $D_{-1}(\mathcal{S}) = \mathcal{P} - \mathcal{S}$, let $D_0(\mathcal{S})$ be the set of non-deep points of $\mathcal{S}$, and for $i \geq 0$ define

$$D_{i+1}(\mathcal{S}) = \{p \in \mathcal{P} \mid p \quad \begin{array}{l}\text{is collinear to a point of } D_i(\mathcal{S}) \\ \text{but not to any point of } D_{i-1}(\mathcal{S})\}.\end{array}$$

We then set

$$D_i^*(\mathcal{S}) = \bigcup_{j=i}^{\infty} D_j(\mathcal{S}).$$

Note that, for $i \geq 0$, the set $D_i(\mathcal{S})$ is contained in $\mathcal{S}$. The *depth* of $\mathcal{S}$ is the integer $d = \max\{i \mid D_i(\mathcal{S}) \neq \emptyset\}$ if it is finite and ∞ otherwise.

We call a point-line geometry *connected* if its collinearity graph is connected. The *distance* $\mathrm{d}(x, y)$ between points x and y is the length of a shortest path from x to y in the collinearity graph of Γ. The *diameter* is the integer $\mathrm{diam} = \max\{\mathrm{d}(x, y) \mid x, y \in \Gamma\}$ if it is finite, and $\mathrm{diam} = \infty$ otherwise. Clearly the depth of $\mathcal{S}$ is bounded by the diameter.

We say that a class B of geometries *respects short lines* if for any two elements $\Gamma_i \in \mathsf{B}$, $i = 1, 2$, and subspaces $\mathcal{S}_i$ and any isomorphism $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ a line l is short in $\Gamma_1 - \mathcal{S}_1$ if and only if l^ε is short in $\Gamma_2 - \mathcal{S}_2$. We say that a geometry Γ respects short lines if the isomorphism class of Γ does.

We say that two lines l and m are *concurrent* if they intersect in a point; we write $l{*}m$. The main tool to study question (Q) will be the following.

Definition 1.4 A *layer extendable* or *LE*-class is a class B of point-line geometries with the following properties:

(LE1) Every element of B is a connected thick partial linear space,

(LE2) for every $\Gamma \in \mathsf{B}$ with subspace $\mathcal{S} \subseteq \Gamma$ the set $D_i^*(\mathcal{S})$ is a subspace of Γ for every $i \in \mathbb{N}$,

(LE3) given $\Gamma_i \in \mathsf{B}$ with subspace $\mathcal{S}_i$ ($i = 1, 2$) and some isomorphism $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$, for any two non-intersecting lines $l_1, l_2 \in \Gamma_1 - \mathcal{S}_1$ we have

(LE3.1) $l_1 \neq \overline{l_1}$ if and only if $l_1^\varepsilon \neq \overline{l_1^\varepsilon}$,

(LE3.2) $\overline{l_1} * \overline{l_2}$ if and only if $\overline{l_1^\varepsilon} * \overline{l_2^\varepsilon}$, and

(LE3.3) for any line h_1 with $|h_1 \cap \mathcal{N}(\mathcal{S}_1)| \geq 2$ there is a line h_2 with $|h_2 \cap \mathcal{N}(\mathcal{S}_2)| \geq 2$ such that $\overline{l_1} - l_1 \in h_1$ if and only if $\overline{l_1^\varepsilon} - l_1^\varepsilon \in h_2$.

The name "layer extendable" stems from the fact that by (LE2) there exist, possibly infinitely many, layers $\Gamma - \mathcal{S} = \Gamma - D_0^*(\mathcal{S})$, $\Gamma - D_1^*(\mathcal{S})$, $\ldots$, $\Gamma - \emptyset = \Gamma$ of subspace complements, and condition (LE3) will turn out to be all that is needed to extend an isomorphism $\Gamma_1 - D_i^*(\mathcal{S}_1) \longrightarrow \Gamma_2 - D_i^*(\mathcal{S}_2)$ uniquely to an isomorphism $\Gamma_1 - D_{i+1}^*(\mathcal{S}_1) \longrightarrow \Gamma_2 - D_{i+1}^*(\mathcal{S}_2)$. Note that (LE3.1) implies that B respects short lines.

Example 1.5 It is fairly easy to see that the class of all projective planes and all projective spaces of dimension $n \geq 3$ not defined over $\mathbb{F}_2$ is an *LE*-class. In Section 4, we give modified versions of results by Pasini and Shpectorov, by Pralle, and by Cohen and Shult to see that the collection of non-degenerate polar spaces forms an *LE*-class P (Theorem 4.1).

The motivation for defining *LE*-classes is the following main result.

Theorem 1 *Let* B *be a class of point-line geometries satisfying* (LE1) *and* (LE2). *Then* B *is an LE-class if and only if it is affinely rigid.*

The scope of this theorem is enhanced by the fact that one can create new *LE*-classes from old ones by taking certain unions (Lemma 3.3) and by using a local property which implies (LE3) (Theorem 3.4).

We will now discuss the particular geometries we will work with. Before we can do so properly we will need some fairly well-known definitions. We will assume familiarity with polar spaces. There are many references for these geometries in the literature, for instance Buekenhout and Shult [3], Tits [16], Cameron [5], Pasini [11], and many more. Those familiar with dual polar spaces and parapolar spaces can skip to the statement of Theorem 2.

A *singular subspace* of a point-line geometry is a subspace any two points of which are collinear. A *gamma space* is a point-line geometry such that whenever l is a line and p is a point, then p is collinear to either none, one, or all points on

l. Thus, singular subspaces and polar spaces are gamma spaces. A set of points C is called *convex* if any geodesic in the collinearity graph between two points of the subspace is entirely contained in the collinearity graph of that subspace. The *convex closure* of a set of points X is the smallest convex subspace containing X.

A *symplecton* is a subspace isomorphic to a non-degenerate polar space of rank at least 2 that is the convex closure of any two of its points at distance 2.

A *dual polar space* is a point-line geometry whose points are the maximal singular subspaces of some non-degenerate polar space of rank at least 2. The lines are the collections of maximal singular subspaces containing a common singular subspace of codimension 1. These geometries were characterized in terms of points and lines in Cameron [4]. The symplecta of a dual polar space are polar spaces of rank 2, often called *quads*.

Definition 1.6 A *parapolar space* is a connected partial linear gamma space together with a family of geodetically closed subspaces, each isomorphic to a non-degenerate polar space of rank at least 2, called symplecta, such that any line is contained in a symplecton and any quadrangle is contained in a unique symplecton. A *strong parapolar space* is a parapolar space in which any two points at distance 2 are contained in a symplecton.

The above definition was first given in Cohen [7]. This is a unified definition for dual polar spaces and strong parapolar spaces as defined in Cooperstein [8], or Shult [14]. Similarly, the present definition of parapolar space unifies the former concepts of dual polar spaces and parapolar spaces.

Examples of parapolar spaces other than dual polar spaces are all polar spaces of rank at least 3 as well as the Lie incidence geometries of type $A_{n,i}$ with $1 < i < n$ (Grassmannians), $D_{n,n-1}$, $D_{n,n}$ (half-spin geometries), $E_{6,1}$, $E_{6,6}$, and $E_{7,7}$ (Bourbaki labeling [2]) (see also Cooperstein [8] and Hanssens [10] for characterizations).

We prove the following results about these geometries. In Section 4, we prove that the thick non-degenerate polar spaces which are not a grid, form an LE-class (Theorem 4.1). For affine polar spaces, we show in Section 5 that most of them have a certain convexity property (Theorem 5.1). As a consequence, any isomorphism between subspace complements of strong parapolar spaces will send affine symplecta to affine symplecta. Using this result and Theorem 4.1, we check in Section 6 that a large class of strong parapolar spaces satisfies the conditions of Theorem 3.4. More precisely, we obtain the following main result on strong parapolar spaces. Here, as in the remainder of the paper, we indicate most geometries using the name of the associated Lie group.

Theorem 2 *Let* PP *be the class of thick strong parapolar spaces with the following properties*

(PP1) *the symplecta of rank* 2*, if any, have lines of length at least* 4 *and are not a grid,*

(PP2) *the symplecta of rank* 3*, if any, are not isomorphic to* $O_7(2)$ *or* $O_6^+(2)$ *(the Klein quadric over* $\mathbb{F}_2$*).*

Then PP *forms an LE-class. As a consequence, if two elements* $\Gamma_i \in \mathsf{PP}$*,* $i = 1, 2$*, have a subspace* $\mathcal{S}_i$*, then any isomorphism* $\Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ *extends uniquely to an isomorphism* $\Gamma_1 \longrightarrow \Gamma_2$*.*

Thus the theorem applies for instance to all dual polar spaces with lines of length at least 4 and whose quads are not grids. It also applies to all Lie incidence geometries of type $A_{n,i}$ ($1 < i < n$) not defined over $\mathbb{F}_2$, and all Lie incidence geometries of type $D_{n,n-1}$, $D_{n,n}$, $E_{6,1}$, $E_{6,6}$, and $E_{7,7}$ (Bourbaki labeling). The proof of Theorem 2 given here does not work for grassmannians over $\mathbb{F}_2$ or for the dual polar space associated to $\mathrm{SP}_{2n}(2)$. As for the conclusion of this theorem, it is likely that it is indeed false for the dual polar space associated to $\mathrm{SP}_{2n}(2)$. Partial results on the grassmannians of type $A_{n,i}$, $n \in \mathbb{N}$, $1 \leq i \leq n$, over $\mathbb{F}_2$ show that, to maintain the conclusion of Theorem 2 one must impose the (probably sufficient) restriction that $2 < i < n - 1$.

2 Deep points

The following result, proved by Shult in [14], gives a criterion for condition (LE2) in Definition 1.4 to be satisfied.

Lemma 2.1 *(i) If* Γ *is connected and* $\mathcal{S}$ *is a proper subspace of* Γ*, then*

$$\mathcal{S} = \biguplus_{i=0}^{\infty} D_i(\mathcal{S}),$$

where $\uplus$ *denotes disjoint union.*

(ii) If, in addition, Γ *has the property that any two points at distance* 2 *from each other are contained in a subspace that is a non-degenerate polar space of rank at least* 2*, then the sets*

$$D_j^*(\mathcal{S}) = \biguplus_{i=j}^{\infty} D_i(\mathcal{S})$$

are subspaces of Γ*.*

If $\mathcal{S}$ is clear from the context we'll write D_j^* and D_i for $D_j^*(\mathcal{S})$ and $D_i(\mathcal{S})$. Note that $D_1^*(\mathcal{S})$ is the set of deep points of $\mathcal{S}$.

We note that the conclusion of the second assertion was proved, in the form of Lemma 2.2 in [14], under the assumption that Γ is a strong parapolar space in the classical sense and not under the assumption presented here. However, the (classical) strong parapolar condition was used to ensure that the convex closure of two points at distance 2 is a non-degenerate polar space of rank at least 2. In fact, the existence of any subspace isomorphic to a non-degenerate polar space of rank at least 2 containing these two points suffices.

3 Extensions of affine isomorphisms

Definition 3.1 For a point-line geometry Γ with a subspace $\mathcal{S}$, two lines l and m of $\Gamma - \mathcal{S}$ are called *parallel* if and only if $l = m$ or $\overline{l} - l = \overline{m} - m \neq \emptyset$. The resulting equivalence relation is often called a parallelism. The equivalence class containing l is called the *parallel class* of l and is denoted $[l]$.

We now prove Theorem 1.

Proof: Let B be an LE-class. Let $\Gamma_j \in \mathsf{B}$ and $\mathcal{S}_j$ $(j = 1, 2)$ be such that there is an isomorphism $\varepsilon_0 \colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$.

We will prove the following claim by finite induction on i:

(C) There exists a unique isomorphism $\varepsilon_i \colon \Gamma_1 - D_i^*(\mathcal{S}_1) \longrightarrow \Gamma_2 - D_i^*(\mathcal{S}_2)$ extending $\varepsilon_0 \colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$.

Once we've proved this, it follows that any isomorphism $\varepsilon_0 \colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ extends uniquely to an isomorphism $\varepsilon \colon \Gamma_1 \longrightarrow \Gamma_2$. Namely, for any point $p \in \mathcal{S}_1$ there exists a unique i such that $p \in D_{i-1}(\mathcal{S}_1)$ by Lemma 2.1. Define $p^\varepsilon = p^{\varepsilon_i}$. Then ε is a unique extension of ε_0 because ε_i is a unique extension of ε_0 to $\Gamma_1 - D_i^*(\mathcal{S}_1)$. Note that also by uniqueness we have $p^{\varepsilon_j} = p^{\varepsilon_i}$ for all $j \geq i$.

It is clear that $\varepsilon \colon \Gamma_1 \longrightarrow \Gamma_1^\varepsilon$ is an isomorphism that uniquely extends ε_0. If $\Gamma_1^\varepsilon \neq \Gamma_2$, then Γ_1^ε is a subspace complement of Γ_2. Hence Γ_1^ε contains a short line and so does Γ_1. This is clearly impossible and so $\varepsilon \colon \Gamma_1 \longrightarrow \Gamma_2$ is the unique isomorphism extending ε_0. Thus B is affinely rigid, as desired.

The statement (C) is trivially true for $i = 0$. We prove the induction step by showing the following.

(C') Any isomorphism $\varepsilon \colon \Gamma_1 - D_{i-1}^*(\mathcal{S}_1) \longrightarrow \Gamma_2 - D_{i-1}^*(\mathcal{S}_2)$ extends uniquely to an isomorphism $\eta \colon \Gamma_1 - D_i^*(\mathcal{S}_1) \longrightarrow \Gamma_2 - D_i^*(\mathcal{S}_2)$.

For all $i = 0, 1, 2, \ldots$, and $j = 1, 2$, let $\Gamma_j^i = \Gamma_j - D_i^*(\mathcal{S}_j) = (\mathcal{P}_j^i, \mathcal{L}_j^i)$.

For $j = 1, 2$, we now define a partition Π_j^{i-1} on the short lines of Γ_j^{i-1}. Since $D_{i-1}^*(\mathcal{S}_j)$ is a subspace, we can consider the collection of short lines of Γ_j^{i-1}. Note that Γ_j^{i-1} is the complement of the subspace $D_{i-1}^*(\mathcal{S}_j)$ in the geometry Γ_j. Define a parallelism on the lines of Γ_j^{i-1} in the sense of Definition 3.1 and let Π_j^{i-1} be the partition whose elements are the equivalence classes of this parallelism.

We now prove (C'). Define a map $\pi\colon D_{i-1}(\mathcal{S}_1) \longrightarrow D_{i-1}(\mathcal{S}_2)$ as follows. For every $p \in D_{i-1}(\mathcal{S}_1)$ there exists a line l in Γ_j^{i-1} such that $p = \overline{l} - l$. Choose any such line l and let $p^\pi = \overline{l^{\varepsilon_{i-1}}} - l^{\varepsilon_{i-1}}$.

We claim that π is a well-defined bijection. It is clear that $p^\pi \in D_{i-1}(\mathcal{S}_2)$. By (LE3.1) and since ε is an isomorphism, the map $l \mapsto l^\varepsilon$ is a bijection between the collection of short lines of Γ_1^{i-1} and the collection of short lines of Γ_2^{i-1}. Also by (LE3.2), two short lines l and m are parallel in Γ_1^{i-1} if and only if l^ε and m^ε are parallel in Γ_2^{i-1}. Finally, by definition of parallel, two short lines l and m of Γ_j^{i-1} satisfy $\overline{l} - l = \overline{m} - m$ if and only if l and m are parallel. Thus π is well-defined and injective. Clearly π has an inverse and so it is a bijection.

We have established that the extension of ε by π is a bijection $\eta\colon \Gamma_1^i \longrightarrow \Gamma_2^i$ between the subspace complements Γ_1^i and Γ_2^i. We now prove that η is in fact an isomorphism that uniquely extends ε. For this it suffices to check that under η, lines of Γ_1^i correspond to lines of Γ_2^i and vice versa. More precisely, as $D_{i-1}^*(\mathcal{S}_j)$ and $D_i^*(\mathcal{S}_j)$ are both subspaces of Γ_j, for every line l of Γ_j^i, $\overline{l} - l$ is at most one point, and l is of one of the types (L1-L3) below:

(L1) l is long in Γ_j^{i-1},

(L2) $|l \cap D_{i-1}(\mathcal{S}_j)| = 1$, in which case $l = l \cap \Gamma_j^{i-1}$ is short in Γ_j^{i-1} and $l = \overline{l}$, or

(L3) $l \subseteq D_{i-1}(\mathcal{S}_j)$ in which case l has at least two points since Γ_j is thick and $\overline{l} - l$ is at most a single point in the subspace $D_i^*(\mathcal{S}_j)$.

Let l be a line of Γ_1^i. We check the cases (L1-L3).

(L1): If l is a long line of Γ_1^{i-1}, then by (LE3.1) $l^\eta = l^\varepsilon$ is a long line of Γ_2^{i-1}.

(L2): If $l = \overline{k}$ for some short line k of Γ_1^{i-1}, then k^ε is a short line of Γ_2^{i-1} and $l^\eta = k^\varepsilon \cup (\overline{k} - k)^\pi = k^\varepsilon \cup (\overline{k^\varepsilon} - k^\varepsilon) = \overline{k^\varepsilon}$, which is a line of type (L2) in Γ_2^i.

(L3): In this case $l = \{\overline{k} - k \mid \overline{k} - k \in l\}$, where the k's are short lines of Γ_j^{i-1}. Note that l is a line of Γ_1^i with $|l \cap D_{i-1}(\mathcal{S}_1)| \geq 2$. Now as $\mathcal{N}(D_{i-1}^*(\mathcal{S}_j)) = D_{i-1}(\mathcal{S}_j)$ for $j = 1, 2$, by (LE3.3) there is a line l_2 of Γ_2^i with $|l_2 \cap D_{i-1}(\mathcal{S}_2)| \geq 2$ such that $\overline{k} - k \in l$ if and only if $\overline{k^\varepsilon} - k^\varepsilon \in l_2$. Thus $l^\eta = l_2$ is a line of type (L3) in Γ_2^i.

By the same reasoning applied to the inverse of η, defined as the extension of ε^{-1} by π^{-1}, we find that every line of Γ_2^i is the η image of a line in Γ_1^i. This establishes that η is an isomorphism extending ε.

It remains to prove that η is the unique isomorphism extending ε to an isomorphism $\Gamma_1^i \longrightarrow \Gamma_2^i$. Suppose η' is another such isomorphism. Then $\zeta = \eta' \circ \eta^{-1}$ is an automorphism of Γ_2^i fixing Γ_2^{i-1} point-wise. Consider any point $p \in D_{i-1}(\mathcal{S}_2)$. Then there is a line l on p intersecting Γ_2^{i-1} by definition of D_{i-1}. In fact $k = l \cap \Gamma_2^{i-1}$ is a short line with $\overline{k} = l$ and $\overline{k} - k = \{p\}$. Now clearly ζ fixes k point-wise and since Γ_2 is partial linear and thick, it also fixes l. Hence ζ fixes $l - k = \{p\}$ and ζ is the identity on Γ_2^i. Thus $\eta' = \eta$ and η is unique.

Thus we have established the 'if' part of the theorem. We shall now prove the 'only if' part.

Suppose that B is affinely rigid, that is, it satisfies (AR). We must check that B satisfies (LE3.1)-(LE3.3). Let $\Gamma_j \in \mathsf{B}$ have subspace $\mathcal{S}_j$ for $j = 1, 2$ such that there is an isomorphism $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$. Furthermore, let l_1, l_2 be non-intersecting lines of $\Gamma_1 - \mathcal{S}_1$. By (AR) there is a unique isomorphism $\eta\colon \Gamma_1 \longrightarrow \Gamma_2$ extending ε. Clearly if l is a short line of $\Gamma_1 - \mathcal{S}_1$, then $\overline{l}^{\eta} = \overline{l^{\varepsilon}}$ since η preserves lines and extends ε and since Γ_j is a thick partial linear space. Hence $(\overline{l} - l)^{\eta} = \overline{l^{\varepsilon}} - l^{\varepsilon}$.

Thus (LE3.1) and (LE3.2) follow at once.

As for (LE3.3), since η is an isomorphism extending ε, we have $\mathcal{N}(\mathcal{S}_1)^{\eta} = \mathcal{N}(\mathcal{S}_2)$. Hence given a line h_1 with $|h_1 \cap \mathcal{N}(\mathcal{S}_1)| \geq 2$, the line $h_2 = h_1^{\eta}$ satisfies $|h_2 \cap \mathcal{N}(\mathcal{S}_2)| \geq 2$ and as $(\Gamma_1 - \mathcal{S}_1)^{\eta} = \Gamma_2 - \mathcal{S}_2$ and $(\overline{l} - l)^{\eta} = \overline{l^{\varepsilon}} - l^{\varepsilon}$, clearly $\overline{l_1} - l_1 \in h_1$ if and only if $\overline{l_1}^{\varepsilon} - l_1^{\varepsilon} \in h_2$. Thus (LE3.3) is satisfied. $\Box$

We now present two ways of creating new *LE*-classes from old ones.

Lemma 3.2 *Any subset of an LE-class is again an LE-class.* $\Box$

Lemma 3.3 *Let B_i, $i = 1, 2$, be LE-classes and suppose that there is no isomorphism $\Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$, where Γ_i belongs to B_i and has proper subspace $\mathcal{S}_i$. Then $\mathsf{B} = \mathsf{B}_1 \cup \mathsf{B}_2$ is again an LE-class.* $\Box$

Theorem 3.4 *Let B be a class of point-line geometries satisfying* (LE1) *and* (LE2) *of Definition 1.4. Suppose in addition that there is an LE-class $\mathcal{T}$ of subspaces of elements of B such that for every $\Gamma \in \mathsf{B}$ we have*

(L) *every line of Γ is contained in some element of $\mathcal{T}$, and*

(IL) *any two distinct intersecting lines of Γ are contained in an element of $\mathcal{T}$,*

(T) *given $\Gamma_i \in \mathsf{B}$ with subspace $\mathcal{S}_i$ ($i = 1, 2$) and some isomorphism $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$, for every $T_1 \in \mathcal{T}$ with $T_1 \cap (\Gamma_1 - \mathcal{S}_1) \neq \emptyset$, there is $T_2 \in \mathcal{T}$ with $T_2 \cap (\Gamma_2 - \mathcal{S}_2) \neq \emptyset$ such that $(T_1 - \mathcal{S}_1)^{\varepsilon} = T_2 - \mathcal{S}_2$.*

Then B is an LE-class.

Proof: We only have to check that (LE3.1-LE3.3) are satisfied in B.

Let $\Gamma_j \in \mathsf{B}$ have subspace $\mathcal{S}_j$ ($j = 1, 2$) and suppose there is an isomorphism $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$. Now let l_1, l_2 be short lines of $\Gamma_1 - \mathcal{S}_1$. (LE3.1): By (L) applied to l_1, there exists $T_1 \in \mathcal{T}$ such that T_1 contains l_1. Since $T_1 \cap (\Gamma_1 - \mathcal{S}_1) \neq \emptyset$, by (T) there is $T_2 \in \mathcal{T}$ with $T_2 \cap (\Gamma_2 - \mathcal{S}_2) \neq \emptyset$ such that $\varepsilon(T_1 - \mathcal{S}_1) = T_2 - \mathcal{S}_2$. Now $\varepsilon\colon T_1 - \mathcal{S}_1 \longrightarrow T_2 - \mathcal{S}_2$ is an isomorphism and by (LE3.1) applied to the LE-class $\mathcal{T}$, $\overline{l_1} \neq l_1$ if and only if $\overline{l_1^\varepsilon} \neq l_1^\varepsilon$. Hence (LE3.1) is satisfied.

(LE3.2): If $\overline{l_1} * \overline{l_2}$, then by (IL) there is an element $T_1 \in \mathcal{T}$ containing $\overline{l_1}$ and $\overline{l_2}$. Since $T_1 \cap (\Gamma_1 - \mathcal{S}_1) \neq \emptyset$, by (T) there is $T_2 \in \mathcal{T}$ with $T_2 \cap (\Gamma_2 - \mathcal{S}_2) \neq \emptyset$ such that $\varepsilon(T_1 - \mathcal{S}_1) = T_2 - \mathcal{S}_2$. Now $\varepsilon\colon T_1 - \mathcal{S}_1 \longrightarrow T_2 - \mathcal{S}_2$ is an isomorphism and by (LE3.1) applied to the LE-class $\mathcal{T}$, also $\overline{l_1^\varepsilon} * \overline{l_2^\varepsilon}$. Now the same argument applied to the isomorphism ε^{-1} shows that also $\overline{l_1^\varepsilon} * \overline{l_2^\varepsilon}$ implies $\overline{l_1} * \overline{l_2}$. Thus (LE3.2) is satisfied.

(LE3.3): Suppose h_1 is a line of Γ_1 with $|h_1 \cap \mathcal{N}(\mathcal{S}_1)| \geq 2$. Then there is a line m intersecting h_1 with $m \cap (\Gamma_1 - \mathcal{S}_1) \neq \emptyset$. By (IL) there is an element $T_1 \in \mathcal{T}$ containing h_1 and m. Since $T_1 \cap (\Gamma_1 - \mathcal{S}_1) \neq \emptyset$, by (T) there is $T_2 \in \mathcal{T}$ with $T_2 \cap (\Gamma_2 - \mathcal{S}_2) \neq \emptyset$ such that $\varepsilon(T_1 - \mathcal{S}_1) = T_2 - \mathcal{S}_2$.

Note that $m \cap h_1$ is a non-deep point of the subspace $T_1 \cap \mathcal{S}_1$ of T_1. Also note that since $\mathcal{D}(T_1 \cap \mathcal{S}_1)$ is a subspace of T_1, at least two points of h_1 are non-deep points of $T_1 \cap \mathcal{S}_1$ in T_1 and hence are non-deep points of $\mathcal{S}_1$ also. Now $\varepsilon\colon T_1 - \mathcal{S}_1 \longrightarrow T_2 - \mathcal{S}_2$ is an isomorphism and so by (LE3.3) applied to the LE-class $\mathcal{T}$, there is a line h_2 in Γ_2 with $|h_2 \cap \mathcal{N}(\mathcal{S}_2)| \geq 2$ such that for all short lines l_1 of $T_1 - \mathcal{S}_1$, $\overline{l_1} - l_1 \in h_1$ if and only if $\overline{l_1^\varepsilon} - l_1^\varepsilon \in h_2$. Now let $l_2 \neq l_1$ be any other short line of $\Gamma_1 - \mathcal{S}_1$ with $\overline{l_2} - l_2 = \overline{l_1} - l_1$. Then since Γ_1 is a partial linear space, $l_1 \cap l_2 = \emptyset$ and since ε is an isomorphism, also $l_1^\varepsilon \cap l_2^\varepsilon = \emptyset$. By (LE3.2) since $\overline{l_1} * \overline{l_2}$, also $\overline{l_1^\varepsilon} * \overline{l_2^\varepsilon}$ and so $\overline{l_2^\varepsilon} - l_2 = \overline{l_1^\varepsilon} - l_1^\varepsilon \in h_2$. The fact that $\overline{l_1^\varepsilon} - l_1^\varepsilon \in h_2$ implies $\overline{l_1} - l_1 \in h_1$, follows by applying the same argument to the isomorphism ε^{-1}.

Since B has properties (LE3.1)-(LE3.3), it is an LE-class. □

Corollary 3.5 *A class of geometries that satisfies the assumptions of Theorem 3.4, is affinely rigid.* □

Example 3.6 The class $\mathsf{A}_{2,1}$ of projective planes is clearly an LE-class. (LE1) is true by definition of a projective plane. (LE2) is satisfied because any subspace $\mathcal{S}$ is either a single point or a line and $D_i(\mathcal{S}) = \emptyset$ for $i \geq 1$. (LE3.1)+(LE3.2) follow from the fact that any two lines that do not intersect in $\Gamma - \mathcal{S}$, must intersect in $\mathcal{S}$. (LE3.3) is satisfied because there is at most one such line h_i which exists precisely if there is more than one parallel class.

The collection $\mathsf{A}_{n,1}$ of all n-dimensional projective spaces with at least four points per line forms an LE-class. This follows from Theorem 3.4 where $\mathsf{A}_{2,1}$ plays the role of $\mathcal{T}$. Clearly (L) and (IL) are satisfied. We check (T): For $i = 1, 2$, let $\Gamma_i \in \mathsf{A}_{n,1}$ have subspace $\mathcal{S}_i$ and suppose there is an isomorphism $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$. Now if $T_1 - \mathcal{S}_1 \neq \emptyset$, then one easily shows that, since the lines have at least four points, $T_1 - \mathcal{S}_1$ is the subspace of $\Gamma_1 - \mathcal{S}_1$ generated by any three points $x, y, z \in T_1 - \mathcal{S}_1$ not all on one line. Clearly then there is $T_2 \in \mathcal{T}$ such that $T_2 - \mathcal{S}_2$ is the subspace of $\Gamma_2 - \mathcal{S}_2$ generated by the three points x^ε, y^ε, and z^ε. It follows that $(T_1 - \mathcal{S}_1)^\varepsilon = T_2 - \mathcal{S}_2$.

4 Polar spaces

In this section we will discuss some general properties of polar spaces we will need later. Let P be the collection of thick non-degenerate polar spaces of rank at least 2 that are not a grid.

The main result is the following.

Theorem 4.1 *The collection* P *is an LE-class.*

Large portions of this statement can be found in the literature in a different form. We will prove it here and indicate where and how it overlaps with known results.

Note that by definition polar spaces are partial linear, so by our thickness assumption P satisfies (LE1). By Lemma 2.1 P satisfies (LE2). We prove that P satisfies (LE3) by means of Theorem 1 in verifying property (AR).

Recall that non-degenerate polar spaces of rank 2 are generalized quadrangles. Let Q be the collection of thick generalized quadrangles that are not a grid.

Proposition 4.2 *The collection* Q *is an LE-class.*

A special case of this statement was proved by Pasini and Shpectorov [12, Lemma 2.3]. Another result close to this can be found in Pralle [13]. Our setting differs from theirs in that in (LE3) we do not a priori assume that Γ_1 and Γ_2 are isomorphic. However, in proving this result we use some of their techniques.

We say that a generalized quadrangle has *order* (s, t) if every point is contained in $t + 1$ lines and every line contains $s + 1$ points. Not every generalized quadrangle has an order, but one can prove that any thick generalized quadrangle that is not a grid has an order (s, t); clearly then $s, t \geq 2$. The following observation explains why we have to exclude grids ($t = 1$) from Q.

Lemma 4.3 *For* $i = 1, 2$, *let* Γ_i *be a generalized quadrangle with subspace* $\mathcal{S}_i$. *Let* $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ *be an isomorphism. Then either both* Γ_1 *and* Γ_2 *are grids, possibly of different order, or there exist* s, t, *possibly infinite, such that the* Γ_i *are of order* (s, t).

Proof: Given any point $p \in \Gamma_i - \mathcal{S}_i$, the map $l \mapsto l \cap (\Gamma_i - \mathcal{S}_i)$ is a bijection between the set of lines of Γ_i on p and the set of lines of $\Gamma_i - \mathcal{S}_i$ on p since Γ_i has thick lines. Therefore there is a bijection between the lines on a point $p \in \Gamma_1 - \mathcal{S}_1$ and the lines on $p^\varepsilon \in \Gamma_2 - \mathcal{S}_2$. In particular, Γ_1 is a grid if and only if Γ_2 is a grid. Note that if Γ_1 and Γ_2 are grids, it is possible that $\mathcal{S}_1 = \emptyset$ and $\mathcal{S}_2$ consists of one line or two intersecting lines; in particular then Γ_1 and Γ_2 are not necessarily isomorphic.

If Γ_i is not a grid for $i = 1, 2$, then it has order (s_i, t) with $s_i, t \geq 2$, possibly infinite.

If s_1 is infinite, then so is s_2, and vice versa and they are the same cardinal number.

Now suppose s_1 and s_2 are finite and not equal. By reversing the isomorphism if necessary, we may assume that $s_1 < s_2$. Since the lines of $\Gamma_1 - \mathcal{S}_1 \cong \Gamma_2 - \mathcal{S}_2$ have length s_1 or $s_1 + 1$ and also have length s_2 or $s_2 + 1$, all lines of $\Gamma_1 - \mathcal{S}_1$ must have length $s_1 + 1 = s_2$. Then $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2$ is a hyperplane, and $\Gamma_1 \cong \Gamma_2 - \mathcal{S}_2$ is a generalized quadrangle of order $(s_2 - 1, t)$. By the one-or-all axiom in the generalized quadrangles $\Gamma_2 - \mathcal{S}_2$ and Γ_2, no two lines l and m meeting $\Gamma_2 - \mathcal{S}_2$ can meet $\mathcal{S}_2$ in the same point, for then none of the points of l would be collinear to any of the points of m in $\Gamma_2 - \mathcal{S}_2$. Since $t \geq 2$, the subspace $\mathcal{S}_2$ is a generalized quadrangle of order $(s_2, t-1)$. If l, m would be two lines of $\Gamma_2 - \mathcal{S}_2$ concurrent in a point of $\mathcal{S}_2$, then no point of l would be collinear to any point of m in $\Gamma_2 - \mathcal{S}_2$ in contradiction to the assumption that $\Gamma_2 - \mathcal{S}_2$ is a generalized quadrangle. Hence there is at most one line in $\Gamma_2 - \mathcal{S}_2$ through a given point of $\mathcal{S}_2$ whence $\mathcal{S}_2$ is a generalized quadrangle of order $(s_2, t-1)$. Consider a line L of $\mathcal{S}_2$. Every point of L belongs to exactly one line that meets $\Gamma_2 - \mathcal{S}_2$ in s_2 points, whereas every point of $\Gamma_2 - \mathcal{S}_2$ is collinear with exactly one point of L. Thus $s_2(s_2 + 1) = |\Gamma_2 - \mathcal{S}_2|$. On the other hand, $|\Gamma_2 - \mathcal{S}_2| = ((s_2 - 1)t + 1)s_2$, as $\Gamma_2 - \mathcal{S}_2$ is a generalized quadrangle of order $(s_2 - 1, t)$. Hence $s_2^2 + s_2 = s_2^2 t - s_2 t + s_2$. This forces $s_2 = (s_2 - 1)t$, so that $s_2 = t = 2$. Thus Γ_2 is isomorphic to $O_5(2)$ and $\mathcal{S}_2$ is isomorphic to $O_4^+(2)$. But then $s_1 = 1$ contradicting that Γ_1 is thick. Thus again $s_1 = s_2$. □

Note 4.4 Note that for a generalized quadrangle Γ with a subspace $\mathcal{S}$, two lines l and m of $\Gamma - \mathcal{S}$ are parallel in the sense of Definition 3.1 if and only if one of the following holds: (1) $l = m$ or (2) no point on l is collinear to a point of m.

The following observation explains the use of Note 4.4.

Lemma 4.5 *For $i = 1, 2$, let Γ_i be a generalized quadrangle with subspace $\mathcal{S}_i$. Let $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ be an isomorphism. Then ε preserves parallelism. In particular it provides a bijection between the parallel classes of $\Gamma_1 - \mathcal{S}_1$ and the parallel classes of $\Gamma_2 - \mathcal{S}_2$.* □

Denote collinearity by $\perp$ and, for any subset $X \subseteq \mathcal{P}$, let $X^\perp$ be the set of points collinear with all points of X (a point is assumed to be collinear to itself). The following two elementary lemmas set short lines and their parallel classes in relation.

Lemma 4.6 *Let $\Gamma \in \mathsf{Q}$ have a subspace $\mathcal{S}$. If $l \in \mathcal{L}(\Gamma - \mathcal{S})$ with $|[l]| \geq 2$, then l is short.*

Proof: Let l and m be parallel in $\Gamma - \mathcal{S}$. Since each point of m is collinear with a point in $\bar{l}$, each point of m is collinear to a point of $\bar{l} - l$ which is just one point x since $\mathcal{S}$ is a subspace. Hence $\bar{l} - l = x = \overline{m} - m$ and l is short. ☐

Lemma 4.7 *Let $\Gamma \in \mathsf{Q}$ have a subspace $\mathcal{S}$. Let $l \in \mathcal{L}(\Gamma - \mathcal{S})$ be short and let x be the unique point $\bar{l} \cap \mathcal{S}$. Then*

$$[l] = \{m \in \mathcal{L}(\Gamma - \mathcal{S}) \mid \overline{m} \cap \mathcal{S} = \{x\}\}.$$

Proof: "$\subseteq$": If $|[l]| = 1$, this is trivial. Suppose $|[l]| \geq 2$, then this follows from the last remark in the proof of Lemma 4.6.

"$\supseteq$":Let $m \in \mathcal{L}(\Gamma - \mathcal{S})$ be such that $m \cap \mathcal{S} = \{x\}$. Then there is no point on l collinear to any point of m by the one-or-all axiom in the generalized quadrangle Γ. Thus $m \in [l]$. ☐

The following lemma characterizes the situations in which short lines exist.

Lemma 4.8 *Let $\Gamma = (\mathcal{P}, \mathcal{L}) \in \mathsf{Q}$ with a subspace $\mathcal{S}$. Then $\Gamma - \mathcal{S}$ has a short line if and only if one of the following holds.*

(a) *There is a line l with $|[l]| \geq 2$. In this case l is short.*

(b) *There is a line l with $|[l]| = 1$ and $l = \{p \in \Gamma - \mathcal{S} \mid p \in m \in \mathcal{L}(\Gamma - \mathcal{S}) \implies |[m]| = 1\}$. In this case l is the unique short line with $|[l]| = 1$ and $\mathcal{S} = x^\perp - l$, where x is the unique point of $\bar{l} \cap \mathcal{S}$.*

(c) *Γ is isomorphic to $O_5(2)$ and $\mathcal{S}$ is isomorphic to $O_4^+(2)$. In this case every line l of $\Gamma - \mathcal{S}$ is short and satisfies $|[l]| = 1$.*

We note that in case (b) there will also be lines as described in (a). However, in case (c) there are no lines as described in (a) or (b).

Lemma 4.8 is proved using Lemmas 4.6, 4.7, 4.9, and 4.10. In the interest of presentation we prove it here.

Proof: Clearly any line l of $\Gamma - \mathcal{S}$ is either long or short and either $|[l]| = 1$ or $|[l]| \geq 2$. If $|[l]| \geq 2$, then l is necessarily short by Lemma 4.6. The two cases left are the ones where $|[l]| = 1$ and l is either long or short. Lemma 4.7 is used to prove Lemma 4.9 which tells us that if l is a short line with $|[l]| = 1$, then either $\Gamma = O_5(2)$ and $\mathcal{S} = O_4^+(2)$ or $\mathcal{S} = x^\perp - l$ where x is the unique point of $\bar{l} - l$. One easily checks that in the former case all lines l of $\Gamma - \mathcal{S}$ are short and satisfy $|[l]| = 1$. Finally, by Lemma 4.10 the latter case occurs precisely if l is as described in case (b) of Lemma 4.8. □

The following lemma tells for which Γ and $\mathcal{S}$ short lines l with $|[l]| = 1$ may exist.

Lemma 4.9 *Let $\Gamma \in \mathsf{Q}$ have a subspace $\mathcal{S}$. Suppose that l is a short line of $\Gamma - \mathcal{S}$ with $|[l]| = 1$. Let x be the unique point on $\bar{l} \cap \mathcal{S}$. Then*

(a) *$\mathcal{S} = x^\perp - l$, or*

(b) *Γ is isomorphic to $O_5(2)$ and $\mathcal{S}$ is isomorphic to $O_4^+(2)$. Every line l of $\Gamma - \mathcal{S}$ is short and satisfies $|[l]| = 1$.*

Proof: Let Γ have order (s, t), where $s, t \geq 2$ possibly infinite.

First note that $\mathcal{S}$ contains t lines on x. If this were not the case, then x would be a point of $\mathcal{S}$ contained in at least two lines meeting $\Gamma - \mathcal{S}$, whence $|[l]| \geq 2$ by Lemma 4.7.

If all points of $\mathcal{S}$ are collinear to x, then $\mathcal{S} \subseteq x^\perp$ and so $\mathcal{S}$ is the set of all but one of the lines on x. Thus the situation is as in (a).

Now assume $\mathcal{S}$ does contain a point that is not collinear to x. Then it follows that $\mathcal{S}$ itself is a non-degenerate generalized quadrangle.

The following elegant argument is due to Jonathan I. Hall. We first show that every point y of $\mathcal{S}$ is on exactly one line not contained in $\mathcal{S}$. Let $L = \bar{l}$. First let $y \in \mathcal{S}$ be not collinear to x. Then since y is collinear to a point of L that does not belong to $\mathcal{S}$ there is at least one line M on y not contained in $\mathcal{S}$. Note that any other such line on y would have to intersect L as well since x is collinear with a point on that line not in $\mathcal{S}$. Thus creating a triangle, there is no further line not in $\mathcal{S}$ through y and M. Since $\mathcal{S}$ is non-degenerate and $s \geq 2$, its non-collinearity graph is connected so that each of its points is on a unique line not contained in $\mathcal{S}$. In particular $\mathcal{S}$ has order $(s, t-1)$.

We will show that in fact $s = t = 2$. Suppose there exist collinear u and v in $\mathcal{S}$ that are both not collinear to either x or y. Let K and N be the unique lines on u and v not contained in $\mathcal{S}$. Now K must intersect L and M in their intersection point and the same is true of N. Thus we find a triangle, a contradiction.

It is easy to verify that this configuration x, y, u, v occurs unless $s = t = 2$. Thus Γ is isomorphic to $O_5(2)$. Also $\mathcal{S}$ is isomorphic to $O_4^+(2)$. Thus the situation is as described in (b). Since every point of $\mathcal{S}$ is on exactly one line not contained in $\mathcal{S}$ and every line of $\Gamma - \mathcal{S}$ meets $\mathcal{S}$ it follows from Lemma 4.7 that every line l of $\Gamma - \mathcal{S}$ is short and satisfies $|[l]| = 1$. □

Lemma 4.10 *Let $\Gamma \in \mathsf{Q}$ have a subspace $\mathcal{S}$. Then $\mathcal{S} = x^\perp - l$, where l is a short line l with $|[l]| = 1$ and x is the unique point on $\bar{l} - l$ if and only if*

$$B = \{p \in \Gamma - \mathcal{S} \mid p \in m \in \mathcal{L}(\Gamma - \mathcal{S}) \implies |[m]| = 1\}$$

is a line.

Proof: " $\implies$ ": We show that $l = B$. Let $L = \bar{l}$ and let p be a point on l. We show that $p \in B$. Since Γ does not contain triangles, p is not collinear to any point of $\mathcal{S} - \{x\}$ so any other line on p is long and hence only parallel to itself. Now let q be a point of $\Gamma - \mathcal{S}$ not on l. We show that $q \notin B$. Since q is not collinear to x, given any line k on x contained in $\mathcal{S}$ there is a line m on q meeting k in some point $z \neq x$. Since k is the only line on z in $\mathcal{S}$ and Γ is not a grid, there is some line $n \neq m$ on z not contained in $\mathcal{S}$. Thus by Lemma 4.7, l is contained in a parallel class of size at least 2 so that $q \notin B$.

"$\Longleftarrow$": Let $l = B$ and let $L = \bar{l}$. We show that l is short and satisfies $|[l]| = 1$.

We first claim that $L \cap \mathcal{S}$ is a point x. Suppose that $a \in \Gamma - (\mathcal{S} \cup B)$ belongs to a line intersecting B in a point p. Then by assumption there is a line m on a that belongs to a parallel class of size at least 2 and so by Lemmas 4.6 and 4.7 a is collinear to some point $y \in \mathcal{S}$. Let $M = \overline{m}$. Now y is not on a line N intersecting B in a point q because then $n = N - \mathcal{S}$ is a line on $q \in B$ parallel to $m \neq n$ contradicting the definition of B. By the one-or-all axiom in the generalized quadrangle Γ, y must be collinear to at least one point of L and so this point must belong to $\mathcal{S}$. Call this point x.

Now suppose there is a line $K \neq L$ on x not contained in $\mathcal{S}$. Take some point a on $K - \mathcal{S}$. Clearly $a \notin B$ so there is a line m on a that belongs to a parallel class of size at least 2 and as before it is collinear to some point $y \in \mathcal{S} - \{x\}$. Again, this point y must be collinear to x. But then x, a, y form a triangle which is impossible since Γ is a generalized quadrangle.

Thus all lines on x, except L, are contained in $\mathcal{S}$. Hence l is a short line with $|[l]| = 1$. Since l is the only short line with $|[l]| = 1$, it follows that $\mathcal{S} = x^\perp - l$ by Lemma 4.9. □

We are now ready to prove Proposition 4.2

Proof: Note that by definition generalized quadrangles are partial linear, so by our thickness assumption Q satisfies (LE1). Also, by Lemma 2.1 Q satisfies (LE2). We now check that Q satisfies (LE3). Let $\Gamma_j \in \mathsf{Q}$ have subspace $\mathcal{S}_j$, for $j = 1, 2$ and suppose there is an isomorphism $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$. Let l_1, l_2 be lines of $\Gamma_1 - \mathcal{S}_1$.

(LE3.1): We check cases (a-c) in Lemma 4.8. (a): Suppose $|[l_1]| \geq 2$. Then by Lemma 4.5 also $|[l_1^\varepsilon]| \geq 2$ and so l_1^ε is short.

(b): Suppose $|[l_1]| = 1$ and $l_1 = \{p \in \Gamma_1 - \mathcal{S}_1 \mid p \in m \in \mathcal{L}(\Gamma_1 - \mathcal{S}_1) \implies |[m]| = 1\}$. By Lemma 4.5 for every line $l \in \Gamma_1 - \mathcal{S}_1$ we have $|[l]| = 1$ if and only if $|[l^\varepsilon]| = 1$. This also holds for ε^{-1} and so $l_1^\varepsilon = \{p \in \Gamma_2 - \mathcal{S}_2 \mid p \in m \in \mathcal{L}(\Gamma_2 - \mathcal{S}_2) \implies |[m]| = 1\}$. Hence l_1^ε is short also.

(c): Suppose $\Gamma_1 \cong O_5(2)$ and $\mathcal{S}_1 \cong O_4^+(2)$. Then by Lemma 4.3 and the uniqueness of a generalized quadrangle of order $(2, 2)$, also $\Gamma_2 \cong O_5(2)$. Now clearly every line of $\Gamma_2 - \mathcal{S}_2$ is short also. Applying the above reasoning to ε^{-1} we find that in all cases $l_1 \in \Gamma_1 - \mathcal{S}_1$ is short if and only if $l_1^\varepsilon \in \Gamma_2 - \mathcal{S}_2$ is short.

(LE3.2): Since l_1 and l_2 do not intersect in $\Gamma_1 - \mathcal{S}_1$, but $\overline{l_1}$ and $\overline{l_2}$ do intersect, these lines are short. By Lemma 4.7 $\overline{l_1} - l_1 = \overline{l_2} - l_2$ if and only if $[l_1] = [l_2]$. By Lemma 4.5 this happens if and only if $[l_1^\varepsilon] = [l_2^\varepsilon]$. This proves (LE3.2).

(LE3.3): In a generalized quadrangle the lines are the cliques in the collinearity graph. Therefore it suffices to show that for any two short lines l_1 and m_1 of $\Gamma_1 - \mathcal{S}_1$ the points $p_1 = \overline{l_1} - l_1$ and $q_1 = \overline{m_1} - m_1$ are collinear if and only if $p_2 = \overline{l_1^\varepsilon} - l_1^\varepsilon$ and $q_2 = \overline{m_1^\varepsilon} - m_1^\varepsilon$ are collinear. We note that since Γ_j is a generalized quadrangle, p_j and q_j are non-collinear if and only if there are lines L_j and M_j on p_j and q_j that intersect in $\Gamma_j - \mathcal{S}_j$. That is p_j and q_j are non-collinear in Γ_j if and only if there are short intersecting lines l'_j and m'_j in $\Gamma_j - \mathcal{S}_j$ such that $l'_j \in [l_j]$ and $m'_j \in [m_j]$. Using (LE3.2) and that ε is an isomorphism, we find that p_1 and q_1 are (non-) collinear if and only if p_2 and q_2 are (non-) collinear. □

Let P_3 be the class of thick non-degenerate polar spaces of rank at least 3.

Note 4.11 If a polar space $\Gamma \in \mathsf{P}_3$ has a subspace $\mathcal{S}$, then a plane of $\Gamma - \mathcal{S}$ is either a projective plane, an affine plane, or a projective plane minus a point. These planes are preserved by automorphisms of $\Gamma - \mathcal{S}$. In case the lines of Γ have at least four points, this is because these planes are the subspaces generated by three pairwise collinear points not on a single line and if the lines of Γ have three points, these planes are the intersections of size 4 or 6 of maximal singular subspaces.

The parallelism in $\Gamma - \mathcal{S}$ in the sense of Definition 3.1 can be made explicit as follows. Two lines l and m are parallel if and only if one of the following holds: (1) $l = m$, or (2) l and m are non-intersecting lines contained in an affine plane or projective plane minus a point of $\Gamma - \mathcal{S}$, or (3) l and m are related by a finite sequence of relations as in (2).

The following result is due to Cohen and Shult [6, Proposition 2.7] and Shult [15].

Proposition 4.12 *Let $\Gamma_i \in \mathsf{P}_3$ have subspace $\mathcal{S}_i$ for $i = 1, 2$. Then any isomorphism $\varphi\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ extends uniquely to an isomorphism $\overline{\varphi}\colon \Gamma_1 \longrightarrow \Gamma_2$.*

We sketch the proof exactly as in Shult [15].
Proof: The case that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are hyperplanes is covered by Proposition 2.7 of [6]. However, the proof for arbitrary subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ is essentially the same. The non-deep points of $\mathcal{S}_i$, $i = 1, 2$, correspond bijectively to the parallel classes of lines. The union of the lines in an extended parallel class $[l]$ containing the short line l is the set

$$\Delta(l) = \{a \in \Gamma_1 - \mathcal{S}_1 \mid a^\perp \cap l = \emptyset \text{ or } l\}.$$

This follows essentially since $m \in [l] - \{l\}$ if and only if $\overline{m}$ and $\bar{l}$ intersect in the point $\bar{l} - l = \overline{m} - m$. Now by the one-or-all axiom, given a point $a \in m$, either $a^\perp \supseteq \bar{l}$ or $a^\perp \cap \bar{l} = \overline{m} \cap \bar{l} \in \mathcal{S}$.

If $\mathcal{S}_1$ has a deep point p_1, then necessarily $\mathcal{S}_1 = p_1^\perp$. Hence $\mathcal{S}_1$ is a hyperplane. Consequently, all lines of $\Gamma_1 - \mathcal{S}_1$ are short. Now, as Γ_1 has rank > 2, a line l is short if and only if $|[l]| \geq 2$. So, $|[l]| \geq 2$ for every line in $\Gamma_1 - \mathcal{S}_1$. Since ε is an isomorphism, we then also have $|[m]| \geq 2$ for every line m of $\Gamma_2 - \mathcal{S}_2$. Hence all lines of $\Gamma_2 - \mathcal{S}_2$ are short and $\mathcal{S}_2$ is a hyperplane. Thus we are led back to Proposition 2.7 of loc. cit.. $\square$

Corollary 4.13 *The class P_3 is an LE-class.*

Proof: Note that by definition non-degenerate polar spaces are partial linear, so by our thickness assumption P_3 satisfies (LE1). By Lemma 2.1 P_3 satisfies (LE2). Now by Proposition 4.12 and Theorem 1, P_3 also satisfies (LE3). $\square$

We now prove Theorem 4.1.

Proof: Clearly $\mathsf{P} = \mathsf{Q} \cup \mathsf{P}_3$. Both Q and P_3 are *LE*-classes. Furthermore, if $\Gamma_1 \in \mathsf{Q}$, $\Gamma_2 \in \mathsf{P}_3$, then for any subspace $\mathcal{S}_i$ of Γ_i there can be no isomorphism $\Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$. This is because the largest singular subspaces of $\Gamma_1 - \mathcal{S}_1$ are the lines, whereas $\Gamma_2 - \mathcal{S}_2$ always contains projective, punctured projective, or affine planes. Thus by Lemma 3.3, P is an *LE*-class. $\square$

Corollary 4.14 *The collection $\mathsf{A}_{n,1} \cup \mathsf{P}$ of all projective spaces with lines of size at least 4 together with the collection of all non-degenerate polar spaces other than grids forms an LE-class.*

Proof: This follows from Lemma 3.3. By Example 3.6 and Theorem 4.1 both $\mathsf{A}_{n,1}$ and P are LE-classes. Suppose $\Gamma_1 \in \mathsf{A}_{n,1}$ and $\Gamma_2 \in \mathsf{P}$ have subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Then $\Gamma_1 - \mathcal{S}_1$ is singular and $\Gamma_2 - \mathcal{S}_2$ is not, hence no isomorphism $\Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ exists. $\square$

Finally we mention a result on polar spaces that will be needed in the next section.

Lemma 4.15 *The complement of a proper subspace in a non-degenerate thick polar space of rank at least* 2 *is connected, non-degenerate, and has diameter at most* 3.

Proof: Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a thick non-degenerate polar space. Let $\mathcal{S}$ be a proper subspace and set $\mathcal{A} = \mathcal{P} - \mathcal{S}$. Denote collinearity by $\perp$.

For any two points $p, q \in \mathcal{A}$ we show that there is a path of points and lines in $\mathcal{A}$ from p to q.

If p and q are on a common line l, then p, l, q is a path in $\mathcal{A}$ connecting p and q and we are done.

If p and q are not collinear, then let l be an arbitrary line on p. Since Γ is non-degenerate one can find a line m on q opposite to l, that is, such that $l^\perp \cap m = \emptyset$. Then because lines of Γ are thick and intersect the subspace $\mathcal{S}$ in none, one, or all of their points, one of the following must happen: (1) the unique point r on m collinear to p belongs to $\mathcal{A}$, (2) the unique point s on l collinear to q belongs to $\mathcal{A}$, or (3) there exist points $t, u \in \mathcal{A}$ different from p, q, r, s and on l and m, respectively, that are collinear.

In all cases there is a path of points and lines in $\mathcal{A}$ connecting p to q. It also follows that $\Gamma - \mathcal{S}$ has diameter at most 3.

Next we show that $\Gamma - \mathcal{S}$ is non-degenerate. Consider an arbitrary point $x \in \Gamma - \mathcal{S}$. Since Γ is non-degenerate, there exist lines L and M on x such that no point of $L - x$ is collinear to a point of $M - x$. Now since Γ is thick and at most one point of both L and M belongs to the subspace $\mathcal{S}$, there exist points $y \in L - (x \cup \mathcal{S})$ and $z \in M - (x - \mathcal{S})$. Clearly y and z are non-collinear. $\square$

We note that this type of argument works for dual polar spaces and in fact also for the metasymplectic space F_4.

5 Convexity of affine polar spaces

To prove Theorem 2 in Section 6, we apply Theorem 3.4 to a class PP of thick strong parapolar spaces where the collection P of the symplecta of the elements of PP plays the role of $\mathcal{T}$. In order to satisfy condition (T) we need to ensure that any isomorphism between affine parapolar spaces takes affine symplecta, which are affine polar spaces, to affine symplecta. This follows from the fact that such (affine) symplecta are uniquely determined by any two of their points at distance 2 in the sense of Theorem 5.1 below which we will prove here.

Given a point-line geometry A we call a set of points X 2-*convex* (in A) if it has the property that, for any $x, y \in X$ at distance at most 2, all points on a geodesic of A from x to y are also contained in X. The 2-*convex closure* of X is the smallest 2-convex subspace containing X. We will frequently investigate the 2-convex closure of subspace complements. Note that if A is such a subgeometry of Γ, then the 2-convex closure of a point subset of A means the closure in A, but not in Γ. Note that in that case geodesics of A need not be geodesics of Γ, but geodesics of length 2 are.

Theorem 5.1 *Let Γ be a non-degenerate thick polar space of rank at least 2. Then any non-empty subspace complement is the 2-convex closure of any two of its points at distance 2, except possibly if Γ is "sparse", that is, if Γ has rank 2 and some lines of length 3, or Γ is isomorphic to $O_6^+(2)$ or $O_7(2)$.*

The exceptional cases of Theorem 5.1 lead to the excluded geometries in Theorem 2. Namely our proof of Theorem 2 in Section 6 does not work for these geometries since they have subspace complements that are not 2-convex closed. We do not have for the assertion of Theorem 2 for these exceptional geometries, only that our proof does not work for them. As remarked after Theorem 2 in the Introduction, the author is investigating these exceptional cases and conjecture there are examples not satisfying the assertion of Theorem 2.

We will first address the case where Γ is a non-degenerate polar space of rank 2, that is, a generalized quadrangle.

Proposition 5.2 *Let Γ be a generalized quadrangle. If every line of Γ has at least four points, then any non-empty subspace complement is the 2-convex closure of any two of its points at distance 2.*

Proof: Let $\mathcal{S}$ be a proper subspace of $\Gamma = (\mathcal{P}, \mathcal{L})$ and let A be the point-line geometry induced on $\mathcal{A} = \mathcal{P} - \mathcal{S}$. Let a and b be any two points at distance 2 in A. Such points exist: simply look at two lines intersecting at a point c of $\mathcal{A}$.

Let C be the 2-convex closure of $\{a, b\}$ in A. We will show that $C = \mathcal{A}$.

Let L and M be lines on a and b, respectively, intersecting in $c \in \mathcal{A}$. Also, let $l = L \cap \mathcal{A}$ and $m = M \cap \mathcal{A}$.

Clearly $l, m \subseteq C$.

Now take a point $x \in \mathcal{A} - (l \cup m)$. We show that x also belongs to C. There are four cases:

Case 1: x is collinear to points $y \in l$ and $z \in m$. Then x lies on a minimal path from y to z and as $y, z \in C$ also $x \in C$.

Case 2: x is collinear to $y \in l$ and the unique point of M in $\mathcal{S}$. Let M' be the line on x and y. Now every point of M' is collinear to precisely one point on M. Then, as there are at least four points on M', there is at least one point x' in $M' - (l \cup \mathcal{S})$ that is collinear to some point z in m. By the previous case, $x' \in C$ and so $x \in M' \subseteq C$.

Case 3: x is collinear to the points $L \cap \mathcal{S}$ and $M \cap \mathcal{S}$. Let L' and M' be the lines of Γ on x intersecting M and L respectively. Let y be a point of $L' - M$ different from x and let z be a point of $M' - L$ different from x. Then y is collinear to a point of l and z is collinear to a point of m, hence both belong to C by the previous case. Clearly x lies on a minimal path from y to z and therefore belongs to C.

Case 4: x is collinear to c. Consider any point $z \in \mathcal{A}$ collinear to x, but not to c. Since z is collinear to points on L and M different from c, the preceding cases show that $z \in C$. Clearly also $c \in C$ and therefore x being on a minimal path from c to z also belongs to C.

This concludes the proof. □

Proposition 5.3 *Let Γ be a non-degenerate polar space of rank at least 3. If Γ has rank at least 4 or every line of Γ has at least four points, then any non-empty subspace complement is the 2-convex closure of any two of its points at distance 2.*

Proof: Let $\mathcal{S}$ be a subspace of $\Gamma = (\mathcal{P}, \mathcal{L})$ and let A be the point-line geometry induced on $\mathcal{A} = \mathcal{P} - \mathcal{S}$. Let a and b be any two points at distance 2 in A. Such points exist as A is non-degenerate by Lemma 4.15.

Let C be the 2-convex closure of $\{a, b\}$ in A. We will show that $C = \mathcal{A}$.

Let a' be any neighbor of a in A. We will prove that $a' \in C$ and that there exists a point $b' \in C$ at distance 2 from a'. By Lemma 4.15 we then are done.

Consider the polar space $P = \{a, b\}^{\perp}$. Since a and b have a common neighbor in $\mathcal{A}$, the subspace $P \cap \mathcal{S}$ of P is not equal to P. We use the notation of definition 1.3 for the geometry P with subspace $P \cap \mathcal{S}$ and, for $i = -1, 0, 1, \ldots$, set $D_i = D_i(P \cap \mathcal{S})$ and $D_i^* = D_i^*(P \cap \mathcal{S})$. Note that since P has diameter 2, we have $D_2 = \emptyset$, so that, for instance, $D_0^* = D_0 \cup D_1$. Furthermore, remark that points of P belong to $D_{-1}(\mathcal{S})$ or $D_0(\mathcal{S})$, but that there may exist points in $D_1 = D_1(P \cap \mathcal{S})$ which are deep with respect to $P \cap \mathcal{S})$ in P, but non-deep with respect to $\mathcal{S}$ in Γ.

Let l_0 be the line on a and a' and denote the point $l_0 \cap P$ by p. We will prove by induction on $i = -1, 0, 1$, that if $p \in D_i$, then a' belongs to C.

If $p \in D_{-1}$, then p belongs to $\mathcal{A}$, whence to C, and so $a' \in C$.

Now let $i \geq 0$. The induction hypothesis is that if l' is a line of $\mathcal{A}$ through a such that $\overline{l'}$ meets P in a point of D_{i-1}, then every point x on l' belongs to C.

We distinguish two partially overlapping cases.

Case 1: Γ has rank at least 4. Consider the polar geometry P_p of lines and planes of P on p. Since D_i^* is a subspace of P and $p \in D_i^*$, also the geometry $D_{i,p}^*$ of lines and planes of D_i^* on p is a subspace of P_p. It is not equal to P_p since, by definition of D_i, at least one line of P on p intersects D_{i-1}.

We show that there are two lines l and m of P on p that are not coplanar and meet D_{i-1}. The lines l and m correspond to non-collinear points of $P_p - D_{i,p}^*$. These clearly exist as P_p has rank at least 2 and is non-degenerate.

Consider points $q \in l \cap D_{i-1}$ and $r \in m \cap D_{i-1}$. Let $q' \in \mathcal{A}$ be a neighbor of a on the line aq and define r' likewise on ar. By induction q' and r' both belong to C. Note that q' and a' are on the plane $\langle a, p, q \rangle_\Gamma$ and r' and a' are on the plane $\langle a, p, r \rangle$. Thus q' and r' are at distance 2 in A and a' is a common neighbor of q' and r'. Thus a' belongs to C.

Case 2: Γ has rank at least 3 and is defined over a field $\mathbb{F}$ of size at least 3. We know that at least one line of P on p meets D_{i-1}. Call this line l and let π be the plane of Γ on a and l. Since $l - \{p\} \subseteq D_{i-1}$, it follows by induction that all points q' of $\pi \cap \mathcal{A}$ such that the line aq' meets $l - \{p\}$, belong C. Then since the lines of Γ have at least 4 points, any subspace containing these points in fact contains $\pi \cap A$. Therefore all points of $\pi \cap A$, including a', belong to the subspace C of A.

Thus in all cases a' belongs to C.

Now we must find $b' \in C$ at distance 2 from a'. Since P is non-degenerate, it contains a point z not collinear to p. As Γ is thick, there is at least one point on the line az that does not belong to $\mathcal{S}$ and is not equal to a. Call this point b'. Then by the above also b' belongs to C. Clearly b' is at distance 2 from a' so we are done. $\square$

The remainder of this section is devoted to handling the case where Γ is a non-degenerate polar space of rank 3 over $\mathbb{F}_2$.

We will need the next result only for rank $n = 3$, but it is more natural to state it for general rank. It could be extracted from the classification of polar spaces of rank at least 3 due to (in alphabetical order) F. Buekenhout, A. Cohen, H. Cuypers, P. Johnson, C. Lefèvre, A. Pasini, E. Shult, and J. Tits. This special case can also be found in Cameron [5] where it is proved using an argument due to J.I. Hall.

Theorem 5.4 *(Theorem 7.5.1. in Cameron [5]) A non-degenerate polar space of finite rank at least 2 all lines of which have three points is the polar space of a quadratic form over $\mathbb{F}_2$ on a vector space of finite dimension.*

Lemma 5.5 *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be an embeddable non-degenerate polar space of finite rank $n \geq 3$ over $\mathbb{F}_2$. Let V be its natural orthogonal embedding and let $\mathcal{S} \subseteq \mathcal{P}$ be any subspace of Γ. If $\mathcal{S}$ contains a line, then*

$$\langle \mathcal{S} \rangle_V = V \text{ if and only if } \mathcal{S} = \Gamma.$$

Proof: The "if" part is clear. Now assume that $\langle \mathcal{S} \rangle_V = V$. We first verify that $\mathcal{S}$ is a connected non-degenerate polar space of finite rank. We use the Buekenhout-Shult axioms (see [3]).

(BS1) Every line of $\mathcal{S}$ contains at least three points because its lines are lines of the thick polar space Γ.

(BS2) No point is collinear with all other points. If this were the case, then since $\langle \mathcal{S} \rangle_V = V$ we would find a point p of Γ that is collinear to all points of Γ contradicting that Γ is non-degenerate.

The axiom on finiteness of chains of singular subspaces (BS3) and the one-or-all axiom (BS4) are directly inherited from Γ since the points, lines and singular subspaces of $\mathcal{S}$ are points, lines and singular subspaces of Γ and Γ satisfies these axioms. Thus $\mathcal{S}$ is a non-degenerate polar space of finite rank.

Since $\mathcal{S}$ contains a line L, and $\mathcal{S}$ satisfies the one-or-all axiom, every point of $\mathcal{S}$ is collinear to some point of L. Thus $\mathcal{S}$ is connected.

By Theorem 5.4 Γ is a quadratic polar space on a vector space of finite dimension. Since an anisotropic quadratic form over $\mathbb{F}_2$ does not exist if $\dim(V) \geq 3$, the Witt index of a non-degenerate quadratic form over $\mathbb{F}_2$ is at least $\lfloor \frac{\dim(V)-1}{2} \rfloor$. Thus Γ is one of $O^+_{2n}(2)$, $O^-_{2n+2}(2)$, or $O_{2n+1}(2)$. Also if $\dim(V)$ is even, then $\mathcal{S}$ is one of $O^+_{2k}(2)$, $O^-_{2k}(2)$ and if $\dim(V)$ is odd, then $\mathcal{S}$ is $O_{2k+1}(2)$, where $k = \lfloor \frac{\dim(V)-1}{2} \rfloor$. By looking at the dimensions of the embedding we see that if Γ is $O_{2n+1}(2)$, then $\Gamma = \mathcal{S}$. Next, if Γ is $O^-_{2n+2}(2)$, then also $\Gamma = \mathcal{S}$ by considerations of the embedding dimension and the Witt index. Finally, if Γ is $O^+_{2n}(2)$, then by considerations of the embedding dimension we see that $\mathcal{S}$ must be $O^+_{2n}(2)$ or $O^-_{2(n-1)+2}(2)$. In Γ the totally singular subspaces of dimension $(n-1)$ and n on a common $(n-2)$-space X form the points and lines of a 3×3 grid. If $\mathcal{S}$ were $O^-_{2n}(2)$, then at least two of the five $(n-1)$-spaces of $\mathcal{S}$ on X must be contained in a common n-space of Γ. As $\mathcal{S}$ is a subspace, this n-space should entirely belong to $\mathcal{S}$ contradicting that $\mathcal{S}$ has Witt index $n-1$. Thus again $\Gamma = \mathcal{S}$. $\square$

Note 5.6 Although we do not need it here, a statement similar to Lemma 5.5 can be made for arbitrary polar spaces in finite dimension. In that case, for V one should take the dominant embedding of Γ (c.f. Tits [16]). The proof, albeit somewhat more technical, then proceeds roughly along the same lines.

Corollary 5.7 *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a non-degenerate polar space of rank $n \geq 3$ over $\mathbb{F}_2$. Let V be its natural orthogonal embedding and let $\mathcal{S} \subseteq \mathcal{P}$ be a proper subspace of Γ. Then $\mathcal{S}$ is contained in a hyperplane of Γ or it is a set of pairwise non-collinear points.*

Proof: Suppose that $\mathcal{S}$ is not a set of pairwise non-collinear points. Then $\mathcal{S}$ contains a line and, by Proposition 5.5, since $\mathcal{S} \neq \Gamma$ we know that $U = \langle \mathcal{S} \rangle_V$ is a proper subspace of V. Take some hyperplane H of V such that $H \supseteq U$. Then $H \cap \mathcal{P}$ is a hyperplane of Γ containing $\mathcal{S}$. $\square$

According to the above results the only non-degenerate polar spaces of rank 3 with three points per line are the quadratic polar spaces of type $O_6^+(2)$, $O_7(2)$, and $O_8^-(2)$. We will now study the 2-convexity of the related generalized affine polar spaces.

Lemma 5.8 *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a polar space of rank 3 over the field $\mathbb{F}_2$. Then the complement of a subspace $\mathcal{S}$ is the 2-convex closure of any two of its points at distance 2, except possibly if*

(1) Γ is isomorphic to $O_7(2)$ and $\mathcal{S}$ is contained in a hyperplane of type $O_6^-(2)$, or

(2) Γ is isomorphic to $O_6^+(2)$ and $\mathcal{S}$ is contained in a hyperplane of type $O_5(2)$.

If, in the exceptional cases, $\mathcal{S}$ is the hyperplane itself, then A is not the 2-convex closure of any two of its points at distance 2.

Note that we can characterize all exceptional cases by the fact that the hyperplane regarded as a polar space has Witt index 2 rather than 3. Note also that $O_8^-(2)$ does not have such hyperplanes.

Proof: The setup of the proof is identical to that of Proposition 5.3. Let $\mathcal{S}$ be a subspace of $\Gamma = (\mathcal{P}, \mathcal{L})$ and let A be the point-line geometry induced on $\mathcal{A} = \mathcal{P} - \mathcal{S}$. Let a and b be any two points at distance 2 in A. Such points exist since A is non-degenerate by Lemma 4.15.

Let C be the 2-convex closure of $\{a, b\}$ in A. We will show that $C = \mathcal{A}$. Let a' be any neighbor of a in A. We will prove that $a' \in C$ and that there exists a point $b' \in C$ at distance 2 from a'. By Lemma 4.15 we then are done.

Consider the rank 2 polar space $P = \{a, b\}^\perp$. Since a and b have a common neighbor in $\mathcal{A}$, the subspace $P \cap \mathcal{S}$ of P is not equal to P. We use the notation of definition 1.3 for the geometry P with subspace $P \cap \mathcal{S}$ and, for $i = -1, 0, 1, \ldots$, set $D_i = D_i(P \cap \mathcal{S})$ and $D_i^* = D_i^*(P \cap \mathcal{S})$. Note that since P has diameter 2, we have $D_2 = \emptyset$, so that, for instance, $D_0^* = D_0 \cup D_1$.

Let l_0 be the line on a and a' and denote the point $l_0 \cap P$ by p. We will prove by induction on $i = -1, 0, 1$, that if $p \in D_i$, then a' belongs tot C. If $p \in D_{-1}$, then p belongs to $\mathcal{A}$, whence to C, and so $a' \in C$.

Now let $i \geq 0$. The induction hypothesis is that if l' is a line of $\mathcal{A}$ through a such that l' meets P in a point of D_{i-1}, then every point x on l' belongs to C. It suffices to show the following for $i = 0, 1$:

(*) any point p of D_i is on two lines contained in P and meeting D_{i-1}.

Namely, given lines l and m on $p \in D_i$ as in (*), consider points $q \in l \cap D_{i-1}$ and $r \in m \cap D_{i-1}$. Let $q' \in \mathcal{A}$ be a neighbor of a on the line aq and define r' likewise on ar. Note that if $q \in D_{-1}$, then $q' = q$ and if $q \in D_0$, then $q' = aq - \{a, q\}$, and similarly for r'. By induction q' and r' both belong to C. Note that q' and a' are on the plane $\langle a, p, q \rangle_\Gamma$ and r' and a' are on the plane $\langle a, p, r \rangle$. Thus q' and r' are at distance 2 in A and a' is a common neighbor of q' and r'. Thus a' belongs to C.

It then remains to find $b' \in C$ at distance 2 from a'. Since P is non-degenerate, it contains a point z not collinear to p. As Γ is thick there is at least one point on the line az that does not belong to $\mathcal{S}$ and is not equal to a. Call this point b'. Then by the above also b' belongs to C. Clearly b' is at distance 2 from a'. This proves our claim.

We will now concentrate on (*). If $i = 1$, then p is the unique point of D_1 since P is a non-degenerate generalized quadrangle with a proper subspace $P \cap \mathcal{S}$ which can have at most one deep point. Hence any line on p meets D_0. Thus it suffices to show that any non-deep point p of $P \cap \mathcal{S}$ is contained in at least two lines contained in P and meeting $P \cap \mathcal{A}$. Let V be the natural embedding for Γ. By Corollary 5.7, $\mathcal{S}$ is either a set of pairwise non-collinear points or it is contained in some hyperplane $\mathcal{H}$ of Γ of the form $H \cap \mathcal{P}$ for some hyperplane H of V.

In the former case any two lines in $P \cap \mathcal{P}$ on p will do.

In the latter case let $\mathcal{H} \supseteq \mathcal{S}$ be a hyperplane. In fact we may assume that $\mathcal{H} = \mathcal{S}$ because $\mathcal{P} - \mathcal{S}$ is the 2-convex closure of $\mathcal{P} - \mathcal{H}$ in $\mathcal{P} - \mathcal{S}$. Namely, if $x \in \mathcal{P} - \mathcal{S}$ is non-deep for $\mathcal{H}$, then there is a line on x meeting $\mathcal{P} - \mathcal{H}$ in two points and we are done. If $x \in \mathcal{H} - \mathcal{S}$ is deep in $\mathcal{H}$, then it is the common neighbor of two non-collinear non-deep points of $\mathcal{H}$ in $\mathcal{P} - \mathcal{S}$ so by the previous case we are done.

So given a hyperplane $\mathcal{H}$ we have to decide if we can find two lines in P meeting $\Gamma - \mathcal{H}$ that contain the non-deep point p. We address this problem for each of the geometries $O_6^+(2)$, $O_7(2)$, and $O_8^-(2)$, individually.

(1) Let Γ be isomorphic to $O_8^-(2)$. Then P is of type $O_6^-(2)$. So p is contained in five lines of P. Checking the possible geometries $P \cap \mathcal{H}$ and using that p is non-deep one verifies that at most three of these lines are contained in $P \cap \mathcal{H}$. Thus we can find the desired two lines on p.

(2) Let Γ be isomorphic to $O_7(2)$. Then P is of type $O_5(2)$. One verifies that one can find the desired two lines provided $P \cap \mathcal{H}$ is not of type $O_4^+(2)$.

Assume now that $P \cap \mathcal{H}$ is of type $O_4^+(2)$. Let H be the 6-dimensional subspace of V supporting $\mathcal{H}$ and let $U \subseteq H$ be the subspace of V of dimension 4 spanned by $P \cap \mathcal{H}$. Then $U^\perp$ supports a geometry of type $O_3(2)$. It contains the non-collinear points a, b, and the subspace $W = U^\perp \cap H$ of dimension 2. Since a and b do not belong to H and $\{a, b\}^\perp \cap H = U$, it follows from the geometry of $U^\perp$ that W contains no points of Γ. Thus $H = U \oplus W$ is of type $O_6^-(2)$. One can verify, for instance by explicitly creating the graph $\Gamma - \mathcal{H}$, that in this case the 2-convex closure of any two points at distance 2 in A is a set of 9 points whose collinearity graph is the tri-partite graph $\overline{K_{3:3:3}}$. Thus A having 36 points is not the 2-convex closure of any two of its points at distance 2.

(3) Let Γ be isomorphic to $O_6^+(2)$. Suppose that $\mathcal{H}$ has a deep point x. Let H be the 5-dimensional subspace of V supporting $\mathcal{H}$ and let $U \subseteq H$ be the subspace of V of dimension 3 supporting $P \cap \mathcal{H}$. It is clear that $U^\perp = \langle a, b, x\rangle_V$ supports a geometry of type $O_3(2)$.

Considering that $U \cap U^\perp$ is the 1-dimensional non-singular radical of both $U^\perp$ and U, we find that also U supports a geometry of type $O_3(2)$. In particular, $P \cap \mathcal{H}$ does not contain lines.

The geometry supported by P is of type $O_4^+(2)$ and so p is contained in two lines of P both of which intersect $P \cap \mathcal{A}$. It now follows that A is the 2-convex closure of any two of its points at distance 2.

If $\mathcal{H}$ does not have a deep point, then one can verify, for instance by explicitly creating the graph $\Gamma - \mathcal{H}$, that the 2-convex closure of any two points of A is a set of 6 points whose collinearity graph is the tri-partite graph $\overline{K_{2:2:2}}$. Thus A having 20 points is not the 2-convex closure of any of its points at distance 2. □

We now prove Theorem 5.1.

Proof: This follows from Propositions 5.2, 5.3, and Lemma 5.8. □

6 Strong parapolar spaces

In this section we will prove Theorem 2.

Recall that PP is the class of thick strong parapolar spaces with the following properties

(PP1) the symplecta of rank 2, if any, have lines of length at least 4 and are not a grid,

(PP2) the symplecta of rank 3, if any, are not isomorphic to $O_7(2)$ or $O_6^+(2)$ (the Klein quadric over $\mathbb{F}_2$).

We will now prove Theorem 2.

Proof: Let PP be the class of strong parapolar spaces satisfying (PP1) and (PP2) above.

We will use Theorem 3.4 to prove that PP forms an LE-class. It then follows from Theorem 1 that PP is affinely rigid, as desired.

(LE1): By definition of a parapolar space, PP satisfies (LE1).

(LE2): By Lemma 2.1 the class of all strong parapolar spaces satisfies (LE2).

Let $\mathcal{T}$ be the collection of symplecta of the members of PP. By definition a symplecton is a non-degenerate polar space of rank at least 2 and so $\mathcal{T}$ forms an LE-class by Theorem 4.1. We now check properties (L), (IL), and (T) of Theorem 3.4.

(L),(IL): These are true by Definition 1.6 of a strong parapolar space.

(T): The argument will rely on the following two observations. Let $i = 1, 2$.

(1) By definition of a strong parapolar space, a symplecton T_i is convex in Γ_i, so any geodesic in Γ_i between points of T_i is contained in T_i.

(2) Moreover, any geodesic in $\Gamma_i - \mathcal{S}_i$ between points at distance 2 is a geodesic in Γ_i and the same holds if we replace Γ_i by T_i. Combining this with the previous observation, we find that the 2-convex closure in $T_i - \mathcal{S}_i$ of a set of points equals the 2-convex closure of that set of points in $\Gamma_i - \mathcal{S}_i$.

Let $\Gamma_i \in \mathsf{PP}$ with subspace $\mathcal{S}_i$ $(i = 1, 2)$ and let $\varepsilon\colon \Gamma_1 - \mathcal{S}_1 \longrightarrow \Gamma_2 - \mathcal{S}_2$ be some isomorphism. Suppose $T_1 \in \mathcal{T}$ with $T_1 \cap (\Gamma_1 - \mathcal{S}_1) \neq \emptyset$.

By assumption, if T_1 has rank 2, then the lines have at least 4 points and if it is of rank at least 3, then it is not isomorphic to $O_7(2)$ or $O_6^+(2)$. Therefore, by Theorem 5.1 applied to the polar space T_1, the geometry $T_1 - \mathcal{S}_1$ is the 2-convex closure of any two of its points. By observation (2), this is true also if we consider the 2-convex closure in $\Gamma_1 - \mathcal{S}_1$ instead of in $T_1 - \mathcal{S}_1$. Let x, y be points at distance 2 in $T_1 - \mathcal{S}_1$. Such points exist since $T_1 - \mathcal{S}_1$ is non-degenerate by Lemma 4.15. Then, x^ε, y^ε are points at distance 2 in $\Gamma_2 - \mathcal{S}_2$. Hence, they are also at distance 2 in Γ_2. By (IL), there is a symplecton $T_2 \in \mathcal{T}$ of Γ_2 containing x^ε and y^ε. By observation (1), the points x^ε and y^ε are at distance 2 in $T_2 - \mathcal{S}_2$. Now by observation (2), $T_2 - \mathcal{S}_2$ is 2-convex closed in $\Gamma_2 - \mathcal{S}_2$. Therefore, since ε is an isomorphism, we find that $(T_1 - \mathcal{S}_1)^\varepsilon \subseteq T_2 - \mathcal{S}_2$. However, using ε^{-1} we also find that $(T_2 - \mathcal{S}_2)^{\varepsilon^{-1}} \subseteq T_1 - \mathcal{S}_1$ and so $(T_1 - \mathcal{S}_1)^\varepsilon = T_2 - \mathcal{S}_2$. Thus (T) is satisfied. □

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