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Far from a point in the $F_4(q)$ geometry

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Abstract

We take the long-root geometry associated to the Chevalley group $F_4(q)$, q even, and consider the subgeometry induced on the set of points at maximal distance from a given point. We shall describe this geometry and in particular determine the parameters of a 12-class association scheme on its point set obtained by joining certain classes of a group scheme.

1. Introduction

Let Δ be the building of type F_4 obtained from the Chevalley group $F_4(q)$ and let its Dynkin diagram be labeled as below.

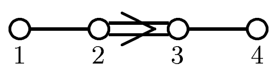

Figure 1: The Dynkin diagram of type F_4

Then $\Gamma = (\mathcal{P}, \mathcal{L})$ will be the point-line geometry whose points and lines are the objects of Δ of type 1 and 2 respectively with inherited incidence (long-root geometry); we will call it the $F_{4,1}(q)$ geometry. We fix a point ∞ and let $\Gamma_\infty = (\mathcal{P}_\infty, \mathcal{L}_\infty)$ be the subgeometry whose point set $\mathcal{P}_\infty$ consists of the points of Γ at maximal distance from ∞ and whose line set $\mathcal{L}_\infty$ consists of the lines in $\mathcal{L}$ that meet $\mathcal{P}_\infty$ in at least 2 points. Our aim is to describe this geometry in some detail. More precisely, we will prove the following result.

Theorem 1.1: *Let Γ be the long-root geometry of the Chevalley group $F_4(q)$ with q even. Then, for any given point ∞, there exists a 12-class association scheme A on the q^{15} points at maximal distance from ∞ whose parameters are as depicted in the diagram in Section 7.*

The proof of this theorem will be given in Section 7. During its preparation we will gather several nice properties of the geometry Γ_∞ and its substructures.

In his dissertation [4], Riebeek determined the parameters of the scheme A in the special case where $q = 2$ with the aid of a computer. One of the motivations behind the present paper is to provide a geometric argument for his result.

We will now give an outline of this paper. In Section 2 we gather some elementary properties of this $F_{4,1}(q)$ geometry, notably on distances and projections. The subgeometries of points far from a given point in the polar space related to $\mathrm{Sp}_6(q)$ and in the dual polar space related to $\mathrm{O}_7(q)$ respectively are subgeometries of Γ_∞. These are studied in Sections 3 and 4.

In Section 5 the classes of the scheme are naturally defined as (unions of) orbitals of the parabolic group stabilizing the point ∞. Here we use a classification of the orbits of the stabilizer of a hyperbolic line.

In Section 6, for a point x in any given class and any line incident with x, we determine the possible distributions of the point set of this line among the classes of the scheme. Furthermore, we determine on how many lines on x a given distribution occurs. This is done in a purely geometric way.

Remark The assumption that q be even only takes effect from Section 5 onwards for the following reason. Whereas in the treatment of the geometries associated to $\mathrm{Sp}_6(q)$ and $\mathrm{O}_7(q)$ it is easy to deal with q even and odd at once, it seems that for $F_4(q)$ this is not so. Although rather similar, the schemes for q even and q odd do not have the same classes (this already transpires from the table of orbits in Cooperstein [3]). Furthermore the use of hyperbolic lines inside symplecta (see Section 5 and Subsection 6.1) and the use of hyperbolic quadrics in the dual of point-residues (see Subsection 6.5 and Section 4) in the case of q even, which quite shortens and clarifies the exposition, doesn't seem to have any kind of counterpart in the case of q odd.

2. Preliminaries

Let Δ be a spherical building with diagram M defined over an index set I. We will regard a building as a diagram geometry rather than as a chamber system. Given $i \in I$ the i-shadow space of Δ is the point-line geometry whose points are the i-elements of Δ and whose lines are the $I^\bullet$-flags, where $I^\bullet$ is the set of labels of the nodes that are adjacent to the i-node in M. We call the i-shadow space of a building with diagram M an *M_i geometry.*

Many incidence properties of Δ can be uncovered by studying an apartment. An apartment of the building Δ is a substructure which is isomorphic to the Coxeter complex belonging to the same diagram as Δ. Given any two flags R and S of Δ there is an apartment A containing both R and S (see Tits [8] Chapter 3). From the convexity of apartments (see Tits loc. cit. Corollary 3.5) it follows not only that the relation between R and S in Δ is the same as in A, but also that the projection of S onto R belongs to A. The *projection* of S onto R is a flag called $\text{proj}_R(S)$ incident with R with the following property: The i-object on $\text{proj}_R(S)$ is the unique object of that type at minimal distance from S among all i-objects incident with R. (Thus, if there are two or more objects of type i at minimal distance from S, then the flag $\text{proj}_R(S)$ has no i-object.)

Let us turn to the case that $M = F_4$. We will describe the Coxeter complex with diagram F_4 by first presenting its 1-shadow space (see Brouwer et al. [1] §10.3).

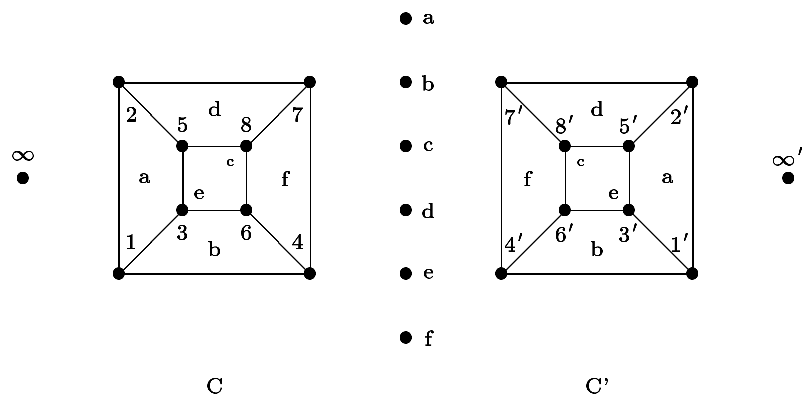

Figure 2: The 1-shadow space of the Coxeter complex with diagram F_4

Complete the graph in Figure 2 by inserting edges in the following places: between ∞ and every vertex of the cube C and likewise for ∞' and C', between the vertices i and i' for $1 \leq i \leq 8$ and, finally, for all $x \in \{a, b, c, d, e, f\}$, between the vertex x and all vertices that lie on the face of C or C' with label x.

The Coxeter complex with diagram F_4 can be recovered from this graph as follows. The 1-cliques, 2-cliques, 3-cliques and octahedra in this graph are the objects of type 1, 2, 3 and 4 respectively and two objects are incident whenever one of them is contained in the other.

Both in the Coxeter complex and in Δ, an object of type 1, 2, 3 and 4 will be called a *point*, *line*, *plane* and *symplecton* respectively.

We define distance relations 0, 1, 2s, 2ns, 3 between points of a Coxeter complex in the following way (by the discussion above this also applies to Δ): The distances 0, 1 and 3 are just the distances in the collinearity graph. Further, two points at distance 2 in the collinearity graph are said to be at distance 2s (resp. 2ns) from one another if they are (not) incident to a common symplecton. For $\lambda = 0, 1, 2, 3, 4$ and any point p let $d_\lambda(p)$ be the collection of points at distance 0, 1, 2s, 2ns, 3 from p, respectively.

LEMMA 2.1: *Given a point p, a line l, a plane V and a symplecton S,*

(i) if $d_4(p) \cap l \neq \emptyset$ then $d_3(p) \cap l$ consists of a point and $l \setminus d_3(p) \subset d_4(p)$.

(ii) if $d_4(p) \cap V \neq \emptyset$ then $d_3(p) \cap V$ consists of a line and $V \setminus d_3(p) \subset d_4(p)$.

(iii) *if* $d_4(p) \cap S \neq \emptyset$ *then* $d_0(p) \cap S = d_1(p) \cap S = \emptyset$, $d_2(p) \cap S$ *consists of a point* s, $d_3(p) \cap S = d_1(s) \cap S$ *and* $d_4(p) \cap S = d_2(s) \cap S$.

(iv) *if* $d_0(p) \cap S = d_4(p) \cap S = \emptyset$, *then* $d_1(p) \cap S$ *is a line* m, $d_2(p) \cap S = \cap_{q \in m} d_1(q)$.

The proof follows rather easily by studying the presentation of the apartment of type F_4 above. Related properties were studied by Cohen in [2] when axiomatizing metasymplectic spaces.

We can view the $F_{4,1}(q)$ geometry, as well as the natural polar space associated to the group $\mathrm{Sp}_{2n}(q)$, as long-root geometries (see Cooperstein [3]). In this way every point of the geometry can be viewed as a long-root subgroup of the corresponding Chevalley group and the geometric relations (distances) between two points can be recognized as algebraic relations in the group. All algebraic relations of this kind are given in the following theorem. A proof for the exceptional Chevalley groups can be found in [3].

THEOREM 2.2: *Let* G *be a finite Chevalley group of rank at least* 2 *and not equal to* ${}^2F_4(q)$. *Let* X *and* Y *be the centers of root subgroups of order* q. *Then one of the following holds:*

1. $\langle X, Y \rangle$ *is elementary abelian, and is the union of* $q+1$ *long-root subgroups that pairwise intersect trivially.*
2. $\langle X, Y \rangle$ *is elementary abelian, and the only long-root subgroups it contains are* X *and* Y.
3. $\langle X, Y \rangle$ *is isomorphic to a Sylow subgroup of order* q^3 *in* $\mathrm{SL}_3(q)$ *and* $Z = Z(\langle X, Y, \rangle)$ *has relation 1 both to* X *and* Y.
4. $\langle X, Y \rangle \cong \mathrm{SL}_2(q)$ *(or* $\mathrm{PSL}_2(q)$ *in* $\mathrm{PO}_4^+(q)$*).*

For each of the distance relations between points of the geometry, the corresponding algebraic relation between the root-groups is given in the following table. Note that points X, Y with root group relation 4 are at maximal distance

Root-group relation	Distance in the $F_{4,1}(q)$ geometry	Distance in the $\mathrm{Sp}_{2n}(q)$ geometry
0	0	0
1	1	-
2	2s	1
3	2ns	-
4	3	2

(opposite). The $q+1$ long-root subgroups in $\langle X, Y \rangle$ form what is usually called the *hyperbolic line* spanned by X and Y. The description of the distances in the $F_{4,1}(q)$ geometry was given above; the description for the $\mathrm{Sp}_{2n}(q)$ geometry is given in the next secion.

3. The geometry far from a point in $\mathrm{Sp}_{2n}(q)$

Let Δ be the building of type C_n obtained from the Chevalley group $\mathrm{Sp}_{2n}(q)$ and let its Dynkin diagram be labeled as below. Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the 1-shadow

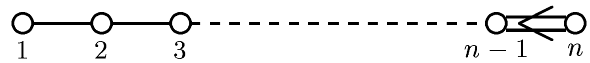

Figure 3: The Dynkin diagram of type C_n

space of Δ (long-root geometry); this is the natural polar geometry associated to the group $\mathrm{Sp}_{2n}(q)$. We will call it the $\mathrm{Sp}_{2n}(q)$ geometry. Fix a point ∞ and let $\Gamma_\infty = (\mathcal{P}_\infty, \mathcal{L}_\infty)$ be the point-line geometry whose point-set comprises all points at maximal distance (distance 2) from ∞ and whose line-set comprises those lines that meet $\mathcal{P}_\infty$ in at least two points. We prove the following result.

Theorem 3.1: *Consider the natural polar geometry Γ for the group $G = \mathrm{Sp}_{2n}(q)$. Fix a point ∞.*

(i) The association scheme S for the group G_∞ acting on the q^{2n-1} points far from ∞ has 3 or 5 classes according as q is even or odd.

(ii) The parameters of S are as depicted in Figures 4 and 5.

The classes C_0, C_1 and $\bar{C}_1$, $\bar{C}_0$ (q even) and $\bar{C}_{1,\mathrm{sq}}$, $\bar{C}_{1,\mathrm{nsq}}$, $\bar{C}_{0,\mathrm{sq}}$, $\bar{C}_{0,\mathrm{nsq}}$ (q odd) of the scheme S are described later in this section. For q odd, by joining the classes $\bar{C}_{i,\mathrm{sq}}$ and $\bar{C}_{i,\mathrm{nsq}}$ into $\bar{C}_i$ ($i = 0, 1$) we obtain the same scheme as for q even.

For $\lambda \in \{0, 1, 2\}$, let d_λ be the collection of pairs of points of Γ at distance λ in the collinearity graph. Given two non-collinear points x and y we can define a *hyperbolic line xy* on them in three ways:

(i) by its natural embedding into $\mathrm{PG}(2n-1, q)$: it is the set of $q+1$ points on the projective line $x \oplus y$.

(ii) geometrically: let $A^\perp$ be the collection of points collinear to every element of the point set A. Then $xy = \{x, y\}^{\perp\perp}$.

(iii) group-theoretically: Given a point x we have a group T_x of transvections $t_x(\lambda) : y \mapsto y + \lambda(y, x)x$ ($\lambda \in \mathbb{F}_q$). Two groups T_x, T_y generate a group $L = \langle T_x, T_y \rangle$ that is isomorphic to $\mathrm{SL}_2(q)$ if and only if x and y are non-collinear; in that case L contains $q+1$ groups T_z (one for each point z on $x \oplus y$) and these points z form xy.

We note that the groups of transvections mentioned in (iii) are precisely the long-root subgroups of the Chevalley group $\mathrm{Sp}_{2n}(q)$. We will use each of these presentations. The subgroup of G generated by the root-groups associated to the points of a point set X will be denoted by $\langle X \rangle$.

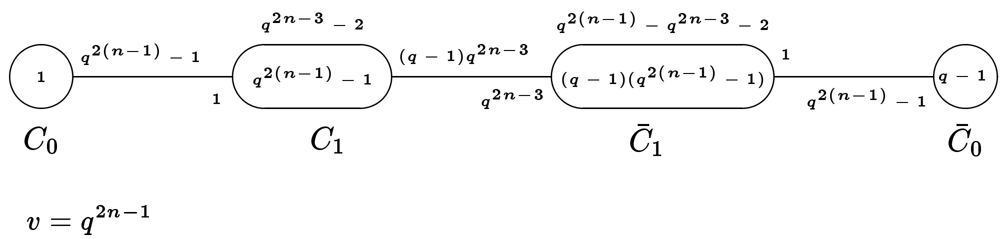

Figure 4: The scheme S for q even.

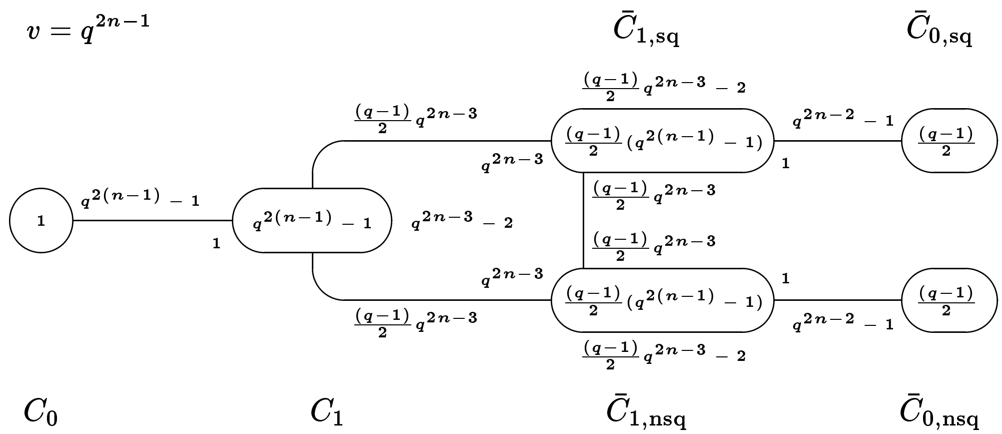

Figure 5: The scheme S for q odd.

Fix a hyperbolic line θ. The stabilizer of θ (or normalizer of $\langle\theta\rangle$) in G has three orbits on the points. We call them X_0, X_1 and $X_{1'}$. In table 1, for each orbit X, $x \in X$ and $\lambda = 0, 1, 2$ we give $|d_\lambda(x) \cap \theta|$ and $|X|$. Recalling that $y \in d_1(x)$,

Table 1: The orbits of the stabilizer of a hyperbolic line in the group $\mathrm{Sp}_{2n}(q)$

orbit	0	1	2	Size of orbit
X_0	1	-	q	$q+1$
X_1	-	1	q	$(q+1)(q^{2n-2}-1)$
$X_{1'}$	-	$q+1$	-	$(q^{2n-2}-1)/(q-1)$

$y \in d_2(x)$ means (i) $(x, y) = 0$ resp. $(x, y) \neq 0$, (in the embedding $\mathrm{PG}(2n-1, q)$, with $(\cdot,\cdot)$ being the symplectic form) (ii) $y \in x^\perp$ resp. $y \notin x^\perp$ (in the geometry) and (iii) $[T_y, T_x] = \{1\}$ resp. $[T_x, T_y] > \{1\}$ (in the group), we can interpret this table accordingly.

Now we are ready to define the classes of the scheme S. For $i = 0, 1$ and $t \in \theta$ let $X_{i,t}$ be the set of points in X_i whose unique closest point on θ is t; a point in $X_{i,t}$ is said to have *position* $[i, t]$ with respect to θ. By transitivity of G_θ on the points of θ we have $|X_{0,t}| = |X_0|/|\theta| = 1$ and $|X_{1,t}| = |X_1|/|\theta| = q^{2n-2} - 1$.

Let

$$C_0(o) = X_{0,o} \ (= \{o\}),$$
$$C_1(o) = X_{1,o},$$
$$\bar{C}_1(o) = \cup_{t\in\theta\setminus\{\infty,o\}} X_{1,t},$$
$$\bar{C}_0(o) = \cup_{t\in\theta\setminus\{\infty,o\}} X_{0,t} \ (= \theta \setminus \{\infty, o\}).$$

For q odd we have $\bar{C}_{i,\epsilon}(o) = \cup_t X_{i,t}$, where t ranges over all 'squares' and 'non-squares' of $\theta \setminus \{\infty\}$ (o corresponding to 0) in the respective cases $\epsilon = \mathrm{sq}, \mathrm{nsq}$. The classes of the schemes are defined as follows: $C_i = \cup_{o\in\mathcal{P}_\infty} \{o\} \times C_i(o)$, $\bar{C}_i = \cup_{o\in\mathcal{P}_\infty} \{o\} \times \bar{C}_i(o)$, $\bar{C}_{i,\epsilon} = \cup_{o\in\mathcal{P}_\infty} \{o\} \times \bar{C}_{i,\epsilon}(o)$, for $i = 0, 1$ and $\epsilon = \mathrm{sq}, \mathrm{nsq}$. Clearly C_i, $\bar{C}_i$ with $i = 0, 1$ are all symmetric. Since G_∞ is transitive on the set $\mathcal{P}_\infty$ these classes are (unions of) G_∞-orbitals.

Proof: (of part i of Theorem 3.1.) We clearly have $d_2(\infty) = \cup_{t\in\theta\setminus\{\infty\}}(X_{0,t} \cup X_{1,t}) = \cup_{i=0,1}(C_i(o) \cup \bar{C}_i(o))$. We first show that $G_{\infty,t,o}$ $(t \in \theta)$ is transitive on the set $X_{1,t}$.

Put $C_\theta = \langle T_u \mid \forall t \in \theta \ (u, t) = 0\rangle$ i.e. the subgroup generated by the long-root subgroups that centralize every long-root subgroup in $\langle\theta\rangle$. This group, which is naturally isomorphic to $\mathrm{Sp}_{2n-2}(q)$, fixes every point of θ and acts transitively on the points of $\theta^\perp$. Thus it acts transitively on the collection of lines containing any given point $t \in \theta$.

Let t be such a point and let l be a line on t. Let η be a hyperbolic line in $\theta^\perp$ (so that $\langle\eta\rangle \le C_\theta$) that meets l in a point u. Then the stabilizer $\langle\eta\rangle_u$ fixes every vector on θ and acts transitively on the vectors of u. Hence $\langle\eta\rangle_u$ is transitive on the points of $l \setminus \{t, u\}$. Thus C_θ is transitive on $X_{1,t}$.

The group G_θ acts as $\mathrm{SL}_2(q)$ on the points of θ. If q is even, this action is sharply 3-transitive, and if q is odd it is 2-transitive and the stabilizer of two points has two orbits on the remaining points.

Thus if q is even, the orbits of $G_{\infty,o}$ on $d_2(\infty)$ are $C_i(o)$, $\bar{C}_i(o)$ $(i = 0, 1)$ and if q is odd, the sets $\bar{C}_i(o)$ $(i = 0, 1)$ split into two orbits $\bar{C}_{i,\mathrm{sq}}(o)$ and $\bar{C}_{i,\mathrm{nsq}}(o)$. □

Finally we consider the lines of Γ_∞. Given a (hyperbolic) line l, a point p is collinear to either one or all points of l. It follows that the feasible distributions of the points of a singular line among the sets of points with position $[0, t]$, $[1, t]$ and $[1']$ are those listed in the table below.

In the following table the DP-entry is the number of points in position P on a line with distribution D. All these distributions occur, except that for $n \le 2$

Distribution	$[0,t]$	$[1,t]$	$[1']$	$\forall t' \in \theta$ $[1,t']$
$[\mathbf{Sp}:\mathbf{i},t]$	1	$q-1$	1	-
$[\mathbf{Sp}:\mathbf{ii}]$	-	-	$q+1$	-
$[\mathbf{Sp}:\mathbf{iii},t]$	-	q	1	-
$[\mathbf{Sp}:\mathbf{iv}]$	-	-	-	1

the distributions [**Sp** : **ii**] and [**Sp** : **iii**, t] do not occur for lack of planes.

In the following two tables the DP-entry is the number of lines with distribution D on a point in position P.

Distribution	[0,t]	[1,t]	[1']
[**Sp** : **i**, t]	$(q^{2(n-1)}-1)/(q-1)$	1	1

Distribution	$[1,t]$	$[1']$
[**Sp** : **ii**]	-	$(q^{2(n-2)}-1)/(q-1)$
[**Sp** : **iii**, t]	$q(q^{2(n-2)}-1)/(q-1)$	$(q^{2(n-2)}-1)$
[**Sp** : **iv**]	q^{2n-3}	-

Now one easily proves part (ii) of Theorem 3.1.

4. The geometry far from a point in $\mathrm{DO}_7(q)$

Let Δ be the building of type B_3 obtained from the Chevalley group $\mathrm{O}_7(q)$. For $i = 1, 2, 3$, its i-elements are the totally singular i-spaces with respect to a non-degenerate quadratic form on a vector space V of dimension 7 defined over the field $\mathbb{F}_q$ where incidence is symmetrized inclusion. Let Γ be the 3-shadow space of Δ; it is the dual polar space associated to $\mathrm{O}_7(q)$. We will call it the $\mathrm{DO}_7(q)$ geometry. In this geometry, for $i = 1, 2, 3$, we refer to the i-elements as 'points', 'lines' and 'quads' of Γ. However, we will use this terminology only in the formulation of the next theorem; In the remainder of the section we will work in the 7-dimensional embedding of the polar space and call the elements of type $i = 1, 2, 3$ *points*, *lines* and *planes* of Δ respectively, as customary.

THEOREM 4.1: *Let Γ be the dual polar space associated to the group $G = \mathrm{O}_7(q)$. Fix a point ∞.*

(i) There exists a 4-class association scheme D on the q^6 points far from the point ∞ whose parameters are as depicted in Figure 6.

(ii) The classes of the scheme D are G_∞-orbits

The classes of D are denoted by C_i $(i = 0, 1, 2\mathrm{g}, 2\mathrm{h}, 3)$ and will be described later in this section.

For $\lambda \in \{0, 1, 2, 3\}$ let d_λ be the set of pairs of planes at distance λ. Any two planes at distance 3 span a subspace of V of dimension 6 on which the quadratic form is non-degenerate. We call such a subspace a 'hyperbolic hyperplane' because the quadratic form induced on it is hyperbolic. The points and planes on it are the elements of the building of type D_3 associated to the group $\mathrm{O}_6^+(q)$. Recall that there are two classes of planes; two planes belong to the same class if and only if the codimension of their intersection in either plane is even. Clearly, two planes that are disjoint must belong to different classes.

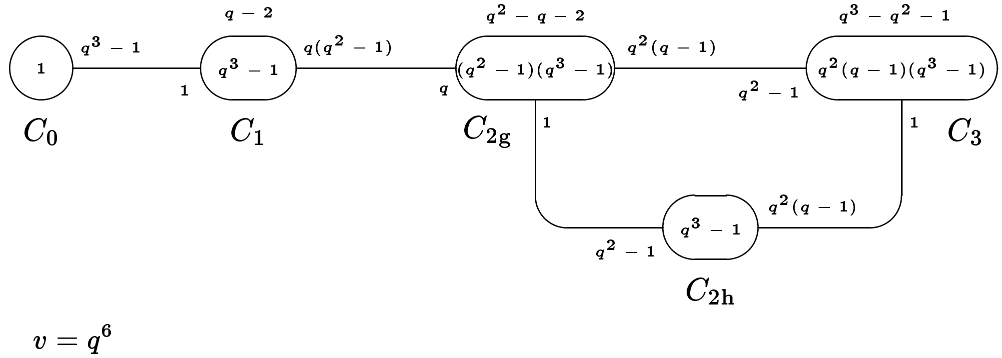

Figure 6: The scheme D.

We will now define what we call the position of a plane with respect to a pair of planes at maximal distance. Fix two disjoint planes O and ∞ and let Θ be the hyperbolic hyperplane on these two planes. Let U be a plane. We say that U has position $[i, j]$, $(i, j = 0, 1, 2, 3)$ with respect to the pair (O, ∞) if $U \in d_i(O)$ and $U \in d_j(\infty)$. Further, if $i + j = 5$ we say that U has position $[i, j]_h$ (resp. $[i, j]_g$) if $U \subseteq \Theta$ (resp. $U \not\subseteq \Theta$). We denote the set of planes in position *pos* by X_{pos}. It is clear that the sets $X_{[i,j]}$ with $i + j = 3, 4, 6$ together with the sets $X_{[2,3]_g}$, $X_{[2,3]_h}$, $X_{[3,2]_g}$, $X_{[3,2]_h}$ partition the set of all planes.

We can define the classes C_i $(i = 0, 1, 2\mathrm{g}, 2\mathrm{h}, 3)$ of the scheme D from the sets $C_i(O)$ in the same way as we did for the $\mathrm{Sp}_{2n}(q)$ geometry. These classes are easily seen to be symmetric.

$$C_0(O) = X_{[0,3]}(O),$$
$$C_1(O) = X_{[1,3]}(O),$$
$$C_{2\mathrm{g}}(O) = X_{[2,3]_g}(O),$$
$$C_{2\mathrm{h}}(O) = X_{[2,3]_h}(O),$$
$$C_3(O) = X_{[3,3]}(O).$$

Next, we describe the action of the point-wise stabilizer $G_{\infty,O}$ of $\{O, \infty\}$ on these classes.

Proof: (of Theorem 4.1 part (ii)) We only consider the action of $G_{\infty,O}$ on the set of planes in $C_3(O)$. The other cases are similar. Let $\perp$ denote the orthogonality relation associated to the quadratic form. Let A be a plane disjoint from ∞ and O. Let Θ be the hyperbolic hyperplane containing ∞ and O. Put $l_A := A \cap \Theta$ and let $p_\infty = l_A^\perp \cap \infty$ and $p_O = l_A^\perp \cap O$. Since G is transitive on pairs of opposite flags $((p_O, O), (p_\infty, \infty))$, every $G_{\infty,O}$ orbit on $C_3(O)$ has a plane A' with $A' \cap \{p_O, p_\infty\}^\perp = A' \cap \Theta$. The geometry of points and lines in $\{p_\infty, p_O\}^\perp$ is the dual of an $\mathrm{Sp}_4(q)$ geometry so we can apply the results from the section on the symplectic geometry here (we can use this to study the $G_{\infty,O}$ orbits on $C_{2\mathrm{g}}(O)$

and $C_{2h}(O)$ as well). Let $l_\infty = p_O^\perp \cap \infty$ and let $l_O = p_\infty^\perp \cap O$. Then the lines l_∞, l_O and l_A are parallel lines inside a grid T ($\subseteq \Theta$). Now G_{l_O,l_∞} acts as $O_3(q)$ on the points of $T^\perp$, namely 3-transitively. Further, the point-wise stabilizer of $T^\perp$ acts as $O_4^+(q)$ on T and hence acts 3-transitively on any set of pairwise parallel lines. In particular, its subgroup stabilizing l_∞ and l_0 is transitive on the lines of T parallel to these lines. Thus $G_{\infty,O}$ is transitive on $C_3(O)$. □

We note that Theorem 4.1 remains valid if we replace $O_7(q)$ by the special orthogonal group $SO_7(q)$.

In the remaining part of this section, for any line of Γ_∞, we will determine the distribution of its point-set among the classes of D.

Lines on O have distribution $[\mathbf{DO} : \mathbf{i}, O]$.

Distribution	$[0,3]$	$[1,3]$	$[1,2]$
$[\mathbf{DO} : \mathbf{i}, O]$	1	$q-1$	1

Let p be a point on O. The geometry of planes and lines on p has type $DO_5(q) \cong Sp_4(q)$. Thus, we can apply the results from Section 3 to this situation. Let ∞_p be the unique plane on p with position $[2,1]$. Let θ be the hyperbolic line $O\infty_p$ of the $Sp_4(q)$ geometry. Then the correspondence between the positions of the planes in the $DO_7(q)$ geometry on p and the points in $Sp_4(q)$ is as follows: where

$DO_7(q)$	$[0,3]$	$[1,3]$	$[1,2]$	$[2,1]$	$[2,2]$	$[2,3]_h$	$[2,3]_g$
$Sp_4(q)$	$[0,O]$	$[1,O]$	$[1']$	$[0,\infty_p]$	$[1,\infty_p]$	$[0,t]$	$[1,t]$

t is some point on θ different from O and ∞_p.

The lines of $DO_7(q)$ correponding to the lines of $Sp_4(q)$ with distribution $[\mathbf{Sp} : \mathbf{ii}, o]$ and $[\mathbf{Sp} : \mathbf{iii}]$ will be said to have distribution $[\mathbf{DO} : \mathbf{ii}, O]$ and $[\mathbf{DO} : \mathbf{iii}]$ respectively.

Let l be a line disjoint from O and ∞. Then either $l \subseteq \Theta$ or l intersects Θ in a point. In the first case l has distribution $[\mathbf{DO} : \mathbf{iv}, O]$. A plane with position $[2,3]_h$ is contained in Θ so each of its q^2 lines missing O has this distribution. A plane x with position $[3,3]$ lies on one such line: it is $x \cap \Theta$.

Distribution	$[2,2]$	$[2,3]_g$	$[2,3]_h$	$[3,3]$	$[3,2]_h$	$[3,2]_g$
$[\mathbf{DO} : \mathbf{iv}, O]$	-	-	1	$q-1$	1	-
$[\mathbf{DO} : \mathbf{v}, O]$	-	1	-	$q-1$	-	1
$[\mathbf{DO} : \mathbf{vi}, O]$	1	-	-	q	-	-

If l meets Θ in a point there are two possibilities. Consider the planes on l that meet $O \cup \infty$. There are either two or one of these. We then get the distributions $[\mathbf{DO} : \mathbf{v}, O]$ and $[\mathbf{DO} : \mathbf{vi}, O]$. The q^2 lines on a plane with position $[2,3]_g$ that do not meet O necessarily have distribution $[\mathbf{DO} : \mathbf{v}, O]$.

Now consider a plane x with position $[3,3]$. Put $l_x = x \cap \Theta$. Then $T = (l_x^\perp \cap O, l_x^\perp \cap \infty)^\perp \cap \Theta$ is a grid that meets O and ∞ in lines l_O and l_∞. Thus on each point a of l_x there is a line m that meets both O and ∞. For each such point a, this line m is unique, otherwise we would find a singular subspace in Θ that meets both O and ∞ in a line. Lines like m are contained in Θ and so no point of $x \setminus l_x$ is contained in such a line.

Consider the unique point $c = T^\perp \cap x$. Clearly this point is coplanar to all lines in T so in particular it is contained in $q+1$ planes with position $[2,2]$ meeting x in a line. If a plane has position $[2,2]$ and contains the point a of l_x, then it must lie on m. Hence, if there is more than one line with distribution $[\mathbf{DO} : \mathbf{vi}]$ on x meeting a, then we find a singular 4-space, a contradiction. Thus we find precisely $q+1$ lines with distribution $[\mathbf{DO} : \mathbf{vi}]$ and the remaining q^2-1 lines must have distribution $[\mathbf{DO} : \mathbf{v}]$.

For later reference we will say that points on a plane with position $[3,3]$ such as a have type (a), points such as c have type (c) and the remaining points on the plane have type (b).

Now one can compute for any two classes of the scheme D how many neighbours a plane of one class has in the other class. After that, some double counting yields the sizes of the classes.

5. The points far from a point in the $F_4(q)$ geometry

In the remainder of this paper q will be even. In this section we will define the classes of the scheme A of Theorem 1.1. Let G be the Chevalley group $F_4(q)$. Let $o \in \mathcal{P}_\infty$. Then o and ∞ determine a *hyperbolic line* θ. Let $N = N(\langle\theta\rangle)$ and $C = C(\langle\theta\rangle)$ be the normalizer and centralizer respectively of the subgroup $\langle\theta\rangle$ in G.

The relations of A will be constructed from the orbits of points under the action of N. These orbits have been studied in detail by Cooperstein [3]. In table 2, for each orbit X, $x \in X$ and $\lambda = 0,1,2,3,4$ we give $|d_\lambda(x) \cap \theta|$ and $|X|$.

Table 2: The orbits of the stabilizer of a hyperbolic line in the group $F_4(q)$ (q even)

orbit	0	1	2	3	4	size of orbit
X_0	1	-	-	-	q	$q+1$
X_1	-	1	-	-	q	$(q-1)(q+1)^2(q^2+1)(q^3+1)$
$X_{1'}$	-	1	-	q	-	$(q+1)^2(q^2+1)^2(q^3+1)$
$X_{2\mathrm{h}}$	-	-	1	-	q	$(q+1)(q^6-1)$
$X_{2\mathrm{g}}$	-	-	1	-	q	$(q+1)(q^4-1)(q^6-1)$
$X_{2'}$	-	-	1	q	-	$(q+1)^2(q^2+1)^2(q^6-1)$
$X_{2''}$	-	-	$q+1$	-	-	$(q^6-1)/(q-1)$
X_3	-	-	-	1	q	$q^3(q+1)(q^4-1)(q^6-1)$
$X_{3'}$	-	-	-	2	$q-1$	$q^7(q+1)(q^4-1)(q^3+1)/2$
$X_{3''}$	-	-	-	$q+1$	-	$q^3(q^6-1)(q^2+1)(q+1)$
X_4	-	-	-	-	$q+1$	$q^7(q-1)(q^4-1)(q^3-1)/2$

We say that a point of X_i ($i \in \{0, 1, 1', 2\text{h}, 2\text{g}, 2', 2'', 3, 3', 3'', 4\}$) is *in position* $[i]$ (with respect to θ). We often refine this by specifying (some of) the points on θ that are closest to it. Thus, for instance, a point x is in position $[1, t]$ if $d_1(x) \cap \theta = \{t\}$ and $\theta \setminus d_1(x) \subset d_4(x)$ and it is in position $[3', t_1, t_2]$ if $d_3(x) \cap \theta = \{t_1, t_2\}$ and $\theta \setminus d_3(x) \subset d_4(x)$. Further a point x is in position $[3', t]$ if it is in position $[3']$ and $t \in d_3(x) \cap \theta$. We denote the set of points in position *pos* by X_{pos}.

Clearly we have the following partitions:

$$\begin{aligned} X_i &= \bigcup_{t\in\theta} X_{i,t} \qquad (i = 0, 1, 1', 2, 2', 3) \\ X_{3'} &= \bigcup_{t_1,t_2} X_{3',t_1,t_2} \end{aligned}$$

Since the group $\langle\theta\rangle$ ($\cong SL_2(q)$) is 2-transitive on the $q + 1$ points of the hyperbolic line θ we have $(q + 1)|X_i(t)| = |X_i|$ ($t \in \theta$, $i = 0, 1, 1', 2, 2', 3$) and $\binom{q+1}{2}|X_{3'}(t_1, t_2)| = |X_{3'}|$ ($t_1, t_2 \in \theta$).

Let us point out the difference between the sets $X_{2\text{h},t}$ and $X_{2\text{g},t}$ ($t \in \theta$). Fix $t \in \theta$ and let $y \in X_{2\text{g},t} \cup X_{2\text{h},t} \subseteq d_2(t)$. Then there is a unique symplecton S containing t and y. Let $s = \text{proj}_S(t')$ for some $t' \in \theta$. Then there is no other possibility but that s has position $[2'']$ so that $s = \text{proj}_S(t')$ for every $t' \in \theta \setminus \{t\}$. Now the position of any other point in S is entirely determined by its position with respect to t and s (cf. Lemma 2.1).

Since S is a geometry of type $\mathrm{O}_7(q)$ with q even, we can also view it as a geometry of type $\mathrm{Sp}_6(q)$. Let st be the symplectic hyperbolic line on s and t. Then $X_{2\text{h},t} \cap S = st \setminus \{s, t\}$ and $X_{2\text{g},t} \cap S = X_{2,t} \cap S \setminus X_{2\text{h},t}$ ('h' for 'hyperbolic', 'g' for 'generic'). We have $|X_{2\text{h},t} \cap S| = q - 1$ and $|X_{2\text{g},t} \cap S| = (q^4 - 1)(q - 1)$. Since there are $(q^6 - 1)/(q - 1)$ symplecta on t we find $|X_{2\text{g},t}| = (q^4 - 1)(q^6 - 1)$ and $|X_{2\text{h},t}| = (q^6 - 1)$ (cf. Table 2).

The classes of A are denoted by C_i, $\bar{C}_i$ ($i = 0$, 1, 2g, 2h, 3, $3'$, 4) and are defined from the sets $C_i(o)$ in the same way as we did for the $\mathrm{Sp}_{2n}(q)$ geometry. Again these classes are symmetric.

$$\begin{aligned} C_i(o) &= X_{i,o} && (i = 0, 1, 2\text{g}, 2\text{h}, 3) \\ \bar{C}_i(o) &= \bigcup_{t\in\theta-\{o,\infty\}} X_{i,t} && (i = 0, 1, 2\text{g}, 2\text{h}, 3) \\ C_{3'}(o) &= \bigcup_{t\in\theta-\{o,\infty\}} X_{3',o,t} \\ \bar{C}_{3'}(o) &= \bigcup_{t_1,t_2\in\theta-\{o,\infty\}} X_{3',t_1,t_2} \\ C_4(o) &= X_4. \end{aligned}$$

Note that in case $q = 2$, the set $\bar{C}_{3'}$ is empty.

6. The lines far from a point in the $F_4(q)$ geometry

Let θ be the hyperbolic line spanned by ∞ and o. The *distribution* of a line L will be the distribution of its point set among the sets X_{pos}, where $[pos]$ is one of $[0, t]$, $[1, t]$, $[1', t]$, $[2\text{h}, t]$, $[2\text{g}, t]$, $[2', t]$, $[2'']$, $[3, t]$, $[3', t_1, t_2]$, $[3'']$, $[4]$, for certain $t, t_1, t_2 \in \theta$.

In this section we determine all distributions that occur among the lines that contain a point at maximal distance (d_4) from some point of θ. Moreover, for any point x at maximal distance from some point of θ and for any distribution, we determine the number of lines on x having that distribution.

First we explain the strategy we will follow. Let O be a line, plane or symplecton and let $O_t = \mathrm{proj}_O(t)$ for every $t \in \theta$. If we know the distance between t and O_t and we also know the mutual arrangement of the projections O_t, then, in view of Lemma 2.1, we can determine the position of all points on O and the distribution of all lines on O simply by considering their distance to the projections O_t. For this we only need some knowledge of the geometry O.

Since we know the geometry of O very well, we will often be content with determining the distance between t and O_t and the arrangement of the projections O_t relative to one another only.

6.1. Lines inside a symplecton meeting θ

Let S be a symplecton meeting θ in a point t. Let $s = \mathrm{proj}_S(t')$ for some $t' \in \theta \setminus \{t\}$. Then s has position $[2'']$ and hence $s = \mathrm{proj}_S(t')$ for all $t' \in \theta \setminus \{t\}$. Using the fact that q is even, we view S as the symplectic geometry $\mathrm{Sp}_6(q)$ and observe that we can determine the distribution of any line in S by looking at its position with respect to the symplectic hyperbolic line spanned by s and t.

In the following table the DP-entry is the number of points in position P on a line with distribution D.

Distribution	$[0,t]$	$[1,t]$	$[1',t]$	$[2\mathrm{g},t]$	$[2\mathrm{h},t]$	$[2',t]$	$[2'']$
$[\mathbf{i},t]$	1	$q-1$	1	-	-	-	-
$[\mathbf{ii},t]$	-	-	$q+1$	-	-	-	-
$[\mathbf{iii},t]$	-	q	1	-	-	-	-
$[\mathbf{iv},t]$	-	-	1	$q-1$	1	-	-
$[\mathbf{v},t]$	-	-	1	q	-	-	-
$[\mathbf{vi},t]$	-	1	-	$q-1$	-	1	-
$[\mathbf{vii},t]$	-	-	1	-	-	$q-1$	1
$[\mathbf{viii},t]$	-	-	1	-	-	q	-

In the following two tables the DP-entry is the number of lines with distribution D on a point in position P.

Distribution	$[0,t]$
$[\mathbf{i},t]$	$(q^3+1)(q^2+1)(q+1)$

6.2. Lines with a point at distance 1 from θ

Let $x \in d_1(t)$ for some $t \in \theta$. We consider lines L on x that are not contained in a symplecton on t.

Distribution	$[1,t]$	$[2\mathrm{g},t]$	$[2\mathrm{h},t]$
$[\mathbf{i},t]$	1	-	-
$[\mathbf{iii},t]$	$q(q^3-1)/(q-1)$	-	-
$[\mathbf{iv},t]$	-	1	$(q^4-1)/(q-1)$
$[\mathbf{v},t]$	-	$q(q+1)$	-
$[\mathbf{vi},t]$	1	q^3	-

LEMMA 6.1: *Let L be a line on $x \in d_1(t)$ for some $t \in \theta$ that is not contained in a symplecton on t. Then $L \setminus \{x\}$ is contained in one of $X_{1'}$, $X_{2'}$, $X_{3''}$, $X_{3,t}$ or $X_{3',t}$ and the latter holds if and only if $x \in X_{1,t}$.*

Proof: The group $\langle t \rangle$ fixes t and x and acts sharply 1-transitively on the points of both $\theta \setminus \{t\}$ and $L \setminus \{x\}$ (see Cooperstein [3]). This proves the first part. If $x \in X_{1,t}$, then $x \in d_4(t')$ for every $t' \in \theta$. This means that we can only have $L \setminus \{x\} \subset X_{3,t} \cup X_{3',t}$. Let $y \in L \setminus \{x\}$ such that $\theta \subset d_3(y) \cup d_4(y)$. Clearly $\{t\} \subseteq d_3(y) \cap \theta$, but if we have equality, then the action of $\langle t \rangle$ shows that $L \subset d_4(t')$ for all $t' \in \theta \setminus \{t\}$. This contradiction implies $y \in X_{3',t}$.

Conversely, if $y \in X_{3',t}$, then there exists $t' \in \theta$ with $y \in d_4(t') \cap L$. By transitivity the unique point of $d_3(t') \cap L$ is in the $\langle t \rangle$ orbit of y. Hence $x \in d_4(t')$ and we are done. □

We find the following (possible) distributions for lines:

Distribution			$\forall t' \neq t$	$\forall t' \neq t$		$\forall t' \neq t$	
	$[1,t]$	$[1',t]$	$[1',t']$	$[2',t']$	$[3,t]$	$[3',t,t']$	$[3'']$
$[\mathbf{ix}]$	-	1	1	-	-	-	-
$[\mathbf{x},t]$	-	1	-	1	-	-	-
$[\mathbf{xi},t]$	-	1	-	-	-	-	q
$[\mathbf{xii},t]$	-	1	-	-	q	-	-
$[\mathbf{xiii},t]$	1	-	-	-	-	1	-

A point $y \in d_3(t)$ is collinear to precisely one point of $d_1(t)$. Also, a point $x \in X_{1,t}$ lies on q^6 lines not contained in a symplecton on t. In the following table the DP-entry is the number of lines with distribution D on a point in position P.

Distribution	$[1,t]$	$[3,t]$	$[3',t,t']$
$[\mathbf{xii},t]$	-	1	-
$[\mathbf{xiii},t]$	q^6	-	1

6.3. Lines with a point at distance 2 from θ

Let $x \in d_2(t)$ for some $t \in \theta$. Let S be the symplecton on t and x and let L be a line on x not contained in S. Let V be the unique plane on L that meets S in a line and call this line M. We assume that $V \setminus M$ contains a point y in

$d_4(t')$ for some $t' \in \theta$. Let $z \in M$ be the unique point of $d_1(t) \cap V$. If $z \in X_{1',t}$, then it follows from Lemma 6.1 that $V \setminus M \subseteq X_{3''} \cup X_{3,t}$. By assumption then $V \cap X_{3,t} \neq \emptyset$. Hence $d_3(t') \cap V$ is a line for every $t' \in \theta \setminus \{t\}$. In fact the lines $d_3(t') \cap V \subset d_3(t)$ all coincide; call this line M'. If M has distribution $[\mathbf{viii}, t]$ then $M' = M$. Otherwise it has distribution $[\mathbf{xi}, t]$. In the former case we only find lines with distribution $[\mathbf{xvi}, t]$ on V and in the latter case we find lines with distribution $[\mathbf{xiv}, t]$ or $[\mathbf{xv}, t]$.

Distribution	$[2', t]$	$[2\mathrm{g}, t]$	$[2\mathrm{h}, t]$	$[3, t]$	$[3'']$
$[\mathbf{xiv}, t]$	-	-	1	$q-1$	1
$[\mathbf{xv}, t]$	-	1	-	$q-1$	1
$[\mathbf{xvi}, t]$	1	-	-	q	-

Suppose $z \in X_{1,t}$. Then M has distribution $[\mathbf{vi}, t]$ (because of the point y). It follows from Lemma 6.1 that $V \setminus M \subset X_{3',t}$. Hence the lines $M_{t'} = d_3(t') \cap V$ with $t' \in \theta \setminus \{t\}$ are distinct and have distribution $[\mathbf{xvii}, t, t']$. The remaining lines have distribution $[\mathbf{xviii}, t]$ (or $[\mathbf{xiii}, t]$).

				$\forall t' \neq t$
Distribution	$[2', t]$	$[2\mathrm{g}, t]$	$[3', t, t']$	$[3', t, t']$
$[\mathbf{xvii}, t, t']$	1	-	q	-
$[\mathbf{xviii}, t]$	-	1	-	1

As for the number of lines L with a certain distribution on $x \in d_2(t)$, we note that L uniquely determines M and the distribution of M. On the other hand M lies on q^2 planes V outside S and we know how many lines on x and V have a certain distribution. Thus by a double count of the lines M and L on x with the appropriate distribution we can see that a point of $X_{2\mathrm{h},t}$ lies on $q^3(q^4-1)/(q-1)$ lines with distribution $[\mathbf{xiv}, t]$ and that a point of $X_{2\mathrm{g},t}$ lies on $q^3(q^3-1)/(q-1)$ lines with distribution $[\mathbf{xv}, t]$ and on q^6 lines with distribution $[\mathbf{xviii}, t]$.

The following lemma is included for use in Subsection 6.7.

LEMMA 6.2: *Let $x \in X_{2''}$. Then the collection of symplecta on x meeting a point of θ forms a (symplectic) hyperbolic line of the $\mathrm{Sp}_6(q)$ geometry that is the residue of x. In particular, every symplecton S on x satisfies $d_4(t) \cap S = \emptyset$ for one or all $t \in \theta$.*

Proof: The residue of x is a geometry of type $\mathrm{Sp}_6(q)$. For $t \in \theta$ let S_t be the unique symplecton on x and t.

Suppose S is a symplecton containing planes V_i, $(i = 1, 2)$ such that $V_i = S \cap S_{t_i}$ for certain $t_i \in \theta$ $(i = 1, 2)$. Let $L_i \subset V_i$ be the line contained in $d_1(t_i)$. These lines span a grid in which every line parallel to L_i has distribution $[\mathbf{ii}, t]$ for some

$t \in \theta$. Thus for every $t \in \theta$, both S and S_t contain the plane on x and the line with distribution $[\mathbf{ii}, t]$. This proves the first part of the lemma.

As a consequence, a symplecton on x has a plane in common with S_t for one or all $t \in \theta$. □

6.4. Lines with a point at distance $3'$ from θ.

Let $x \in X_{3',t_1,t_2}$ for certain $t_1, t_2 \in \theta$. Let z_i $(i = 1, 2)$ be the unique point in $d_1(x) \cap X_{1,t_i}$ (see Lemma 6.1). We consider all lines L on x that do not contain a point of X_4.

LEMMA 6.3: *Given a point $t \in \theta$ let $x \in d_3(t)$, let z be the point in $d_1(t) \cap d_1(x)$ and let L be a line on x. Then $L \cap d_4(t) = \emptyset$ if and only if L is contained in a symplecton on z; otherwise $L \cap d_4(t) = L \setminus \{x\}$.*

Proof: Let A be an apartment on t and the flag (x, L, S) for some symplecton S. Then A also contains the unique point $z \in d_1(t) \cap d_1(x)$ because $zx = \mathrm{proj}_x(t)$ and $z = \mathrm{proj}_{zx}(t)$. We see that we could have chosen S on z if and only if $L \cap d_4(t) = \emptyset$. Clearly, if $d_4(t) \cap L \neq \emptyset$ then x is the unique point of $L \cap d_3(t)$. □

We claim that L is contained in a symplecton S on x and z_1 or z_2. By Lemma 6.3, if L has a point in X_4, then L is not contained in a symplecton on z_i $(i = 1, 2)$. Conversely, if L is not contained in a symplecton on z_1 or z_2, we find that $L \setminus \{x\}$ has a unique point in $d_3(t)$ for every $t \in \theta \setminus \{t_1, t_2\}$, hence it has at least one point in X_4.

By symmetry assume that S is a symplecton on L and z_1. In view of Lemma 2.1, we first want to locate the projections of $t \in \theta$ on S. Since xz_1 has distribution $[\mathbf{xiii}, t_1]$ we have $\emptyset \neq d_4(t) \cap L \subset S$ for every $t \in \theta \setminus \{t_1\}$ and $\emptyset \neq d_1(t_1) \cap S$ $(t_1 \notin S)$. Hence $\pi(t) = d_2(t) \cap S$ is a point for every $t \in \theta \setminus \{t_1\}$ and $M = d_1(t_1) \cap S$ is a line with distribution $[\mathbf{iii}, t_1]$. The unique point p on M with position $[1', t_1]$ is collinear with the point $\pi(t)$ for every $t \in \theta$. Let Π be the line on p and $\pi(t)$, for some $t \in \theta$. Then Π has distribution $[\mathbf{x}, t_1]$ and so its point set is $\{p, \pi(t) \mid t \in \theta \setminus \{t_1\}\}$.

Now using Lemma 2.1 we can easily determine the distribution of any line in S. We will simply list the results.

On a plane containing xz_1 we find a line with distribution $[\mathbf{vi}, t_1]$, hence the plane is described in Subsection 6.3. On x there is one line with distribution $[\mathbf{xvii}, t_1, t_2]$, namely $d_3(t_2) \cap V$, and the remaining $q - 1$ lines have distribution $[\mathbf{xviii}, t_1]$.

A line with distribution $[\mathbf{xviii}, t_1]$ or $[\mathbf{xvii}, t_1, t]$ $(t \in \theta \setminus \{t_1\})$ is contained in a unique plane on xz_1. Hence on $x \in X_{3',t_1,t_2}$ there are $(q^3 - 1)/(q - 1)$ lines with distribution $[\mathbf{xvii}, t_1, t_2]$ and $q^3 - 1$ lines with distribution $[\mathbf{xviii}, t_1]$.

Let V be a plane of S on Π. The line $d_1(z_1) \cap V$ is contained in $d_2(t_1)$ and has distribution $[\mathbf{viii}, t_1]$. We have $V \setminus (d_2(t_1) \cup \Pi) \subset X_{3''}$. Hence we find lines on V with distribution $[\mathbf{xix}, t_1, t']$.

Distribution	$[2',t]$	$[2',t']$	$[3'']$	$[3',t,t']$	$\forall t'\neq t$ $[3',t,t']$	$[3,t]$
$[\mathbf{xix},t,t']$	1	1	$q-1$	-	-	-
$[\mathbf{xx},t,t']$	-	-	1	q	-	-
$[\mathbf{xxi},t]$	-	-	-	-	1	1

Let V be a plane on $\pi(t_2)$ and x. The line on $\pi(t_2)$ and x has distribution $[\mathbf{xvii},t_2,t_1]$. The line $d_1(p)\cap V$ lies on $\pi(t_2)$ and has distribution $[\mathbf{xix},t_1,t_2]$. We have $V\setminus d_1(p)\subset X_{3',t_1,t_2}$. Thus in V and on x we find $q-1$ lines with distribution $[\mathbf{xx},t_1,t_2]$.

If L is a line on x not coplanar to z_1 or $\pi(t_2)$, then it lies opposite to Π (in S) and we see that it has distribution $[\mathbf{xxi},t_1]$.

A line L with distribution $[\mathbf{xx},t]$ or $[\mathbf{xxi},t]$ lies in a unique symplecton on xz_1. Inside such a symplecton we know how many lines on x have that distribution. Using this and a double count we find that on $x\in X_{3',t_1,t_2}$ there are $(q^3-1)(q+1)$ lines with distribution $[\mathbf{xx},t_1,t_2]$ and $q^2(q^3-1)$ lines with distribution $[\mathbf{xxi},t_1]$.

The lines on x containing a point of X_4 will be treated in Subsections 6.6 and 6.7.

6.5. Lines with a point at distance 3 from θ.

Let $x\in X_{3,t}$ for some $t\in\theta$. We consider all lines L on x that do not contain a point from X_4.

Firstly, there is precisely one line on x containing a point of $d_1(t)$ and it has distribution $[\mathbf{xii},t]$.

We next consider a line L on x that contains a point $y\in d_2(t)$. In Subsection 6.3 we have seen that L has distribution $[\mathbf{xiv},t]$, $[\mathbf{xv},t]$ or $[\mathbf{xvi},t]$. We determine how often each distribution occurs among the lines on x.

Let z be the unique point in $d_1(x)\cap d_1(t)$. Then these lines L lie in a plane V on the line xz. Let M be the unique line of V that is contained in some symplecton S on zt (there is only one such S). Then the distribution of M (which is $[\mathbf{iv},t]$, $[\mathbf{v},t]$ or $[\mathbf{viii},t]$) determines the distribution of the lines in V as described in Subsections 6.2 and 6.3.

Let E be a line with distribution $[\mathbf{ix}]$ on z (in fact it is unique). Let $M_x=M$ and let M_E be the line on S that is coplanar to E. We will see that the distribution of M_E determines that of M_x.

Regarding the geometry of symplecta, planes and lines on z as a geometry of type $Sp_6(q)$, call these objects Points, Lines and Planes respectively. Furthermore two objects sharing a Line will be called co-Linear. Using that q is even, we regard this geometry as a $\mathrm{DO}_7(q)$ geometry and apply the results of Section 4. The Planes E, zt and zx are pairwise disjoint. Then we know from Section 4 that the Point S has one of three possible positions: (a) The Planes M_x and M_E share a Line that meets zt (in S), zx and E in a Point. (b) The Plane M_x is not co-Linear to M_E nor is it co-Linear to all Planes that are co-Linear to zt and

M_E. (c) The Plane M_x is not co-Linear to M_E but is co-Linear to all Planes that are co-Linear to zt and M_E.

Now we note that M_E being coplanar to E contains the unique point s of S in position $[2'']$. It follows that M_E has distribution $[\mathbf{vii}, t]$. We note that we can characterize case (c) (resp. (b)) by the fact that they occur precisely when M_x has (no) point on the symplectic hyperbolic line ts of S.

Thus in case (a) M_x has distribution $[\mathbf{viii}, t]$; in case (b) M_x has distribution $[\mathbf{v}, t]$; in case (c) M_x has distribution $[\mathbf{iv}, t]$.

In Section 4 we have seen that case (a), (b) and (c) occur $q+1$, q^2-1 and 1 time(s) respectively. Hence on x, in total, there are $(q+1)q$ lines with distribution $[\mathbf{xvi}, t]$, there are $(q^2-1)q+(q-1)=q^3-1$ lines with distribution $[\mathbf{xv}, t]$ and there is 1 line with distribution $[\mathbf{xiv}, t]$ with respect to θ and t.

Let us finally consider a line L on x that has a point in $d_3(t)$ but none in $d_2(t)$. Then L is contained in $d_3(t)$. By considering an apartment on the point t and the flag (x, L) we see that L is contained in a unique symplecton S on zx (remember z is the unique point in $d_1(t)\cap d_1(x)$), but not in a plane on zx (cf. Lemma 6.3).

Let M_t and M_E be the lines on S that are coplanar to zt and E respectively. Then M_E has distribution $[\mathbf{vii}, t]$ or $[\mathbf{x}, t]$. For every $t' \in \theta\setminus\{t\}$, $\mathrm{proj}_S(t') = d_2(t')\cap S$ is a point because $\emptyset \neq d_4(t')\cap L \subset S$. Further, $t \notin S$ but $d_1(t)\cap S \neq \emptyset$ and $\mathrm{proj}_S(t) = M_t$. Since L is opposite to M_E (in S) the distribution of M_E determines that of L by projection. It follows that if M_E has distribution $[\mathbf{x}, t]$, then L has distribution $[\mathbf{xxi}, t]$. Moreover, if M_E has distribution $[\mathbf{vii}, t]$, then L has distribution $[\mathbf{xxii}, t]$.

Distribution	$[3'']$	$[3, t]$
$[\mathbf{xxii}, t]$	1	q

There are essentially two cases for the configuration formed by the lines E, zt and the symplecton on zx, corresponding to case (a) and cases (b),(c) of Section 4 respectively. In case (a) M_E and M_t lie in a common plane V of S. Thus M_E shares a symplecton with zt and has distribution $[\mathbf{vii}, t]$. In case (b-c) M_E is not coplanar to M_t. Thus M_E is not contained in a symplecton on zt and hence has distribution $[\mathbf{x}, t]$.

In each symplecton on zx there are q^3 lines on x that are not coplanar to zt. Hence on x, in total (cf. again Section 4), there are $(q+1)q^3$ lines with distribution $[\mathbf{xxii}, t]$, and there are q^5 lines with distribution $[\mathbf{xxi}, t]$.

6.6. Somewhat far planes

A plane is *somewhat far* with respect to θ if, and only if for every $t\in\theta$ the plane contains a point of $d_4(t)$ and has no points in $\cap_{t\in\theta} d_3(t)$ (that is, with position $[3'']$).

Suppose that V is a somewhat far plane. Clearly, for every $t \in \theta$ the set $L_t = \mathrm{proj}_V(t) = d_3(t) \cap V$ is a line and no three of these lines lie on a common point. In particular no two of these lines coincide, so that V has one point with position $[3', t_1, t_2]$ for every pair $t_1, t_2 \in \theta$ ($\binom{q+1}{2}$ in total), one point with position $[3, t_1]$ for every $t_1 \in \theta$ ($q+1$ in total) and $\binom{q}{2}$ remaining points which have position $[4]$.

Let us look at the lines on V. Clearly, the line L_t has distribution $[\mathbf{xxi}, t]$. The collection of lines $\mathcal{O} = \{L_t \mid t \in \theta\}$ is an ovoid or $(q+1)$-arc in the dual of the projective plane V. Since q is even it follows from Theorem 3 in §1.3 of Thas [7] (see also Segre [5] and Thas [6]) that there is a line N such that $\mathcal{O} \cup \{N\}$ is a hyperoval of the dual of the projective plane V. The line N meets the lines in $\mathcal{O}$ in $q+1$ distinct points and hence has distribution $[\mathbf{xxiv}]$. Any other line L on V has the property that each of its points lies on either 0 or 2 lines in $\mathcal{O} \cup \{N\}$. Hence L has distribution $[\mathbf{xxiii}, t, \pi]$, where π is a partition of $\theta \setminus \{t\}$.

	$\forall \{t_1, t_2\} \in \pi$		$\forall t \in \theta$	
Distribution	$[3', t_1, t_2]$	$[3, t]$	$[3, t]$	$[4]$
$[\mathbf{xxiii}, t, \pi]$	1	1	-	$q/2$
$[\mathbf{xxiv}]$	-	-	1	-

In the sequel of this subsection we will determine, for any point in $X_3 \cup X_{3'}$, the number of neighbours in X_{3',t_1,t_2}, X_{3,t_1} ($t_1, t_2 \in \theta$) and X_4 using these somewhat far planes.

Let $x \in X_{3,t}$ for some $t \in \theta$ and let L be a line with distribution $[\mathbf{xxi}, t]$ on x. We first determine the number of somewhat far planes on L. Let z be the unique point in $d_1(x) \cap d_1(t)$. Then L and zx are contained in a unique symplecton S. Let V be a plane on L. Clearly, for $t' \in \theta \setminus \{t\}$ we have $x \in d_4(t') \cap V$. Further, V has a point in $d_4(t)$ if and only if $V \not\subseteq S$ and in that case we have $X_{3''} \cap V \subseteq X_{3''} \cap L_t = \emptyset$ so that V is somewhat far.

Thus a line with distribution $[\mathbf{xxi}, t]$ is contained in q^2 planes that are somewhat far from θ. We saw in Subsection 6.5 that x lies on q^5 lines with distribution $[\mathbf{xxi}, t]$ and so x lies on q^7 somewhat far planes.

Now let $y \in d_4(t)$ be a point collinear to x. We determine how many somewhat far planes lie on the line xy. Any symplecton S' on zx contains a unique line L' that lies on a plane V' with y. The line L' has distribution $[\mathbf{xxi}, t]$ or $[\mathbf{xxii}, t]$ (see Subsection 6.5) and as V already contains $y \in d_4(t)$, this plane is somewhat far precisely in the former case.

Since L' is a line of S' that is not coplanar to xz we conclude from the end of Subsection 6.5 (case (b-c)) that there are q^2 symplecta S' such that L' has distribution $[\mathbf{xxi}, t]$. Hence there are q^2 planes on x and y that are somewhat far.

It follows that $x \in X_{3,t}$ ($t \in \theta$) is collinear to $q^7/q^2 = q^5$ points with position

$[3, t_1]$ or $[3', t_1, t_2]$ for given $t_1, t_2 \in \theta \setminus \{t\}$, and is collinear to $q^7 \cdot (q(q-1)/2)/q^2 = q^5(q(q-1)/2)$ points with position [4].

One can now compute the number of lines with distribution $[\mathbf{xxiii}, t]$ and $[\mathbf{xxiv}]$ on $x \in X_{3,t}$.

Now let $x \in X_{3',t_1,t_2}$ for certain $t_1, t_2 \in \theta$. We have seen in Subsection 6.4 that x lies on $q^2(q^3-1)$ lines with distribution $[\mathbf{xxi}, t_i]$ $(i = 1, 2)$. Hence x lies on $q^4(q^3-1)$ planes that are somewhat far from θ (see above).

Now let $y \in d_4(t_1) \cap d_4(t_2)$ be a point collinear to x. The line L on x and y satisfies $d_4(t) \cap L \neq \emptyset$ for all $t \in \theta$ so that $d_3(t) \cap L$ is a single point.

We determine how many planes V' on L are somewhat far from θ. For this it remains to determine how many planes V' on L contain a point in position $[3'']$. As L contains a point in $d_4(t)$ for every $t \in \theta$, the set $L'_t = d_3(t) \cap V'$ is a line of V' for every $t \in \theta$. Now V' contains a point with position $[3'']$ if and only if the lines L_{t_1} and L_{t_2} coincide.

Let L_i $(i = 1, 2)$ be the line on x and z_i. The line L'_{t_i} $(i = 1, 2)$ is the unique line of V' that is contained in a symplecton on L_i. By considering again the dual of the residue of x, which is a geometry of type $\mathrm{O}_7(q)$, and using the results of Section 4, we see that there are $q+1$ planes on L that contain a line that is contained in a symplecton both on L_1 and L_2, that is where $L_{t_1} = L_{t_2}$; these were the $q+1$ Lines on the unique Point of type (c) (see the end of Section 4). The remaining q^2 planes on x and y are apparently somewhat far from θ.

Hence the point $x \in X_{3',t_1,t_2}$ is collinear to $q^4(q^3-1) \cdot 1/q^2 = q^2(q^3-1)$ points in either one of the positions $[3, t_3]$, $[3', t_3, t_4]$, with $t_3, t_4 \in \theta \setminus \{t_1, t_2\}$. Moreover $x \in X_{3',t_1,t_2}$ is collinear to $q^4(q^3-1)(q(q-1)/2)/q^2 = q^2(q^3-1)(q(q-1)/2)$ points with position [4].

One can now compute the number of lines with distribution $[\mathbf{xxiii}, t]$ and $[\mathbf{xxiv}]$ on $x \in X_{3',t_1,t_2}$.

6.7. Somewhat far symplecta

We say that a symplecton is *somewhat far* from the hyperbolic line θ if it contains a point with position [4]. As a side remark, we note that it is easy to see that a symplecton is somewhat far if and only if it contains a somewhat far plane.

LEMMA 6.4: *Assume that q is even. Let S be a somewhat far symplecton with respect to a hyperbolic line θ. Let π_t be the projection of t onto S $(t \in \theta)$ and put $\Pi = \{\pi_t \mid t \in \theta\}$. View S as embedded into the natural $\mathrm{Sp}_6(q)$ module V. Then, no three elements of Π are on a singular or hyperbolic line and* $\dim(\langle \Pi \rangle_{\mathsf{V}}) = 3$.

Let $\perp$ denote the orthogonality relation with respect to the symplectic form on V and put $\mathsf{P} = \langle \Pi \rangle_V \subseteq \mathsf{V}$.

Proof: Let $x \in S$ be a point with position [4]. Then, for every $t \in \theta$, $\pi_t = d_2(t) \cap S$. For every $t \in \theta$, the points in $d_1(\pi_t) \cap S$ (resp. $S \setminus (\{\pi_t\} \cup d_1(\pi_t))$ are precisely the points in $d_3(t) \cap S$ (resp. $d_4(t) \cap S$). Since for every $t \in \theta$ a point π_t is in position $[2'']$, $[2', t]$ or $[2, t]$ and for every $t' \in \theta \setminus \{t\}$ and $i \in \{0, 1, 2\}$ we have $\pi_t \in d_i(\pi_{t'})$ if and only if $\pi_{t'} \in d_i(\pi_t)$, the following are the cases that may occur:

(i) All π_t's coincide.

(ii) The π_t's are distinct but pairwise collinear.

(iii) The π_t's are pairwise non-collinear.

From Lemma 6.2 we may conclude that case (i) never occurs.

The points of Π do not form a single projective line because otherwise we would have $\pi_t \in d_1(x)$ for some $t \in \theta$, contradicting $x \in d_4(t)$. Then, recalling that a point of S that is collinear to three points of Π must be collinear to all points of Π, we find that P has dimension 3. □

For a point in X_4 we determine the number of its neighbours in X_{3,t_1}, X_{3',t_1,t_2} and X_4 ($t_1, t_2 \in \theta$). Let x be a point in $S \cap X_4$. Then $x \notin \mathsf{P}$ so that $\dim(x \oplus \mathsf{P}) = 4$, and hence the number of points in $X_{3''} \cap S$ that are collinear to x (i.e. in $(x \oplus \mathsf{P})^\perp$) is $q + 1$. Then by a double count of the pairs of lines on x containing a point from $X_{3''}$ and symplecta containing these lines, we find that a point in X_4 is collinear to $(q + 1)(q^3 + 1)$ points in $X_{3''}$. Here we use the fact that a line on x contains precisely one point from $d_3(t)$ for every $t \in \theta$. Similarly one can see that a point in X_4 is collinear to $q^2(q^3 + 1)$ points of X_{3',t_1,t_2} and of X_{3,t_1} for any given $t_1, t_2 \in \theta$.

The remainder of this subsection is devoted to finding the distributions of the remaining lines of Γ_∞. We consider lines L that contain a point in $d_4(t)$ for every $t \in \theta$. Note that $|d_3(t) \cap L| = 1$ and $L \setminus d_3(t) \subset d_4(t)$, for any $t \in \theta$.

If $d_3(t) \cap L$ is the same point for all $t \in \theta$, then L has distribution [**xxv**]. From the previous we conclude that a point $x \in X_4$ lies on $(q + 1)(q^3 + 1)$ such lines.

Distribution	$[3'']$	[4]
[**xxv**]	1	q

If we assume $L \cap X_{3''} = \emptyset$, then, considering the results from Subsection 6.6, we are apparently looking at those lines of a somewhat far plane V not having distribution [**xxi**, t] for some $t \in \theta$. These lines are described in Subsection 6.6 and we are done.

Using the results on somewhat far symplecta at the beginning of this subsection one can compute the number of lines with distribution [**xxiii**, t] and [**xxiv**] on points in X_4.

7. Proof of the main theorem

Proof: (of Theorem 1.1) The classes of the scheme partition the set $\mathcal{P}_\infty \times \mathcal{P}_\infty$ and are unions of orbitals under the action of the stabilizer G_∞ of the point ∞. Table 2 lists all orbits of the stabilizer of a hyperbolic line θ for q even (cf. Cooperstein [3]). Let $o \in \mathcal{P}_\infty$ together with ∞ span the hyperbolic line θ. Then it is clear that the sets $C_i(o)$ and $\bar{C}_i(o)$ ($i = 0, 1, 2\mathrm{g}, 2\mathrm{h}, 3, 3', 4$) as defined in Section 5 partition $\mathcal{P}_\infty$ and we are done because G_∞ is transitive on $\mathcal{P}_\infty$. Thus we are dealing with an association scheme obtained by joining certain classes of a group scheme (for G_∞) on the point set $\mathcal{P}_\infty$.

The sizes of the classes are easily computed again using Table 2. As for the other parameters, for every point $x \in \mathcal{P}_\infty$, we have determined all possible distributions of the point-set of a line on x among the G_∞-orbitals and for each such distribution we have determined how many lines with that distribution contain x. From these facts it is easy, though tedious, to compute for any pair of classes how many neighbours a point in the one class has in the other class. □

In Figure 7 the classes are represented as follows (from bottom left to top right): $C_0(o)$, $\bar{C}_0(o)$; $C_1(o)$, $C_{3'}(o)$, $\bar{C}_1(o)$; $C_{2\mathrm{g}}(o)$, $\bar{C}_{2\mathrm{g}}(o)$; $C_{2\mathrm{h}}(o)$, $\bar{C}_{2\mathrm{h}}(o)$; $C_3(o)$, $\bar{C}_3(o)$; $C_4(o)$; $\bar{C}_{3'}(o)$.

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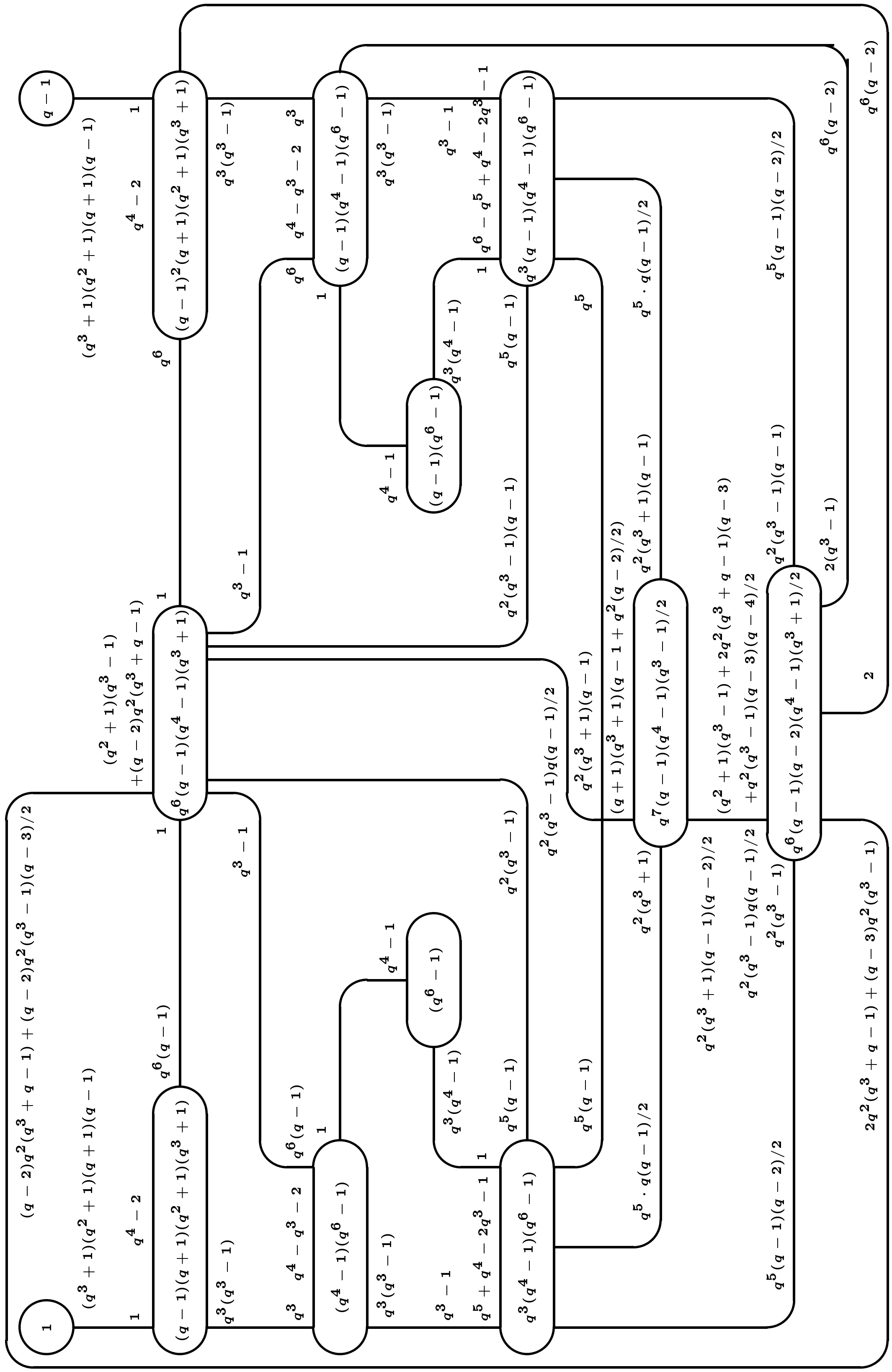

Figure 7: The scheme A.