On Geometries related to Buildings

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PROEFSCHRIFT

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Rieuwert Jan BLOK

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Samenstelling promotiecommissie:

Rector Magnificus,	voorzitter
Prof. dr. ir. T.H.M. Smits,	Technische Universiteit Delft
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Preface

The theory of buildings developed by Tits and the theory of diagram geometries developed by Buekenhout provide two frameworks to deal with geometric structures and related groups in a uniform way. A key concept here are the (Buekenhout-Tits) diagrams that can be used on the one hand to describe the incidence structure of a geometry and on the other hand to describe the structure of certain groups. This approach has been exploited by many mathematicians to work on a wide area of geometries and groups. The geometries we study in this thesis form an important part of this area. The rôle of the diagram also emerges clearly: Several results presented in this thesis are stated solely in terms of the building, its diagram and sometimes a condition on the field involved.

Of late there has been some interest in affine geometries and the classification of hyperplanes. Many examples of such affine geometries arise as the geometry far from a point p, i.e. as the complement of the hyperplane consisting of the points not far from the point p. One of the important properties to study in this context is the connectedness of these affine geometries.

This thesis has two main subjects. In Chapter 2 we deal with the first subject, which is the spanning of shadow spaces of buildings by 'small' sets of points. This chapter contains material from Blok and Brouwer [4]. Chapters 3 and 4 are devoted to the second subject, which is the subgeometry of a building consisting of flags far away from a given flag. In Chapter 3 we have included material from Blok and Brouwer [3].

In Chapter 1 we introduce concepts that play a central role in this thesis. We first introduce the notion of a diagram geometry and illustrate this with several classical examples. Then we turn to the chamber system, the geometry and the shadow spaces of a spherical building. Some attention is paid to substructures as roots and apartments. We show how these objects can be obtained from groups with a (B, N)-pair. In the remaining part of this chapter we sketch how one constructs the standard embedding for a shadow space into a Lie algebra module using Chevalley groups.

Chapter 2 is devoted to a study of minimal spanning sets for shadow spaces of spherical buildings belonging to a single node of the diagram. We prove that for a shadow space belonging to a node of the diagram that corresponds to a so-called minimal weight the set of points on an apartment forms a minimal spanning set. This is shown using a computation in the Coxetergroup. Here the main ingredient is the fact that in this special case the Coxetergraphs have small diameter. By looking at the dimension of the standard embedding of a shadow space into a Lie algebra module, we conclude that there exist no other shadow spaces that are spanned by the points on an apartment. In the second part of this chapter we focus on embeddable polar spaces of finite rank. We show that if a non-degenerate polar space is embedded into a vector space V that has a subspace U such that the induced polar space is again non-degenerate and spanned by $\dim(U) + d$ points, then the polar space itself is spanned by $\dim(V) + d$ points. Here the techniques are more classical.

In Chapter 3 we consider the subgeometry of the shadow space of a spherical building on the flags far away from a given flag. We focus on its connectness properties. It is shown that in almost all cases the subgeometry far from a flag is connected; the exceptions occur only in geometries with relatively short lines ($q = 2, 3$) and in certain freely constructed generalized polygons. We first give an overview of the known results on generalized polygons and then proceed to study buildings of higher rank. It turns out that the subgeometry far away from a flag is disconnected if and only if it contains a disconnected rank 2 geometry. This result is almost entirely established with purely geometric methods. A uniform proof is found by determining the stabilizer of a connected component inside the stabilizer of the entire subgeometry. By computing the index of the former group in the latter group we are able to compute the number of connected components in all cases.

In Chapter 4 the parameters of several association schemes are computed. We are mainly interested in the scheme on the set of points $\mathcal{X}$ far away from a fixed point ∞ inside the metasymplectic geometry $F_4(q)$ (q even), where the classes are defined by distance relations. We proceed as follows. We choose a point o in $\mathcal{X}$. Using the presentation of the geometry as a long-root geometry it is possible to determine the orbits inside the set $\mathcal{X}$ under the action of the stabilizer of the pair (∞, o); the corresponding orbitals form the actual classes of the association scheme. Next, we consider all lines that meet $\mathcal{X}$ in at least two points and determine their intersection with the orbits defined above. The strategy is to find a suitable object O that we know the line to be incident with and then determine the position of that line with respect to the projection of the hyperbolic line ∞o onto the object O. As a preliminary to, and indeed as part of, the treatment of the $F_4(q)$ case, separate subsections are included to deal with the subgeometries of type $\mathrm{Sp}_6(q)$ and $\mathrm{DO}_7(q)$.

In the first place I would like to express my gratitude towards my supervisor and promotor Andries Brouwer for many inspiring discussions, his support in general and his patience. Furthermore I thank my promotor Theo Smits for his support during the period of my research.

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Amsterdam, April 1999 Rieuwert J. Blok.

Chapter 1

Buildings

The aim of this chapter is to provide basic material needed in Chapters 2, 3 and 4. The main topic is buildings and the geometries and groups related to them.

1.1 Diagram geometries

An *incidence geometry* Γ is a quadruple $(\mathcal{O}, \star, \tau, I)$ consisting of a set $\mathcal{O}$ of *objects*, a symmetric relation $\star$ on $\mathcal{O}$ called the *incidence relation*, a finite set I of *types*, and a *type map* $\tau\colon \mathcal{O} \longrightarrow I$. The following axiom is required:

(B1) Two objects of the same type are incident if and only if they are equal.

The *rank* of Γ is the size of I. In this booklet we will only consider geometries of finite rank. A *flag* is a set of pairwise incident objects. A flag containing an object of each type is called a *chamber*. The *transversality property* is the following.

(B2) Every flag is contained in a chamber.

Geometries satisfying this property are called *transversal*. Given a flag F, the *residue* of F, denoted by Γ_F, is a quadruple $(\mathcal{O}_F, \star_F, \tau_F, I_F)$ where $\mathcal{O}_F$ is the collection of objects in $\mathcal{O} \setminus F$ that are incident with every object on F, $I_F = I \setminus \tau(F)$ and $\star_F$ and τ_F are the restrictions of $\star$ and τ to $\mathcal{O}_F$. The *type* of F (resp. Γ_F) is the set $\tau(F)$ (resp. $I \setminus \tau(F)$). The *cotype* of F (resp. Γ_F) is the type of Γ_F (resp. F). It is clear that if Γ is a (transversal) geometry, then so is Γ_F.

We say that a geometry is *thick*, *firm* or *thin* if every residue of type $\{i\}$ $(i \in I)$ has at least three, at least two or precisely two objects of type i.

A property holds *residually* if it holds in every residue of rank at least 2. A geometry is *connected* if the graph induced by $\star$ on $\mathcal{O}$ is connected. Thus, a geometry is *residually connected* if every residue of rank at least 2 is connected.

An isomorphism between two geometries $\Gamma = (\mathcal{O}, \star, \tau, I)$ and $\Gamma' = (\mathcal{O}', \star',$ $\tau', I')$, is a pair of maps $(\phi_{\mathcal{O}}, \phi_I)$ where $\phi_{\mathcal{O}}\colon \mathcal{O} \longrightarrow \mathcal{O}'$ and $\phi_I\colon I \longrightarrow I'$ are bijections such that $\phi_I \circ \tau = \tau' \circ \phi_{\mathcal{O}}$ and for $X, Y \in \mathcal{O}$ we have $\phi_{\mathcal{O}}(X) \star' \phi_{\mathcal{O}}(Y)$ if and only if $X \star Y$. The isomorphism is *type preserving* if $I = I'$ and $\phi_I = \mathrm{id}.$.

A *diagram* M defined over a finite set I is a collection of classes C_J $(J \subseteq I, |J| = 2)$ of geometries with type set J (up to isomorphism). The *rank* of the diagram M is the size of

I. We usually depict a diagram as a set of nodes, one for each element in I, and then, for $i, j \in I$, join the nodes with label i and j by an edge adorned with a symbol for the class C_{ij}.

A geometry belongs to the diagram M if (up to isomorphism) it has type set I and if for every $J \subseteq I$ with $|J| = 2$, every residue of type J is type preservingly isomorphic to some member of C_J.

Thus we only need to describe some classes of rank two geometries and assign a diagram to them. An important series of classes is formed by the *generalized polygons*. Given an integer $n \geq 2$, the class of generalized n-gons consists of all rank two geometries that have at least two objects of both types and satisfy the following two axioms:

(GP1) The diameter of the incidence graph is n.

(GP2) The girth of the incidence graph is $2n$.

The simplest but most important class of such geometries are the generalized 2-gons, often called (generalized) *digons*. In this case the axioms (GP1) and (GP2) say that every two objects of different type are incident.

In general the diagram for the class of generalized n-gons is the one shown in Fig. 1.1.

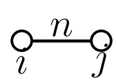


Figure 1.1: The diagram for the class of generalized n-gons.

The edges in a diagram carrying the symbol for the 'digon' class are usually omitted, for clarity's sake. Often in the diagram for the class of generalized 3, 4, 6 and 8-gons the label 'n' is omitted and the edge is replaced by 1, 2, 3 and 4 strokes respectively. We will see examples of geometries belonging to such diagrams later on.

Theorem 1.1 (Buekenhout [9], Lemma 7.2) *Let Γ be a transversal, residually connected geometry with diagram M. Let the diagram M be the disjoint union of two diagrams M_1 and M_2, with no edges between the nodes of M_1 and M_2. Then any object belonging to a node in M_1 is incident to any object belonging to a node in M.*

In the proof, which is not entirely trivial, it is essential that Γ is residually connected.

Notes

The idea of incidence geometries or diagram geometries was developed by Buekenhout in [9]. The material covered here is largely adapted from Cameron [12]. For more details we refer to Buekenhout [10] or Pasini [35].

1.1.1 Some classical geometries

The A_n-geometry

Let V be a vector space of finite dimension $n+1$ defined over a field $\mathbb{F}$. We construct a geometry from the lattice of linear subspaces of V. Let $\mathcal{O}$ be the collection of non-trivial $\mathbb{F}$-linear subspaces of V ($\{0\}$ and V are excluded) and let $I = \{1, 2, \ldots, n\}$. Then $\tau\colon \mathcal{O} \longrightarrow I$ maps each linear subspace to its dimension over $\mathbb{F}$ and for $X, Y \in \mathcal{O}$ we have $X \star Y$ if $X \subseteq Y$ or $Y \subseteq X$. This geometry is a projective space of dimension n and it belongs to the following diagram. More generally, we obtain geometries with the same diagram if we replace the field

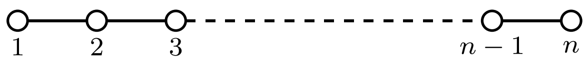


Figure 1.2: The diagram of type A_n

$\mathbb{F}$ by a division ring D.

The B_n-geometry

Let V be a vector space of finite dimension $2n+1$ defined over a field $\mathbb{F}$. Let $\mathcal{Q}$ be a non-degenerate quadratic form on V of Witt index n. A subspace U of V is called *totally singular* (t.s.) with respect to $\mathcal{Q}$ if $\mathcal{Q}(u) = 0$ for all $u \in U$. The Witt index of $\mathcal{Q}$ is, by definition, the highest dimension such a subspace can have.

Let $\mathcal{O}$ be the collection of all subspaces of V that are totally singular with respect to $\mathcal{Q}$. Further let I, τ and $\star$ be defined as in the previous example. Then $\Gamma = (\mathcal{O}, \star, \tau, I)$ is (the incidence geometry of) a parabolic quadric. It has the following diagram:

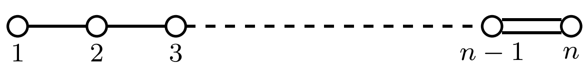


Figure 1.3: The diagram of type B_n/C_n

The C_n-geometry

Let V be a vector space of finite dimension $2n$ defined over a field $\mathbb{F}$. Let s be a non-degenerate symplectic form on V. A subspace U of V is called *totally isotropic* with respect to s if $\mathsf{s}(u, v) = 0$ for all $u, v \in U$. The highest dimension such a subspace can have is n.

Let $\mathcal{O}$ be the collection of all subspaces of V that are totally singular with respect to $\mathcal{Q}$. Further let I, τ and $\star$ be defined as in the previous example. Then $\Gamma = (\mathcal{O}, \star, \tau, I)$ is the symplectic geometry. It also has the diagram shown in Figure 1.3. In fact all polar spaces have such a diagram (cf. Tits [45]).

The D_n-geometry

Let V be a vector space of finite dimension $2n$ defined over a field $\mathbb{F}$. Let $\mathcal{Q}$ be a non-degenerate quadratic form on V of Witt index n. We can make a geometry with a B_n-diagram like we did above.

We note that here, unlike in the B_n-geometry, a t.s. $(n-1)$-space is contained in precisely two t.s. n-spaces. Moreover one can show that the graph whose vertices are the t.s. n-spaces and whose edges are the t.s. $(n-1)$-spaces is bipartite. Given a t.s. n-space U these classes comprise those t.s. n-spaces whose intersection with U has even (resp. odd) codimension in both n-spaces.

Let $\mathcal{O}$ be the collection of all subspaces of V that are totally singular with respect to $\mathcal{Q}$, except those of dimension $n-1$.

Let $I = \{1, 2, \ldots, n-1, n\}$. We define τ in the following way. The map τ takes all t.s. subspaces of dimension $i \leq n-2$ to i. Further τ takes all t.s. n-spaces of one class to $n-1$ and those in the other class to n. Finally for $X, Y \in \mathcal{O}$ we have $X \star Y$ if $X \subseteq Y$ or $Y \subseteq X$ or if X and Y are t.s. n-spaces such that $X \cap Y$ is a (t.s.) $(n-1)$-space. The geometry $\Gamma = (\mathcal{O}, \star, \tau, I)$ is sometimes called the oriflamme geometry of a hyperbolic quadric. It has the following diagram:

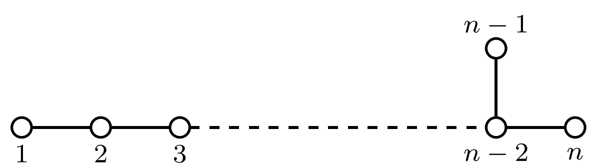


Figure 1.4: The diagram of type D_n

1.1.2 Geometries from groups

In each of the previous examples there exists a subgroup of $\mathrm{GL}(V)$ that preserves the structure of the geometry. For the geometries of type A_n, B_n, C_n and D_n described above, examples of such groups are $\mathrm{GL}(n, \mathbb{F})$ (or $\mathrm{SL}(n, \mathbb{F})$), $O(2n+1, \mathbb{F})$, $\mathrm{Sp}(2n, \mathbb{F})$ and $O^+(2n, \mathbb{F})$, respectively, and their projective analogues. In fact these groups determine the geometries involved uniquely. We now briefly describe a way to look at geometric structures implicitly present in such groups.

Coxeter systems

A *Coxeter diagram* defined over an index set I is a matrix $M = (m_{ij})_{i,j \in I}$ satisfying $m_{ij} = m_{ji} \geq 2$ for $i \neq j$ and $m_{ii} = 1$ for all $i, j \in I$. We often represent M by a picture consisting of a collection of nodes, labeled by the elements of I, one node for each element, and for each pair $(i, j) \in I \times I$ the nodes labeled by i and j are joined by a stroke labeled with m_{ij}. By convention we omit the stroke if $m_{ij} = 2$, and write a single, double or triple stroke (without label) if m_{ij} equals 3, 4 or 6, respectively.

A *Coxeter system* with diagram M is a pair (W, R) where W is a group with generator set R. The elements from R are indexed by I as follows $R = \{r_i \mid i \in I\}$ and satisfy $(r_i r_j)^{m_{ij}} = 1$. These relations define the group W, which is called the *Coxeter group of type (with diagram) M*. If W is finite then M and W are said to be *spherical*.

We shall need the *length function* $l\colon W \longrightarrow \mathbb{N}$, where $l(w)$ is the smallest l for which there is an expression $w = r_{i_1} \cdots r_{i_l}$ with $i_j \in I$ for $j = 1, \ldots, l$.

The diagram M is called *simply-laced* if it has only single bonds (i.e. $m_{ij} \leq 3$ for all $i, j \in I$). For any subset $J \subseteq I$, M_J denotes the subdiagram of M defined over J and W_J will be the subgroup of W having generator set $\{r_i\}_{i \in J}$. The pair $(W_J, \{r_i\}_{i \in J})$ then is a Coxeter system with diagram M_J. The following result can be found as Theorem 2.5.6. in Carter [14].

Lemma 1.2 *For $J, K \subseteq I$ we have*

(i) $W_J \cap W_K = W_{J \cap K}$ *and*

(ii) $\langle W_J, W_K \rangle = W_{J \cup K}$.

Geometries from Coxeter groups

Let $(W, \{r_i\}_{i \in I})$ be a Coxeter system with a diagram M defined over a set I. From this Coxeter system we construct a geometry $\Gamma(\mathcal{O}, \star, \tau, I)$ as follows. Let $\mathcal{O}$ be the collection of left cosets of $W_{I-\{i\}}$ $(i \in I)$ (clearly, we can do the same with right cosets). Let X, Y be left cosets of $W_{I-\{i\}}$ and $W_{I-\{j\}}$ respectively. We define $X \star Y$ if $X \cap Y$ is a left coset of $W_{I-\{i,j\}}$. Finally, τ takes left cosets of $W_{I-\{i\}}$ to i.

Proposition 1.3 *The geometry Γ is a transversal, residually connected geometry and M, interpreted as the diagram of a geometry, is its diagram.*

The flags of type J in this geometry are in one-to-one correspondence with the left cosets of the subgroup W_{I-J}.

1.1.3 Groups with a (B, N)-pair

We say that a group G has a (B, N)-pair or Tits-system if there are subgroups $B, N \leq G$ such that

(BN0) $G = \langle B, N \rangle$,

(BN1) $H = B \cap N \lhd N$ and $W = N/H$ is a Coxeter group with distinguished generators $r_1, \ldots, r_n$,

(BN2) $BrBwB \subseteq BwB \cup BrwB$ for each $w \in W$ and $r = r_i$,

(BN3) $rBr^{-1} \neq B$ for each $r = r_i$.

More information on (B, N)-pairs can be found in e.g. Tits [45], Ronan [38], Brouwer et al. [8] or Carter [14]. Most of the results and their proofs in this section can be found in Ronan [38] Ch. 5. Examples of such groups are the Chevalley groups (see Carter [14], Ch. 8). We will come back to this later.

Lemma 1.4

(i) (Bruhat decomposition) *If, for $w, w' \in W$ we have $BwB = Bw'B$, then $w = w'$ and therefore G is the disjoint union $\cup_{w \in W} BwB$.*

(ii) In (BN2) above we have $BrBwB = BrwB$ if $l(rw) > l(w)$.

Parabolic subgroups

For any subset $J \subseteq I$, let $W_J = \langle r_i \mid i \in J \rangle$. It follows from (BN2) that the set $BW_JB = \cup_{w \in W_J} BwB$, is a subgroup of G containing B; This subgroup is denoted by P_J and is called the standard parabolic subgroup of G of type J. The conjugates of P_J are called parabolic subgroups of G of type J. The following is due to Tits and can be found in [45] and Ronan [38] as Theorem 5.4.

Theorem 1.5

The subgroups of G containing B are precisely the groups P_J.

(i)(ii) $P_J \cap P_K = P_{J \cap K}$ and $\langle P_J, P_K \rangle = P_{J \cup K}$.

(iii) $N_G(P_J) = P_J$

(iv) There is a bijection of double coset spaces $W_J \backslash W / W_K \longrightarrow P_J \backslash G / P_K$ defined by $W_J w W_K \mapsto P_J w P_K$.

Geometries from groups with a (B, N)-pair

Let G be a group with a Tits system (B, N). From this group we construct a geometry $\Gamma(\mathcal{O}, \star, \tau, I)$ as follows. Let $\mathcal{O}$ be the collection of left cosets of $P_{I-\{i\}}$ $(i \in I)$ (clearly, we can do the same with right cosets). Let X, Y be left cosets of $P_{I-\{i\}}$ and $P_{I-\{j\}}$ respectively. We define $X \star Y$ if $X \cap Y$ is a left coset of $P_{I-\{i,j\}}$ (or, equivalently if $X \cap Y \neq \emptyset$). Finally, τ takes left cosets of $P_{I-\{i\}}$ to i.

Proposition 1.6 *The geometry Γ is a transversal, residually connected geometry and M, interpreted as the diagram of a geometry, is its diagram.*

The flags of type J in this geometry are in one-to-one correspondence with the left cosets of the subgroup P_{I-J}.

Let us define the geometry $\Gamma_C = (\mathcal{O}_C, \star_C, \tau_C, I)$ as follows. Let $\mathcal{O}_C$ be the collection of conjugates of $P_{I-\{i\}}$ $(i \in I)$. For conjugates X and Y of $P_{I-\{i\}}$ and $P_{I-\{j\}}$ define $X \star_C Y$ if $X \cap Y$ is a conjugate of $P_{I-\{i,j\}}$. Further let τ take conjugates of $P_{I-\{i\}}$ to i.

Lemma 1.7 *The geometries Γ and Γ_C are isomorphic.*

Proof: We consider the action of G on Γ by left multiplication. Define ${}^gP_J = gP_Jg^{-1}$.

Let $\phi \colon \cup_{J \subseteq I, g \in G} gP_J \longrightarrow \cup_{J \subseteq I, g \in G} {}^gP_J$ defined by $gP_J \mapsto {}^gP_J$. That is we map the left coset of a parabolic group gP_J to its stabilizer ${}^gP_J$ in G. By Lemma 1.5 this map is well defined, injective and surjective. Moreover it clearly preserves containment. Thus restricted to Γ it is a map onto Γ_C that preserves incidence and it obviously sends τ to τ_C.

Each example given in Subsection 1.1.1 arises as the geometry corresponding to a group with a (B, N)-pair. All Chevalley groups have a (B, N)-pair (Carter [14] §8.2) and hence are a source of geometries. Examples of these groups are not only the groups $\mathrm{SL}(n+1, \mathbb{F})$, $O(2n+1, \mathbb{F})$, $\mathrm{Sp}(2n, \mathbb{F})$ and $O^+(2n, \mathbb{F})$, but also the groups corresponding to Lie algebras of exceptional type such as $E_6(\mathbb{F})$, $E_7(\mathbb{F})$, $E_8(\mathbb{F})$, $F_4(\mathbb{F})$ and $G_2(\mathbb{F})$. In addition, the twisted Chevalley groups also have a (B, N)-pair. In this way one can also obtain geometries from the unitary groups $U(2n, \mathbb{F})$ and $U(2n+1, \mathbb{F})$, the twisted orthogonal groups $O^-(2n+2, \mathbb{F})$, and ${}^3D_4(\mathbb{F})$, ${}^2E_6(\mathbb{F})$, ${}^2F_4(\mathbb{F})$. □

Notes

There is a wealth of books on classical geometries but I will name only a few. The classical geometries are described from a geometric and combinatorial point of view in Brouwer et al. [8] and Cameron [12]. In Tits [45] and Ronan [38] these geometries are dealt with in the context of (B, N)-pairs and buildings. Material on the classical groups involved can be found in Dieudonné [28] and Taylor [44]. Most topics are covered in Buekenhout [10].

1.1.4 Chamber systems

A *chamber system* over an index set I is a set $\mathcal{C}$, together with a collection $(\Pi_i)_{i \in I}$ of partitions of $\mathcal{C}$. We often (slightly incorrectly) refer to $\mathcal{C}$ as the chamber system. An element of $\mathcal{C}$ is called a *chamber* and an element of Π_i is called an *i-panel*. Two chambers in an i-panel are called *i-adjacent*.

Example: Consider a transversal geometry $\Gamma = (\mathcal{O}, \star, \tau, I)$. Let $\mathcal{C}(\Gamma)$ be the collection of all chambers (i.e. maximal flags) in Γ. For $i \in I$ let Π_i be the partition of $\mathcal{C}(\Gamma)$ in which two chambers c and c' are in one element of Π_i if, for $j \in I - \{i\}$ the j-objects on x and x' are equal. Then $(\mathcal{C}(\Gamma), (\Pi_i)_{i \in I})$ is a chamber system, called the chamber system of Γ.

We say that a chamber system is *thick*, *firm* or *thin* if every panel has at least three, at least two or precisely two chambers, respectively.

A *gallery of type* $(i_1, \ldots, i_l)$ ($i_j \in I$, for $j = 1, \ldots, l$) is a sequence of chambers $c_0, c_1, \ldots, c_l$ such that c_{j-1} and c_j are i_j-adjacent. The gallery is a gallery of type J (or J-gallery) ($J \subseteq I$) if $i_j \in J$ for all $j = 1, \ldots, l$. A subset $X \subseteq \mathcal{C}$ is *(J-)connected* if any two chambers in X can be joined by a (J-)gallery. The J-connected components of $\mathcal{C}$ are called *J-residues* or residues of type J. The *rank* of a J-residue is the cardinality of J. The *cotype* and *corank* of a J-residue are the set $I - J$ and its cardinality respectively.

Lemma 1.8 *Let Γ be a transversal geometry and let $\mathcal{C}$ be its chamber system.*

(i) If Γ is residually connected, then $\mathcal{C}$ is connected, and

(ii) If $\mathcal{C}$ is connected, then Γ is connected,

but the converse to neither of these implications holds.

The proof is easy. An almost trivial but useful consequence of Lemma 1.8 is the following.

Corollary 1.9 *Let Γ be a transversal geometry of finite rank and let $\mathcal{C}$ be its chamber system. Then Γ is residually connected if and only if the chamber system of every residue of rank at least 2 is connected.*

Let $(\mathcal{C}, (\Pi_i)_{i \in I})$ be a chamber system over a set I. The *geometry of the chamber system* $\mathcal{C}$ is the geometry $\Gamma(\mathcal{C}) = (\mathcal{O}, \star, \tau, I)$. Here, $\mathcal{O}$ is the set of residues of type $I - \{i\}$ (i ranging over I), two objects being incident if, as subsets of $\mathcal{C}$, they have nonempty intersection and τ is the map that takes residues of type $I - \{i\}$ to $i \in I$.

The following result is largely covered by the discussion of §2.2 in Tits [46].

Lemma 1.10 *Let Γ be a transversal geometry and let $\mathcal{C}$ be a connected chamber system.*

(i) If $\mathcal{C} = \mathcal{C}(\Gamma)$ and Γ is firm and residually connected, then $\Gamma(\mathcal{C}) \cong \Gamma$.

(ii) If $\Gamma = \Gamma(\mathcal{C})$ and $\mathcal{C}$ is firm and any three residues of type J_1, J_2 and J_3 that intersect pairwise intersect in a non-empty residue of type $J_1 \cap J_2 \cap J_3$, then $\mathcal{C}(\Gamma) \cong \mathcal{C}$.

Proof: (i) Since Γ is transversal, an object X (of type i, say) is on at least one chamber, hence corresponds to at least one residue R of type $I - \{i\}$ in $\mathcal{C}$. Since Γ is residually connected, any two chambers on X are joined by a gallery whose chambers are all on X (Corollary 1.9) and (hence) this gallery lies in R. Thus X corresponds to precisely one residue of type $I - \{i\}$. Since Γ is firm, every panel has at least two chambers and so residues of different types can never coincide. Thus R represents a unique object of Γ and this object is X. Again since Γ is transversal, two objects are incident if and only if there is a chamber containing both. Hence two objects in Γ are incident if and only if the corresponding residues in $\mathcal{C}$ share a chamber.

(ii) As we saw above if $\mathcal{C}$ is firm, then residues of different type are distinct. Hence to a chamber of $\mathcal{C}$ there corresponds at least one chamber of Γ. Suppose that we have a chamber of Γ, that is there are residues X_i of type $i \in I$ that intersect pairwise. Then by assumption the intersections $X_1 \cap X_i$ ($i \in I \setminus \{1\}$) are residues of type $I - \{1, i\}$ that intersect pairwise in residues of type $I - \{1, i, j\}$. Applying induction we see that $\cap_{i \in I} X_i$ is a nonempty residue of type $\emptyset$, that is, a chamber. Thus each chamber in Γ is represented by a unique chamber in $\mathcal{C}$.

Two chambers of $\mathcal{C}$ on an i-panel are contained in the same residue of type $I - \{j\}$ for all $j \in I - \{i\}$, but their $I - \{i\}$-residues are different. Hence they correspond to chambers of Γ that are i-adjacent. □

Notes

The notion of chamber systems was developed by Tits in [45] and [46] (See also Buekenhout [9]). The definition given here is taken from Ronan [38].

1.1.5 Shadow spaces of geometries

Let $\Gamma = (\mathcal{O}, \star, \tau, I)$ be a geometry. Fix $i \in I$. The *i-shadow space* of Γ is the point-line geometry $\Gamma^{(i)} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is the collection of i-objects and $\mathcal{L}$ consists of the sets $\{p \in \mathcal{P} \mid p \star F\}$, where F is a flag of cotype i.

EXAMPLE: Given a diagram M defined over a set I and a Coxeter system $(W, \{r_i\}_{i \in I})$ with this diagram. For $i \in I$ we construct the i-shadow space of the geometry obtained from this Coxeter system as follows: For $\mathcal{P}$ take the collection of left cosets of $W_{I \setminus \{i\}}$ and for the elements of $\mathcal{L}$ take the set of points incident with a pair of elements $\{u, v\}$ with $u^{-1}v = r_i$. This is a thin geometry. Its collinearity graph is called the Coxeter graph of type M_i (cf. Brouwer et al. [8], §10.3).

More generally, let G be a group with a (B, N)-pair of type M. For $i \in I$ we construct the i-shadow space of the geometry obtained from this group as follows: For $\mathcal{P}$ take the collection of left cosets of $P_{I \setminus \{i\}}$ and for the elements of $\mathcal{L}$ take the set of points incident with the elements of a coset $gP_{\{i\}}$, $(g \in G)$ (i.e. the set $\{glP_{I-\{i\}} \mid l \in P_{\{i\}}\}$).

From (BN3) it follows that this is a thick geometry (cf. Ronan [38], p. 58, Remark 1.).

1.2 Buildings

The development of the theory of buildings is almost entirely due to Tits. The theory covered in this section can be found, in much greater detail, in Tits [45]. Here we will follow the presentation of the material as it is done in Ronan [38].

1.2.1 Coxeter buildings

Let M be a Coxeter diagram defined over a finite set I and let $(W, \{r_i\}_{i \in I})$ be a Coxeter system with diagram M. The set W, together with the partitions $\Pi_i = \{\{w, wr_i\} \mid w \in W\}$ $(i \in I)$ is a chamber system called the *Coxeter chamber system of type M*.

Lemma 1.11 (see Ronan [38], Lemma 2.2) *The automorphism group of the Coxeter chamber system is the Coxeter group. Its action (by left multiplication) is sharply 1-transitive.*

Words and galleries

Given a word $f = i_1 \dots i_l$ in the free monoid on I, we set $r_f = r_{i_1} \cdots r_{i_l}$; if $\emptyset$ denotes the null word, then $r_\emptyset = 1$. For distinct $i, j \in I$ with m_{ij} finite, we write $p(i, j)$ to mean $\dots ijij$ (m_{ij} factors). An *elementary homotopy* is an alteration from a word of the form $f_1 p(i, j) f_2$ to the word $f_1 p(j, i) f_2$. Two words are called *homotopic* if one can be transformed into the other by a series of elementary homotopies, and we write $f \simeq g$ to mean f and g are homotopic.

An *elementary contraction* (or *expansion*) is an alteration from a word of the form $f_1 ii f_2$ to the word $f_1 f_2$ (or from $f_1 f_2$ to $f_1 ii f_2$). We define two words to be *equivalent* if one can be transformed into the other by a sequence of elementary homotopies, expansions and contractions.

A word is called *reduced* if it is not homotopic to a word of the form $f_1 ii f_2$.

Note that there is a gallery of type f from x to y if and only if $x^{-1}y = r_f$ (the gallery being e.g. $(x, xr_1, \ldots, xr_f = y)$). A *minimal* gallery with end points x and y is a gallery that has minimal length among all galleries joining x and y. The *distance* between x and y then is the length of a minimal gallery from x to y and is denoted by $\mathrm{d}(x, y)$.

Theorem 1.12 (see Ronan [38], Lemma 2.3 and Theorem 2.11)

(i) Two words f and g are equivalent if and only if $r_f = r_g$.

(ii) A gallery of type f is minimal if and only if f is reduced.

(iii) Any two reduced words f and g which are equivalent must be homotopic.

Roots

Let r be a *reflection*; this is a conjugate of r_i for some $i \in I$. The *wall* M_r of r is the collection of residues of W that are stabilized by r. We say that a gallery $(c_0, \ldots, c_l)$ *crosses* the wall M_r whenever r interchanges c_{i-1} with c_i for some i, $1 \leq i \leq l$.

Lemma 1.13 (see Ronan [38], Lemma 2.4)

(i) A minimal gallery crosses a given wall at most once.

(ii) Given chambers x and y, the number of times modulo 2 that a gallery from x to y crosses a given wall only depends on the chambers x and y.

Given a wall M_r and a chamber c, there is a partition of W into two sets consisting of chambers that can be connected to c by a gallery that crosses M_r an even (resp. odd) number of times. This partition does not depend on the choice of c. The two elements of this partition are called *roots* and are said to be *opposite* one another; if one is denoted α, the other is often denoted $-\alpha$, and if r is the reflection we often let $\pm\alpha_r$ denote the two roots.

Given a panel π, there is a unique reflection r interchanging the two chambers of π. This reflection determines the wall M_r and the two roots $\pm\alpha$. Suppose that $c \in \pi$ lies in α. Then the panel π together with the chamber c determine α uniquely. We say that α is the root determined by π and c.

A set X of chambers is called *convex* if any minimal gallery joining two chambers in X is entirely contained in X.

Lemma 1.14 (see Ronan [38], Proposition 2.6)

(i) Roots are convex.

(ii) If x, y are adjacent chambers and α is a root with $x \in \alpha$, $y \notin \alpha$, then

$$\alpha = \{c \mid d(c, x) < d(c, y)\}$$

Residues

The residues of type J in W are the left cosets of the subgroup W_J.

Theorem 1.15 (see Ronan [38], Theorem 2.9) *Given $w \in W$ and a residue R, there is a unique chamber of R nearest to w (denoted by $\text{proj}_R(w)$), and for any chamber $x \in R$, there is a minimal gallery from w via $\text{proj}_R(w)$ to x.*

The chamber $\text{proj}_R(w)$ is called the *projection* of w onto R.

Proposition 1.16 (see Ronan [38], Lemma 2.10) *If x and y are chambers in a common J-residue, then any minimal gallery between x and y is a J-gallery. That is, residues are convex.*

Sphericity

A Coxeter diagram M is called *spherical* if the Coxeter group with diagram M is finite; we also use the word spherical for the Coxeter group, the Coxeter system and the Coxeter chamber system.

Two chambers are called *opposite* if their distance equals $\text{diam}(W)$.

Theorem 1.17 (see Ronan [38], Theorem 2.15) *For spherical W we have the following.*

(i) diam(W) equals half the number of roots.

(ii) Two chambers are opposite if and only if they lie in no common root.

(iii) Each chamber has a unique opposite.

(iv) If x and y are opposite chambers then any chamber lies on a minimal gallery from x to y.

We note that *(ii)* can be used to define 'opposite' also in the case that M is not spherical.

Coxeter buildings

The Coxeter chamber system can be given a W-valued distance function

$$\delta: W \times W \longrightarrow W$$

defined by $\delta(x, y) = x^{-1}y$ $(x, y \in W)$. We note that $\delta(x, y) = r_{i_1} \cdots r_{i_l}$ with $i_j \in I$ for $j = 1, \ldots, l$ if and only if there is a gallery of type $(i_1, \ldots, i_l)$ joining x to y. The pair (W, δ) is called the *Coxeter building* of type M.

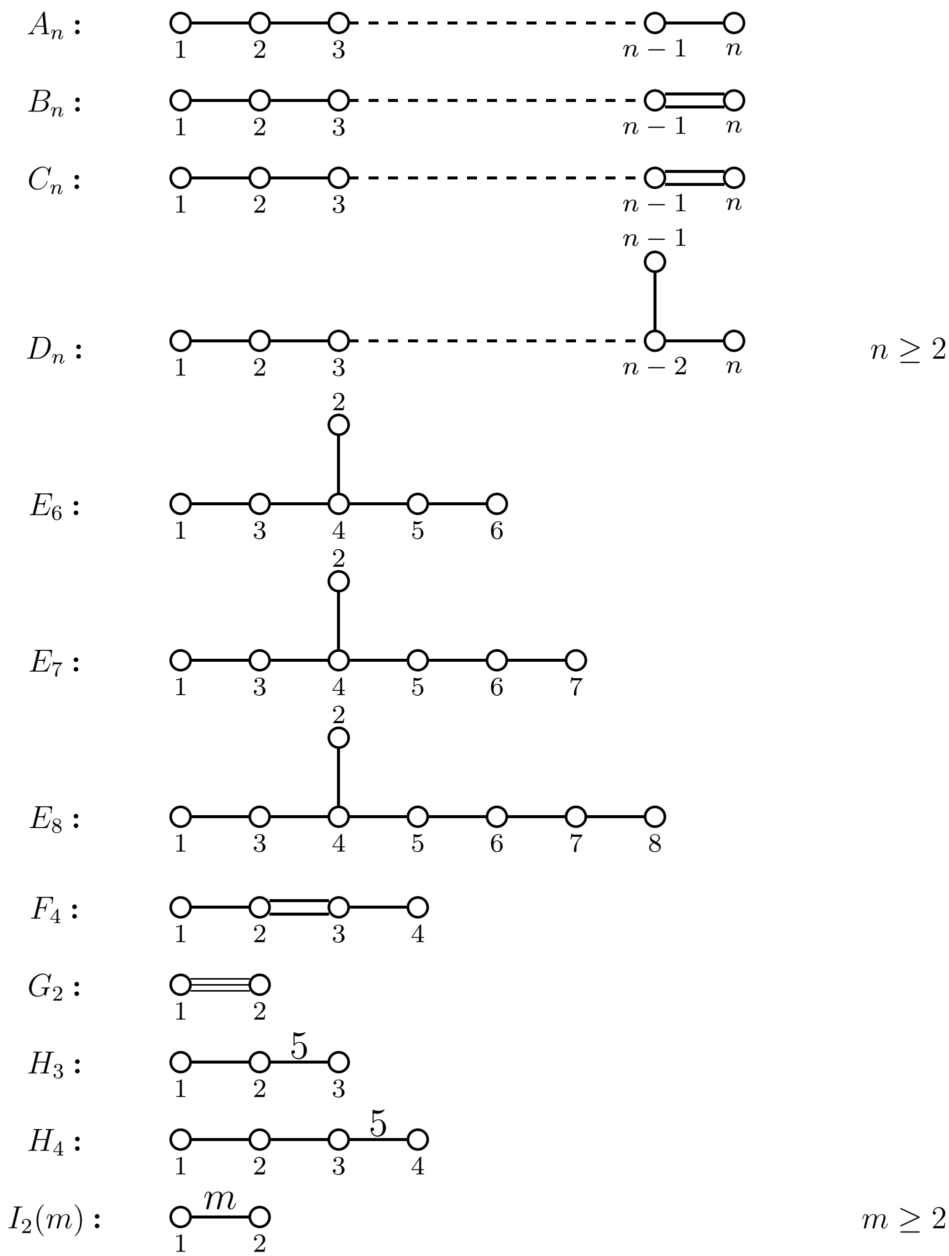


Figure 1.5: The complete list of spherical Coxeter diagrams.

1.2.2 Buildings

Let M be a Coxeter diagram defined over a finite set I and let $(W, \{r_i\}_{i \in I})$ be a Coxeter system with diagram M. A *building* of type M is a pair $(\mathcal{C}, \delta)$, where $\mathcal{C}$ is a chamber system defined over I in which each panel contains at least two chambers, and where δ is a W-valued distance-function

$$\delta: \mathcal{C} \times \mathcal{C} \longrightarrow W$$

such that $\delta(x,y) = r_f$ if and only if there exists a gallery of reduced type f joining x and y. We note that in a small number of cases a Coxeter group may belong to Coxeter systems of distinct types, depending on the choice for the generator set. In the sequel this choice will be clear from the context.

The chamber system of a building is firm by definition; we say that a building is *thick* or *thin* whenever its chamber system is thick or thin, respectively.

Examples

The Coxeter chamber system W together with the distance function defined by $\delta(x,y) = x^{-1}y$ $(x, y \in W)$ is a thin building with diagram M.

A building of rank 2 with diagram I_m is a generalized m-gon and vice versa. More precisely, the chamber system of the building is the chamber system of the generalized m-gon. We note here that the Coxeter group with diagram I_m is the dihedral group of order $2m$. The reduced words are the finite alternating sequences $iji\dots$ of length $\leq m$; they give distinct group elements except that $iji\dots$ and $jij\dots$ (m factors) give the same element.

Residues, apartments and roots

In this section Δ will be a building with diagram M.

Theorem 1.18 (see Ronan [38], Theorem 3.5) *Let $J \subseteq I$. A residue of Δ of type J is a building with diagram M_J.*

Given a subset $X \subseteq W$ we define a map $\phi\colon X \longrightarrow \Delta$ to be an *isometry* if it preserves the W-distance. Thus if for the moment we denote the distance functions of the buildings W and Δ by δ_W and δ_Δ respectively, we require

$$\delta_\Delta(\phi(x), \phi(y)) = \delta_W(x,y)$$

for all $x, y \in X$.

An *apartment* is an isometric image of W into Δ. A *root* of Δ is a root in an apartment of Δ. A *wall* of Δ is the collection of residues meeting an apartment Σ in a residue of a wall of Σ (see Lemma 1.21 below). We denote the wall of a root α in Δ as $\partial\alpha$. We note that if X is a root in an apartment Σ of Δ, then it is a root in any apartment of Δ that contains X. Moreover by the following theorem an isometric image of a root of W is a root of Δ.

Theorem 1.19 (see Ronan [38], Theorem 3.6) *Any isometry of a subset $X \subset W$ into Δ extends to an isometry of W into Δ.*

An important consequence, which will be used frequently, is the following.

Corollary 1.20 (see Ronan [38], Corollary 3.7) *Any two chambers lie in a common apartment.*

The following lemma shows that there is a 1-1 correspondence between the residues of type J in Δ that meet an apartment Σ and the residues of type J in Σ.

Lemma 1.21 *Let R be a residue of type J in Δ, for some subset $J \subseteq I$, and let Σ be an apartment of Δ. Then either R is disjoint from Σ or $R \cap \Sigma$ is a residue of type J in Σ.*

Theorem 1.22 (see Ronan [38], Theorem 3.8) *Let Σ be an apartment containing a chamber c and meeting a residue R. Then every minimal gallery from c to R lies in Σ; in particular apartments are convex.*

Corollary 1.23 (cf. Ronan [38], Corollary 3.9 and its proof) *If R is any residue of Δ and c is any chamber, then there is a unique chamber (call it $\text{proj}_R(c)$) on R nearest to c. Moreover if Σ is any apartment meeting c and R, then $\text{proj}_R(c)$ lies in Σ.*

The chamber $\text{proj}_R(c)$ is called the *projection* of c onto R.

Direct products of buildings

Thinking of M as a graph, let $M = M_1 \sqcup \ldots \sqcup M_k$ be the decomposition of the diagram into connected components, where M_t is defined over the set I_t $(1 \leq t \leq k)$. In particular I is the disjoint union $I_1 \sqcup \ldots \sqcup I_k$, where $m_{ij} = 2$ if i and j belong to different components. Fix some chamber c of a building Δ with diagram M and let Δ_t be the I_t-residue on c.

Theorem 1.24 (see Ronan [38], Theorem 3.10) *With the notation above, Δ is isomorphic to the direct product $\Delta_1 \times \cdots \times \Delta_k$.*

Clearly an apartment Σ for Δ is isomorphic to $\Sigma_1 \times \cdots \times \Sigma_k$, where Σ_t is an apartment for Δ_t $(1 \leq t \leq k)$. We note that since a reflection r_i with $i \in I_t$ only affects Δ_t, the roots of Σ are of the form $\Sigma_1 \times \cdots \times \Sigma_{t-1} \times \alpha_t \times \Sigma_{t+1} \times \cdots \times \Sigma_k$, where α_t is a root of Σ_t.

Another definition

One can define buildings without explicitly mentioning a W-valued distance function using the following theorem.

Theorem 1.25 (see Ronan [38], Theorem 3.11) *Let $\mathcal{C}$ be a chamber system containing subsystems (called* apartments*) isomorphic to a given Coxeter chamber system (over the same indexing common apartment. Then $\mathcal{C}$ is a building if, given two apartments Σ and Σ' containing a common chamber x and a common chamber or panel y, Σ and Σ are isomorphic via an isomorphism that fixes x and y.*

In fact this characterization is a predecessor of the one we use as a definition.

1.2.3 Groups with a (B, N)-pair

The relation between buildings and groups with a (B, N)-pair can be understood via the notion of strong transitivity. Let G be a group acting on the building Δ. The group G is said to act *strongly transitively* on Δ if

(ST0) for each $w \in W$ the group G is transitive on ordered pairs of chambers (x, y) such that $\delta(x, y) = w$,

(ST1) there is some apartment Σ whose stabilizer in G is transitive on the chambers of Σ.

We will take the action of G from the left.

Theorem 1.26 (cf. Ronan [38], Theorem 5.2) *Let Δ be a thick building having a strongly transitive automorphism group G, let Σ be an apartment as in the definition of strong transitivity above, and let W be the Coxeter group corresponding to Σ. Let c be a given chamber of Σ, and let $B = Stab_G(c)$ and $N = Stab_G(\Sigma)$. Then (B, N) is a Tits system for G and*

$$\delta(c, d) = w \iff d \in BwB \cdot c$$

There is an almost converse to the theorem above:

Theorem 1.27 (Ronan [38], Theorem 5.3) *Every Tits system (B, N) in a group G defines a building, the chambers being left cosets of B, with i-adjacency given by*

$$gB \sim_i hB \iff g^{-1}h \in B\langle r_i \rangle B.$$

Proposition 1.28 (see Ronan [38], Theorem 5.4) *The parabolic subgroup P_J ($J \subset I$) is the stabilizer in G of the J-residue containing c.*

Hence the J-residues in the building of Theorem 1.27 are the left cosets of the parabolic subgroup P_J.

1.2.4 Related geometries

Let Δ be a building with chamber system $\mathcal{C}$. The *geometry of* Δ is the geometry $\Gamma(\Delta)$ obtained from its chamber system (cf. Subsection 1.1.4).

Lemma 1.29 *Let Δ be a building with chamber system $\mathcal{C}$ and let $\Gamma(\Delta)$ be the geometry obtained from $\mathcal{C}$. Then there exists an isomorphism*

$$\mathcal{C}(\Gamma(\Delta) \cong \mathcal{C}.$$

Proof: We use Lemma 1.10 part (ii). By definition the chamber system $\mathcal{C}$ is firm. The condition that the intersection of three pairwise intersecting residues is non-empty is also satisfied. □

Lemma 1.30 *Let Δ be a building with chamber system $\mathcal{C}$ and let $\Gamma(\Delta)$ be the geometry obtained from $\mathcal{C}$. Then there is an incidence preserving bijection between the collection of flags in $\Gamma(\Delta)$ and the collection of residues in $\mathcal{C}$.*

Proof: A flag F of $\Gamma(\Delta)$ is a set of pairwise incident objects $\{X_j\}_{j \in J}$, where X_j is an object of type j. As in the proof of Lemma 1.29 part (ii), we find that the intersection $R_F = \cap_{j \in J} X_j$ is a residue of type $I - J$. It is not difficult to see that the map sending F to R_F is an incidence preserving bijection between the collection of flags in $\Gamma(\Delta)$ and the collection of residues in $\mathcal{C}$. □

Suppose now that the diagram of Δ is defined over the set I. For $i \in I$ we define the *i-shadow space of* Δ to be the point-line geometry $\Delta^{(i)} = (\mathcal{P}, \mathcal{L})$ by taking for $\mathcal{P}$ the collection of residues of type $I - \{i\}$; we define the elements of $\mathcal{L}$ as the sets of points incident to an i-panel. Alternatively, we can obtain the geometry $\Delta^{(i)} = (\mathcal{P}, \mathcal{L})$ as the i-shadow space of the geometry of Γ, i.e. $\Gamma^{(i)} \cong \Delta^{(i)}$.

Suppose that Δ is obtained from a group G with a (B, N)-pair (cf. Theorems 1.26 and 1.27). Then by Proposition 1.28, there is a one-to-one correspondence between the collection of parabolic subgroups of type J in G and the residues of type J in Δ. It is clear that this correpondence preserves containment. If we denote the geometry obtained from the group G (cf. page 6) by $\Gamma(G)$ and its i-shadow space (cf. Subsection 1.1.5) by $\Gamma^{(i)}(G)$, then we have the following equalities.

Lemma 1.31

$$\begin{aligned} \Gamma(\Delta) &\cong \Gamma(G) \\ \Delta^{(i)} &\cong \Gamma^{(i)}(\Delta) \cong \Gamma^{(i)}(G) \end{aligned}$$

1.2.5 Moufang buildings

This subsection is meant to give a brief introduction to buildings of Moufang type. We will assume that the buildings are thick and spherical as we are dealing only with spherical buildings in this booklet. Let Δ be thick and spherical.

Opposite

Let $d = \mathrm{diam}(W)$. Recall that two chambers x and y are *opposite*, by definition, if $\mathrm{d}(x, y) = d$. We will extend this notion to arbitrary residues, starting with panels.

Lemma 1.32 (Ronan [38], Lemma 6.1) *Let π be a panel (of type i) on chambers x and x' in an apartment Σ of Δ. If y and y' denote the chambers of Σ opposite to x and x' respectively, then y and y' are adjacent. Moreover if $\hat{\pi}$ is the panel (of type $\hat{\imath}$) common to y and y', then π and $\hat{\pi}$ determine the same wall of Σ and $r_{\hat{\imath}} = w_I^{-1} r_i w_I$ where w_I is the longest word of W.*

For an apartment Σ of Δ let $\mathrm{op}_\Sigma \colon \Sigma \longrightarrow \Sigma$ denote the map sending each chamber of Σ to its opposite in Σ; it is called the *opposition involution*. This does not always preserve types; it sends i-panels to $\hat{\imath}$-panels where $r_{\hat{\imath}} = w_I^{-1} r_i w_I$.

Consider the map that sends i to j whenever an i-object (resp. i-panel) is opposite a j-object (resp. j-panel). This map is well defined and we denote it by w_I. It induces a symmetry on M, also denoted by w_I.

Theorem 1.33 *The element w_I acts as the identity on all diagrams, but flips A_n, D_{2m+1}, E_6 and I_{2m+1}.*

This result already appears in entry XI of the tables at the end of Bourbaki [6].

We now define two residues of Σ to be opposite if they are interchanged by op_Σ. More generally, two residues of a spherical building are *opposite* if their intersections with some apartment Σ are opposite in Σ (cf. Lemma 1.21). It can be shown that this definition does not depend on the choice of the apartment Σ.

Lemma 1.34 (see Ronan [38], Lemma 6.2) *Given opposite panels π and $\hat{\pi}$ in a spherical building, and chambers $x \in \pi$ and $y \in \hat{\pi}$ one has $d(x,y) = d$ unless $x = \mathrm{proj}_\pi(y)$ in which case $d(x,y) = d-1$. In particular $\mathrm{proj}_{\hat{\pi}}$ and proj_π are eachother's inverse.*

Recall that a *root* of a building is a root in an apartment of that building.

Lemma 1.35 (see Ronan [38], Lemma 6.3) *Let α be a root in a spherical building, and x a chamber having a panel π in $\partial\alpha$. Then there is a unique root containing x on $\partial\alpha$ and if $x \notin \alpha$ there is a unique apartment containing α and x.*

For a chamber c let $E_1(c)$ be the set of chambers lying in some rank 1 residue (panel) on c.

Proposition 1.36 (see Ronan [38], Proposition 6.4) *Let c and b be opposite chambers of a spherical building (assumed to be thick), and suppose that ϕ is an automorphism fixing b and all chambers of $E_1(c)$. Then ϕ is the identity.*

Root groups

For any root α in a spherical building Δ, let

$$U_\alpha = \{g \in \mathrm{Aut}(\Delta) \mid g \text{ fixes every chamber having a panel in } \alpha - \partial\alpha\}.$$

This will be called a *root group* if the diagram has no isolated nodes. This condition on the diagram ensures that there is a chamber $c \in \alpha$ such that every chamber of $E_1(c)$ has a panel in $\alpha - \partial\alpha$. Since any apartment Σ containing α contains a chamber opposite to c, it follows from Proposition 1.36 that only the identity fixes Σ. The building is called *Moufang* if, for every root α the group U_α acts transitively (hence sharply 1-transitively) on the set of apartments containing α. In fact it suffices to assume this only for the set Φ of roots α of a given apartment Σ. If g sends α to β then $(U_\alpha)^g = U_\beta$. Moreover we will see that the group $\langle U_\alpha \mid \alpha \in \Phi\rangle$ has a (B,N)-pair (See Theorem 1.43) and is transitive on the set of apartments, hence on the set of roots, and each U_β is seen to act as required.

For a chamber c let $E_2(c)$ denote the set of chambers lying in one of the rank 2 residues on c.

Theorem 1.37 (Tits [45], Theorem 4.16) *Let Σ and Σ' be apartments containing chambers c and c', in spherical buildings Δ and Δ' respectively. Then any isomorphism from $E_2(c) \cup \Sigma$ to $E_2(c') \cup \Sigma'$ extends to an isomorphism from Δ to Δ'.*

Corollary 1.38 (see Ronan [38], Corollary 6.7) *If Δ is a spherical building such that each connected component of the diagram has rank ≥ 3, then Δ is Moufang.*

Not all spherical buildings of rank 2 are Moufang. Many non-Moufang projective planes (generalized 3-gons) are known and the free constructions of which we will see an example in Section 3.2.2 are hardly ever Moufang. The following theorem shows that Moufang buildings of rank ≥ 3 don't have non-Moufang residues.

Theorem 1.39 (cf. Tits [45], Addenda or Ronan [38], Theorem 6.8) *If Δ is a Moufang building then so is every residue whose residual diagram contains no isolated nodes (so that the term Moufang applies).*

The following theorem deals with the rank 2 case.

Theorem 1.40 (Tits and Weiss 1976/79) *Moufang generalized m-gons can exist only for $m = 3, 4, 6$ and 8.*

We note that there exist examples in each case. Combining Theorems 1.38, 1.39 and 1.40 we find the following result.

Corollary 1.41 *The diagram of a thick building has no H_3-subdiagram.*

A (B, N)-pair

Let Δ be a thick spherical building of Moufang type and let Σ be an apartment. Suppose that Φ is the root system of Σ. Let U_α ($\alpha \in \Phi$) be a system of root groups. Let G be the group generated by the U_α.

Lemma 1.42 (see Ronan [38], Ch. 6 §4) *For every root α and any element $u \in U_\alpha$, there exists an element $m(u) \in U_{-\alpha}uU_{-\alpha}$ that stabilizes Σ (i.e. it interchanges α and $-\alpha$).*

Then let N be the group generated by the $m(u)$ for $u \in U_\alpha$, as α ranges over Φ, and let H be the subgroup of N fixing all chambers of Σ. Given some chamber $c \in \Sigma$, let Φ^+ be the collection of roots in Φ containing c, called the *positive roots*, and define

$$B = \langle H, U_\alpha \mid \alpha \in \Phi^+ \rangle.$$

Proposition 1.43 *With the notation above (B, N) is a Tits system for G and $B \cap N = H$.*

Now let R_J be the J-residue of Σ on c. We define Φ_J^ϵ for $\epsilon = +, -, 0$ respectively to be the set of roots $\alpha \in \Phi$ such that $R_J \subseteq \alpha$, $R_J \subseteq -\alpha$ or $R_J \cap \alpha \neq \emptyset \neq R_J \cap -\alpha$. Now put

$$\begin{aligned} U_J = &\ \langle U_\alpha \mid \alpha \in \Phi_J^+ \rangle \\ L_J = &\ \langle H, U_\alpha \mid \alpha \in \Phi_J^0 \rangle \end{aligned}$$

Theorem 1.44 (see Ronan [38], Theorem 6.18) *For a Moufang building of spherical type we have*

$$P_J = U_J \rtimes L_J.$$

Moreover if $\hat{R}_J$ is the residue of Σ opposite to R_J, then L_J is the subgroup fixing R_J and $\hat{R}_J$ and U_J acts sharply 1-transitively on the residues opposite to R_J.

The decomposition in this theorem is sometimes called the *Levi decomposition*.

1.3 Chevalley groups

In the presentation of some elements from the theory of Chevalley groups, we mainly follow Carter [14], although much is also present in Steinberg [43]. As for the basic facts on Lie algebras we refer to Humphreys [31]. An overview can also be found in Cohen [17].

1.3.1 Weyl groups

Systems of roots

Let $\mathcal{E}$ be a euclidean space of finite dimension n with inner product $(\cdot,\cdot)$. For a vector r let w_r be the reflection $s \longrightarrow s - 2\frac{(r,s)}{(r,r)}r$. A subset Φ of $\mathcal{E}$ is called a *root system* in $\mathcal{E}$ if the following axioms are satisfied.

(i) Φ is a finite set of non-zero vectors,

(ii) Φ spans $\mathcal{E}$,

(iii) if $\mathsf{r}, \mathsf{s} \in \Phi$, then $w_{\mathsf{r}}(\mathsf{s}) \in \Phi$,

(iv) if $\mathsf{r}, \mathsf{s} \in \Phi$, then $2\frac{(\mathsf{r},\mathsf{s})}{(\mathsf{r},\mathsf{r})} \in \mathbb{Z}$,

(v) if $\mathsf{r}, a\mathsf{r} \in \Phi$ $(a \in \mathbb{R})$, then $a = \pm 1$.

A root system is called *reducible* if it can be partitioned into two subsets such that any two vectors from different subsets are orthogonal; a root system is called *irreducible* if it is not reducible.

Theorem 1.45 (Carter [14], Proposition 2.1.2) *Every root system Φ has a basis $\Pi = \{\mathsf{r}_1, \ldots, \mathsf{r}_n\}$ such that any root $\mathsf{r} \in \Phi$ can be written as $\sum_{i=1}^{n} \lambda_i \mathsf{r}_i$, where either all λ_i are non-negative or all λ_i are non-positive.*

The subset Π is called a *fundamental system* for Φ. A root in Φ is called *positive* (resp. *negative*) with respect to a fundamental system Π if it can be written as a linear combination of the roots in Π with only non-negative (resp. non-positive) coefficients. By Theorem 1.45 we can partition Φ into two sets Φ^+ and Φ^- comprising the positive and negative roots with respect to Π respectively.

Weyl groups and the Dynkin diagram

The *Weyl group* $W = W(\Phi)$ of the root system Φ is the group of linear transformations generated by the reflections $\{w_{\mathsf{r}} \mid \mathsf{r} \in \Phi\}$.

From now on we fix a fundamental system $\Pi = \{\mathsf{r}_1, \ldots, \mathsf{r}_n\}$. The reflections w_{r} with $\mathsf{r} = \mathsf{r}_i$ are called *fundamental*.

Proposition 1.46 (Carter [14], Proposition 2.1.8)

(i) $W(\Pi) = \Phi$,

(ii) $W = \langle w_{\mathsf{r}_i} \mid i = 1, \ldots, n \rangle$.

Thus, if we have Π, then we can construct W from it and then obtain Φ by letting W act on the elements of Π.

Theorem 1.47 (Carter [14], Theorem 2.2.4) *The group $W(\Phi)$ acts sharply 1-transitively on the collection of fundamental systems in Φ.*

Thus, up to unitary transformations (from W) of $\mathcal{E}$, there is just one fundamental system in Φ.

In turn, the fundamental system Π is (up to a scalar) determined by what is called its *Dynkin diagram*. The Dynkin diagram is obtained from Π in the following way: For each element of Π introduce a node and, for each pair of elements $\mathsf{r}, \mathsf{s} \in \Pi$ join the corresponding nodes by a stroke adorned with the label 'i' if the angle between r and s equals $\frac{i-1}{i}\pi$. Moreover if r and s do not have equal length insert an arrow pointing towards the shortest of them. A number of conventions are used to improve readability: strokes with label '2' are deleted and strokes with label '3', '4' and '6' are replaced by a single, double and triple stroke without a label respectively.

Theorem 1.48 *All Dynkin diagrams corresponding to an irreducible root system of rank n are displayed in Figure 1.6.*

A sketch of a proof of this theorem can be found in Carter et al. [13] and a full proof is given in Bourbaki [6]. The diagrams in Figure 1.6 are labeled as in Bourbaki [6]. We will use this labeling throughout the text.

Now let M be the Dynkin diagram. It is labeled by the set $I = \{1, \ldots, n\}$. If we disregard its arrows, the Dynkin diagram can be read as a Coxeter diagram.

Theorem 1.49 (Carter [14], Theorem 2.4.1) *The pair $(W, \{w_{\mathsf{r}_i}\}_{i \in I})$ is a Coxeter system with Coxeter diagram M.*

1.3.2 Simple Lie algebras

Definition and examples

A *Lie algebra* $\mathfrak{L}$ is a vector space over a field $\mathbb{F}$ together with a product $[\cdot, \cdot]$, called a *bracket product*, that satisfies the following conditions:

(i) $[\cdot, \cdot]$ is linear in both arguments,

(ii) $[x, x] = 0$ for all $x \in \mathfrak{L}$,

(iii) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in \mathfrak{L}$.

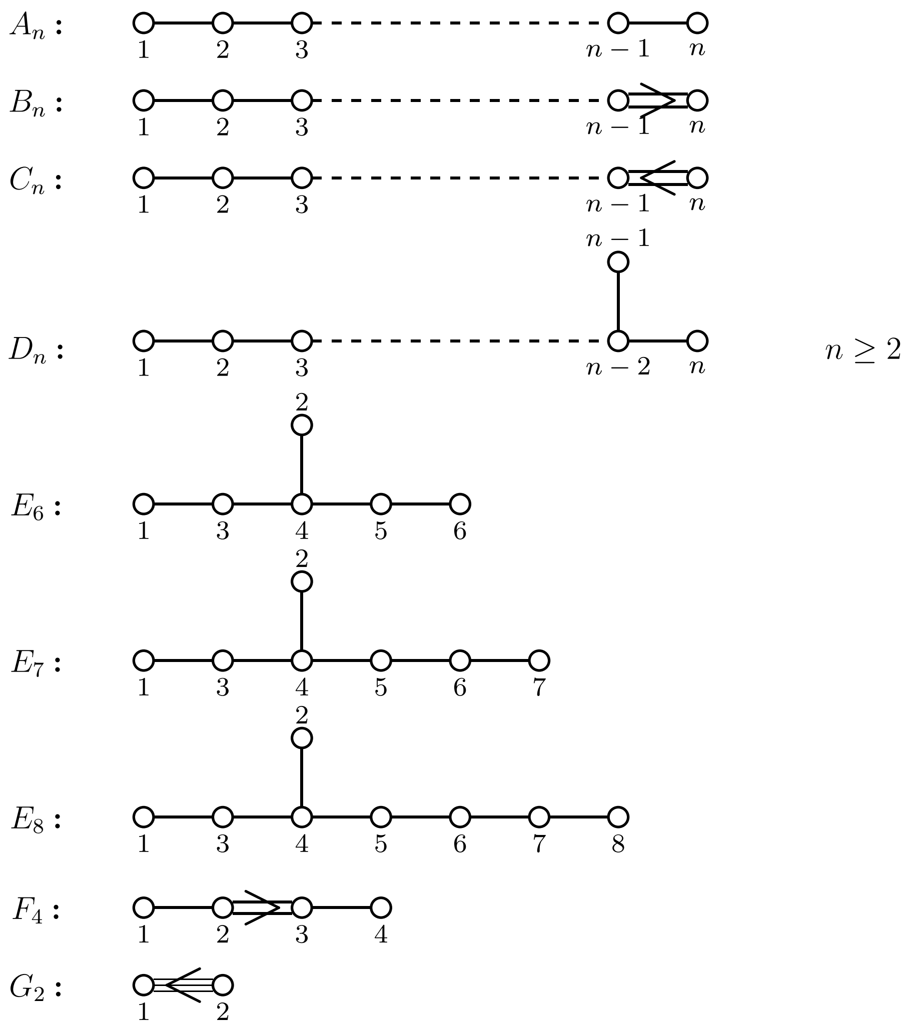


Figure 1.6: The Dynkin diagrams of the irreducible root systems

Condition (iii) is called the *Jacobi identity*. An immediate consequence of these conditions is that $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{L}$.

EXAMPLE: From an associative algebra one can construct a Lie algebra by defining $[x, y] = xy - yx$. Many important examples of Lie algebras arise in this way.

EXAMPLE: Given a vector space V of dimension n over the field $\mathbb{F}$ the Lie algebra $\mathfrak{gl}(V)$ is (by definition) obtained from the associative algebra of linear endomorphisms of V via the construction described in the previous example. We write $\mathfrak{gl}(n, \mathbb{F})$ for $\mathfrak{gl}(V)$ if we think of the endomorphisms of V as $n \times n$ matrices over $\mathbb{F}$.

EXAMPLE: The Lie algebra $\mathfrak{sl}(2, \mathbb{F})$ consists of all 2×2 matrices with trace 0. It is generated by the following three matrices:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The relations between these elements are

$$\begin{aligned} [x,y] &= xy - yx &= h \\ [h,x] &= 2x \\ [h,y] &= -2y \end{aligned}$$

Homomorphisms, isomorphisms and automorphisms for Lie algebras are defined in the obvious way.

Subalgebras and ideals

Let $\mathfrak{L}$ be a Lie algebra. For two subspaces $\mathfrak{M}, \mathfrak{N} \subseteq \mathfrak{L}$ we define $[\mathfrak{M}, \mathfrak{N}]$ to be the subspace spanned by $[x, y]$ ($x \in \mathfrak{M}$, $y \in \mathfrak{N}$). A subspace $\mathfrak{M}$ is a *subalgebra* if $[\mathfrak{M}, \mathfrak{M}] \subseteq \mathfrak{M}$.

EXAMPLE: If $\mathfrak{L} = \mathfrak{gl}(n, \mathbb{F})$, then $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{sl}(n, \mathbb{F})$; the latter is defined as the subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ consisting of matrices of trace 0 (see the example of $\mathfrak{sl}(2, \mathbb{F})$ above).

A subalgebra $\mathfrak{M} \subseteq \mathfrak{L}$ is called *abelian* if $[\mathfrak{M}, \mathfrak{M}] = \{0\}$.

EXAMPLE: Each one-dimensional subspace of a Lie algebra $\mathfrak{L}$ is an abelian subalgebra because $[x, x] = 0$ and the bracket product is bilinear.

A subalgebra $\mathfrak{M}$ of $\mathfrak{L}$ is an *ideal* if $[\mathfrak{L}, \mathfrak{M}] \subseteq \mathfrak{M}$. Given a Lie algebra $\mathfrak{L}$ and an ideal $\mathfrak{I}$, we can define the factor algebra $\mathfrak{L}/\mathfrak{I}$ in the obvious way, and we have a canonical homomorphism $\mathfrak{L} \longrightarrow \mathfrak{L}/\mathfrak{I}$.

A Lie algebra is called *simple* if its dimension is at least 2 and has no ideals other than the Lie algebra itself and $\{0\}$.

EXAMPLE: If the field $\mathbb{F}$ has characteristic other than 2, then $\mathfrak{sl}(2, \mathbb{F})$ is simple.

Representations and modules

Let $\mathfrak{L}$ denote a Lie algebra defined over a field $\mathbb{F}$. A *representation* of $\mathfrak{L}$ is a homomorphism $\phi\colon \mathfrak{L} \longrightarrow \mathfrak{gl}(V)$, where V is a vector space (over $\mathbb{F}$).

A vector space V together with a map $\mathfrak{L} \times V \longrightarrow V$ denoted by $(x, v) \mapsto xv$ is called an $\mathfrak{L}$-*module* if we have

(i) $(ax + by)v = a(xv) + b(yv)$

(ii) $x(av + bw) = a(xv) + b(xw)$

(iii) $[x, y]v = xyv - yxv$ ($x, y \in \mathfrak{L}$, $v, w \in V$, $a, b \in \mathbb{F}$)

Given the representation ϕ the vector space V together with the map $(x, v) \mapsto \phi(x)(v)$ is an $\mathfrak{L}$-module and, conversely an $\mathfrak{L}$-module V defines a representation $\phi\colon \mathfrak{L} \longrightarrow \mathfrak{gl}(V)$.

Let U be a subspace of V and let $\mathfrak{M}$ be a subspace of $\mathfrak{L}$. Let $\mathfrak{M}U$ be the subspace of V spanned by all elements xu for $x \in \mathfrak{M}$ and $u \in U$. The subspace U is called a *submodule* of V if $\mathfrak{L}U \subseteq U$. An $\mathfrak{L}$-module V is called *irreducible* if it has no other submodules than V itself and $\{0\}$.

Now $\mathfrak{L}$ is itself an $\mathfrak{L}$-module under the multiplication $\mathfrak{L}\times\mathfrak{L} \longrightarrow \mathfrak{L}$ given by $x, y \mapsto [x, y]$. In this guise $\mathfrak{L}$ is called the *adjoint module.* The linear map $\operatorname{ad} x\colon \mathfrak{L} \longrightarrow \mathfrak{L}$ defined by $y \mapsto [x, y]$ is called the *adjoint map* and the corresponding representation $x \mapsto \operatorname{ad} x$ from $\mathfrak{L}$ into $\mathfrak{gl}(\mathfrak{L})$ is called the *adjoint representation.*

Notice that in this context a submodule is an ideal and consequently 'irreducible' means 'simple'.

The Cartan decomposition

We will describe the decomposition of (the adjoint module of) a simple Lie algebra into an abelian subalgebra and a collection of eigenspaces for this subalgebra (called root spaces).

We assume that the field involved is algebraically closed and has characteristic 0. Remember from linear algebra that any endomorphism $x \in \mathfrak{gl}(V)$ can be written as $x = x_s + x_n$, where x_s is diagonalizable (= semisimple) and x_n is nilpotent (the so-called Jordan-Chevalley decomposition). In particular we can write $\operatorname{ad} x = \operatorname{ad} x_s + x_n$. In case of a simple Lie algebra $\mathfrak{L}$ it can be shown that there exist unique x_s and x_n in $\mathfrak{L}$ with $\operatorname{ad} x_s = (\operatorname{ad} x)_s$, and $\operatorname{ad} x_n = (\operatorname{ad} x)_n$ (Humphreys [31], Theorem 6.4).

The elements x_s and x_n are called the *semisimple* part and the *nilpotent* part of x. The element x is called semisimple (resp. nilpotent) if $x = x_s$ (resp. $x = x_n$).

Thus, if a Lie algebra does not consist of nilpotent elements only, then we can find a subalgebra consisting of semisimple elements.

A subalgebra of a Lie algebra is called *toral* if it consists only of semisimple elements.

Let us fix a maximal toral subalgebra $\mathfrak{H}$ of $\mathfrak{L}$.

Lemma 1.50 (see Humphreys [31], Lemma 8.1)

$$[\mathfrak{H}, \mathfrak{H}] = \{0\}.$$

Corollary 1.51 *Given $h_1, h_2 \in \mathfrak{H}$ and $y \in \mathfrak{L}$, we have* $\operatorname{ad} h_1 \cdot \operatorname{ad} h_2 \cdot y = \operatorname{ad} h_2 \cdot \operatorname{ad} h_1 \cdot y$.

This means that the elements $\operatorname{ad} h$ $(h \in \mathfrak{H})$ act as commuting semisimple endomorphisms on the vector space $\mathfrak{L}$. It is well known from linear algebra that we can diagonalize these elements simultaneously. Thus we can decompose $\mathfrak{L}$ as a direct sum of eigenspaces $\mathfrak{L}_{\mathsf{r}^*}$, where $\mathsf{r}^* \in \mathfrak{H}^*$, the dual space of $\mathfrak{H}$, so $\mathfrak{L}_{\mathsf{r}^*} = \{x \in \mathfrak{L} \mid \forall h \in \mathfrak{H} : [h, x] = \mathsf{r}^*(h)x\}$. For some positive integer m we can write

$$\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{i=1}^{m} \mathfrak{L}_{\mathsf{r}_i{}^*}.$$

This decomposition is called the *Cartan decomposition* of $\mathfrak{L}$. The non-zero subspaces $\mathfrak{L}_{\mathsf{r}_i{}^*}$ are called the *root spaces* of $\mathfrak{L}$ (with respect to $\mathfrak{H}$).

Note that $\mathfrak{H} \subseteq \mathfrak{L}_0$ (0 denoting the zero functional on $\mathfrak{H}$). It can be shown for simple Lie algebras $\mathfrak{L}$ over $\mathbb{C}$, that $\mathfrak{H} = \mathfrak{L}_0$ (Humphreys [31], Proposition 8.2); in fact one can show that the collection of maximal toral subalgebras of $\mathfrak{L}$ coincides with the collection of so-called Cartan subalgebras of $\mathfrak{L}$ (Humphreys [31], Corollary 15.3).

The Chevalley basis

From now on we assume that $\mathfrak{L}$ is simple and that the field is algebraically closed and has characteristic 0.

We will examine the Cartan decomposition somewhat closer to establish the relationship with root systems as discussed before. Eventually we find a basis (called a Chevalley basis) for $\mathfrak{L}$ that generates a Lie algebra over the ring $\mathbb{Z}$. This basis plays a crucial role when defining Chevalley groups.

Lemma 1.52 (see Humphreys, [31], Proposition 8.1) $[\mathfrak{L}_{r^*}, \mathfrak{L}_{s^*}] \subseteq \mathfrak{L}_{r^*+s^*}$.

The *Killing form* is the scalar product k:

$$\mathsf{k}(x, y) = \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y).$$

Proposition 1.53 (Humphreys [31], Proposition 5.1) *k is non-singular.*

This is proved by showing that the *radical* $\{x \in \mathfrak{L} \mid \forall y \in \mathfrak{L} : \mathsf{k}(x, y) = 0\}$ of k is an ideal hence, by simplicity, must be $\mathfrak{L}$ itself or $\{0\}$. It then follows that we must have the latter.

Lemma 1.54 (Humphreys [31], Proposition 8.1) *Let $\mathfrak{L}_{r^*}$ and $\mathfrak{L}_{s^*}$ be root spaces. Then $k(\mathfrak{L}_{r^*}, \mathfrak{L}_{s^*}) = 0$ if $r^* + s^* \neq 0$.*

Corollary 1.55 *k restricted to $\mathfrak{H}$ is non-singular.*

As a consequence, for each root space $\mathfrak{L}_{\mathsf{r}^*}$ we can find an element $\mathsf{r} \in \mathfrak{H}$ such that $\mathsf{r}^*(h) = \mathsf{k}(\mathsf{r}, h)$ for all $h \in \mathfrak{H}$. The element r is called a *root* of the Lie algebra $\mathfrak{L}$; the set of all roots is called the *root system* of the Lie algebra $\mathfrak{L}$.

This root system is indeed a root system in the euclidean space $\mathfrak{H}$ in the sense of Subsection 1.3.1. We give references for the details of this important observation.

Following the literature we will identify in our notation each $\mathsf{r}^* \in \mathfrak{H}^*$ with the corresponding $\mathsf{r} \in \mathfrak{H}$. Proposition 8.4 of Humphreys [31] tells us that all root spaces $\mathfrak{L}_{\mathsf{r}^*}$ are one-dimensional, and that $[\mathfrak{L}_{\mathsf{r}}, \mathfrak{L}_{-\mathsf{r}}] \subseteq \langle \mathsf{r} \rangle \subseteq \mathfrak{H}$. Theorem 8.5 of Humphreys [31] shows us that k is positive definite on the space spanned by the roots in $\mathfrak{H}$ (over $\mathbb{R}$) and that the roots r form an irreducible root system Φ in $\mathfrak{H}$ with innerproduct k. In particular there exists a fundamental system Π for Φ.

It can be shown that the root system uniquely determines $\mathfrak{L}$ and that there exists a simple Lie algebra for each irreducible root system.

It will be convenient to fix a basis for such a Lie algebra, depending only on the type of the root system. We already know that we can choose an element from each root space and a basis for $\mathfrak{H}$ to find a basis for $\mathfrak{L}$. Following Carter (loc. cit.) we exploit the freedom left in choosing a basis for $\mathfrak{L}$, to find a basis with some very practical properties.

Let us first write down what kind of properties for a basis for $\mathfrak{L}$ we can easily find. We abbreviate $\mathsf{k}(\cdot, \cdot)$ by $(\cdot, \cdot)$.

For each root r, the scalar multiple $h_{\mathsf{r}} = 2\frac{\mathsf{r}}{(\mathsf{r},\mathsf{r})}$ is called its *coroot*. Following Carter we put $A_{\mathsf{r},\mathsf{s}} = (2\frac{\mathsf{r}}{(\mathsf{r},\mathsf{r})}, \mathsf{s}) = 2\frac{(\mathsf{r},\mathsf{s})}{(\mathsf{r},\mathsf{r})}$.

Let us choose an element e_{r} in each $\mathfrak{L}_{\mathsf{r}}$ with $\mathsf{r} \in \Phi^+$ and take $e_{-\mathsf{r}} \in \mathfrak{L}_{-\mathsf{r}}$ such that $[e_{\mathsf{r}}, e_{-\mathsf{r}}] = h_{\mathsf{r}}$. We now have a basis $\{h_{\mathsf{r}}(\mathsf{r} \in \Pi), e_{\mathsf{r}}(\mathsf{r} \in \Phi)\}$ of $\mathfrak{L}$. The multiplication in terms of these basis elements satisfies:

$$\begin{aligned} [h_{\mathsf{r}}, h_{\mathsf{s}}] &= 0 \\ [h_{\mathsf{r}}, e_{\mathsf{s}}] &= \tfrac{2(\mathsf{r},\mathsf{s})}{(\mathsf{r},\mathsf{r})} e_{\mathsf{r}} = A_{\mathsf{r},\mathsf{s}} e_{\mathsf{r}} \\ [e_{\mathsf{r}}, e_{-\mathsf{r}}] &= h_{\mathsf{r}} \\ [e_{\mathsf{r}}, e_{\mathsf{s}}] &= 0 \text{ if } \mathsf{r}+\mathsf{s} \notin \Phi \end{aligned}$$

Let us write $[e_{\mathsf{r}}, e_{\mathsf{s}}] = N_{\mathsf{r},\mathsf{s}} e_{\mathsf{r}+\mathsf{s}}$. The numbers $N_{\mathsf{r},\mathsf{s}}$ are called the *structure constants* of the basis. The structure constants depend, up to scalars, on the choice for the e_{r}. They satisfy a number of conditions.

Given two roots r and s, the largest integer p such that $-p\mathsf{r}+\mathsf{s}$ is a root is denoted by $p(\mathsf{r},\mathsf{s})$; the largest integer q such that $q\mathsf{r}+\mathsf{s}$ is a root is denoted by $q(\mathsf{r},\mathsf{s})$. It turns out that we can choose the basis $\{h_{\mathsf{r}}(\mathsf{r} \in \Pi), e_{\mathsf{r}}(\mathsf{r} \in \Phi)\}$ in such a way that

$$\begin{aligned} [h_{\mathsf{r}}, h_{\mathsf{s}}] &= 0, \\ [h_{\mathsf{r}}, e_{\mathsf{s}}] &= A_{\mathsf{r},\mathsf{s}} e_{\mathsf{r}}, \\ [e_{\mathsf{r}}, e_{-\mathsf{r}}] &= h_{\mathsf{r}}, \\ [e_{\mathsf{r}}, e_{\mathsf{s}}] &= 0 \text{ if } \mathsf{r}+\mathsf{s} \notin \Phi, \\ [e_{\mathsf{r}}, e_{\mathsf{s}}] &= \pm(p+1) e_{\mathsf{r}+\mathsf{s}} \text{ if } \mathsf{r}+\mathsf{s} \in \Phi, \end{aligned}$$

Such a basis is called a *Chevalley basis* (see Carter [14], Theorem 4.2.1). An important feature of this basis is that all structure constants are integral so that the Chevalley basis in fact spans what we might call a Lie algebra over $\mathbb{Z}$.

1.3.3 The Chevalley groups

Let us assume that $\mathfrak{L}$ is a simple Lie algebra over $\mathbb{C}$ and that we have fixed a maximal toral subalgebra $\mathfrak{H}$. Choose a Chevalley basis $\{h_{\mathsf{r}}(\mathsf{r} \in \Pi), e_{\mathsf{r}}(\mathsf{r} \in \Phi)\}$ for $\mathfrak{L}$; in this section we will refer to its elements as 'the basis elements for $\mathfrak{L}$'.

After the previous section we find ourselves ready to find automorphisms for $\mathfrak{L}$ that will move all basis elements of $\mathfrak{L}$ to sums of integral multiples of the basis elements. Because of this property we will be able to define automorphisms for Lie algebras over arbitrary fields. The group generated by these elements is a Chevalley group.

The exponential map

Lemma 1.56 (Carter [14], Section 4.3) $\operatorname{ad} e_{\mathsf{r}}$ *is a nilpotent map.*

For $t \in \mathbb{F}$ and $\mathsf{r} \in \Phi$ define

$$\begin{aligned} x_{\mathsf{r}}(t) = &\ \exp(t \operatorname{ad} e_{\mathsf{r}}) = \\ = &\ 1 + t \operatorname{ad} e_{\mathsf{r}} + \tfrac{1}{2!} t^2 (\operatorname{ad} e_{\mathsf{r}})^2 + \tfrac{1}{3!} t^3 (\operatorname{ad} e_{\mathsf{r}})^3 + \cdots + \tfrac{1}{m!} t^m (\operatorname{ad} e_{\mathsf{r}})^m \end{aligned}$$

where m is the smallest integer such that $(\operatorname{ad} e_{\mathsf{r}})^{m+1} = 0$.

Proposition 1.57 (Carter [14], Lemma 4.3.1) *$x_r(t)$ is an automorphism of the Lie algebra $\mathfrak{L}$.*

We will now concentrate on the precise action of $x_{\mathsf{r}}(t)$ on the Chevalley basis.

Lemma 1.58 (Carter [14], Section 3.3) *The sequence*

$$-pr+s, -(p-1)r+s, \ldots, s, \ldots, (q-1)r+s, qr+s$$

consist entirely of roots.

The sequence in the above lemma is called the *r-chain of roots through s*.

Lemma 1.59 (Carter [14], Section 3.4) $A_{r,s} = 2\frac{(r,s)}{(r,r)} = p(r,s) - q(r,s)$.

Lemma 1.60 (Carter [14], Section 4.3)

$$\begin{aligned}
x_r(t) \cdot e_r &= e_r \\
x_r(t) \cdot h_r &= h_r - 2te_r \\
x_r(t) \cdot e_{-r} &= e_{-r} + th_r - t^2 e_r \\
x_r(t) \cdot h_s &= h_s - A_{s,r} te_r \\
x_r(t) \cdot e_s &= e_s + N_{r,s} te_{r+s} + \tfrac{1}{2!} N_{r,s} N_{r,2r+s} t^2 e_{2r+s} \\
&\quad + \tfrac{1}{3!} N_{r,s} N_{r,2r+s} N_{r,3r+s} t^3 e_{3r+s} + \cdots + \tfrac{1}{q!} N_{r,s} \cdots N_{r,(q-1)r+s} t^q e_{qr+s},
\end{aligned}$$

where $q = q(r,s)$.

We abbreviate: $M_{\mathsf{r},\mathsf{s},i} = \frac{1}{i!} N_{\mathsf{r},\mathsf{s}} N_{\mathsf{r},\mathsf{r}+\mathsf{s}} N_{\mathsf{r},2\mathsf{r}+\mathsf{s}} \cdots N_{\mathsf{r},(i-1)\mathsf{r}+\mathsf{s}}$.

Thus we can rewrite the last equation of the lemma above as

$$x_{\mathsf{r}}(t)(e_{\mathsf{s}}) = \sum_{i=0}^{q} M_{\mathsf{r},\mathsf{s},i} t^i e_{\mathsf{r},i\mathsf{r}+\mathsf{s}}$$

Since $N_{\mathsf{r},\mathsf{s}} = \pm(p+1)$, we derive that $M_{\mathsf{r},\mathsf{s},i} = \pm\binom{p+i}{i}$, and we conclude that

Theorem 1.61 (Carter [14], Section 4.3) *Each element of the Chevalley basis is transformed by $x_r(t)$ into a linear combination of Chevalley basis elements whose coefficients are products of integers and integral powers of t.*

Chevalley groups over an arbitrary field

Now let $\mathfrak{L}_{\mathbb{Z}}$ be the subset of $\mathfrak{L}$ of all linear combinations of the basis elements with coefficients in $\mathbb{Z}$. It is clear that this set is closed under the bracket product.

For any field $\mathbb{F}$ let $\mathfrak{L}_{\mathbb{F}}$ be the tensor product $\mathbb{F} \otimes \mathfrak{L}_{\mathbb{Z}}$ over the prime field of $\mathbb{F}$.

We can extend the bracket product to $\mathfrak{L}_{\mathbb{F}}$ by putting

$$[1_{\mathbb{F}} \otimes x), (1_{\mathbb{F}} \otimes y)] = 1_{\mathbb{F}} \otimes [x, y]$$

and expanding it linearly. Thus $\mathfrak{L}$ becomes a Lie algebra over the field $\mathbb{F}$.

We define the action of the formal expression $x_{\mathsf{r}}(t)$ ($t \in \mathbb{F}$) on $\mathfrak{L}_{\mathbb{F}}$ in the obvious way. For instance:

$$x_{\mathsf{r}}(t) \cdot (1_{\mathbb{F}} \otimes e_{\mathsf{s}}) = \sum_{i=0}^{q} M_{\mathsf{r},\mathsf{s},i} t^i (1_{\mathbb{F}} \otimes e_{i\mathsf{r}+\mathsf{s}}).$$

Thus, we obtain a set of automorphisms $\{x_{\mathsf{r}}(t) \mid \mathsf{r} \in \Phi, t \in \mathbb{F}\}$ for $\mathfrak{L}_{\mathbb{F}}$.

The group $G = \mathfrak{L}(\mathbb{F})$ generated by the linear transformations $x_{\mathsf{r}}(t)$ ($\mathsf{r} \in \Phi$, $t \in \mathbb{F}$) is called the (adjoint) *Chevalley group* of type $\mathfrak{L}$ over the field $\mathbb{F}$.

Root groups and commutator relations

Lemma 1.62 (Carter [14], Section 5.1) *The set $X_{\mathsf{r}} = \{x_{\mathsf{r}}(t) \mid t \in \mathbb{F}\}$ ($\mathsf{r} \in \Phi$) is a subgroup of G which is isomorphic to the additive group of $\mathbb{F}$.*

The subgroup X_{r} is called the *root group* of the root r. A detailed case by case study produces the following useful result (known as the Chevalley commutator relations).

Theorem 1.63 (see Carter [14], Theorem 5.2.2)
Let $G = \mathfrak{L}(\mathbb{F})$ be a Chevalley group over an arbitrary field $\mathbb{F}$, r, s be linearly independent roots of $\mathfrak{L}$ and t, u elements of $\mathbb{F}$. Define the commutator

$$[x_{\mathsf{s}}(u), x_{\mathsf{r}}(t)] = x_{\mathsf{s}}^{-1}(u) x_{\mathsf{r}}^{-1}(t) x_{\mathsf{s}}(u) x_{\mathsf{r}}(t).$$

Then we have

$$[x_{\mathsf{s}}(u), x_{\mathsf{r}}(t)] = \prod_{i,j>0} x_{i\mathsf{r}+j\mathsf{s}}(C_{ij\mathsf{rs}}(-t)^i u^j),$$

where the product is taken over all pairs of positive integers i, j for which $ir + js$ is a root in order of increasing $i + j$. The constants C_{ijrs} are given by

$$\begin{aligned} C_{i1\mathsf{rs}} &= M_{\mathsf{r},\mathsf{s},i}, \\ C_{1j\mathsf{rs}} &= (-1)^j M_{\mathsf{r},\mathsf{s},j}, \\ C_{32\mathsf{rs}} &= \tfrac{1}{3} M_{\mathsf{r}+\mathsf{s},\mathsf{r},2}, \\ C_{23\mathsf{rs}} &= -\tfrac{2}{3} M_{\mathsf{s}+\mathsf{r},\mathsf{s},2}. \end{aligned}$$

Each $C_{ij\mathsf{rs}}$ is one of ± 1, ± 2, ± 3.

Let us now deal with subgroups generated by X_{r} and $X_{-\mathsf{r}}$ for some $\mathsf{r} \in \Phi$. These groups are closely related to the groups $\mathrm{SL}(2, \mathbb{F})$.

First note that the special linear group $\mathrm{SL}(2, \mathbb{F})$ can be generated, as a matrix group, by the matrices

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

where t runs through $\mathbb{F}$.

Now we can find a morphism from $\mathrm{SL}(2, \mathbb{F})$ onto $\langle X_{\mathsf{r}}, X_{-\mathsf{r}} \rangle$.

Theorem 1.64 (Carter [14], Proposition 6.5.2 and Theorem 6.3.1) *For each root* r *there exists a morphism* ϕ_{r} *from* $SL(2,\mathbb{F})$ *onto* $\langle X_{\mathsf{r}}, X_{-\mathsf{r}}\rangle$, *with*

$$\begin{aligned} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} &\mapsto x_{\mathsf{r}}(t), \text{ and} \\ \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} &\mapsto x_{-\mathsf{r}}(t). \end{aligned}$$

We will now examine the images under ϕ_{r} of certain special elements in $\mathrm{SL}(2,\mathbb{F})$.

For $t \neq 0$ define

$$\begin{aligned} h_{\mathsf{r}}(t) &= \phi(\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}), \\ n_{\mathsf{r}} &= \phi(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}). \end{aligned}$$

Proposition 1.65 (Carter [14], Propositions 6.4.1 and 6.4.2)

$$\begin{aligned} h_{\mathsf{r}}(t) \cdot h_{\mathsf{s}} &= h_{\mathsf{s}}, & \mathsf{s} \in \Pi, \\ h_{\mathsf{r}}(t) \cdot e_{\mathsf{s}} &= t^{A_{\mathsf{r},\mathsf{s}}} e_{\mathsf{s}}, & \mathsf{s} \in \Phi, \\ n_{\mathsf{r}}(t) \cdot h_{\mathsf{s}} &= h_{w_{\mathsf{r}}\mathsf{s}}, & \\ n_{\mathsf{r}}(t) \cdot e_{\mathsf{s}} &= \eta_{\mathsf{r},\mathsf{s}} e_{w_{\mathsf{r}}\mathsf{s}}, & \text{where } \eta_{\mathsf{r},\mathsf{s}} = \pm 1. \end{aligned}$$

A (B, N)**-pair**

Let U and V be the groups generated by the root groups X_{r} with $\mathsf{r} \in \Phi^+$ and $\mathsf{r} \in \Phi^-$ respectively.

We define three special subgroups B, N and H of G.

$$\begin{aligned} H &= \langle h_{\mathsf{r}}(t) \mid \mathsf{r} \in \Phi, t \in \mathbb{F} \setminus \{0\}\rangle, \\ N &= \langle H, n_{\mathsf{r}} \mid \mathsf{r} \in \Phi\rangle, \\ B &= UH. \end{aligned}$$

Theorem 1.66 (see Carter [14], Proposition 8.2.1) *The pair* (B,N) *is a Tits system for* G.

1.3.4 Chevalley groups and Moufang buildings

The relation between two descriptions of apartments

Let Φ be a root system with Dynkin diagram M and let $\Pi = \{\mathsf{r}_1, \ldots, \mathsf{r}_n\}$ be a fundamental system for Φ. Let W be the Weyl group generated by the fundamental reflections w_{r_i} $(i \in I = \{1, 2, \ldots, n\})$.

Lemma 1.67 (Carter [14], Theorem 2.2.4) *The group* W *acts sharply 1-transitively on the collection of fundamental root systems for* Φ.

Let $\mathcal{C}(W)$ be the Coxeter chamber system and let $\mathcal{C}(\Phi)$ be the collection of fundamental root systems in Φ. Then the map $\psi: \mathcal{C}(W) \longrightarrow \mathcal{C}(\Phi)$ given by $w \mapsto w\Pi$ is a bijection. This map is an isomorphism of chamber systems if we declare $x\Pi$ and $y\Pi$ to be i-adjacent if $x^{-1}y = w_{\mathsf{r}_i}$ $(1 \leq i \leq n)$.

Lemma 1.68 *For any root $r \in \Phi$ let $\alpha_r = \{\Pi \mid r \in \Phi^+(\Pi)\}$. Then the map $r \mapsto \alpha_r$ is a bijection between the set of roots in Φ and the set of roots in $\mathcal{C}(\Phi)$.*

Lemma 1.69 (Carter [14], Lemma 2.1.5) *Given a root system Φ and a fundamental system Π and a root $r \in \Pi$. Then w_r transforms every root from Π into a positive root, except that it transforms r into the negative root $-r$.*

Corollary 1.70 *Let $\Pi = \{r_1, \ldots, r_n\}$ and let $\Pi' = w_{r_i}\Pi$ ($i \in \{1, \ldots, n\}$). Suppose that $r \in \Phi^\epsilon(\Pi)$ ($\epsilon = \pm$). Then also $r \in \Phi^\epsilon(\Pi')$, except if $r = r_i$ (resp. $r = -r_i$).*

Proof: Suppose $r \in \Phi^+$, that is $r = \sum_{j=1}^n \lambda_j r_j$ with $\lambda_j \geq 0$ ($j = 1, \ldots, n$). We first show that $w_{r_i} r$ is in $\Phi^+(\Pi)$, except if $r = r_i$.

Clearly, if $\lambda_j = 0$ for all $j \neq i$, then $r = r_i$ and $w_{r_i} r \in \Phi^-(\Pi)$. Suppose that $\lambda_j > 0$ for some $j \neq i$. Then in view of Lemma 1.69 the coefficient of r_j in the expression for $w_{r_i} r$ will be positive, but this is impossible unless $r \in \Phi^+$.

Finally, $w_{r_i} r = \sum_{j=1}^n \mu_j r_j$ if and only if $r = \sum_{j=1}^n \mu_j w_{r_i} r_j$ and we are done.

The case $r \in \Phi^-$ is dealt with likewise. □

Proof: (of Lemma 1.68) If $\Pi_0, \ldots, \Pi_l$ is a gallery and r is a root, and $r \in \Phi^+(\Pi_0)$, then $r \in \Phi^+(\Pi_l)$ if and only if the gallery crosses the wall determined by the reflection w_r an even number of times. Thus (cf. Lemma 1.13) α_r is a root of $\mathcal{C}(\Phi)$.

Conversely, let α be a root of $\mathcal{C}(\Phi)$. Take a chamber Π of α having a panel p on the wall of α. Then, for some $r \in \Pi$, the reflection w_r interchanges α and $-\alpha$ and $p \supset \{\Pi, w_r\Pi\}$. The root α_r contains the chamber Π, but not $w_r\Pi$.

Thus both α and α_r are roots determined by the i-panel and contain Π, hence (cf. page 10) they coincide. □

Now let Δ be the building corresponding to the Tits-system (B, N) of G. We recall that its chambers are the left cosets of B in G. The group G acts by left multiplication on Δ as a group of automorphisms. We show that the root groups X_r are root groups in the sense of Section 1.2.5.

Let Σ be the apartment whose chambers are the left cosets wB ($w \in W = N/H$). We see that the chamber systems Σ and $\mathcal{C}(\Phi)$ are isomorphic via the map $wB \mapsto w\Pi$.

Choose a root α of Σ. Then by Lemma 1.68 there is a root $r \in \Phi$ such that α consists of those chambers wB for which $r \in \Phi^+(w\Pi)$.

Let $s \in \Phi^+(\Pi)$ be such that $r = ws$. Let $n \in N$ be such that $nH = wH$. We have $X_{ws} = nX_s n^{-1}$ (cf. Proposition 1.65). Since $B = HU$, $U = \langle X_r \mid r \in \Phi^+(\Pi)\rangle$, we have $nX_s n^{-1} nB = (nX_s)B = n(X_s B) = wB$. Thus X_r stabilizes all chambers of the root α_r and hence is a root group in the sense of Section 1.2.5 (whenever the term applies).

1.3.5 Embeddings for shadow spaces

Let Δ be a spherical building defined over a field $\mathbb{F}$ with a diagram M defined over an index set I. Fix $i_0 \in I$ and let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the i_0-shadow space of Δ. We describe how Γ can be embedded into the irreducible module whose highest weight is the fundamental weight λ_{i_0} of the semisimple Lie algebra with diagram M.

This subsection is meant as a reformulation of (part of) the theory on embeddings of shadow spaces presented in Cohen [17]. We first sketch how one constructs an irreducible Lie algebra module with given highest weight. In the beginning we shall assume that the field is $\mathbb{C}$, but later on we will also consider arbitrary fields. We then indicate how one constructs Chevalley groups acting on these modules. Finally we show how a shadow space of a building can be embedded into such a module. More on this subject can be found in Humphreys [32], Ch. XI or [31], Ch. VII or Cohen [17].

We will mainly follow Carter et al. [13] and occasionally need details, for which we refer to Humphreys [31].

The universal enveloping algebra

Let $\mathfrak{L}$ be a finite dimensional Lie algebra over $\mathbb{C}$. Let $\mathfrak{L}^{\otimes}$ be the tensor algebra of $\mathfrak{L}$.

$$\mathfrak{L}^{\otimes} = \mathbb{C}1 \otimes \mathfrak{L} \otimes (\mathfrak{L} \otimes \mathfrak{L}) \otimes (\mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L}) \otimes \dots$$

The vector space $\mathfrak{L}^{\otimes}$ defined over $\mathbb{C}$ has a natural multiplication. Let $\mathfrak{I}$ be the two-sided ideal of $\mathfrak{L}^{\otimes}$ generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y] \quad \text{for } x, y \in \mathfrak{L}$$

Let $\mathfrak{U} = \mathfrak{L}^{\otimes}/\mathfrak{I}$. Then $\mathfrak{U}$ is an associative algebra called the *universal enveloping algebra* of $\mathfrak{L}$.

A basis for $\mathfrak{U}$ can be obtained as follows. If $x_1, \dots, x_n$ is a basis for $\mathfrak{L}$, then the set of elements

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad i_r \in \mathbb{Z}, i_r \geq 0$$

forms a basis for $\mathfrak{U}$. This is called the Poincaré-Birkhoff-Witt basis theorem. We note that we have

$$x_i x_j - x_j x_i = [x_i, x_j]$$

An important property of the enveloping algebra $\mathfrak{U}$ is that is has the same representation theory as $\mathfrak{L}$. If V is an $\mathfrak{L}$-module then V can be regarded as an $\mathfrak{L}^{\otimes}$-module in a natural way. Since

$$[x, y]v = x(yv) - y(xv)$$

for $x, y \in \mathfrak{L}$, $v \in V$ we see that

$$(x \otimes y - y \otimes x)v - [x, y]v = 0$$

for all $v \in V$. Thus elements $x \otimes y - y \otimes x - [x, y]$ lie in the the kernel of the action on the module V. This kernel is a two-sided ideal of $\mathfrak{L}^{\otimes}$ so contains $\mathfrak{I}$.

Conversely every $\mathfrak{U}$-module may be regarded as an $\mathfrak{L}$-module under the map

$$\mathfrak{L} \longrightarrow \mathfrak{L}^{\otimes} \longrightarrow \mathfrak{U}.$$

This map is injective by the PBW-basis theorem, and so $\mathfrak{L}$ may be regarded as a subspace of $\mathfrak{U}$.

Verma modules

We now suppose that $\mathfrak{L}$ is a non-trivial simple Lie algebra. Let $\mathfrak{H}$ be a maximal toral subalgebra (Cartan subalgebra) of $\mathfrak{L}$ and

$$\mathfrak{L} = \mathfrak{H} \bigoplus_{\mathsf{r} \in \Phi} \mathbb{C} e_{\mathsf{r}}$$

be the Cartan decomposition of $\mathfrak{L}$ with respect to $\mathfrak{H}$. We recall that the Killing form gives a bijection $\mathfrak{H} \leftrightarrow \mathfrak{H}^*$. In the preceding sections we used this to actually identify $\mathfrak{H}$ with its dual. In this section however we will make the distinction. Let $\Pi = \{\mathsf{r}_1, \ldots, \mathsf{r}_n\}$ be a fundamental system for Φ and let h_i be the element corresponding to $\frac{2\mathsf{r}_i}{(\mathsf{r}_i,\mathsf{r}_i)} \in \mathfrak{H}^*$ under this bijection. Then

$$\mathsf{r}_j(h_i) = \frac{2(\mathsf{r}_i, \mathsf{r}_i)}{(\mathsf{r}_i, \mathsf{r}_i)} = A_{\mathsf{r}_i,\mathsf{r}_j} \in \mathbb{Z}.$$

Thus all the fundamental roots $\mathsf{r}_i, \ldots, \mathsf{r}_n$ take integer values at h_i. The special vectors $h_1, \ldots, h_n$ form a basis for $\mathfrak{H}$. They are called the *fundamental coroots*.

Let $\lambda \in \mathfrak{H}^*$ and let $\mathfrak{J}(\lambda)$ be the left ideal of $\mathfrak{U}$ generated by the elements e_{r} ($\mathsf{r} \in \Phi^+$) and $h_i - \lambda(h_i)1$ for $i = 1, \ldots, n$. Thus

$$\mathfrak{J}(\lambda) = \bigoplus_{\mathsf{r} \in \Phi^+} \mathfrak{U} e_{\mathsf{r}} \oplus \bigoplus_{i=1}^{n} \mathfrak{U}(h_i - \lambda(h_i)1).$$

Then $\mathfrak{J}(\lambda)$ is a $\mathfrak{U}$-submodule of $\mathfrak{U}$. Let $M(\lambda) = \mathfrak{U}/\mathfrak{J}(\lambda)$. Then $M(\lambda)$ is also a $\mathfrak{U}$-submodule, called the *Verma module* detemined by λ. We have a natural homomorphism

$$\mathfrak{U} \xrightarrow{\theta} M(\lambda)$$

of left $\mathfrak{U}$-modules. Let $m_\lambda = \theta(1)$. Then we have

$$\begin{aligned} e_{\mathsf{r}} m_\lambda &= 0 \text{ for all } \mathsf{r} \in \Phi^+ \\ h_i m_\lambda &= \lambda(h_i) m_\lambda \text{ for } i = 1, \ldots, n. \end{aligned}$$

Since each element $u \in \mathfrak{U}$ satisfies $u = u1$, each element of $M(\lambda)$ has the form $u m_\lambda$ for some $u \in \mathfrak{U}$.

Thus

$$M(\lambda) = \mathfrak{U} m_\lambda$$

is a *cyclic* $\mathfrak{U}$-module.

We may regard $M(\lambda)$ as an $\mathfrak{H}$-module. Then $M(\lambda)$ decomposes into the direct sum of one-dimensional $\mathfrak{H}$-submodules. The one-dimensional representations obtained from these submodules are called the *weights* of $M(\lambda)$. We note that λ is a weight of $M(\lambda)$ since

$$h_i m_\lambda = \lambda(h_i) m_\lambda.$$

All the weights of $M(\lambda)$ have the form

$$\lambda - m_1\mathsf{r}_1 - \cdots - m_n\mathsf{r}_n$$

where $m_1, \ldots, m_n$ are non-negative integers. Thus λ is in a natural sense the highest weight of $M(\lambda)$. We call $M(\lambda)$ the Verma module with highest weight λ.

It can be shown that $M(\lambda)$ has a unique maximal irreducible submodule $K(\lambda)$. Let $L(\lambda) = M(\lambda)/K(\lambda)$. Then $L(\lambda)$ is an irreducible $\mathfrak{U}$-module.

We thus have a procedure for constructing irreducible $\mathfrak{L}$-modules. For each $\lambda \in \mathfrak{H}^*$ we have obtained an irreducible $\mathfrak{L}$-module $L(\lambda)$ as the top quotient of the Verma module $M(\lambda)$. The module $L(\lambda)$ is not necessarily finite dimensional –this depends on the choice of λ.

Fundamental dominant weights

Theorem 1.71 (cf. Humphreys [31], §21.1 and §21.2) *The dimension of $L(\lambda)$ is finite if and only if $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, n$.*

We call $\lambda \in \mathfrak{H}^*$ *integral* if $\lambda(h_i) \in \mathbb{Z}$ for all i and *dominant integral* if in addition $\lambda(h_i) \geq 0$ for all i.

Theorem 1.72 (see Humphreys [31], §21.2) *Every finite dimensional irreducible $\mathfrak{L}$-module has the form $L(\lambda)$ for some dominant integral $\lambda \in \mathfrak{H}^*$.*

The dominant integral weights can be descibed conveniently in the following way. Let $\omega_i \in \mathfrak{H}^*$ satisfy $\omega_i(h_j) = \delta_{ij}$ (Kronecker δ). The elements $\omega_1, \ldots, \omega_n$ of $\mathfrak{H}^*$ uniquely determined in this way are called the *fundamental weights*. They form a basis of $\mathfrak{H}^*$. It is clear that the dominant integral weights are precisely the weights $\lambda = \sum_{i=1}^n m_i\omega_i$ with all $m_i \in \mathbb{Z}_{\geq 0}$

We consider the relation between the fundamental weights $\omega_1, \ldots, \omega_n$ and the fundamental roots $\mathsf{r}_1, \ldots, \mathsf{r}_n$. Let

$$\mathsf{r}_i = \sum_{j=1}^{n} m_{ij}\omega_j.$$

Then we have $\mathsf{r}_i(h_j) = m_{ij}\omega_j(h_j) = m_{ij}$. By definition of h_j we have

$$m_{ij} = \mathsf{r}_i(h_j) = (\mathsf{r}_i, \frac{2\mathsf{r}_j}{(\mathsf{r}_j, \mathsf{r}_j)}) = A_{\mathsf{r}_j,\mathsf{r}_i}.$$

Thus

$$\mathsf{r}_i = \sum_{j=1}^{n} A_{\mathsf{r}_i,\mathsf{r}_j}\omega_j.$$

Thus the transpose of the Cartan matrix transforms the fundamental weights into the fundmental roots.

A fundamental weight ω is called *minimal* (sometimes *minuscule*) if there are non-negative integers m_i $(i = 1, \ldots, n)$ with $m_i > 0$ for at least one i, such that

$$\omega - m_1\mathsf{r}_1 - \cdots - m_n\mathsf{r}_n$$

is positive.

Proposition 1.73 (cf. Bourbaki [6], Ch. 8, §7, Proposition 6) *Let λ be a fundamental dominant weight and let $\mathcal{H}$ be the collection of weights of the irreducible module $L(\lambda)$. Let W denote the Weyl group. Then $\mathcal{H} = W\lambda$ if and only if λ is a minimal weight.*

Arbitrary fields

Let us fix some dominant weight λ and let V be the irreducible module with highest weight λ. Let $\mathbb{F}$ be any field. We will sketch how one can construct a Lie algebra over $\mathbb{F}$, which we shall denote by $\mathfrak{L}_V(\mathbb{F})$, that acts on a module $V(\mathbb{F})$ obtained from the module V.

Regard $\mathfrak{L}$ as a subalgebra of its universal enveloping algebra $\mathfrak{U}$. Let $\mathfrak{U}_{\mathbb{Z}}$ be the subring with 1 of $\mathfrak{U}$ generated by the elements of the form $\frac{e_{\mathsf{r}}^m}{m!}$. A lattice in a $\mathfrak{L}$-module is called *admissible* if it is invariant under the action of $\mathfrak{U}_{\mathbb{Z}}$. By Theorem 27.1 of Humphreys [31] such a lattice always exists.

Let us choose an admissible lattice, Λ, in V. The stabilizer of this lattice in $\mathfrak{L}$ is the Lie algebra over $\mathbb{Z}$ generated by the Chevalley basis, possibly extended by a subalgebra of $\mathfrak{H}$.

The Lie algebra $L_V(\mathbb{F})$ is obtained by tensoring the stabilizer in $\mathfrak{L}$ of the lattice Λ with the field $\mathbb{F}$. Similarly the module $V(\mathbb{F})$ is obtained by tensoring the lattice Λ with the field $\mathbb{F}$. It is clear that the action of $L_V(\mathbb{F})$ on $V(\mathbb{F})$ is well defined and induced by the action of $\mathfrak{U}_{\mathbb{Z}}$ on V.

We now examine the weights of this module. Suppose $\mathcal{H}$ is the set of weights of V and for $\mu \in \mathcal{H}$ we denote by V_μ its weight space. As a consequence of Theorem 27.1 of Humphreys [31], we have $\Lambda = \bigoplus_{\mu\in\mathcal{H}} \Lambda \cap V_\mu$ and therefore $V(\mathbb{F}) = \bigoplus_{\mu\in\mathcal{H}} V_\mu(\mathbb{F})$ where $V_\mu(\mathbb{F}) = (\Lambda \cap V_\mu) \otimes_{\mathbb{Z}} \mathbb{F}$.

Chevalley groups acting on Lie algebra modules

1.3.3. We keep the notation of the previous subsubsection. Let $\{h_{\mathsf{r}} \mid \mathsf{r} \in \Pi\} \cup \{e_{\mathsf{r}} \mid \mathsf{r} \in \Phi\}$ be the Chevalley basis for $\mathfrak{L}$. Let ϕ be the $\mathfrak{L}$-representation corresponding to the module V. Then the Chevalley group $G(V)$ corresponding to the module V is the subgroup of $\mathrm{GL}(V)$ generated by the elements of the form $\exp t\phi e_{\mathsf{r}} = \sum_{k=0}^{\infty} t^k \phi \left[\frac{e_{\mathsf{r}}^k}{k!}\right]$ with $t \in \mathbb{C}$.

Similarly let $\phi_{\mathbb{F}}$ be the $\mathfrak{L}_V(\mathbb{F})$-representation corresponding to the module $V(\mathbb{F})$. Then the Chevalley group $G_{\mathbb{F}}(V)$ corresponding to the module $V(\mathbb{F})$ is the subgroup of $\mathrm{GL}(V(\mathbb{F}))$ generated by the elements of the form $\exp t\phi e_{\mathsf{r}} = \sum_{k=0}^{\infty} t^k \phi \left[\frac{e_{\mathsf{r}}^k}{k!}\right]$ with $t \in \mathbb{F}$.

An example: $\mathfrak{sl}_2$

Let $\mathfrak{L} = \mathfrak{sl}_2(\mathbb{C})$. Taking $e_{\mathsf{r}} = x$, $h_{\mathsf{r}} = h$ and $e_{-\mathsf{r}} = y$, we see that the basis given in Subsection 1.3.2 is a Chevalley basis for $\mathfrak{L}$. We see that $\mathfrak{H}$ has dimension 1 and so there is only one fundamental weight ω, say. Let $V = L(\lambda)$ with $\lambda = m\omega$. In this case Theorem 1.71 can be rephrased as $\dim(L(m\omega)) = m + 1$ (cf. Humphreys [31], Ch. 1, §7.).

Let us look at the case where $m = 1$, i.e. λ is a fundamental dominant weight. Then V is a two-dimensional vector space over $\mathbb{C}$ with basis elements $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$ and we may assume that $e_{\mathsf{r}}, h_{\mathsf{r}}$ and $e_{-\mathsf{r}}$ act on this basis in the obvious way. It is clear that the lattice Λ spanned by e_1 and e_2 is invariant under the action of e_{r}, h_{r} and $e_{-\mathsf{r}}$. Hence Λ is admissible.

The Chevalley group of $\mathfrak{L}$ belonging to the module L has elements

$$x_{\mathsf{r}}(t) = \begin{pmatrix}1 & t\\0 & 1\end{pmatrix},\ x_{-\mathsf{r}}(t) = \begin{pmatrix}1 & 0\\t & 1\end{pmatrix}.$$

The subgroup generated by the $x_{\mathsf{r}}(t)$ ($t \in \mathbb{C}$) is called the *root subgroup* belonging to the root r.

In this case we can repeat this construction for an arbitrary field $\mathbb{F}$ only by replacing '$\mathbb{C}$' with $\mathbb{F}$. Notice that we do not get into trouble with the characteristic since $e_{\mathsf{r}}^k = 0$ already for $k = 2$.

Embedding a shadow space into a Lie algebra module

Let Δ be a spherical building defined over a field $\mathbb{F}$ with a diagram M defined over an index set I. Fix $i_0 \in I$ and let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the i_0-shadow space of Δ.

We describe how Γ can be embedded into the irreducible module whose highest weight is the fundamental weight λ_{i_0} of the semisimple Lie algebra with diagram M. The relation between the geometry and the geometry induced on them by these embeddings was studied by Cohen and Cooperstein in [20]

Let $\mathfrak{L}$ be the semisimple Lie algebra over $\mathbb{C}$ with root system Φ of type M. Let λ_{i_0} be the fundamental dominant weight corresponding to the node of M with label i_0 and suppose $V = L(\lambda_{i_0})$ is the unique irreducible $\mathfrak{L}$-module of highest weight λ_{i_0} with maximal vector v^+.

From this module we can construct a module $V' = V(\mathbb{F})$ over $\mathbb{F}$ and a Chevalley group $G_V(\mathbb{F})$ of type M acting on V' (see page 25). In this procedure we make sure that the admissible lattice involved is minimal; the reason for this will become apparent in the proof of Lemma 1.74 below. Since G has a (B, N)-pair (see e.g. Steinberg [43], §3. and Carter [14], Ch. 8), there exists a building Δ_G with diagram M whose chamber system can be identified with the collection G/B of left cosets of B (see Theorems 1.26 and 1.27). If M is simply-laced, then this is the unique ($|I| \geq 3$) spherical building over $\mathbb{F}$ with diagram M, and if $M = B_n$ or C_n, it is the building obtained from $O_{2n+1}(\mathbb{F})$ or $Sp_{2n}(\mathbb{F})$ respectively. In the sequel, we will identify Δ with Δ_G. The sets of points and lines of Γ will then be identified with the set G/P of left cosets of the standard parabolic subgroup $P = P_{I-\{i_0\}}$ and the collection of subsets $gP_{\{i_0\}}P$ ($g \in G$), respectively.

Lemma 1.74 *The mapping $\zeta : G/P \longrightarrow \mathbb{P}(V')$ sending gP to the projective point $[gv^+]$ is an embedding such that $\langle \zeta(G/P) \rangle = \mathbb{P}(V')$.*

Proof: We will first prove that ζ is well-defined and that it sends points to points and lines to lines, preserving incidence. As G is a subgroup of $\mathrm{GL}(V')$, it suffices to consider the image of P and $P_{\{i_0\}}P$ only. In order to determine the action of the standard parabolic subgroups P and $P_{\{i_0\}}$ on $[v^+]$, we will first examine the action of a minimal standard parabolic subgroup $P_{\{i\}}$ for arbitrary $i \in I$ on $[v^+]$. Using the Levi decomposition of a standard parabolic subgroup, we write $P_{\{i\}} = U_{\{i\}} \cdot L_{\{i\}}$, where $L_{\{i\}} = \langle H, U_{\alpha_i}, U_{-\alpha_i} \rangle$ and $U_{\{i\}} = \langle U_\alpha | \alpha \in \Phi^+ - \{\alpha_i\} \rangle$. Here, U_α denotes the root group of a root $\alpha \in \Phi$ and $H = N \cap B$. Furthermore, as H normalizes each U_α, we have $L_{\{i\}} = \langle U_{\alpha_i}, U_{-\alpha_i} \rangle H$. By definition, $[v^+]$ is stabilized by B and hence in particular by H and $U_{\{i\}}$ which implies that in fact $P_{\{i\}}[v^+] = \langle U_{\alpha_i}, U_{-\alpha_i} \rangle [v^+]$.

We now turn to the group $\langle U_{\alpha_i}, U_{-\alpha_i} \rangle$. Since $U_\alpha = \{x_\alpha(t) | t \in \mathbb{F}\}$ ($\alpha \in \Phi$), this is the Chevalley group of the Lie subalgebra $\mathfrak{l}$ with basis $\{h_i, e_{\alpha_i}, e_{-\alpha_i}\}$, acting on $\mathbb{P}(V')$ by the restriction of ϕ to $\mathfrak{l}$. Let S be the irreducible $\mathfrak{l}$-submodule of V' containing v^+. This module has dimension $\lambda_{i_0}(h_i) + 1$ (see page 33).

Since λ_{i_0} is a fundamental weight, we have $\lambda_{i_0}(h_i) \leq 1$, implying that $S = \mathbb{F}\langle U_{\alpha_i}, U_{-\alpha_i}\rangle v^+$. Looking again at the dimension of S we find that P stabilizes $[v^+]$ and that $P_{\{i_0\}}P[v^+]$ is the set of all points on a line. Thus, ζ is well-defined and sends points to points and lines to lines. As a line in G/P is a set of points, ζ preserves incidence.

Injectivity of ζ follows from the fact that the stabilizer $\mathrm{Stab}_G([v^+])$ of $[v^+]$, by containing the maximal standard parabolic subgroup P, is itself standard parabolic and hence must be equal to P. Finally, by the minimal choice of the admissible lattice L, the $\mathbb{F}G$-module V' is cyclic (see Humphreys [31], §27.5), whence the equality $\langle\zeta(G/P)\rangle = \mathbb{P}(V')$ follows. □

Chapter 2

Spanning sets for shadow spaces of buildings

In this chapter we find spanning sets for many shadow spaces of spherical buildings. In Section 2.1 we define the basic notions. In Section 2.2 we consider the shadow spaces of buildings of type A_n, B_n, C_n, D_n, E_6 and E_7 that correspond to a minimal weight of the corresponding Lie algebra. The arguments are based on general building theory. In Section 2.3 we find spanning sets for all polar spaces associated to a reflexive sesquilinear or pseudo-quadratic form. Here the arguments are more geometric and based on the orthogonal decomposition of the embedding and some fiddling with sigmas and epsilons. Some of the results from this chapter were published earlier [4].

2.1 Point-line geometries, their embeddings and spanning sets

We explore some general properties of spanning sets, projective embeddings and hyperplanes of point-line geometries and the relation between them.

Point-line geometries

Let $\Gamma = (\mathcal{O}, \star, \tau, I)$ be a geometry of rank 2 (i.e. $|I| = 2$). Call the objects of type 1 and 2 *points* and *lines* respectively, and let $\mathcal{P}$ and $\mathcal{L}$ be the set of points and lines respectively. With this convention, we write $\Gamma = (\mathcal{P}(\Gamma), \mathcal{L}(\Gamma))$ or even $\Gamma = (\mathcal{P}, \mathcal{L})$ if no confusion can arise.

We say that $\Gamma = (\mathcal{P}, \mathcal{L})$ is a *partial linear space* if it satisfies the following two additional conditions.

(L0) any element from $\mathcal{L}$ is incident with at least two elements from $\mathcal{P}$,

(L1) any two elements from $\mathcal{P}$ (resp. $\mathcal{L}$) are incident with at most one element from $\mathcal{L}$ (resp. $\mathcal{P}$).

It is not difficult to see that we can view (define) the lines of a partial linear space as subsets of the point set (with the incidence relation as (symmetrized) '$\in$'). A partial linear space in this guise will be called a *point-line geometry.*

We now give a direct definition. A *point-line geometry* Γ is a pair $(\mathcal{P}, \mathcal{L})$, (or $(\mathcal{P}(\Gamma), \mathcal{L}(\Gamma))$) consisting of a set $\mathcal{P}$ whose members are called points, and a collection $\mathcal{L}$ of subsets of $\mathcal{P}$ of cardinality at least two, whose members are called lines, satisfying the condition that no two points are on (in) two different lines. Incidence is defined by '$\in$'.

Subspaces and spanning sets

For any subset X of $\mathcal{P}$ let $\mathcal{L}(X)$ be the set $\{L \cap X \mid L \in \mathcal{L}, |L \cap X| > 1\}$. A *subspace* is a point set X such that every line that intersects X in at least two points is entirely contained in X. Note that $(X, \mathcal{L}(X))$ is a point-line geometry that is a subspace if and only if $\mathcal{L}(X) \subseteq \mathcal{L}$. A *hyperplane* is a subspace that intersects every line. Thus for any hyperplane H and any line L we have $L \subseteq H$ or $|L \cap H| = 1$.

For a point set S, let $\langle S\rangle_\Gamma$ be the smallest subspace of Γ (with respect to inclusion) that contains S; it is the intersection of all subspaces containing S. If $X = \langle S\rangle_\Gamma$, then S is said to *span*, or to be a *spanning set* for, the subspace X. If $\langle S\rangle_\Gamma = \mathcal{P}(\Gamma)$, we simply say that S is a spanning set. A finite spanning set S is *minimal* if $\langle S\rangle_\Gamma = \langle S'\rangle_\Gamma$ implies $|S| \leq |S'|$.

Given a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, and a point $p \in \mathcal{P}$ we denote by $p^\perp$ the set of points collinear to p. We extend this definition to subsets $X \subseteq \mathcal{P}$ by putting $X^\perp = \cap_{p \in X} p^\perp$.

Embeddings

An *embedding* of a point-line geometry Γ into another point-line geometry Γ' is an injective map $\mathcal{P}(\Gamma) \xrightarrow{\alpha} \mathcal{P}(\Gamma')$ that sends lines onto lines. The embedding α is called *projective* if Γ' is the geometry of points and lines of some desarguesian projective space $\mathbb{P}$ such that $\langle \alpha(\mathcal{P}(\Gamma))\rangle_\mathbb{P} = \mathbb{P}$. The dimension of the embedding is the dimension of the vector space underlying $\mathbb{P}$.

Organising the class of projective embeddings of Γ into a category, we define a morphism of embeddings $\Gamma \xrightarrow{\hat\alpha} \hat{\mathbb{P}}$, $\Gamma \xrightarrow{\alpha} \mathbb{P}$ to be a map $\hat{\mathbb{P}} \xrightarrow{\pi} \mathbb{P}$ induced by a semilinear map of the underlying vector spaces, such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\hat\alpha} & \hat{\mathbb{P}} \\ & \overset{\alpha}{\searrow} & \downarrow \pi \\ & & \mathbb{P} \end{array}$$

is commutative. The embedding $\hat\alpha$ is called *universal relative to* α if for any embedding α' and morphism $\alpha' \xrightarrow{\pi'} \alpha$, there exists a morphism $\hat\alpha \xrightarrow{\hat\pi} \alpha'$ such that $\pi = \pi' \circ \hat\pi$. The embedding α is *universal relative to itself* or *relatively universal*[1] if any embedding α' having α as a morphic image, is isomorphic to α. The embedding α is *absolutely universal* or simply *universal* if for every embedding β there is a morphism from α to β.

It is true that a universal embedding is relatively universal, but the converse is certainly not true. A geometry may not even have a universal embedding (see e.g. Batens and Pasini

[1] In Tits [45] Ch. 8 such an embedding is called 'dominant'.

[2]). If a point-line geometry has an embedding then it has a relatively universal embedding as well, as we will see now.

Lemma 2.1 *(i) Let $\alpha_i : \Gamma \longrightarrow \mathbb{P}_i$ $(i = 1, 2)$ be projective embeddings. If there are morphisms $\pi_{12}: \alpha_1 \longrightarrow \alpha_2$ and $\pi_{21}: \alpha_2 \longrightarrow \alpha_1$, then α_1 and α_2 are isomorphic.*

(ii) An embedding that is universal relative to another embedding is universal relative to itself.

Proof: (i) By assumption there exists a set of points $\{p_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{P}(\Gamma)$ such that $\{\alpha_1(p_\lambda)\}_{\lambda\in\Lambda}$ is a basis for $\mathbb{P}_1$. The morphism π_{12} maps $\alpha_1(p_\lambda)$ to $\alpha_2(p_\lambda)$ hence the set $\{\alpha_2(p_\lambda)\}_{\lambda\in\Lambda}$ spans $\mathbb{P}_2$. The morphism π_{21} maps $\alpha_2(p_\lambda)$ to $\alpha_1(p_\lambda)$ hence the set $\{\alpha_2(p_\lambda)\}_{\lambda\in\Lambda}$ is independent in $\mathbb{P}_2$. Thus π_{21} is an isomorphism from $\mathbb{P}_2$ to $\mathbb{P}_1$.

(ii) Let $\alpha, \hat{\alpha}$ and π be as in the picture above and suppose $\hat{\alpha}$ is universal relative to α. Suppose that ϕ is a morphism from an embedding β to $\hat{\alpha}$. Then $\pi\circ\phi$ is a morphism from β to α and hence there exists a morphism ψ from $\hat{\alpha}$ to β (such that $\pi\circ\phi\circ\psi = \pi$). By (i) we are done. □

Proposition 2.2 (cf. Ronan [37] Proposition 3) *Let Γ be a point-line geometry. Then for every projective embedding α of Γ, there is an embedding $\hat{\alpha}$ that is universal relatively to it; this $\hat{\alpha}$ is relatively universal.*

Proof: This is Proposition 3 of Ronan [37] combined with Lemma 2.1. □

Let α be an embedding of $\Gamma = (\mathcal{P}, \mathcal{L})$ into a projective geometry $\mathbb{P}$. Then a hyperplane $\bar{H}$ of $\mathbb{P}$ intersects $\alpha(\mathcal{P})$ in a hyperplane of Γ; such a hyperplane will be called *projectively induced*. Since the preimage of a hyperplane under a morphism of embeddings is again a hyperplane we can use 2.2 to restrict the study of projectively induced hyperplanes to relatively universal embeddings.

There has been some study into geometries all of whose hyperplanes are projectively induced (Ronan [37], Cooperstein and Shult [21], Shult [40], Shult and Thas [42], Hall and Shult [30], Shult [41] and Cooperstein and Shult [23]). In all concrete cases it is shown that all hyperplanes are induced by the same (relatively universal) embedding.

There is in fact the following result, due to Shult, Cooperstein and Thas:

Theorem 2.3 *Let Γ be the shadow space of a building with diagram $A_{n,i}$ $(i = 1, 2, \ldots, n)$, $B_{n,n}$, $C_{n,1}$, $D_{n,1}$, $D_{n,n-1}$, $D_{n,n}$, $E_{6,1}$, $E_{6,6}$ or $E_{7,1}$. Then all hyperplanes of Γ are induced by the natural Lie algebra embedding.*

Lemma 2.4 *Let Γ be a point-line geometry with finite spanning set S and let $\Gamma \stackrel{\alpha}{\longrightarrow} \mathbb{P}(V)$ be a projective embedding. Then,*

(i) $\dim(V) \leq |S|$, *and*

(ii) $\dim(V) = |S|$ *implies that for all spanning sets S' and all embeddings $\Gamma \stackrel{\alpha'}{\longrightarrow} \mathbb{P}(V')$, we have $\dim(V') \leq \dim(V) = |S| \leq |S'|$; In particular, S is a minimal spanning set and α is universal relative to itself.*

Proof: (i) We have $\langle\alpha(S)\rangle_{\mathbb{P}(V)} = \langle\alpha(\langle S\rangle_\Gamma)\rangle_{\mathbb{P}(V)} = \langle\alpha(\Gamma)\rangle_{\mathbb{P}(V)} = \mathbb{P}(V)$.
(ii) Suppose that $\dim(V) = |S|$. The (in-) equalities $\dim(V') \leq \dim(V) = |S| \leq |S'|$ are immediate from part (i). If $\Gamma \xrightarrow{\alpha'} \mathbb{P}(V')$ is a projective embedding, then $\dim(V') \leq \dim(V)$ and since our vector spaces have finite dimension, if a morphism from α' to α exists, it must be an isomorphism. □

2.2 Spanning sets for shadow spaces related to minimal fundamental weights

The shadow spaces of spherical buildings that are related to a minimal fundamental weight of a semisimple Lie algebra are the following point-line geometries: The grassmannians, with diagram $A_{n,i}$ $(i = 1, 2, \ldots, n)$, the dual parabolic polar spaces, with diagram $B_{n,n}$, the symplectic polar spaces, with diagram $C_{n,1}$, the hyperbolic polar spaces, with diagram $D_{n,1}$, the half-spin geometries, with diagram $D_{n,n-1}$ and $D_{n,n}$, and the exceptional geometries with diagram $E_{6,1}$, $E_{6,6}$ and $E_{7,1}$. In this section we will show that, with 'few' exceptions, these geometries have a very natural spanning set, namely the set of points on an apartment. The size of this set equals the dimension of the natural embedding afforded by the Lie algebra module corresponding to the minimal fundamental weight, showing that the spanning set is minimal and providing a proof for the fact that this embedding is relatively universal.

2.2.1 Apartments

In this subsection we produce a list of shadow spaces that are spanned by the set of points on an apartment. The main theorem here is Theorem 2.6.

Let Δ be a building with a spherical diagram M defined over an index set I of rank n. Fix $i_0 \in I$ and let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the i_0-shadow space of Δ. We recall that this is the point-line geometry whose points are the objects of type i_0 and whose lines are the sets of i_0-objects on an i_0-panel. Given a set of chambers $X \subseteq \Delta$, we will denote the set of points incident to at least one of these chambers by $\mathcal{P}(X)$. The collection of objects on X then is denoted by $\mathrm{obj}(X)$.

A computation in Coxeter groups

We let M_i denote M with designated node i, and let $M_i^\bullet$ denote the connected component of M_i containing i.

Lemma 2.5 *For any apartment B in Δ and any pair α, β of roots with $B = \alpha \sqcup \beta$ we have $\mathcal{P}(B) \subseteq \bigcup_{a\in obj(\alpha)} \mathcal{P}(a)$ if $M_{i_0}^\bullet$ is of one of the following types:*

1) A_{n,i_0} with $i_0 \in \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}_{\geq 1}$,

2) $D_{n,1}$, $D_{n,n-1}$, $D_{n,n}$ with $n \in \mathbb{N}_{\geq 3}$,

3) $E_{6,1}$, $E_{6,6}$, $E_{7,7}$,

4) $B_{n,n}$ with $n \in \mathbb{N}_{\geq 3}$

and $\mathcal{P}(B) \subseteq \bigcup_{a \in obj(\alpha)} \mathcal{P}(a) \cup \bigcup_{p \in \mathcal{P}(\alpha) \cap \mathcal{P}(\beta)} p^{\perp}$ *if* $M^{\bullet}_{i_0}$ *is of type*

5) $C_{n,1}$ *with* $n \geq 2$.

Proof: Clearly, $\mathcal{P}(B) = \mathcal{P}(\alpha) \cup \mathcal{P}(\beta)$ and $\mathcal{P}(\alpha) \subseteq \bigcup_{a \in \mathrm{obj}(\alpha)} \mathcal{P}(a)$. Now choose any $p \in \mathcal{P}(\beta)$.

We first show how one sees if p is on an object of α or in $q^{\perp}$ for some point $q \in \mathcal{P}(\alpha) \cap \mathcal{P}(\beta)$. Let r be the reflection interchanging α and β. Assume that $x \in \beta$ is a chamber on p. If $\delta(x, rx) \in {W_J}^{W_{I-\{i_0\}}}$, for some subset $J \subset I$, then p contains a chamber y such that $\delta(y, ry) \in W_J$. Suppose there is an element $i \in I - J$. Then y and ry lie in the same i-object, a say. Since $y \in p \cap a$ and $ry \in a \cap \alpha$ we have $p \in \bigcup_{a \in \mathrm{obj}(\alpha)} \mathcal{P}(a)$.

If $\delta(x, rx) \in {W_{I-\{i_0\}}}^{r_{i_0} W_{I-\{i_0\}}}$, then there is a chamber $y \in p$ such that $\delta(y, ry) \in {W_{I-\{i_0\}}}^{r_{i_0}}$. Let q be the point containing the chamber yr_{i_0}. Then clearly q is collinear to p and it lies both on α and β because $\delta(yr_{i_0}, ryr_{i_0}) \in W_{I-\{i_0\}}$.

We note that $\{\delta(x, rx) \mid x \in \beta\} = \bigcup_{i \in I} {W_{\{i\}}}^W$ and that if $S \subseteq I$ is the index set of a connected simply-laced subdiagram of M then ${W_{\{i\}}}^W = {W_{\{j\}}}^W$ for all $i, j \in S$. The lemma will follow from the fact that in the cases 1), 2) and 3), we have

$$ {W_{\{i_0\}}}^W = \bigcup_{j \in I} {W_{\{j\}}}^{W_{I-\{i_0\}}}, $$

and in cases 4) and 5), we have

$$ \begin{aligned} W^W_{\{n-1\}} &= \left({W_{\{n-1\}}}^{W_{\{n\}}}\right)^{W_{I-\{n\}}}, \\ {W_{\{n\}}}^W &= \{ {r_n}^{r_{n-1} \cdot r_{n-2} \cdots r_{k+1} \cdot r_k} \mid 1 \leq k \leq n-1 \} \cup \{1, r_n\} \quad \text{and} \\ {W_{\{1\}}}^W &= {W_{\{1\}}}^{W_{I-\{1\}}}. \end{aligned} $$

Note that in case 4) we have $W^{W_{\{n\}}}_{\{n-1\}} \subset W_J$ with $J = \{n-1, n\}$ and $n \geq 3$ so that there is an element $i \in I - J$. The computation that establishes the equations for cases 4) and 5) is slightly longer but otherwise completely similar to the one needed to obtain the equation for cases 1), 2) and 3) and is therefore omitted.

We will prove the first equation using induction on $|I|$. For $|I| = 1$ the equality evidently holds. Now let $|I| \geq 2$. We will show that the set $\bigcup_{j \in I} {W_{\{j\}}}^{W_{I-\{i_0\}}}$, which is invariant under conjugation by $W_{I-\{i_0\}}$, is also invariant under conjugation by r_{i_0} so that it equals ${W_{\{i_0\}}}^W$.

Put $I^{\circ} = \{i \in I \mid m_{ii_0} \geq 3\}$. Since M is a forest, each $j \in I^{\circ}$ is in the index set of a different component of $M_{I-\{i_0\}}$. Call this index set I_j. We note that $M_{I_j, j}$ is again in the list of the lemma and $|I_j| < |I|$, hence we can use induction. For $l \in I_j$ we have

$$ \left({W_{\{l\}}}^{W_{I-\{i_0\}}}\right)^{r_{i_0}} = \left({W_{\{j\}}}^{W_{I_j}}\right)^{r_{i_0}} $$

because M_{I_j} is simply-laced and connected. By induction the right hand side equals

$$ \left(\bigcup_{i \in I_j} {W_{\{i\}}}^{W_{I_j - \{j\}}}\right)^{r_{i_0}} $$

and since r_{i_0} commutes with r_i for all $i \in I_j - \{j\}$, this in turn equals

$$ \bigcup_{i \in I_j} {W_{\{i\}}}^{r_{i_0} W_{I_j - \{j\}}} $$

This is a subset of $\bigcup_{j\in I} {W_{\{j\}}}^{W_{I-\{i_0\}}}$ since ${r_i}^{r_{i_0}} = r_i$ for $i \in I_j - \{j\}$ and ${r_j}^{r_{i_0}} = {r_{i_0}}^{r_j}$. It remains to consider the case $l = i_0$. Since the Coxeter graphs of type $\mathrm{M}_{I-\{i_0\},\{j,k\}}$ (with $j, k \in I^\circ$ not necessarily distinct) have diameter less than or equal to 2 (cf. Brouwer et al. [8], §10.4.3), every element of $W_{I-\{i_0\}}$ has a product representation containing at most two generators r_j with $j \in I^\circ$. In other words we have

$$W_{I-\{i_0\}} = \bigcup_{j,k\in I^\circ} W_{I^\bullet} W_j W_{I^\bullet} W_k W_{I^\bullet},$$

where $I^\bullet = I - (I^\circ \cup \{i_0\})$. Now all elements in $W_{I^\bullet}$ commute with r_{i_0} and we already know that ${r_{i_0}}^{r_j} = {r_j}^{r_{i_0}}$ for $j \in I^\circ$. Hence we find

$$\left({W_{\{i_0\}}}^{W_{I-\{i_0\}}}\right)^{r_{i_0}} \subseteq \{1\} \cup \{r_{i_0}\} \cup \bigcup_{j\in I^\circ} {r_j}^{W_{I^\bullet}} \cup \bigcup_{j,k\in I^\circ} {r_k}^{W_{I^\bullet} r_{i_0} r_j r_{i_0} W_{I^\bullet}}$$

and since ${r_k}^{W_{I^\bullet} r_{i_0} r_j r_{i_0} W_{I^\bullet}} = {r_k}^{W_{I^\bullet} r_j r_{i_0} r_j W_{I^\bullet}} = {r_k}^{W_{I_k-\{k\}} r_j r_{i_0} r_j W_{I^\bullet}}$ we can use the fact that r_j centralizes W_{I_k} in case $j \neq k$ and induction otherwise to conclude that ${r_k}^{W_{I_k-\{k\}} r_j} \subset \bigcup_{i\in I_k} {r_i}^{W_{I_k-\{k\}}}$. It follows that $\left(W_{\{i_0\}}^{W_{I-\{i_0\}}}\right)^{r_{i_0}}$ is contained in $\bigcup_{j\in I} {W_{\{j\}}}^{W_{I-\{i_0\}}}$. □

Remark: The proof of the lemma above shows that in the simply-laced cases every point of β is equal to or at least collinear to a point of α.

Spanning the geometry

First, we state the main result of this section.

Theorem 2.6 *We have $\langle \mathcal{P}(A)\rangle_\Gamma = \mathcal{P}$ for any apartment A of Δ if $M^\bullet_{i_0}$ is one of the following*

1) A_{n,i_0} *with* $i_0 \in \{1, 2, \ldots, n\}$ *and* $n \in \mathbb{N}_{\geq 1}$,

2) $D_{n,1}$, $D_{n,n-1}$, $D_{n,n}$ *with* $n \in \mathbb{N}_{\geq 3}$,

3) $E_{6,1}$, $E_{6,6}$, $E_{7,7}$,

4) $B_{n,n}$ *with* $n \in \mathbb{N}_{\geq 2}$, *provided that its rank two residues of type* $B_{2,2}$ *as a building are spanned by the points on an apartment* [2].

5) $C_{n,1}$, *with* $n \in \mathbb{N}_{\geq 2}$, *provided that its rank two residues of type* $C_{2,1}$ *as a building are spanned by the points on an apartment* [3].

Let $\mathcal{A}$ be the collection of apartments of Δ. We also denote by $\mathcal{A}$ the graph whose vertices and edges are the apartments and roots of Δ, respectively, and in which a vertex lies on an edge if the apartment contains the root. In the proof of the theorem above we will use the following property of the graph $\mathcal{A}$.

[2]This holds, for instance, for the buildings obtained from $O_{2n+1}(\mathbb{F})$ with $\mathrm{Char}(\mathbb{F}) \neq 2$.
[3]This holds, for instance, for the buildings obtained from $Sp_{2n}(\mathbb{F})$ with $\mathrm{Char}(\mathbb{F}) \neq 2$.

Lemma 2.7 *The graph $\mathcal{A}$ has finite diameter.*

Proof: Let $A, B \in \mathcal{A}$. It suffices to show that there is a path $A = A_0, \ldots, A_m = B$ of apartments such that A_{i-1} and A_i $(i = 1, \ldots, m)$ share a root. It suffices to show this in case A and B share at least one chamber because otherwise we can find a third apartment that intersects each in a chamber.

Suppose that c is a chamber of $A \cap B$. Let d be the chamber of A opposite to c and let $(c = c_0, c_1, \ldots, c_l = d)$ be a minimal gallery from c to d. This gallery lies in A by convexity of A and it has finite length as Δ is spherical.

Suppose that B already contains c_i for $0 \leq i < k$, but not c_k. Let π be the panel on c_{k-1} and c_k. This panel determines a root β of B that contains c. This root then contains $c_0, \ldots, c_{k-1} \in B$ by convexity of roots, but not c_k. Now there exists an apartment B' on β that contains c_k. We have $c_0, \ldots, c_k \in B'$ and $\beta \subseteq B \cap B'$. By induction on l we find the desired path of apartments. This path has length $\leq l+1$ so the graph has finite diameter. □

Proof: (of Theorem 2.6) We will use induction on $|I|$. If $|I| \leq 2$, then the statement can easily be verified for cases 1), 2) and 3), whereas for cases 4) and 5), this is precisely the assumption. The fact that the buildings obtained from $O_{2n+1}(\mathbb{F})$ and $Sp_{2n}(\mathbb{F})$ with $\mathrm{Char}(\mathbb{F}) \neq 2$, satisfy the conditions of case 4) and 5) respectively, follows from Proposition 2 in Shult and Thas [42].

Now let $|I| \geq 3$. Let p be any point in Γ. Then, p is incident with some apartment B of Δ. By Lemma 2.7 the graph $\mathcal{A}$ has finite diameter. It therefore suffices to show that if the apartments A and B share a root α, then $p \in \langle \mathcal{P}(A) \rangle_\Gamma$. Suppose that p is incident with some object Y on α. If $p = Y$, we are done. If $p \neq Y$, then Y is a building with a diagram N for which $N^\bullet_{i_0}$ is again of one of the types in the list above and $A \cap Y$ is an apartment of Y. Since each point (line) of $Y^{(i_0)}$ is contained in a unique point (line) of Γ, the geometry $Y^{(i_0)}$ is isomorphic to a subspace of Γ and we can apply the induction hypothesis to finish the proof. Suppose that $M^\bullet_{i_0}$ has type $C_{n,1}$ and p is collinear with a point q on $\mathcal{P}(A) \cap \mathcal{P}(B)$. The point q is a building Y of type C_{n-1} over $I - \{1\}$ and $Y^{(2)}$ is the point-line geometry of lines and planes on q. If two lines on Y lie on a plane and are contained in a subspace then this subspace contains an apartment of the plane. This is an $A_{2,1}$ geometry so all points of the plane lie in the subspace. Now it follows from the induction hypothesis applied to $Y^{(2)}$ that $p \in \langle \mathcal{P}(A) \rangle$. □

2.2.2 The spanning set and the standard embedding

In this subsection we will show that roughly speaking there are no other examples of shadow spaces that are spanned by the set of points on an apartment.

Theorem 2.8 *Suppose Δ is a (spherical) building with a diagram M of type A_n, B_n, C_n, D_n or E_n. If M is of type B_n or C_n, assume moreover that Δ is obtained from the group $O_{2n+1}(\mathbb{F})$ or $Sp_{2n}(\mathbb{F})$ respectively, with Char$(\mathbb{F}) \neq 2$. Then $\langle \mathcal{P}(A) \rangle_{\Delta^{(i_0)}} = \Delta^{(i_0)}$ for any apartment A in Δ if and only if $M^{\bullet}_{i_0}$ is of one of the following types* [4]*:*

1) A_{n,i_0} *with* $i_0 \in \{1, 2, \dots, n\}$*, and* $n \in \mathbb{N}_{\geq 1}$*,*

2) $D_{n,1}$*,* $D_{n,n-1}$*,* $D_{n,n}$ *with* $n \in \mathbb{N}_{\geq 3}$*,*

3) $E_{6,1}$*,* $E_{6,6}$*,* $E_{7,7}$*,*

4) $B_{n,n}$ *with* $n \in \mathbb{N}_{\geq 2}$*,*

5) $C_{n,1}$ *with* $n \in \mathbb{N}_{\geq 2}$*.*

The proof will be given at the end of this subsection.

The standard embedding

Suppose that Δ is defined over the field $\mathbb{F}$. Let G be the Chevalley group defined over $\mathbb{F}$ with diagram M. Then the i_0-shadow space $\Gamma = (\mathcal{P}, \mathcal{L})$ of Δ can be obtained from G by taking for $\mathcal{P}$ the collection of left cosets of the maximal (standard) parabolic subgroup $P = P_{I-\{i_0\}}$ of type $I - i_0$ of G and for $\mathcal{L}$ the collection of left cosets $\{glP \mid l \in P_{i_0}\}$ $(g \in G)$, where P_{i_0} is the minimal standard parabolic subgroup of type $\{i_0\}$ of G (this follows from the classification theorem by J. Tits, see Section 1.2).

Let V' be the G-module derived from the module of highest weight λ_{i_0} for the semisimple Lie algebra with diagram M over $\mathbb{C}$ (see Subsection 1.3.5). Then the map

$$\begin{array}{rl} G/P & \xrightarrow{\zeta} \mathbb{P}(V') \\ gP & \mapsto [gv^+] \end{array}$$

is an embedding such that $\langle \zeta(G/P) \rangle = \mathbb{P}(V')$ (see Lemma 1.74). This provides a projective embedding for Γ.

We consider the span of the points of an apartment inside the space $\mathbb{P}(V')$. Let (B, N) be a Tits-system for G. Then N is the stabilizer of an apartment of Δ, A say.

By one of the (B, N)-pair axioms $W \cong N/(N \cap H)$. We note that for $w \in W$, the expressions wB, wP and $\zeta(wP)$ are well-defined since $H \subset B \subset P$. Now the set of chambers resp. points on A are $\{wB \mid w \in W\}$ resp. $\{wP \mid w \in W\}$. Under ζ these points are mapped to the collection $\{[wv^+] \mid w \in W\}$.

Lemma 2.9 *We have $\langle wv^+ \mid w \in W \rangle = V'$ if and only if λ_{i_0} is a minimal fundamental weight.*

[4]The nodes in the diagrams are labeled as in Boubaki [6], Ch. 6 §4.

Proof: A point in $\zeta(WP)$ looks like $[wv^+]$ $(w \in W)$. The action of G on V', hence also of W is induced by the action of the corresponding Lie algebra over $\mathbb{Z}$ on the admissible lattice L involved. Let $V \supset L$ be the irreducible Lie algebra module of highest weight λ_{i_0} over $\mathbb{C}$. Let $\mathcal{H}$ be the set of weights of V and for $\mu \in \mathcal{H}$ let V_μ be its weight space.

Then $L = \bigoplus_{\mu \in \mathcal{H}}(L \cap V_\mu)$ and therefore $V' = \bigoplus_{\mu \in \mathcal{H}} V'_\mu$, where $V'_\mu = (L \cap V_\mu) \otimes_{\mathbb{Z}} \mathbb{F}$. If $\mathbb{F} = \mathbb{C}$, then $V'_\mu = V_\mu$ and for $w \in W$ and $\mu \in \mathcal{H}$, we have $wV_\mu = V_{w\mu}$ and in fact, $w(V_\mu \cap L) = V_{w\mu} \cap L$ $(w \in W)$ (see Humphreys [31], §21.2). From the preceding, it follows that then also $wV'_\mu = V'_{w\mu}$ $(w \in W)$. By Proposition 6 in Ch. 8 §7 of [5], $\mathcal{H} = W\lambda_{i_0}$ if and only if λ_{i_0} is a minimal weight. Since $V_{\lambda_{i_0}}$ is spanned by v^+ only, $V'_{\lambda_{i_0}} = \langle v^+ \rangle$. This proves the lemma. □

We note here that $\zeta(WP)$ is a set of $[W : W_{I-\{i_0\}}]$ independent points in $\mathbb{P}(V')$. We are ready to prove Theorem 2.8.

Proof: (of Theorem 2.8) The 'if' part follows directly from Theorem 2.6. We now prove the 'only if' part. For $|I| \leq 2$, we only need to show that if $M^\bullet_{i_0}$ is of type $B_{2,1}$ or $C_{2,2}$, then Γ is not spanned by the points on an apartment. The natural embedding of Γ, in both cases being the $O_5(\mathbb{F})$ geometry, is 5-dimensional and consequently Γ cannot be spanned by the 4 points that an apartment has.

For $|I| \geq 3$, we may identify Δ with the building obtained from G. We may assume that A is the apartment with chamber set $\{wB \mid w \in W\}$ because G is transitive on the collection of apartments. Consider the projective embedding $\zeta\colon \Gamma \longrightarrow \mathbb{P}(V')$. We have $\langle \zeta(\mathcal{P}(\Gamma)) \rangle = \mathbb{P}(V')$ whereas by Lemma 2.9 we have $\langle \zeta(\mathcal{P}(A)) \rangle = \mathbb{P}(V')$ if and only if λ_{i_0} is a minimal weight.

In the theorem, all diagrams of type A_n, D_n, E_n, B_n and C_n that have a minimal weight, together with the labels i_0 for which the fundamental dominant weight λ_{i_0} is a minimal weight, are listed (see Bourbaki [5], Ch.8 §7 [5]). Hence, if $M^\bullet_{i_0}$ is not of one of the types listed in the theorem, then $\langle \mathcal{P}(A) \rangle$ is a proper subspace of Γ. □

Notes

The main result of Cooperstein and Shult in [22] is rather similar to Theorem 2.8 and Theorem 2.6, but was shown using a case-by-case analysis. In their paper, they define a *frame* of a geometry to be a subset of the point set that spans the geometry and is independent for some projective embedding. They show that apartments are frames in precisely the cases discussed here.

A result very similar to Lemma 2.5 can be found as Lemma 4.5 in Ronan and Smith [39]. Instead of reducing the proof to a computation in the Weyl group, these authors use a case-by-case analysis.

[5] The notation here refers to the Dynkin diagrams of the dual of the root system. For simply-laced diagrams, these diagrams are identical. Note however, that the Dynkin diagram for the dual of the root system of type B_n is of type C_n.

2.2.3 Other shadow spaces

In Subsection 2.2.1, we have seen that the geometry $\Delta^{(i_0)}$ is spanned by the set of points on an apartment if $M^{\bullet}_{i_0}$ is in a certain list. In this section, for a building Δ with diagram of type D_n with $n \geq 4$, we will give a geometric argument showing why the set of points on an apartment in the geometry $\Delta^{(i_0)}$ $(2 \leq i_0 \leq n-2)$ does not span the entire geometry.

Let V be a vector space of dimension $2n$ over a field $\mathbb{F}$ provided with a non-degenerate quadratic form $\mathcal{Q}$ of maximal Witt index. For any subspace U of V, let $U^{\perp}$ denote the subspace that is orthogonal to U with respect to the symmetric bilinear form associated to $\mathcal{Q}$. We will call two subspaces U, W of V of the same dimension *partially orthogonal* if $U \cap W^{\perp} \neq 0$, or, equivalently, $U^{\perp} \cap W \neq 0$.

Recall that (the geometry of) the building Δ is the incidence geometry whose objects are the totally singular subspaces of V of dimension different from $n-1$. There are two types of totally singular n-spaces, two totally singular n-spaces having the same type if their intersection has even codimension in both. Incidence is defined by inclusion with the exception that two totally singular n-spaces of different type are called incident if their intersection has codimension one in both. The building Δ has diagram D_n and all thick buildings of type D_n arise in this way (see Tits [47] and Ronan [38]).

Let $I = \{1, \hat{1}, \ldots, n, \hat{n}\}$ and let $\hat{}: I \longrightarrow I$ be the involution interchanging i and $\hat{i}$. We can choose a basis $B = \{e_i, e_{\hat{i}} \mid 1 \leq i \leq n\}$ of V such that $\mathcal{Q}(x) = x_1 x_{\hat{1}} + \cdots + x_n x_{\hat{n}}$ with respect to B. For any k-set $S \subseteq I$ let E_S denote the k-space $\langle e_i \mid i \in S\rangle$. The collection of totally singular subspaces of dimension different from $n-1$ that are spanned by subsets of B is an apartment of Δ. Call this apartment A. The chambers on A can be identified with the ordered sequences $(i_1, \ldots, i_{n-1})$ for which $E_{\{i_1,\ldots,i_{n-1}\}}$ is a totally singular $(n-1)$-space.

Fix i_0 with $2 \leq i_0 \leq n-2$ so that i_0 does not correspond to an end node of the diagram of Δ. The point-line geometry $\Delta^{(i_0)} = (\mathcal{P}, \mathcal{L}, \star)$ is obtained by taking for $\mathcal{P}$ the set of totally singular i_0-spaces of V, for $\mathcal{L}$ the set of pairs (P, M) where M is a totally singular (i_0+1)-space in V and P is a totally singular (i_0-1)-space in M, and stipulating that for $L \in \mathcal{P}$, $L \star (P, M)$ whenever $P \subseteq L \subseteq M$.

Proposition 2.10 *(i) Let H be any i_0-space in V $(i_0 \geq 2)$. Then the set $\mathcal{P}_H$ of totally singular i_0-spaces that are partially orthogonal to H is a proper hyperplane of $\Delta^{(i_0)}$.*

(ii) Let $\mathcal{H}$ be the collection of i_0-sets in I that contain at least one of the subsets $\{i, \hat{i}\}$ $(1 \leq i \leq n)$. Then, $\langle \mathcal{P}(A)\rangle_{\Delta^{(i_0)}} \subseteq \bigcap_{S\in\mathcal{H}} \mathcal{P}_{E_S}$.

Proof: (i) Let $(P, M) \in \mathcal{L}$. We show that $\mathcal{P}_H$ contains either one or all totally singular i_0-spaces incident with (P, M). If $P \cap H^{\perp} \neq 0$ we are done. If $P \cap H^{\perp} = 0$, then, since $\mathrm{codim}_M(P) = 2$ and $\dim(M \cap H^{\perp}) \geq 1$, either one or all lines incident with (P, M) are in $P + (M \cap H^{\perp})$.

We will now prove that $\mathcal{P}_H$ is a proper hyperplane. Since the automorphism group of Δ is transitive on the collection $\mathcal{A}$ of apartments in Δ, we may assume that $H = E_S$ for some i_0-set $S \subset I$. We have $V = E_{S\cap\hat{S}} + E_{\hat{S}-S} + E_{S-\hat{S}} + E_{I-(S\cup\hat{S})}$. Let $J \subset \{1, \ldots, n\}$ be such that $\{\{j, \hat{\jmath}\} \mid j \in J\}$ is a partition of $S \cap \hat{S}$. Since $i_0 \leq n$ we can find distinct k_j $(j \in J)$ such that $\{k_j, \hat{k_j}\} \subset I - (S \cup \hat{S})$. For $j \in J$, choose $u_j \in \langle e_j, e_{k_j}\rangle - \{e_j, e_{k_j}\}$ and $u_{\hat{\jmath}} \in \langle e_{\hat{\jmath}}, e_{\hat{k_j}}\rangle - \{e_{\hat{\jmath}}, e_{\hat{k_j}}\}$ such that $U_j = \langle u_j, u_{\hat{\jmath}}\rangle$ is a totally singular 2-space. As the

k_j ($j \in J$) are distinct, also $U' = \sum_{j\in J} U_j$ is a totally singular subspace. We note that $E_{S\cap\hat{S}} + E_{I-(S\cup\hat{S})} = U' + E_{I-(S\cup\hat{S})}$. Let U be the totally singular i_0-space $U' + E_{\hat{S}-S}$. Then $U + H^\perp = U' + E_{\hat{S}-S} + E_{S-\hat{S}} + E_{I-(S\cup\hat{S})} = E_{S\cap\hat{S}} + E_{\hat{S}-S} + E_{S-\hat{S}} + E_{I-(S\cup\hat{S})} = V$ and since $\dim(U) = \mathrm{codim}_V(H^\perp)$ we have $U \cap H^\perp = 0$. This shows that U is a totally singular i_0-space which is not in $\mathcal{P}_H$.
(ii) Let $E_T \in \mathcal{P}(A)$ be any totally singular i_0-space. Then for any $S \in \mathcal{H}$ there is at least one $j \in T$ such that $j \in \{i, \hat{i}\}$ and $S \cap \{i, \hat{i}\} = \emptyset$. It follows that $E_{\{j\}} \subseteq E_T \cap E_S^\perp$. This shows that $\mathcal{P}(A)$ is contained in the subspace $\bigcap_{S\in\mathcal{H}} \mathcal{P}_{E_S}$. □

By exhibiting a proper subspace of $\Delta^{(i_0)}$ that contains $\mathcal{P}(A)$, we have shown explicitly why $\mathcal{P}(A)$ cannot span the geometry $\Delta^{(i_0)}$.

2.2.4 A note on hyperplanes

Let Δ be a building with a connected spherical diagram M defined over a finite index set I. Let $i_0 \in I$ correspond to a minimal weight and let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the i_0-shadow space of Δ. This geometry is spanned by the points of an apartment (Theorem 2.6). Recall from Subsection 2.2.2 that there exists an embedding

$$\Gamma \xrightarrow{\zeta} \mathbb{P}(V)$$

whose dimension equals the number N_A of points on an apartment. From now on we identify Γ with $\zeta(\Gamma)$.

For a residue R, let $\mathrm{Far}(R)$ be the collection of chambers opposite some chamber of R and let $\mathrm{Near}(R) = \Delta \setminus \mathrm{Far}(R)$ (see also Section 3.1 and Subsection 1.2.1). Let $\hat{p}$ be the residue on A opposite to p ($\in \mathrm{Far}(p)$).

Lemma 2.11 *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the i_0-shadow space of a spherical building.*

(i) The points on Near($\hat{p}$) form a hyperplane of Γ.

(ii) The hyperplane Near($\hat{p}$) is a maximal subspace of Γ.

Proof: (i) Choose a line L and let π be an i_0-panel meeting all points of L. Then $\mathrm{proj}_\pi(\hat{p})$ is a residue on π. This residue contains either one chamber (of π) or the whole of π. Thus either one or all points of L are in $\mathrm{Near}(\hat{p})$ and hence it is a hyperplane.

(ii) In general, a hyperplane of a geometry whose lines have at least three points is maximal if and only if its complement is connected. Since the type set of the point $\hat{p}$ has size $|I| - 1$, it follows from Theorem 3.30 that the complement of $\mathrm{Near}(\hat{p})$, which is $\mathrm{Far}_\Gamma(\hat{p})$, is connected. □

Remark: In the case that i_0 corresponds to a minimal weight, we can prove part (*ii*) in a different way. Given any point q in $\mathcal{P} \setminus \mathrm{Near}(\hat{p})$, there exists an apartment B on $\hat{p}$ and q. All points of B, except q are in $\mathrm{Near}(\hat{p})$. Hence the subspace $\langle \mathrm{Near}(\hat{p}), q\rangle_\Gamma$ contains all points of the apartment B and hence equals Γ.

For any apartment A and point $p \in \mathcal{P}(A)$ let $\Upsilon_{A,p}$ and $H_{A,p}$ be the subspaces of Γ and $\mathbb{P}$ respectively spanned by $\mathcal{P}(A) \setminus \{p\}$.

Theorem 2.12 *If M is simply-laced, then*

$$\Upsilon_{A,p} = Near(\hat{p}) = H_{A,p} \cap \Gamma.$$

In the proof we use a connectedness argument similar to Lemma 2.7.

For a residue R let $\mathcal{A}(R)$ be the collection of apartments meeting R. We make a graph with vertex set $\mathcal{A}(R)$ (also denoted by $\mathcal{A}(R)$ in which two apartments are adjacent if they share a root.

Lemma 2.13 *The graph $\mathcal{A}(R)$ is connected.*

To prove this lemma we modify the proof for Lemma 2.7 slightly.

Proof: Let $A, B \in \mathcal{A}(R)$ and suppose they have a chamber $c \in R$ in common. Let $(c = c_0, \dots, c_k)$ be a minimal gallery of maximal length (k) in $A \cap B$. Suppose that $c_k = d_{l+1}, \dots, d_1, d_0 = \mathrm{op}_B(c))$ is a gallery in B such that $(c = c_0, \dots, c_k, d_l, \dots, d_1, d_0 = \mathrm{op}_B(c))$ is minimal (and (hence) lies in B). Consider the panel π on c_k and d_l and choose a root α of A determined by π and containing c_k. By minimality the gallery $(c_0, \dots, c_k)$ is entirely contained in α. Now define $c_{k+1} = d_l$ and replace A by the unique apartment on α that contains c_{k+1}. Then we see that we now have apartments A, B such that $A \cap B$ contains the minimal gallery $(c = c_0, \dots, c_{k+1})$ of length $k + 1$. By repeating this step, we eventually reach B by a path in $\mathcal{A}(R)$ (or indeed in $\mathcal{A}(c)$).

Since for any two chambers x and y (in R, if you like) there exists an apartment that contains x and y, we can in fact join any two apartments in $\mathcal{A}(R)$ by a path in $\mathcal{A}(R)$. □

Proof: (of Theorem 2.12) Since $\mathcal{P}(A) - \{p\} \subseteq \mathrm{Near}(\hat{p})$, we have $\Upsilon_{A,p} \subseteq \mathrm{Near}(\hat{p})$ and since the lines of Γ are lines of $\mathbb{P}(V)$ we also have $\Upsilon_{A,p} \subseteq H_{A,p} \cap \Gamma$. We now show that $\Upsilon_{A,p} = \mathrm{Near}(\hat{p})$. Then it follows that $\mathrm{Near}(\hat{p}) \subseteq H_{A,p} \cap \Gamma$ so that by Lemma 2.11 we must have equality here as well.

Using Lemma 2.13 we only have to show that if two apartments B_1, B_2 on $\hat{p}$ share a root, then $(\mathrm{Near}(\hat{p}) \cap \mathcal{P}(B') \subset \langle \mathrm{Near}(\hat{p}) \cap \mathcal{P}(B) \rangle_\Gamma$.

Let β be the root contained in B and B' and let $-\beta$ be the root of B opposite to β. Then $\mathcal{P}(\beta) \subset \mathrm{Near}(\hat{p}) \cap \mathcal{P}(B) \cap \mathcal{P}(B')$. Now choose a point z in $\mathcal{P}(B') \cap \mathrm{Near}(\hat{p}) \setminus \mathcal{P}(B)$. Then since M is simply laced z lies on a line L that meets β and $-\beta$ in points x and y respectively (see the remark following the proof of Lemma 2.5). We have seen that $x \in \mathrm{Near}(\hat{p}) \cap \mathcal{P}(B)$, but since y and z have the same distance to $\hat{p}$, also $y \in \mathrm{Near}(\hat{p}) \cap \mathcal{P}(B)$. Thus $z \in L \subset \langle \mathrm{Near}(\hat{p}) \cap \mathcal{P}(B) \rangle_\Gamma$, as desired. □

2.3 Spanning sets for polar spaces

In this section we find spanning sets for polar spaces consisting of the totally isotropic (resp. totally singular) subspaces of a vector space with respect to a trace-valued sesquilinear (resp. pseudo quadratic) form. This class of polar spaces contains more polar spaces than the ones obtained from a symplectic form or a quadratic form that were considered in the previous section. The main result is Theorem 2.30

2.3.1 Polarities and sesquilinear forms

We give a brief overview of the theory of sesquilinear forms, pseudo-quadratic forms and polarities. No proofs are included; these can be found in Dieudonné [28] and Tits [45]. For a full exposition of this material we refer the reader to Dieudonné [28], Tits [45] and also to Buekenhout [10], Cameron [12] and Taylor [44].

Semilinear maps and collineations

We note that in this section, a field is not necessarily commutative. Let V_i be (right) vector spaces over the fields $\mathbb{F}_i$, $(i = 1, 2)$ and suppose that $\sigma: \mathbb{F}_1 \longrightarrow \mathbb{F}_2$ is an isomorphism. Then a *σ-semilinear map* (or *semilinear map relative to σ*) from V_1 to V_2 is an additive map ϕ satisfying $\phi(va) = \phi(v)a^\sigma$, $(v \in V_1,\ a \in \mathbb{F}_1)$. If $\mathbb{F}_1 = \mathbb{F}_2$ and $\sigma = \text{id.}$, then ϕ is *linear*.

Put $\mathbb{P}_i = \mathbb{P}(V_i)$, $(i = 1, 2)$. A *collineation* from $\mathbb{P}_1$ to $\mathbb{P}_2$ is a bijection that sends any three points on a line to three points on a line. Examples of collineations are the maps induced by a semilinear map from V_1 to V_2.

The following theorem, which we will not prove here, shows that (in vector space dimension at least 3) this is in fact the generic situation:

Theorem 2.14 *Let V_i be vector spaces of the same dimension $n \geq 3$ over the isomorphic fields $\mathbb{F}_i$ $(i = 1, 2)$. Suppose that $\phi: \mathbb{P}(V_1) \longrightarrow \mathbb{P}(V_2)$ is a bijection such that any three points on a line are mapped to three points on a line. Then ϕ is induced by a semilinear bijection of the underlying vector spaces.*

This theorem is known as the 'Fundamental Theorem of Projective Geometry' (abbreviated: FTGP).

Sesquilinear forms

Let V be a (right) vector space over a field $\mathbb{F}$ and let V^* be its *dual* space. We will write (v', v) instead of $v'(v)$ for $v' \in V^*$, $v \in V$. The space V^* is a vector space over $\mathbb{F}$ of the same dimension as V, but with $\mathbb{F}$ acting from the left. Alternatively, V^* can be viewed as a right vector space over the *opposite field* $\mathbb{F}^o$. In order to construct a sesquilinear form on V, we need a semilinear map from V to V^*. Clearly such a map can only exist if $\mathbb{F}$ and $\mathbb{F}^o$ are isomorphic; identifying $\mathbb{F}$ and $\mathbb{F}^o$, this means that there exists an automorphism σ of the additive group of $\mathbb{F}$ satisfying $(xy)^\sigma = y^\sigma x^\sigma$; such an isomorphism is called an *antiautomorphism*. Note that if there exists an antiautomorphism that is an automorphism then the field is commutative, and conversely.

Let ϕ be a σ-semilinear map from V to V^*, for some antiautomorphism σ of $\mathbb{F}$. Then the map $f: V \times V \longrightarrow \mathbb{F}$ given by $f(v, w) = (\phi(v))(w)$ is a *σ-sesquilinear form*, or *sesquilinear form relative to σ*; it satisfies

$$\begin{aligned} f(v_1 + v_2, w_1 + w_2) &= f(v_1, w_1) + f(v_1, w_2) + f(v_2, w_1) + f(v_2, w_2) \\ f(va, wb) &= a^\sigma f(v, w)b \end{aligned}$$

for all $v_1, v_2, v, w_1, w_2, w \in V$ and $a, b \in \mathbb{F}$. Conversely, every function satisfying these equations can be uniquely obtained in this way. In particular, if f is not the zero form, that is the form satisfying $f(v,w) = 0$ for all $v, w \in V$, then f determines σ. Note that if $\sigma = \text{id.}$, then f is *bilinear*. A σ-sesquilinear form f is called *reflexive* if the relation

$$f(v,w) = 0 \qquad (v, w \in V) \tag{2.1}$$

is symmetric. Given a reflexive sesquilinear form f, there is a relation $\perp_f$ (or simply $\perp$ if no confusion can arise) on $V \times V$ given by $v \perp w$ if $f(v,w) = 0$; if $f(v,w) = 0$ for $v, w \in V$, then we say that the vectors v and w are *orthogonal* (w.r.t. f). As f is reflexive, this relation is symmetric. For any subset $U \subset V$ define $U^{\perp} = \langle v \in V \mid \forall u \in U : u \perp v \rangle$. A subspace $U \subset V$ is called *totally isotropic with respect to* f if $U \subseteq U^{\perp}$. The subspace $V^{\perp}$ is called the *radical* of f. The codimension of $V^{\perp}$ is called the *rank* of f. The form f is *non-degenerate* on V if $V^{\perp} = \{0\}$. The form f induces a non-degenerate reflexive form on $V/V^{\perp}$.

If there exists a non-zero constant $\epsilon \in \mathbb{F}$ such that

$$f(w,v) = f(v,w)^{\sigma}\epsilon \qquad (v, w \in V) \tag{2.2}$$

then f is called (σ, ϵ)-*hermitian*.

Lemma 2.15 (see Dieudonné [28] or Tits [45]) *A reflexive σ-sesquilinear form is (σ, ϵ)-hermitian, for some ϵ in the center of $\mathbb{F}$, and conversely.*

Often the terms *symmetric*, *antisymmetric*, *σ-hermitian* and *σ-antihermitian* are used for (σ, ϵ)-hermitian forms with (σ, ϵ) equal to $(\text{id.}, 1)$, $(\text{id.}, -1)$, $(\sigma, 1)$ and $(\sigma, -1)$.

If f is a σ-sesquilinear form and $c \in \mathbb{F}^* = \mathbb{F} \setminus \{0\}$, then cf is a sesquilinear form relative to the antiautomorphism $a \mapsto ca^{\sigma}c^{-1}$. Additivity of cf in both factors is obvious, as is the linearity in the second factor. Furthermore we have $(cf)(va,w) = cf(va,w) = ca^{\sigma}c^{-1}(cf)(v,w)$. Note that cf is σ-sesquilinear in the case that $\mathbb{F}$ is commutative. The forms f and cf are said to be *proportional*. Notably, if f is reflexive, then also cf is reflexive.

An antiautomorphism of order 2 of a field is called an *involution*.

Lemma 2.16 (see Dieudonné [28] or Tits [45]) *A reflexive sesquilinear form that is not antisymmetric is proportional to a symmetric form or a σ-hermitian form, where σ is an involution of the field involved.*

By $\mathbb{F}^{\sigma}$ we will denote the subfield of $\mathbb{F}$ comprising the elements of $\mathbb{F}$ fixed by σ.

Remark: Suppose that f is a σ-hermitian form, for some field involution σ. Let us call elements $x, y \in \mathbb{F}$ such that $x^{\sigma} = x$ and $y^{\sigma} = -y$ symmetric and antisymmetric respectively. So for instance, the elements from $\mathbb{F}^{\sigma}$ are symmetric and elements of the form $\eta - \eta^{\sigma}$ with $\eta \in \mathbb{F} \setminus \mathbb{F}^{\sigma}$ are antisymmetric. Then one can verify that qf is hermitian or antihermitian according to whether q is symmetric or antisymmetric.

Polarities

Let V be a (right) vector space over a field $\mathbb{F}$. Put $\mathbb{P} = \mathbb{P}(V)$. A polarity on $\mathbb{P}$ is an inclusion reversing map on the collection of subspaces of $\mathbb{P}$ sending points of $\mathbb{P}$ to hyperplanes of $\mathbb{P}$. Two polarities π and π' on the spaces $\mathbb{P}$ and $\mathbb{P}'$ are isomorphic if there exists a projective isomorphism $\phi \colon \mathbb{P} \longrightarrow \mathbb{P}'$ such that $\pi' \circ \phi = \pi$.

Given a polarity π on $\mathbb{P}$, there is a symmetric relation $\perp_\pi$ (or simply $\perp$ if no confusion can arise) on $\mathbb{P} \times \mathbb{P}$ given by $p \perp q$ if $p \in \pi(q)$; if $p \perp q$ for $p, q \in \mathbb{P}$, then we say that the points p and q are *orthogonal* (w.r.t. π). For any subset $X \subset \mathbb{P}$ define $X^\perp = \langle p \in \mathbb{P} \mid \forall q \in X : p \perp q \rangle$. A subspace $X \subset \mathbb{P}$ is called *totally isotropic with respect to* π if $X \subset X^\perp$. The subspace $\mathbb{P}^\perp$ is called the *radical* of π. The *rank* of π is the codimension of $\mathbb{P}^\perp$. The polarity π is *non-degenerate* if $\mathbb{P}^\perp = \emptyset$. At the other end, the polarity π that relates every point to $\mathbb{P}$ is called the zero polarity. The polarity π induces a non-degenerate polarity on $\mathbb{P}/\mathbb{P}^\perp$.

Let V^* denote the dual of V. Then a non-degenerate polarity π can be viewed as a collineation between $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, sending a point p to a point representing a functional that defines $\pi(p)$. By the fundamental theorem of projective geometry (FTPG) , such a collineation is induced by a semilinear bijection ϕ of the underlying vector spaces. Let f be the sesquilinear form associated to ϕ. Now let $p, q \in \mathbb{P}(V)$ be represented by vectors $v_p, v_q \in V$. We see that $p \perp_\pi q$ if and only if v_p is in the hyperplane defined by $\phi(v_q)$, that is $(\phi(v_q))(v_p) = f(v_p, v_q) = 0$. Hence the relation $\perp_\pi$ is the projectivised form of $\perp_f$. Thus the symmetry of π corresponds to the reflexivity of f. Note that f is necessarily non-degenerate.

So summarizing this we have the following lemma.

Lemma 2.17 (see Tits [45]) *Every non-degenerate polarity π is induced by a non-degenerate sesquilinear form f, and conversely. Moreover, f is (anti)symmetric or σ-hermitian for some field involution σ.*

Pseudo-quadratic forms

A (σ, ϵ)-hermitian form f is said to be *trace-valued* if

$$f(v, v) \in \{t + t^\sigma \epsilon \mid t \in \mathbb{F}\} \qquad (v \in V)$$

Note that for symmetric bilinear forms $((\sigma, \epsilon) = (\text{id.}, 1))$ in charactistic 2, 'trace-valued' means alternating.

Let $\mathbb{F}$ be a field, $Z(\mathbb{F})$ its center, V a right vector space over $\mathbb{F}$ and $\sigma : \mathbb{F} \longrightarrow \mathbb{F}$ an anti-automorphism and let $\epsilon \in \mathbb{F}$ satisfy

$$\epsilon\epsilon^\sigma = 1 \tag{2.3}$$

$$x^{\sigma^2} = \epsilon x \epsilon^{-1} \quad (x \in \mathbb{F}). \tag{2.4}$$

We assume that if $\sigma = \text{id.}$ and $\mathrm{Char}(\mathbb{F}) \neq 2$, then $\epsilon \neq -1$. Set

$$\begin{aligned} \mathbb{F}_{\sigma,\epsilon} &= \{t - t^\sigma \epsilon \mid t \in \mathbb{F}\} \\ \mathbb{F}^{\sigma,\epsilon} &= \{t \in \mathbb{F} \mid t + t^\sigma \epsilon = 0\}. \end{aligned}$$

As $\mathbb{F}_{\sigma,\epsilon}$ is a subgroup of the additive group of $\mathbb{F}$ we can define

$$\mathbb{F}^{(\sigma,\epsilon)} = \mathbb{F}/\mathbb{F}_{\sigma,\epsilon} \quad .$$

The quotient $\mathbb{F}^{(\sigma,\epsilon)}$ is non-trivial (see the remark below). We define a 'multiplication' on $\mathbb{F}^{(\sigma,\epsilon)}$ by putting, for $a, \bar{b} \in \mathbb{F}$ and $b = \bar{b} + \mathbb{F}_{\sigma,\epsilon} \in \mathbb{F}^{(\sigma,\epsilon)}$ their product equal to $b * a = a^\sigma \bar{b} a + \mathbb{F}_{\sigma,\epsilon} \in \mathbb{F}^{(\sigma,\epsilon)}$.

Remark: The condition that we exclude the case $\mathrm{Char}(\mathbb{F}) \neq 2$, $\sigma = \mathrm{id.}$ and $\epsilon = -1$ is equivalent to requiring that $\mathbb{F}_{\sigma,\epsilon} \neq \mathbb{F}$. This can be seen as follows. Suppose that $\{t - t^\sigma\epsilon \mid t \in \mathbb{F}\} = \mathbb{F}$. Then every element $a \in \mathbb{F}$, being of the form $a = t - t^\sigma\epsilon$ for some $t \in \mathbb{F}$, satisfies $a^\sigma\epsilon = -a$. Put $a = \epsilon$ and use property (2.3) of ϵ, to see that this implies $\epsilon = -1$. But then $a^\sigma = a$ for all $a \in \mathbb{F}$ and we have $\sigma = \mathrm{id.}$. If $\mathrm{Char}(\mathbb{F}) = 2$, then we see that $\mathbb{F}_{\sigma,\epsilon} = \mathbb{F}_{\mathrm{id.},1} = \{0\} \neq \mathbb{F}$, contradicting our assumption, but if $\mathrm{Char}(\mathbb{F}) \neq 2$ we actually have $\mathbb{F}_{\mathrm{id.},-1} = \{t + t \mid t \in \mathbb{F}\} = \mathbb{F}$.

Lemma 2.18 Tits [45]) *For a function $q: V \longrightarrow \mathbb{F}^{(\sigma,\epsilon)}$, the following properties are equivalent :*

(i) there exists a σ-sesquilinear form $g: V \times V \longrightarrow \mathbb{F}$ such that $q(v) = g(v,v) + \mathbb{F}_{\sigma,\epsilon}$ for all $v \in V$,

(ii) for $a \in \mathbb{F}$ and $v \in V$ one has $q(va) = q(v) * a$, and there exists a trace-valued (σ,ϵ)-hermitian form $f: V \times V \longrightarrow \mathbb{F}$ such that $q(v + w) = q(v) + q(w) + (f(v,w) + \mathbb{F}_{\sigma,\epsilon})$ for all $v, w \in V$.

For a proof see Tits [45].

A *(σ,ϵ)-quadratic form* or *pseudo-quadratic form relative to σ and ϵ* is a function $q: V \longrightarrow \mathbb{F}^{(\sigma,\epsilon)}$ that satisfies the conditions *(i)* and *(ii)* of the above lemma. Like before, *σ-quadratic* means $(\sigma, 1)$-quadratic and *quadratic* means id.-quadratic.

Let q be a (σ,ϵ)-quadratic form, let $c \in \mathbb{F}^\star$ and let $\sigma': \mathbb{F} \longrightarrow \mathbb{F}$ be defined by $t^{\sigma'} = ct^\sigma c^{-1}$ and let $\epsilon' = c(c^\sigma)^{-1}\epsilon$. Then $c\mathbb{F}_{\sigma,\epsilon} = \mathbb{F}_{\sigma',\epsilon'}$ and we can define the function $cq: V \longrightarrow \mathbb{F}^{(\sigma',\epsilon')}$ in the obvious way. The form cq is a (σ,ϵ')-quadratic form. The forms q and cq are called *proportional.*

Lemma 2.19 (Tits, [45]) *Every pseudo-quadratic form is proportional to a σ-quadratic form, for suitable σ. Every pseudo-quadratic form which is not quadratic is proportional to a $(\sigma,-1)$-quadratic form, where σ can be chosen in such a way that* $1 \in \mathbb{F}_{\sigma,-1} = \{t + t^\sigma \mid t \in \mathbb{F}\}$.

The following simple observation is quite useful.

Lemma 2.20 (Tits, [45]) *Given a pseudo-quadratic form, let f be a trace-valued (σ,ϵ)-hermitian form such that*

$$q(v + w) = q(v) + q(w) + (f(v,w) + \mathbb{F}_{\sigma,\epsilon}) \quad (v, w \in V).$$

If $U \subseteq V$ is a subspace with $q(v) = 0$ $(v \in U)$, then $f(v,w) = 0$ $(v, w \in U)$.

Applying the above lemma to the case $U = V$ shows that f is uniquely determined by q. The form f is called the *sesquilinearized form* of q; it is denoted by βq.

For a pseudo-quadratic form q, the *orthogonality relation* $\perp_q$ (or simply $\perp$ if no confusion can arise) on $V \times V$ *with respect to* q is the orthogonality relation $\perp_{\beta q}$ with respect to βq.

The linear subspace $V^\perp \cap q^{-1}(0)$ is called the *radical* of q. Its dimension (resp. codimension in V, resp. codimension in $V^\perp$) is called the *corank* (resp. the *rank*, resp. the *defect*) of q. Following Tits [45] we say that a form of corank 0 is called *nondegenerate.*

Proposition 2.21 (Tits, [45]) *The sesquilinearization map β from the set $Q_{\sigma,\epsilon}$ of (σ, ϵ)-quadratic forms on V to the set $S_{\sigma,\epsilon}$ of trace-valued (σ, ϵ)-hermitian forms on V, which maps a pseudo-quadratic form to its sesquilinearized form is well-defined and surjective. A (σ, ϵ)-quadratic form q belonging to the kernel of β takes its values in $\mathbb{F}^{\sigma,\epsilon}/\mathbb{F}_{\sigma,\epsilon}$. The map β is a bijection if and only if $\mathbb{F}_{\sigma,-\epsilon} = \mathbb{F}^{\sigma,-\epsilon}$.*

A subspace $U \subset V$ such that $q(v) = 0$ for all $v \in U$ is called *totally singular with respect to q*.

Lemma 2.22 (Tits, [45]) *Let q be a (σ, ϵ)-quadratic form, let f be its sesquilinearized form and suppose that $\mathbb{F}_{\sigma,\epsilon} = \mathbb{F}^{\sigma,\epsilon}$. Then q and f determine one another uniquely and a subspace $U \subseteq V$ is totally singular for q if and only if it is totally isotropic for f.*

2.3.2 Classical polar spaces

We recall the Tits axioms for polar spaces and state the main classification theorems for embeddable polar spaces, due to Tits for the finite rank ≥ 3 case and to Buekenhout et al. for the rank 2 case. As for the structure of embedded polar spaces we discuss an orthogonal decomposition. Again we omit proofs whenever we can; these can be found in Dieudonné [28], Tits [45], Cameron [12], Buekenhout [10], Taylor [44] and any other textbook on polar spaces.

Axioms and examples

First we introduce a set of axioms defining a polar space. These axioms, which are due to Tits (see Tits [45], Ch. 7), are equivalent to those of Buekenhout and Shult (see page 57 or Buekenhout and Shult [11]).

A *polar space* is a set S together with certain distinguished subsets, called (linear) *subspaces*, that satisfy the axioms (T1) to (T4) below, for some integer $n \geq 1$ called the *rank* of the space S.

(T1) A subspace L, together with the subspaces it contains, is a d-dimensional projective space, with $-1 \leq d \leq n - 1$ (By definition, a (-1)-dimensional projective space is empty, a 0-dimensional projective space consists of a single point, and a 1-dimensional projective space is an arbitrary set of cardinality at least 2, with no additional structure; its proper linear subspaces are its points and the empty set).

The number d is called the *dimension* of the subspace L. A subspace of dimension 1 (resp. 2) is called a *line* (resp. a *plane*). Two points are *collinear* if there is a line containing both of them.

(T2) The intersection of two subspaces is again a subspace.

(T3) Given a subspace L of dimension $n - 1$ and a point $p \in S \setminus L$, there exists a unique subspace M containing p and such that $\dim(M \cap L) = n - 2$; it contains all points of L that are collinear to p.

(T4) There exist two disjoint subspaces of dimension $n - 1$.

Let V be a right vector space over a field $\mathbb{F}$. For brevity we will call a subspace of V of dimension i ($1 \leq i \leq \dim(V)$) an *i-space of V* or simply *i-space*. The geometry whose objects of type i ($1 \leq i \leq \dim(V)$) are the i-spaces of V with incidence defined by (symmetrized) inclusion is called the *projective space of V* and is denoted by $\mathbb{P}(V)$. In this section, *all* projective spaces are of this type.

From now on, F will be a non-degenerate (σ, ϵ)-hermitian (or (σ, ϵ)-quadratic) form on V, f will denote a non-degenerate (σ, ϵ)-hermitian form and q will denote a (σ, ϵ)-quadratic form.

The geometry associated to the pair (V, f) (resp. (V, q)) is the subgeometry of $\mathbb{P}(V)$ whose i-objects are the totally isotropic (resp. totally singular) i-spaces of V and is denoted by $\Gamma(V, f)$ (resp. $\Gamma(V, q)$). We define the geometry associated to (V, F) accordingly.

Theorem 2.23 *(Tits. [45]) Let F be a non-degenerate reflexive sesquilinear (resp. pseudo quadratic) form defined on a vector space V.*

(i) Let X, Y be totally isotropic (resp. totally singular) subspaces of V with respect to F. Then the subspace $X \oplus Y$ of V is totally isotropic (resp. totally singular) if and only if X and Y are totally isotropic (resp. totally singular) and $X \perp Y$.

(ii) All totally isotropic (resp. totally singular) subspaces of V that are maximal with respect to inclusion have the same dimension $n(F)$.

The number $n(F)$ is called the *Witt index* of (V, F).

Theorem 2.24 (cf. Tits, [45]) *The geometry $\Gamma(V, F)$ is a non-degenerate polar space of rank $n(F)$.*

From now on we will call $\Gamma(V, F)$ the *polar space associated to the pair* (V, F).

Theorem 2.25 (Tits) *A polar space of finite rank ≥ 3 whose planes are Desarguesian is isomorphic to a polar space associated to a pair (V, F) where F is a non-degenerate reflexive sesquilinear or pseudo-quadratic form, or to the polar space of rank 3 which is the grassmannian of 2-spaces in a 4-dimensional vector space.*

We mention that a stronger classification result by Johnson also includes polar spaces of infinite rank (Theorem 5.1 of [33]). However, since we are only interested in geometries that have finite spanning sets, the afore mentioned result falls beyond the scope of our exposition. The next proposition takes care of the rank 2 case. Remember the polar spaces of rank 2 are precisely the generalized quadrangles.

Proposition 2.26 *(Buekenhout, Lefèvre, Dienst, Tits) Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a non-degenerate generalized quadrangle and suppose $e : \Gamma \longrightarrow \mathbb{P}(V)$ is a projective embedding. Then there exists a non-degenerate reflexive sesquilinear or pseudo-quadratic form F such that $e(\mathcal{P})$ and $e(\mathcal{L})$ are precisely the totally isotropic or totally singular* 1- *and* 2-*spaces of V respectively–i.e. the embedding is a 'natural classical embedding'.*

Orthogonal decomposition

Let V be a (right) vector space over a field $\mathbb{F}$ and let F be a reflexive sesquilinear or pseudo-quadratic form on V. We call a pair (V, F) after the form F, only replacing 'form' by 'space', adding the phrase 'over the field $\mathbb{F}$' if we want to specify that V is defined over the field $\mathbb{F}$. Thus, for instance, a (σ, ϵ)-*hermitian space over* $\mathbb{F}$ is a pair (V, f) consisting of a (right) vector space V over the field $\mathbb{F}$ and a (σ, ϵ)-hermitian form f.

Suppose (V_i, F_i) $(i = 1, 2)$ is a non-degenerate reflexive sesquilinear or pseudo-quadratic space. Then a *morphism* of (V_1, F_1) into (V_2, F_2) is an injective linear map $u: V_1 \longrightarrow V_2$ such that $F_1 = F_2 \circ u$, that is

$$F_1(v, w) = F_2(u(v), u(w)) \quad (v, w \in V_1),$$

or

$$F_1(v) = F_2(u(v)) \qquad (v \in V_1)$$

according to whether F_i is sesquilinear or pseudo-quadratic, respectively. If u has an inverse which is also a morphism, then it is called a *isomorphism* and the spaces (V_i, F_i) are called isomorphic; if $V = V_i$ and $F = F_i$ $(i = 1, 2)$, then u is called an automorphism of (V, F).

We distinguish several types of reflexive sesquilinear (resp. pseudo-quadratic) spaces (V, F). If $F(v, v) \neq 0$ (resp. $F(v) \neq 0$) for all $v \in V$, it is called *anisotropic*; an anisotropic quadratic space of dimension 2 is sometimes called an *elliptic line*. If it is non-degenerate and V is spanned by two distinct totally isotropic (resp. totally singular) 1-spaces, the space is called a *hyperbolic line*. finally, if V is the direct sum of mutually orthogonal hyperbolic lines, the space (V, F) is called *hyperbolic*.

Proposition 2.27 *Let V be a vector space of finite dimension over a field $\mathbb{F}$ endowed with a trace-valued (σ, ϵ)-hermitian or (σ, ϵ)-quadratic form F. Then*

(i) there is an orthogonal decomposition (w.r.t. F, that is)

$$V = V^{\perp} \oplus V_{\text{hyp}} \oplus V_{\text{an}},$$

where V_{hyp} is the direct sum of $n(F) - \dim(V^{\perp})$ mutually orthogonal hyperbolic lines and V_{an} is an anisotropic subspace of V.

(ii) the dimensions of the orthogonal components are invariants of the pair (V, F).

A decomposition of V as $V^{\perp} \oplus V_{\text{hyp}} \oplus V_{\text{an}}$ is called an *orthogonal decomposition of V*. The subspaces $V^{\perp}$, V_{hyp} and V_{an} are called *orthogonal components* of V.

Proof: (i) If F is sesquilinear (resp. pseudo-quadratic), let f be F itself (resp. its sesquilinearized form). In the proof 'orthogonal' will stand for 'orthogonal with respect to f', which coincides with 'orthogonal with respect to F' in either case.

Let V' be any subspace such that $V = V^{\perp} \oplus V'$ and put $f' = f|_{V'}$. Then this is an orthogonal decomposition of V, and (V', f') is non-degenerate because its radical $V^{\perp}$ is contained in $V^{\perp} \cap V' = \{0\}$. Hence we may assume that (V, f) is non-degenerate.

If V is anisotropic the rank r is zero and $V = V_{\rm an}$ and we are done. Otherwise we can choose an isotropic (resp. singular) vector $v \in V$. We claim that there exists a vector u such that $f(v,u) \neq 0$. For, if no such vector exists, then v is in the radical of (V,f), which equals $\{0\}$ if f is sesquilinear and which contains no singular vectors if f is pseudo-quadratic, because (V,f) is non-degenerate. By multiplying u by an element from $\mathbb{F}^*$ we can ensure that $f(v,u) = 1$.

Let $w = u - va$ $(a \in \mathbb{F})$, then $f(v,w) = 1$ for each value of a. Suppose that $f = f$ is (σ,ϵ)-hermitian. Then, since f is trace-valued, there exists $b \in \mathbb{F}$ such that $f(u,u) = b + b^\sigma \epsilon$. Put $a = \epsilon^{-1} b$. It then follows that $f(w,w) = f(u,u) - f(u,v)a - a^\sigma f(v,u) + a^\sigma f(v,v)a = f(u,u) - (\epsilon a + a^\sigma)$ and using $\epsilon\epsilon^\sigma = 1$, we find that this equals $f(u,u) - (b + b^\sigma \epsilon) = 0$, so that $H = \langle v,w \rangle_V$ is a hyperbolic line. If, on the other hand, f is pseudo-quadratic, then put $a = f(u)$. It then follows that $f(w) = f(u) + (-a)^\sigma f(v)(-a) - f(u,v)a = f(u) - f(u) = 0$, so that again $H = \langle v,w \rangle_V$ is a hyperbolic line.

We claim that if $V' = H^\perp$, then $V = H \oplus V'$. For, if $x \in V$, then $x' = x - wf(v,x) - vf(w,x)$ lies in V': $f(v,x') = f(v,x) - f(v,w)f(v,x) - f(v,v)f(w,x) = f(v,x) - 1f(v,x) - 0f(w,x) = 0$ (using linearity in the second factor and $f(v,w) = 1$). Repeating this procedure with $V = V'$, we find an orthogonal decomposition of the required form.

(ii) Clearly $\dim(V^\perp)$ is an invariant of the pair (V,F). The fact that $\dim(V_{\rm hyp})$ is an invariant of the pair (V,F) follows from Theorem 2.23. This is due to the fact that $V_{\rm hyp} = M_1 \oplus M_2$ where M_i $(i = 1,2)$ is a totally isotropic (resp. totally singular) subspace of dimension $n(F) - \dim(V^\perp)$. Hence also $\dim(V_{\rm an}) = \dim(V) - \dim(V^\perp) - \dim(V_{\rm hyp})$ is an invariant of the pair (V,F). □

Lemma 2.28 *Let V be a right vector space over a field $\mathbb{F}$ and let F be a (σ,ϵ)-hermitian (resp. (σ,ϵ)-quadratic) form on V. Then any two hyperbolic lines are isomorphic.*

Proof: If F is sesquilinear (resp. pseudo-quadratic), let f be F itself (resp. its sesquilinearized form).

Suppose that $u_i, v_i \in V$ $(i = 1,2)$ are isotropic (resp. singular) vectors such that $f(u_i,v_i) = b_i \neq 0$ and put $H_i = \langle u_i, v_i \rangle_V$. We may assume that v_i is scaled such that $b_i = 1$ $(i = 1,2)$. If $f = F$ is (σ,ϵ)-hermitian, then

$$\begin{aligned} f(\lambda_{i1}u_i + \mu_{i1}v_i, \lambda_{i2}u_i + \mu_{i2}v_i) &= \lambda_{i1}^\sigma f(u_i,v_i)\mu_{i2} + \mu_{i1}^\sigma f(v_i,u_i)\lambda_{i2} \\ &= \lambda_{i1}^\sigma \mu_{i2} + \mu_{i1}\epsilon\lambda_{i2}, \end{aligned}$$

and if F is pseudo-quadratic, then

$$\begin{aligned} f(\lambda_i u_i + \mu_i v_i) &= \lambda_i^\sigma f(u_i)\lambda_i + \mu_i^\sigma f(v_i)\mu_i + \lambda_i^\sigma f(u_i,v_i)\mu_i + \mathbb{F}_{\sigma,\epsilon} \\ &= \lambda_i^\sigma \mu_i + \mathbb{F}_{\sigma,\epsilon}. \end{aligned}$$

Thus the linear injection that sends u_1 to u_2 and v_1 to v_2 preserves f and hence is an isomorphism between the hyperbolic lines $(H_i, f|_{H_i})$ $(i = 1,2)$. □

Lemma 2.29 *Let V be a right vector space over $\mathbb{F}$ of dimension 2 and let f be a non-degenerate (σ,ϵ)-hermitian form or a (σ,ϵ)-quadratic form. Then either f is a symplectic form or V contains a 1-space that is not totally singular/isotropic.*

Proof: Clearly if f is symplectic, i.e. (id., -1)-hermitian, then $f(v,v) = 0$ for all $v \in V$. Conversely, suppose that f is a non-degenerate (σ,ϵ)-hermitian form with $f(v,v) = 0$ for all $v \in V$. For $v, w \in V$ such that $f(v,w) \neq 0$, we find $f(v+w, v+w) = f(v,v) + f(v,w) + f(w,v) + f(w,w) = 0$ so that $f(v,w) = -f(w,v)$. Thus, indeed f is bilinear and anti-symmetric. Furthermore, the only (σ,ϵ)-quadratic form such that $f(v) = 0$ for all $v \in V$ is the zero form. □

2.3.3 Spanning sets for classical polar spaces

Introduction

In this section we find minimal spanning sets for polar spaces associated to a non-degenerate reflexive sesquilinear form or a non-degenerate pseudo-quadratic form. We note that in several cases the polar spaces associated to a pseudo quadratic form and its sesquilinearized form coincide, for instance if $\mathrm{Char}(\mathbb{F}) \neq 2$.

Theorem 2.30 *Let V be a (right) vector space over the field $\mathbb{F}$. Let F be a non-degenerate reflexive sesquilinear form or non-degenerate pseudo-quadratic form on V and let Γ be the associated polar space. Suppose that there is a 4-dimensional subspace $V_0 \subseteq V$ such that F restricted to V_0 is non-degenerate and has Witt index 2.*

(i) If the polar space associated to $(V_0, F|_{V_0})$ has a minimal spanning set of $(d+4)$ points, then Γ has a spanning set of $d + \dim(V)$ points.

(ii) Moreover, $\Gamma(V_0, F)$ is spanned by four points ($d = 0$) if F is pseudo-quadratic and is spanned by five points ($d = 1$) if F is sesquilinear and there is no non-degenerate pseudo-quadratic form q such that the polar spaces associated with F and q coincide.

The proof will be given at the end of this subsection.

We keep the notation of the previous subsection. Thus V is a (right) vector space of finite dimension over the field $\mathbb{F}$ and F is a non-degerate (σ,ϵ)-hermitian or (σ,ϵ)-quadratic form on V. Further, Γ will be the polar space associated to the pair (V, F).

Polar spaces as point-line geometries

In order to consider spanning sets for polar spaces, we must view polar spaces as point-line geometries.

A point-line polar space is a point-line geometry satisfying the following axioms, proposed by Buekenhout and Shult [11]. Here 'subspace' means subspace of the point-line geometry; a singular subspace is a subspace, any two of whose points are collinear.

(BS1) Any line contains at least three points.

(BS2) No point is collinear with all others.

(BS3) Any chain of singular subspaces has finite length.

(BS4) If a point p is not on the line L, then p is collinear with one or all points of L.

A point-line geometry satisfying (BS1), (BS3) and (BS4) is called an *point-line polar space*; it is called *degenerate* if it violates (BS2) and *non-degenerate* otherwise. Since this set of axioms is equivalent to (T1)-(T4) given before, we use 'polar space' to mean polar space or point-line polar space, depending on the context.

In the sequel let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a non-degenerate point-line polar space. For any set of points $S \subseteq \mathcal{P}$, the set $S^\perp$ is the set of points in collinear to each point of S. The set $\mathcal{P}^\perp$, comprising the points that are collinear to all points, is called the *radical* of Γ.

If p and q are collinear points, then pq denotes the line containing these points. Recall that $\langle S \rangle_\Gamma$ (or $\langle S \rangle$) is the smallest subspace of Γ containing S and that a *hyperplane* is a subspace that contains at least one point of each line. A subspace S is called *maximal* if it is a proper subspace that is maximal with respect to inclusion. The following lemma is trivial, but worth noting.

Lemma 2.31 *If M is a maximal subspace of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ and $p \in \mathcal{P} \setminus M$, then $\langle M, p \rangle_\Gamma = \Gamma$.*

It is easy to see that a maximal subspace is a hyperplane. In a non-degenerate polar space also the converse is true.

Proposition 2.32 *Let Γ be a non-degenerate point-line polar space. Then every hyperplane is a maximal subspace.*

Proof: Let us denote collinearity by $\sim$. For points p and q in $\Gamma \setminus H$ we show that $\langle H, p \rangle_\Gamma = \langle H, q \rangle_\Gamma$. There are two cases.

1) If $p^\perp \cap H \neq q^\perp \cap H$, then we may assume that there is a point x on H collinear to p but not to q (by possibly interchanging p and q). Let $y = q^\perp \cap px$ and let $z = qy \cap H$. Then $\langle x, y, z \rangle$ is contained both in $\langle H, p \rangle$ and $\langle H, q \rangle$. Since x, y and z are distinct, we also have $p, q \in \langle x, y, z \rangle$ and we are done.

2) If $p^\perp \cap H = q^\perp \cap H$, then by (BS1) we can choose a point q' collinear to q not on H. If $p^\perp \cap H \neq q'^\perp \cap H$, then by part 1) we have $\langle H, p \rangle = \langle H, q' \rangle$ and it is clear that $\langle H, q' \rangle = \langle H, q \rangle$ so that again we are done. We now show that this is indeed the case.

Suppose that $p^\perp \cap H = q'^\perp \cap H$. Choose a maximal singular subspace S containing q but not q'. Then $S \cap H \subset q^\perp \cap H = q'^\perp \cap H$. As also $q \subset q'^\perp$, the subspace $\langle q', S \cap H, q \rangle = \langle q', S \rangle$ is singular (by repeated application of (BS3)) and strictly larger than the maximal subspace S, a contradiction. □

Remark: Note that we have shown that the complement of a hyperplane H in a non-degenerate point-line polar space Γ is connected (See also Chapter 3).

The geometries we are interested in all satisfy the axioms (BS) and so we can use the proposition above.

Chains of subspaces

Next we find chains of subspaces $S_0 \subset S_1 \subset \cdots \subset S_k = \Gamma$ such that S_{i-1} is a maximal subspace in S_i. Clearly if S_0 has a spanning set of size m, then S_k has a spanning set of size $m + k$ (cf. Lemma 2.31).

For any subspace $U \subseteq V$, denote the subspace of Γ induced on U by $\Gamma(U)$ (i.e. $\Gamma(U)$ is the set of 1-spaces of U that are points of Γ).

Proposition 2.33 *Suppose that there is a subspace $V_0 \subset V$ of dimension 4 and Witt index 2. Suppose moreover that* $\dim(V) > 4$.

(i) If F is symplectic, then there exist subspaces U_0, U_1 and U_2 with $V_0 \subseteq U_0 \subset U_1 \subset U_2 = V$ such that F is non-degenerate on U_0, and for $i = 0, 1$, $\Gamma(U_i)$ is a maximal subspace of $\Gamma(U_{i+1})$.

(ii) If F is not symplectic, then there exists a hyperplane U of the vector space V with $V_0 \subseteq U \subset V$ such that F is non-degenerate on U. Then $\Gamma(U)$ is a non-degenerate polar space and it is a maximal subspace of Γ (which is a non-degenerate polar space too).

Moreover, if $\dim(V) > 2n(F)$*, then we can choose U such that it has Witt index $n(F)$ again.*

Proof: Since V_0 has dimension 4 and Witt index 2, we have ${V_0}^{\perp} = \{0\} = V_{0\,\mathrm{an}}$. Since $V^{\perp} \cap V_0 = \{0\}$, we can find an orthogonal decomposition $V^{\perp} \oplus V_{\mathrm{hyp}} \oplus V_{\mathrm{an}}$ for V extending the one for V_0. In particular we may assume that $V_{\mathrm{hyp}} = \oplus_{i=1}^{n} H_i$ where H_i is a hyperbolic line ($i = 1, \ldots, n = n(F) - \dim(V^{\perp})$) and $V_0 = H_1 \oplus H_2$ (note: $n \geq 2$).

(i) Since F is non-degenerate and symplectic we have $V^{\perp} = \{0\}$ and $V_{\mathrm{an}} = \{0\}$ so that $n = n(F) \geq 3$.

Let R be a hyperplane of H_n. Take $U_0 = \oplus_{i=1}^{n-1} H_i$ and $U_1 = R \oplus U_0$. Then $V_0 \subseteq U_0 \subset U_1 \subset U_2 = V$ as required. Put $\Gamma_i = \Gamma(U_i)$ ($i = 0, 1, 2$). Then $\Gamma(U_0)$ is a non-degenerate (symplectic) polar space, in view of part *(ii)* of Proposition 2.27. Further, Γ_1 is a hyperplane of the non-degenerate polar space $\Gamma_2 = \Gamma$ and hence (by Proposition 2.32) a maximal subspace. Thus we only have to show that the hyperplane Γ_0 of the degenerate polar space Γ_1 is a maximal subspace. But this is easy:

Let r be the point of Γ corresponding to R. As $R = {U_1}^{\perp}$ this point is the radical of Γ_1. Then clearly $\langle r, \mathcal{P}(\Gamma_0)\rangle_{\Gamma_1} = \Gamma_1$, and for every $p \in \Gamma_1 \setminus \Gamma_0$, the subspace $\langle p, \mathcal{P}(\Gamma_0)\rangle_{\Gamma_1}$ contains r and Γ_0, whence Γ_1.

(ii) If $\dim(V) > 2n(F)$, then $\dim(V_{\mathrm{an}}) > 0$ and we can choose a hyperplane ${V_{\mathrm{an}}}'$ of V_{an} and take $U = V_{\mathrm{hyp}} \oplus {V_{\mathrm{an}}}'$. Then $V_0 \subseteq U \subset V$, as required.

Now $\Gamma(U)$ is a non-degenerate polar space in view of part *(ii)* of Proposition 2.27. Note that U has Witt index $n(F)$.

If $\dim(V) = 2n(F)$, then $V = V_{\mathrm{hyp}} = \oplus_{i=1}^{n} H_i$. Since Γ is not symplectic, a hyperbolic line contains an anisotropic 1-space A, say (see Lemma 2.29). For U take the hyperplane $\oplus_{i=1}^{n-1} H_i \oplus A$ of V. Then, since $n > 2$, we have $V_0 \subset U \subset V$, as required. Again $\Gamma(U)$ is a non-degenerate polar space in view of part *(ii)* of Proposition 2.27. Note that U has Witt index $n(F) - 1$. □

The case $\dim(V) = 4$

From now on we assume that $\dim(V) = 4$.

Proposition 2.34 *Let f be a non-degenerate reflexive sesquilinear form on V and let q be a non-degenerate pseudo-quadratic form on V. Then $\Gamma(V, q)$ is spanned by four points and $\Gamma(V, f)$ is spanned by four or five points.*

We will prove this at the end of this subsection.

Lemma 2.35 *Let F be a non-degenerate (σ, ϵ)-hermitian or (σ, ϵ)-quadratic form of Witt index 2 and let $\Gamma = \Gamma(V, F)$. Suppose that F' is a non-degenerate (σ', ϵ')-hermitian or (σ', ϵ')-quadratic form of Witt index 2 such that $\Gamma' = \Gamma(V, F')$ is a subspace of Γ. Then the sesquilinearized forms of F and F' are proportional.*

Proof: Let f and f' be the sesquilinarized forms of F and F' respectively.

We can decompose V as $H_1 \oplus H_2$, where H_i $(i = 1, 2)$ are hyperbolic lines for F'. Let $u_i, v_i \in H_i$ $(i = 1, 2)$ be singular or isotropic vectors with respect to F' (and F). Then, for $x, y \in \{u_1, u_2, v_1, v_2\}$, we have $f(x, y) = 0 = f'(x, y)$ unless $x = u_i$ and $y = v_i$ $(i = 1, 2)$. Let $c_i \in \mathbb{F}$ be such that $f(u_i, v_i) = c_i f'(u_i, v_i)$ $(i = 1, 2)$. We first show that $c_1 = c_2$.

Notice that for all $a, b \in \mathbb{F}^*$ the vectors $u = u_1 + u_2 a$ and $v = v_1 + v_2 b$ are isotropic (or singular) with respect to F' (and F), and for each $b \neq 0$, there is a unique $a \neq 0$ such that $f'(u, v) = 0$ (and $f(u, v) = 0$). We have $0 = f(u, v) = f(u_1, v_1) + a^\sigma f(u_2, v_2) b$ which equals $c_1 f'(u_1, v_1) + a^\sigma c_2 f'(u_2, v_2) b = 0$ and multiplying by ${c_1}^{-1}$, we find that also $\hat{u} = u_1 + u_2 c_1^{-1} a^\sigma {c_2}^{\sigma'}$ satisfies $f(\hat{u}, v) = 0$. Since both u and $\hat{u}$ are in the 2-space $\langle u_1, u_2 \rangle_V$ we must have $\hat{u} = ud$ for some $d \in \mathbb{F}^*$ and since their coefficient of u_1 is the same, we have $\hat{u} = u$. Thus $a = c_1^{-1} a^\sigma {c_2}^{\sigma'}$ for all $a \in \mathbb{F}^*$, and clearly for $a = 0$ too. Taking $a = 1$, we see that $c = c_1 = c_2$ and $a^{\sigma'} = c^{-1} a^\sigma c$ $(a \in \mathbb{F})$.

Put $F'' = cF'$ and let f'' be its sesquilinearized form. The form f'' is (σ'', ϵ'')-hermitian and satisfies $f''(u_i, v_i) = f(u_i, v_i)$ $(i = 1, 2)$ and $\sigma'' = \sigma$. Thus, in order to prove $cF' = F$, we only have to show that $\epsilon'' = \epsilon$.

Let $a, b \in \mathbb{F}$ and $x = u_1 + v_2 a$, $y = u_2 + v_1 b$ be such that $f''(x, y) = 0$ (and $f(x, y) = 0$). Then we find that $0 = f(x, y) = f(u_1, v_1 b) + f(v_2 a, u_2) = f(u_1, v_1 b) + f(u_2, v_2 a)^\sigma \epsilon$ which equals $f''(u_1, v_1 b) + (f''(v_2 a, u_2)^\sigma \epsilon'')^\sigma \epsilon$. Using properties 2.3 and 2.4 we find that $\epsilon''^\sigma f''(v_2 a, u_2)^{\sigma^2} \epsilon = f''(v_2 a, u_2) \epsilon''^{-1} \epsilon$ so that we have $0 = f''(x, y) = f''(x, \hat{y})$, where $\hat{y} = u_2 + v_1 b \epsilon''^{-1} \epsilon$. By the argument that showed why $\hat{u} = u$, we now have $\hat{y} = y$, which implies $\epsilon'' = \epsilon$.

Thus $f'' = cf' = f$. □

Lemma 2.36 *Let q be a non-degenerate (σ, ϵ)-quadratic form on V of Witt index 2 and with sesquilinearized form f. Let $\Gamma_q = \Gamma(V, q)$ and $\Gamma_f = \Gamma(V, f)$. Then a subspace S of Γ_f which is a non-degenerate polar space of rank 2 is equal to Γ_q or to Γ_f. In particular, $S \subseteq \Gamma_q$ implies $S = \Gamma_q$.*

Proof: A non-degenerate subspace S of Γ_f is an embedded non-degenerate polar space of rank 2. Hence, by Theorem 2.26, S consists of the totally isotropic (resp. totally singular) 1-spaces and 2-spaces with respect to a reflexive sesquilinear or pseudo-quadratic form. Further, by Lemma 2.35 the reflexive sesquilinear form that defines orthogonality in S is proportional to f. Thus either $S = \Gamma_f$ or there exists a (σ, ϵ)-quadratic form q' with $S = \Gamma_{q'} = \Gamma(V, q')$. Since the sesquilinearized form of q' is proportional to f, we may as well assume that it is f itself.

Now suppose that either $\Gamma_{q'} \subseteq \Gamma_q$ or $\Gamma_q \subseteq \Gamma_{q'}$. Then there are vectors $u_i, v_i \in V$ $(i = 1, 2)$ that are singular with respect to q and q' (and hence isotropic with respect to f) such that $H_i = \langle u_i, v_i \rangle_V$ are hyperbolic lines for q and q' (and hence for f). Let $x = \sum_{i=1}^{2} \lambda_i u_i + \mu_i v_i$

be an arbitrary vector in V. Then $q(x)$ is completely determined by f since the u_i and v_i ($i = 1, 2$) are singular (to be precise: $q(x) = \sum_{i=1}^{2} \lambda_i^\sigma f(u_i, v_i)\mu_i$) and the same holds for q'. Thus $q' = q$ and $\Gamma_{q'} = \Gamma_q$. □

Remark: The situation where $\Gamma_q \neq \Gamma_f$ can occur only under the condition that f does not determine q uniquely, that is $\mathbb{F}_{\sigma,-\epsilon} \neq \mathbb{F}^{\sigma,-\epsilon}$.

Proof: (of Proposition 2.34) Put, for brevity, $\Gamma_f = \Gamma(V, f)$ and $\Gamma_q = \Gamma(V, q)$.

First assume that q is a pseudo-quadratic form whose sesquilinearized form is f.

Choose the four corner points of an ordinary quadrangle in Γ_q. These points span a non-degenerate subspace of Γ_q, and by Lemma 2.36 this subspace is in fact Γ_q itself.

Now, if we can choose q such that $\Gamma_q = \Gamma_f$, then we are done. Otherwise, by Lemma 2.36, Γ_q is a maximal subspace of Γ_f and we only need to extend the spanning set of Γ_q by one extra point to span Γ_f.

Since Γ_q and Γ_f have an embedding into the 4-dimensional vector space V, these geometries have a minimal spanning set of at least four points (cf. Lemma 2.4). □

Remark: Situations in which Γ_f is spanned by five but not four points do occur.

Proof: (of Theorem 2.30) Given V_0, there exists a chain of subspaces $V_0 \subset V_1 \subset \cdots \subset V_k = V$ such that, for $i = 1, \ldots, k$, V_{i-1} is a hyperplane in V_i and $\Gamma(V_{i-1})$ is a maximal subspace in $\Gamma(V_i)$, By Proposition 2.33.

Thus if $\Gamma(V_0)$ is spanned by $d + 4$ points, then Γ is spanned by $d + \dim(V)$ points.

The second part of the theorem is Proposition 2.34. □

Chapter 3

The geometry far from a residue

3.1 The geometry and its diagram

Let I be a finite set and let $M = (m_{ij})_{i,j\in I}$ be a Coxeter diagram over I. We consider a building $\Delta = (\mathcal{C}, \delta)$ with diagram M. Furthermore, we will look at two kinds of related geometries: the geometry $\Gamma = (\mathcal{O}, \star, \tau, I)$ of Δ and, for $i \in I$, we consider the i-shadow space $\Gamma^{(i)} = (\mathcal{P}, \mathcal{L})$ of Γ (or Δ if you like).

This chapter is devoted to a study of the substructures of Δ, Γ and $\Gamma^{(i)}$ consisting of the chambers, flags, or points 'far away from' or 'in general position with respect to' a given residue, flag or point, respectively. We show that the substructure of Γ consisting of flags far from a given flag F is a transversal geometry with a Buekenhout-Tits diagram that can easily be read off from the type of F. This is established in Theorem 3.2. Furthermore, for a spherical building of rank at least 3 we determine precisely in which cases this geometry is residually connected. The result is presented in Theorems 3.29 and 3.30.

3.1.1 What is far?

Let $(W, (r_i)_{i\in I}, M, I)$ be the Coxeter system with diagram M and let w_I be the longest word in W. We call two chambers c and d in Δ *opposite* if $\delta(c, d) = w_I$. We extend this notion to any pair of residues by stipulating that residues R and S are opposite if every chamber of R is opposite some chamber of S and vice versa. We define two residues R and S to be *far* if there exist chambers in R and S that are opposite. Thus 'far' and 'opposite' are different notions in general, the former implying the latter but not conversely. We note that, if in the definitions above R or S is a chamber, then 'opposite' and 'far' do coincide.

For any residue R we define $\mathrm{Far}_\Delta(R)$ to be the collection of chambers in Δ far from (or opposite to) R. Together with the adjacency relations induced by $\mathcal{C}$, this becomes a chamber system, that we will also denote as $\mathrm{Far}_\Delta(R)$.

Note The notion of 'far' can be introduced in any building Δ that possesses an *opposition relation*, such as is the case in spherical and twin buildings. This is a symmetric relation $\mathcal{O}$ on $\mathcal{C} \times \mathcal{C}$ such that $(x, y) \in \mathcal{O}$ if and only if x and y are at maximal distance (or at minimal codistance). Two chambers x, y with $(x, y) \in \mathcal{O}$ are called *opposite*.

Now consider the geometry Γ. Recall from Subsection 1.2.4 that we have an incidence preserving bijection between the collection of flags in Γ and the collection of residues in Δ.

We say that two chambers of Γ are opposite if they are opposite in Δ. Similarly, two flags E and F are opposite if the corresponding residues in Δ are opposite, that is, if every chamber on E is opposite some chamber of F and vice versa. Two flags E and F are *far in* Γ if there exist chambers on E and F that are opposite (that is, if the corresponding residues in Δ are far). So, again 'opposite' and 'far' are different notions, the former implying the latter but not conversely.

For a flag F of Γ we define $\mathrm{Far}_\Gamma(F)$ to be the collection of objects in Γ far from F. Together with the incidence relation and the type function induced by the geometry Γ, this becomes a geometry that we will also denote as $\mathrm{Far}_\Gamma(F)$ (we note here that for us a geometry is not necessarily transversal. See page 1).

Now fix $i \in I$ and consider the point-line geometry $\Gamma^{(i)}$. The distance in $\Gamma^{(i)}$ is understood as the distance in its incidence graph. We say that a point or line is *far* from another point or line if they are at maximal distance in the incidence graph. For a point P, we define $\mathrm{Far}_{\Gamma^{(i)}}(P)$ to be the collection of points and lines far from P in $\Gamma^{(i)}$. Together with the incidence relation induced by $\Gamma^{(i)}$ this is a point-line geometry that we also denote as $\mathrm{Far}_{\Gamma^{(i)}}(P)$.

The natural question, whether the notions of 'far' in Γ and 'far' in $\Gamma^{(i)}$ coincide, can be answered affirmatively.

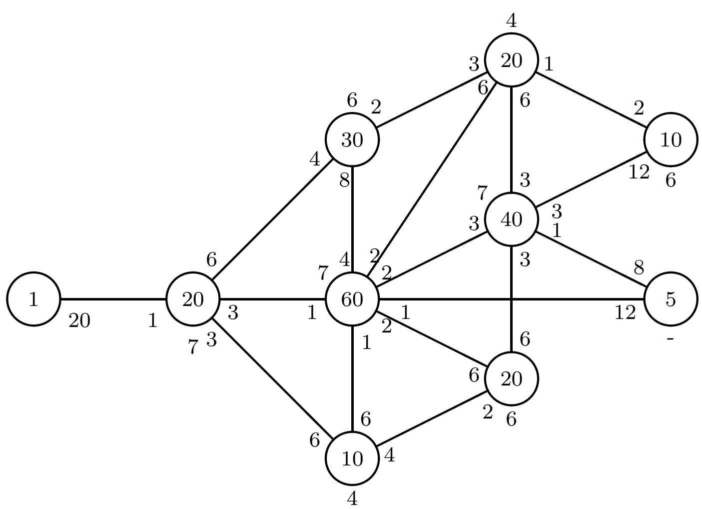
Proposition 3.1 *Let Γ be (the geometry of) a spherical building with a connected diagram labeled by a set I and let P be a point of the i-shadow space $\Gamma^{(i)}$ ($i \in I$) of Γ.*
Then, $Far_{\Gamma^{(i)}}(P) = Far_\Gamma(P) \cap \Gamma^{(i)}$.

Before we prove this we need to point out the relation between the types of opposite residues in Δ (or Γ). For this we recall a few results from Subsection 1.2.5. Note that the opposition relation is symmetric because $w_I^{-1} = w_I$. Consider the map that sends i to j whenever an i-object (resp. i-panel) is opposite a j-object (resp. j-panel). This map is well defined and we denote it by w_I. It induces a symmetry on M, also denoted by w_I. For easy reference we repeat Theorem 1.33.

Theorem 1.33 *The element w_I acts as the identity on all diagrams, but flips A_n, D_{2m+1}, E_6 and I_{2m+1}.*

Proof: (of Proposition 3.1.) First of all, we may restrict ourselves to the thin case. Look at the double coset diagram of the point-line geometry $\Gamma^{(i)}$. The distance from 1 of the double coset represented by w is the minimum number of factors r (where r is the reflection belonging to node i) in any expression of w. Since $w < w_I$ in Bruhat order, for each reduced expression of w_I there is a reduced expression for w obtained from that of w_I by canceling factors. Thus, no double coset is farther from 1 than w_I. If w_I does leave type i invariant, then the double coset diagram of the point-line geometry is reflected by left multiplication by w_I (see Brouwer et al. [8], 10.2.11). Since the double coset of 1 has a unique neighbour in the diagram, this means that also the double coset of w_I has a unique neighbour in the diagram. Consider a double coset represented by w different from that represented by w_I. There is an expression $w_I = wu$ with $l(w_I) = l(w) + l(u)$, and u describes a walk from w to w_I through the double coset diagram in which the distance to 1 never decreases. But it must pass through the unique neighbour of w_I, which is closer to 1, so also w is closer to 1.

On the other hand, if w_I moves type i, then we either have a Grassmann graph, or a dual polar graph, or $E_{6,1}$ (all distance regular), or $E_{6,3}$. Only the latter requires further investigation, and we find that also there there is a unique double coset at maximal distance, see the diagram below. □



$v = 216$

A proof of Proposition 3.1 without case analysis has not been found yet.

Note In general the set $\mathrm{Far}_{\Gamma^{(i)}}(P)$ is the complement of a hyperplane of $\Gamma^{(i)}$ (the hyperplane of points *not* far from P). (see Theorem 2.12 for the proof that these hyperplanes are induced by an embedding in the case that i corresponds to a minimal fundamental weight of the Dynkin diagram.)

3.1.2 The Buekenhout-Tits diagram of $\mathrm{Far}_\Gamma(F)$

Fix a flag F in Γ. In this subsection we show that the geometry $\mathrm{Far}_\Gamma(F)$ induced on the collection of objects far from F is transversal and has a Buekenhout-Tits diagram. More precisely, we prove the following theorem. The phrase 'adding arrows' in the next theorem will be explained below.

Theorem 3.2 *Let Γ be (the geometry of) a building of spherical type, and fix a flag F. The subgeometry $Far_\Gamma(F)$ of Γ consisting of all elements of Γ far away from F (with inherited type and incidence) is a transversal geometry with a Buekenhout-Tits diagram obtained from that of Γ by adding arrows pointing towards the nodes in* $\mathrm{op}(\mathrm{typ}\, F)$*.*

Let F be the flag in the Theorem above. Our first concern is to show that $\mathrm{Far}_\Gamma(F)$ is a transversal geometry. To this end we introduce the collection $\mathcal{F}$ of all flags of Γ far from F.

Lemma 3.3 *The geometry $\mathrm{Far}_\Gamma(F)$ is transversal and $\mathrm{Far}_\Delta(F)$ is its chamber system.*

Proof: If a chamber of Δ is far from (the residue corresponding to) F, then all objects containing it are in $\mathrm{Far}_\Gamma(F)$, so the chamber corresponds to a chamber of $\mathrm{Far}_\Delta(F)$.

Each flag in $\mathcal{F}$ can be identified with a residue in Δ. Since such a flag is far from F, it contains a chamber opposite to some chamber of F and hence the residue is far from the residue corresponding to F. In particular, the chambers of $\mathcal{F}$ correspond to chambers of $\mathrm{Far}_\Delta(F)$.

We now show that $\mathcal{F}$ is the collection of flags of $\mathrm{Far}_\Gamma(F)$. We first consider the inclusion '$\subseteq$'. Let K be a flag in $\mathcal{F}$ and let R_F and R_K be the residues of Δ corresponding to F and K, respectively. Since K is far from F, it is incident with a chamber opposite to a chamber that is incident to F. Thus R_K contains a chamber opposite to some chamber of R_F. Hence all objects containing R_K are far from R_F; these objects correspond precisely to those objects of which K consists.

Next, we consider the inclusion '$\supseteq$'. Note that all objects of $\mathrm{Far}_\Gamma(F)$ are included in $\mathcal{F}$. It only remains to show that if two flags of $\mathcal{F}$ are incident, then their union is also in $\mathcal{F}$. This follows from Lemma 3.4. □

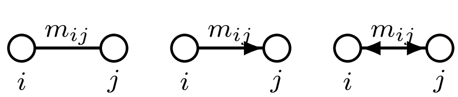
Lemma 3.4 *Let K and L be incident flags of $\mathcal{F}$. Then $K \cup L$ is a flag of $\mathcal{F}$ as well.*

Proof: Let R_F, R_K and R_L be the residues of Δ corresponding to F, K and L respectively. Let A be an apartment incident with R_F and $R_K \cap R_L$. Then in A the three residues $\mathrm{op}_A R_F$, R_K and R_L are pairwise incident, and hence have a common chamber. Hence $R_K \cap R_L$ is far from R_F. It follows that the objects of Δ containing $R_K \cap R_L$ are all far from R_F; these objects correspond precisely to those objects of which $K \cup L$ consists. □

In the remainder of this subsection we will look at the diagram of $\mathrm{Far}_\Gamma(F)$ and prove the second part of the theorem. The reader may skip this since it is not essential to the text.

As always in Buekenhout geometry, the theorem is a claim for geometries of rank at least 3, and is a definition of the strokes involved for geometries of rank 2. Let us repeat this definition explicitly.

The three diagrams



denote, respectively, (i) the class of all generalized m_{ij}-gons, (ii) the class of all subgeometries of a generalized m_{ij}-gon found by taking all elements in the incidence graph at distance $m_{ij}-1$ or m_{ij} from a given i-element when m_{ij} is odd, or from a given j-element when m_{ij} is even, and (iii) the class of all subgeometries of a generalized m_{ij}-gon found by taking all elements in flags at distance m_{ij} from a given flag in the flag graph (the line graph of the incidence graph). As customary, we delete all (arrowed) edges labeled 2 (with or without arrows these all represent generalized digons), we delete labels 3, and use double edges instead of edges labeled 4.

Thus, the diagrams

represent the classes of projective planes, affine planes, dual affine planes and biaffine planes, respectively. Instead of using arrows, one usually uses labels Af and Af* for affine and dual affine planes.

The idea for the definition of the diagram of $\mathrm{Far}_\Gamma(F)$ is based on the following principle.

Lemma 3.5 *Let W be a Coxeter building of spherical type. Then for any two residues R and S we have $\mathrm{op}_R(\mathrm{proj}_R(S)) = \mathrm{proj}_R(\mathrm{op}_W(S))$.*

Proof: Let us view proj_R as a map sending chambers to chambers. Since in the Coxeter buildings W and R for each chamber there is a unique chamber opposite to it, we can view op_W and op_R as a map sending chambers to chambers as well. Therefore it suffices to show $\mathrm{op}_R(\mathrm{proj}_R(x)) = \mathrm{proj}_R(\mathrm{op}_W(x))$ for a chamber $x \in W$. But both projection and opposite are invariant under left multiplication, so we may assume that $R = W_J$ is a subgroup of W. Now $\mathrm{proj}_R(x) = p$ is the element of R such that $p^{-1}x$ is the unique shortest coset representative of the right coset Rx. So we have to show that if a is the shortest element of Ra, i.e., is left R-reduced, then $w_0(R)aw_0$ is shortest in Raw_0. Or again, that $w_0(R)a$ is longest in Ra. But that holds, since $l(ra) = l(r) + l(a)$ for all $r \in R$. □

Lemma 3.6 *Let R and S be far residues of Δ. Then the set of residues meeting $Far_\Delta(S)$ and contained in R is the set of residues of $Far_R(\mathrm{proj}_R(S))$. In particular every residue in R belongs to $Far_\Delta(S)$ if and only if $\mathrm{proj}_R(S) = R$, that is, if and only if R and S are opposite residues.*

Proof: It suffices to prove this for a Coxeter building Σ instead of Γ – then the statements about Γ follow by taking the union over all apartments Σ incident with R and S. By Lemma 3.5 we are done. □

Proof: (of the theorem). We have seen that $\mathrm{Far}_\Gamma(F)$ is a transversal geometry. As for the statement about the diagram, we use induction on the rank of Γ. Consider a residue R of type J far from F. By the previous lemma the set of residues meeting $\mathrm{Far}_\Gamma(F)$ and contained in R is the set of residues of $\mathrm{Far}_R(\mathrm{proj}_R(F))$. By the induction hypothesis, the diagram for this geometry is obtained from that of R by adding arrows that point towards the nodes in $\mathrm{op}_J(\mathrm{typ}(\mathrm{proj}_R(F)))$. This diagram is just the diagram for $\mathrm{Far}_\Gamma(F)$ restricted to the nodes in J because we have $\mathrm{op}_J(\mathrm{typ}(\mathrm{proj}_R(F))) = J \cap \mathrm{op}_I(\mathrm{typ}(F))$ by Lemma 3.5. □

Notes

The geometries obtained from other geometries by removing a hyperplane are often called 'affine geometries'. Such affine geometries have been studied by several people. A few references are Batens and Pasini [2], Brouwer [7] van Bon and Cuypers [48], Blok and Brouwer [3], Cuypers [26], Cohen and Shult [15], Cuypers and Pasini [27] and Pasini [35].

3.2 Connectedness

In Subsection 3.1.2 we have seen that the geometry $\mathrm{Far}_\Gamma(F)$ is transversal. This subsection is devoted to a study of some connectedness properties of $\mathrm{Far}_\Delta(F)$ and $\mathrm{Far}_\Gamma(F)$. For spherical buildings of rank at least 3 we will show precisely when this geometry is residually connected. For those buildings and flags where this is not the case we determine the number of connected components of $\mathrm{Far}_\Delta(F)$. The main results on connectedness and the number of components are presented in Theorems 3.29, 3.30 and 3.33 respectively.

3.2.1 Introduction

We will apply the following corollary to Lemma 1.8 to the transversal geometry $\mathrm{Far}_\Gamma(F)$ and its chamber system $\mathrm{Far}_\Delta(F)$.

Corollary 3.7 *Let Γ be a transversal geometry and let $\mathcal{C}$ be its chamber system. Then Γ is residually connected if and only if the chamber system of every residue of rank at least 2 is connected.*

Thus we will study the connectedness of (the residues in) $\mathrm{Far}_\Delta(F)$. It will turn out that $\mathrm{Far}_\Delta(F)$ is connected if and only if the intersection of every rank 2 residue with $\mathrm{Far}_\Delta(F)$ is connected or empty.

We present two approaches to this problem, one group theoretic and one combinatorial/geometric. They are treated in Subsubsections 3.2.4, 3.2.6 and 3.2.7 respectively. The group theoretic approach yields the more complete answers: It shows exactly for which spherical buildings Δ and flags F the chamber system $\mathrm{Far}_\Delta(F)$ is connected and it enables us to compute the number of connected components in all other cases. The combinatorial approach has the same results. Both approaches reduce the problem of deciding on the connectedness of $\mathrm{Far}_\Delta(F)$ to the rank 2 case.

3.2.2 The rank 2 case

Let Δ (and Γ) be a building of rank 2. We will think of Γ as a point-line geometry, so $\Gamma = (\mathcal{P}, \mathcal{L})$ is a thick generalized n-gon of order (s,t), say (Remember, a generalized n-gon is said to have order (s,t) if every line has $s+1$ points and every point has $t+1$ lines.).

Proposition 3.8 *Let Γ be a thick generalized n-gon with $n \leq 4$ and let F be a flag. Then $\mathrm{Far}_\Gamma(F)$ is (residually) connected except when Γ is the generalized quadrangle of order $(2,2)$ and F is a point-line pair.*

Proof: If $n = 2$ then two elements are 'far' if (and only if) they are not equal. Hence, far from F we find the digon $(\mathcal{P} \setminus (\mathcal{P} \cap F), \mathcal{L} \setminus (\mathcal{L} \cap F))$, which is clearly connected. If $n = 3$ then two elements are 'far' if (and only if) they are not incident. Hence, far from a line, a point or a point-line pair F respectively we find an affine plane, a dual affine plane or a biaffine plane; if Γ is thick, these geometries are connected because every element is incident with at most one element not far from F. For $n = 4$, the result follows from Theorem 3.9. □

If $n = 4$ then two points (lines) are far when they are not collinear (not concurrent) and a point and a line are far when they are not incident. Here we find the smallest disconnected example; it is the subgeometry of the unique generalized quadrangle of order $(2, 2)$ far from a point-line pair. In this quadrangle the points and lines far from a point $p \in \mathcal{P}$ are the vertices and edges of a cube; each line $L \in \mathcal{L}$ on p is concurrent with four 'parallel' lines of this cube such that the geometry far from (p, L) is the union of two quadrangles. This however is the only generalized quadrangle where the geometry far away from a flag is disconnected, as the following theorem shows.

Theorem 3.9 *(Brouwer and Cuypers) Let $(\mathcal{P}, \mathcal{L})$ be a thick generalized quadrangle. Then the subgeometries far from a point or line (or indeed, the complement of an arbitrary geometric hyperplane) and the subgeometry far from a point-line pair are connected, except that in case $(\mathcal{P}, \mathcal{L})$ is the unique generalized quadrangle of order $(2, 2)$, the subgeometry far from a point-line pair is the union of two quadrangles.*

Proof: We first show that the complement of a geometric hyperplane is connected. This includes the subgeometries far away from a point (or dually, a line) p since either one or all points of a line are collinear to, i.e. not far from, p.

We denote collinearity (concurrency) and incidence by $\sim$ and $\star$ respectively. Let H be a geometric hyperplane and let $x, y \in \mathcal{P} \setminus H$. We must find a sequence $x = x_0, \ldots, x_k = y$ of points with $p_i \in \mathcal{P} \setminus H$ and $x_{i-1} {\sim} x_i$ for $i = 1, \ldots, k$. If $x {\sim} y$ we are done. If not then choose mutually non-concurrent lines M and N on x and y respectively and let x', y' be such that $M \star x' {\sim} y$ and $N \star y' {\sim} x$. If x' or y' lies in $\mathcal{P} \setminus H$, then x, x', y or x, y', y is the required sequence. Otherwise, by thickness we can take a point x'' on M distinct from x and x'. Then the point y'' with the property that $N \star y'' {\sim} x''$ is distinct from y and y' and hence lies in $\mathcal{P} \setminus H$. Now x, x'', y'', y is the required sequence.

The part about the generalized quadrangle of order $(2, 2)$ was discussed above. Let $(\mathcal{Q}, \mathcal{M})$ be the geometry far away from an incident point-line pair (p, L). Since the points far from (p, L) are the points far from p and the geometry far from p is connected, it suffices to show that two points $q, r \in \mathcal{Q}$ that are joined by a line K concurrent to L are joined by a chain in $(\mathcal{Q}, \mathcal{M})$. By duality, we may assume that each line has at least four points. Let M, N be lines on q and r distinct from K. Then $M, N \in \mathcal{M}$. A chain $q \star M \star x \star H \star y \star N \star r$ will join q and r in $(\mathcal{Q}, \mathcal{M})$ unless $p {\sim} x$ or $p {\sim} y$ or $L {\sim} H$. Assume that it is impossible to avoid any of these obstructions. Let $A = \{a \mid p {\sim} a {\sim} q\}$ and $B = \{b \mid p {\sim} b {\sim} r\}$. Suppose $x \in N \setminus B$. Then one of the lines on x meets L and, by assumption, all other lines on x meet A. If $x, x' \in N \setminus B$ are distinct, their common neighbours lie on N and certainly not in A and hence $|A| = 2$. Now $t = 2$ so that $s > 3$. Thus we can find a third point $q'' \in N \setminus B$ which must be collinear to at least one element of A, a contradiction. $\square$

This theorem and (almost) this proof appeared in Brouwer [7].

For the purpose of determining the connectedness of the subgeometry far from a flag in spherical buildings of rank at least 3 the result of Proposition 3.8 suffices since the rank 2 residues of such buildings are generalized n-gons with $n \leq 4$.

3.2.3 Generalized n-gons with $n \geq 5$

We keep the notation of the previous subsection. Whereas for $n \leq 4$ we found that $\mathrm{Far}_\Gamma(F)$ is connected except in one case, for $n \geq 5$ the situation is different. The following construction, which is due to Abramenko, illustrates this fact.

Proposition 3.10 (cf. Abramenko [1], Proposition 9)
For every integer $n \geq 5$, there exists a generalized n-gon Γ together with a chamber c of Γ satisfying the following conditions:

(i) *Every element of Γ is contained in infinitely many chambers.*

(ii) *$\mathrm{Far}_\Gamma(c)$ is an infinite disjoint union of trees.*

Proof: Take an arbitrary connected, bipartite graph Γ_0 of girth $2n$ and fix an edge c of Γ_0. For every $i \in \mathbb{N}$ construct a bipartite graph Γ_i containing Γ_{i-1} using the following steps (d_i denotes the usual distance in Γ_i):

(1) If x and y are vertices of Γ_{i-1} with $d_{i-1}(x,y) = n+1$, insert a new path $p(x,y)$ of length $n-1$ connecting them.

(2) If additionally $d_{i-1}(x,c) \geq n-1$ and $d_{i-1}(y,c) \geq n-1$, insert a further path $p(x,y,c)$ of length $n-2$ connecting a central vertex $z(x,y)$ of $p(x,y)$ (for even n there are two possibilities) with c. This is done such that the type function on $\Gamma_{i-1} \cup p(x,y)$ can be extended to $\Gamma \cup p(x,y) \cup p(x,y,c)$.

Then $\Gamma = \bigcup_{i=0}^{\infty} \Gamma_i$ is a generalized n-gon. Clearly Γ is connected and bipartite. For $u, v \in \Gamma_{i-1}$ with $d = d_{i-1}(u,v)$ we have $d_i(u,v) = d$ if $d \leq n$ and $d_i(u,v) \geq d-2$ otherwise. Step (1) joins vertices x and y at distance $n+1$ by a path of length $n-1$. Thus all paths containing x and y, which had length $l \geq n+1$ in Γ_{i-1}, can be replaced by a path of length $l-2$ in Γ_i. Step (2) joins two vertices at distance $\leq n$ in Γ_{i-1} by a path of length at least $\lfloor \frac{n-1}{2} \rfloor + (n-2) \geq n$ (since $n \geq 5$) and hence doesn't reduce the distances between the vertices of Γ_{i-1}. Note that the new vertices are not contained in a cycle of length $< 2n$. Thus the fact that Γ_0 has girth $2n$ implies that Γ has girth $2n$ and diameter at least n. By step (1) the diameter of Γ is at most n.

If Γ_0 contains an edge whose vertices are each contained in at least three edges then Γ is thick. This follows from the well known fact that for any generalized n-gon Γ there is a pair (s,t) (called the order of Γ) of (possibly infinite) numbers such that the vertices on any given edge have valency $s+1$ and $t+1$.

Let $\mathrm{Far}_{\Gamma_i}(c)$ be the subgraph of Γ_i generated by all vertices $z \in \Gamma_i$ such that $d_{i-1}(z,c) \geq n-1$. Then $\mathrm{Far}_\Gamma(c) = \bigcup_{i=0}^{\infty} \mathrm{Far}_{\Gamma_i}(c)$ because of the relation between d_{i-1} and d_i indicated above.

Elements of distinct connected components of $\mathrm{Far}_{\Gamma_0}(c)$ lie in distinct connected components of $\mathrm{Far}_\Gamma(c)$. Moreover, the cycles of $\mathrm{Far}_\Gamma(c)$ are contained in $\mathrm{Far}_{\Gamma_0}(c)$. If in constructing Γ_i from Γ_{i-1} two vertices u, v of $\mathrm{Far}_{\Gamma_{i-1}}(c)$ are joined by a path, this must happen in step (1). By applying step (2) at least the vertex $z(x,y)$ of the path $p(x,y)$ is not far from c in Γ_i and so the vertices u and v are not connected by a path in $\mathrm{Far}_{\Gamma_i}(c)$. So during the construction no pair of connected components of $\mathrm{Far}_{\Gamma_0}(c)$ can become part of a single connected component of $\mathrm{Far}_\Gamma(c)$ and no path in $\mathrm{Far}_{\Gamma_0}(c)$ can be completed to a cycle in $\mathrm{Far}_\Gamma(c)$.

If we let Γ_0 be a connected, bipartite graph of girth $2n$ and c is an edge whose vertices have valency ≥ 3 and $\mathrm{Far}_{\Gamma_0}(c)$ is the disjoint union of k trees, then Γ is a thick generalized n-gon and $\mathrm{Far}_{\Gamma}(c)$ is the disjoint union of at least k trees. An example of such a graph Γ_0 is obtained by glueing k copies of an ordinary (thin) $2n$-gon along an edge c. □

These examples are rather large and seem to have little symmetry so one expects to avoid them by considering finite generalized n-gons or by requiring more symmetry.

For a finite thick generalized n-gon Γ is finite, it is possible to obtain the following result.

Theorem 3.11 (Brouwer) *Let $(X, \mathcal{L})$ be a thick finite generalized n-gon of order (s, t). Then*

(i) *The subgeometry induced by the points and lines in general position w.r.t. a given point x (or, indeed, the complement of an arbitrary geometric hyperplane in $(X, \mathcal{L})$) is connected, except possibly in the cases $(n, s, t) = (6, 2, 2)$, $(8, 2, 4)$.*

(ii) *The subgeometry induced by the points and lines in general position w.r.t. a given flag (x, L) is connected, except possibly in the cases $(n, s, t) = (4, 2, 2)$, $(6, 2, 2)$, $(6, 3, 3)$, $(8, 2, 4)$, $(8, 4, 2)$.*

(iii) *For the stated possibly exceptional parameter sets actual exceptions do occur: in case (i) for $G_2(2)$ and ${}^2F_4(2)$, and in case (ii) also for $B_2(2)$ ($\cong Sp(4, 2)$) and $G_2(3)$.*

Here 'in general position w.r.t.' means 'at maximal distance from'. This theorem together with a description of the (known) exceptions, including the number of connected components appears in Brouwer [7]. The proof uses knowledge of the eigenvalues of the adjacency matrix and a compressed form of this matrix.

One way of assuming more symmetry in Γ is to require that it is of Moufang type. A Moufang polygon is a rank 2 building posessing an automorphism group acting strongly transitively that is generated by the root groups of an apartment (see Subsection 1.2.5). These automorphism groups have been classified and we can describe them by generators and commutator relations. This can be used to prove the following proposition.

Theorem 3.12 (cf. Abramenko [1], Proposition 7)
Let Δ be a thick Moufang generalized n-gon not associated to one of the following four groups:

$$C_2(2) = Sp_4(2),\ G_2(2),\ G_2(3),\ {}^2F_4(2).$$

Then $Far_\Delta(c)$ is connected for any chamber $c \in \Delta$.

The idea of the proof is to determine the stabilizer of a component of $\mathrm{Far}_\Gamma(c)$ and to compute the index of this subgroup in the stabilizer of $\mathrm{Far}_\Gamma(c)$. This gives the number of components in all cases. A more elaborate description of this procedure will be given in Subsubsection 3.2.4, where this strategy, which I happened to discover independently, is applied to an arbitrary flag in an arbitrary spherical building.

This theorem has also been proved by Van Maldeghem and others in a purely geometric way. We mention that all exceptions stated in this theorem are finite and were described in Theorem 3.11.

3.2.4 Higher rank

We assume that Δ is a thick Moufang building of spherical type. Let Σ be an apartment with root system Φ and let c be a chamber on Σ. Let G be the group generated by the root groups U_α ($\alpha \in \Phi$). We know that G acts strongly transitively on Δ and that ($B = \mathrm{Stab}_G(c), N = \mathrm{Stab}_G(\Sigma)$) is a Tits system for G. Put $H = B \cap N$.

Fix a residue R on c. In this subsection we will determine the stabilizer in G of $\mathrm{Far}_\Delta(R)$ and of one of its connected components.

The stabilizer of Far$_\Delta(R)$

For $\epsilon = +, -, 0$, we define $\Phi^\epsilon(R)$ to be the set of roots $\alpha \in \Phi$ such that $R \cap \Sigma \subseteq \alpha$, $R \cap \Sigma \subseteq -\alpha$ or $R \cap \alpha \neq \emptyset \neq R \cap -\alpha$. Now put

$$\begin{aligned} U(R) = &\ \langle U_\alpha \mid \alpha \in \Phi(R) \rangle \\ L(R) = &\ \langle H, U_\alpha \mid \alpha \in \Phi^0(R) \rangle \\ P(R) = &\ \langle L(R), U(R) \rangle \end{aligned}$$

Theorem 3.13 (cf. Ronan [38], Theorem 5.4 and Theorem 6.18) *For a thick Moufang building of spherical type we have*

$$P(R) = U(R) \rtimes L(R).$$

Let $\widehat{R}$ be the residue of Σ opposite to R. Then,

(i) $P(R)$ is the stabilizer of R in G,

(ii) $L(R)$ is the subgroup of G stabilizing R and $\widehat{R}$; it acts transitively on the chambers of both R and $\widehat{R}$,

(iii) $U(R)$ fixes every chamber of R and acts sharply 1-transitively on the residues of Δ opposite to R.

Corollary 3.14 *The stabilizer of Far$_\Delta(R)$ in G is $P(R)$. This group acts transitively on the chambers of Far$_\Delta(R)$ and the stabilizer of the chamber $\hat{c}$ of Σ opposite to c lies in its subgroup $L(R)$.*

Proof: This is immediate from the theorem. □

The stabilizer of a connected component of Far$_\Delta(R)$

In order to describe the stabilizer of a component of $\mathrm{Far}_\Delta(R)$ in the way we defined the stabilizer of $\mathrm{Far}_\Delta(R)$ we need to refine the notions $\Phi^\epsilon(R)$ ($\epsilon = +, -, 0$). For two residues R, T on Σ that have a non-empty intersection, let $\Phi^+(R;T)$, $\Phi^-(R;T)$ and $\Phi^0(R;T)$ be the sets of roots α of $\Phi^0(T)$ such that $R \cap \Sigma \subseteq \alpha$, $R \cap \Sigma \subseteq -\alpha$ or $R \cap \alpha \neq \emptyset \neq R \cap -\alpha$ respectively. This is a refinement of the notation above in the sense that we have $\Phi^\epsilon(R) = \Phi^\epsilon(R;\Delta)$ ($\epsilon = +, 0, -$).

Later on we will need the following lemma, which is easy to prove but a little technical.

Lemma 3.15 *Let R, T be residues on Σ that have a non-empty intersection. Then (with $\epsilon = +, 0, -$)*

(i) $\Phi^0(R;T) = \Phi^0(T;R) = \Phi^0(R) \cap \Phi^0(T) = \Phi^0(R \cap T)$,

(ii) $\Phi^\epsilon(R;T) = \Phi^\epsilon(R \cap T;T)$.

Proof: (i) The first two equalities follow immediately from the definitions. Next we show that $\Phi^0(R) \cap \Phi^0(T) = \Phi^0(R \cap T)$. Let $\alpha \in \Phi$ and let r be the reflection that interchanges α and $-\alpha$. Note that α is in $\Phi^0(X)$ for some residue X on Σ (a) if and only if $rX \cap X \neq \emptyset$ hence (b) if and only if $rX = X$. Consider the inclusion $r(R \cap T) \subseteq rR \cap rT$. Now (a) implies '$\supseteq$' and (b) together with (a) implies '$\subseteq$'.

(ii) For $\epsilon = 0$ this follows from (i) and the observation that, as all roots of $R \cap T$ are roots of T, we have $\Phi^0(R \cap T) = \Phi^0(R \cap T;T)$. As for the cases $\epsilon = +, -$, notice that the complements of $\Phi^0(R;T)$ and $\Phi^0(R \cap T;T)$ in $\Phi^0(T)$ coincide, that is $\Phi^+(R;T) \cup \Phi^-(R;T) = \Phi^+(R \cap T;T) \cup \Phi^-(R \cap T;T)$. The equalities for $\epsilon = +, -$ now follow easily. □

Set $\Phi^{(i)}(R) = \bigcup_T \Phi^+(R;T)$ for $i = 1, 2$, where T ranges over all residues of rank i that intersect R, but are not contained in R. Now put

$$\begin{aligned} U^{(1)}(R) &= \langle U_\alpha | \alpha \in \Phi^{(1)}(R) \rangle \\ P^{(1)}(R) &= \langle U^{(1)}(R), L(R) \rangle \end{aligned}$$

Remark: The groups H, $L(R)$, $P(R)$ and $P^{(1)}(R)$ depend on the choice of Σ because $H = \mathrm{Fix}_G(\Sigma)$. This will not produce confusion because Σ and R will remain fixed throughout the rest of this chapter. The groups do not depend on the choice of the chamber c.

We have the following theorem analogous to the first part of Theorem 3.13.

Theorem 3.16 *If R meets no $I_2(m)$-residue with $m \geq 4$ in more than one chamber we have*

$$P^{(1)}(R) = U^{(1)}(R) \cdot L(R).$$

Proof: We show that $L(R)$ normalizes $U^{(1)}(R)$ under the stated conditions. Let $\alpha \in \Phi^{(1)}(R)$ be the root in $\Phi^0(T)$ for a rank 1 residue T that intersects R in a chamber and let $g \in U_\beta$ for some root $\beta \in \Phi^0(R)$ of R. Since H normalizes each U_α, it suffices to show that $(U_\alpha)^g \subseteq U^{(1)}(R)$. Let Y be a rank 2 residue such that α and β are roots of $\Phi^0(Y)$. Then the Chevalley commutator relations indicate that the commutator $[U_\alpha, U_\beta]$ is contained in the subgroup generated by root groups U_γ of Y such that $\gamma \supseteq \alpha \cap \beta$ and $\gamma \neq \alpha, \beta$ (see Theorem 1.63). If Y has diagram $A_1 \times A_1$ this means that $[U_\alpha, U_\beta] = \{1\}$ and we are done. If Y has diagram A_2, then $[U_\alpha, U_\beta] \leq U_{r\alpha}$, where r is the reflection that interchanges β and $-\beta$. Since $r\alpha$ is a root of $\Phi^0(rT)$ and rT is a residue of rank 1 which intersects, but is not contained in R and $(U_\alpha)^g \subseteq U_\alpha U_{r\alpha}$, we are done also in this case.

Finally, the conditions of the proposition now ensure that the rank 2 residue Y doesn't have a diagram $I_2(m)$ with $m \geq 4$. First note that one can find such a residue Y in the smallest residue Z of Δ that contains R and T.

Now let $m \geq 4$. The fact that R meets no $I_2(m)$ residue in more than one chamber means that the diagram of R is disjoint from any subdiagram of type $I_2(m)$. As T has rank 1, we find that the diagram of Z has no $I_2(m)$ subdiagrams.

The inclusion $U^{(1)}(R) \subseteq U(R)$ implies that $U^{(1)}(R) \cap L(R) = 1$ so that $P^{(1)}(R)$ is the semidirect product $U^{(1)}(R) \cdot L(R)$. $\square$

Remarks

The condition that R should not meet an $I_2(m)$-residue with $m \geq 4$ in more than one chamber seems strange, but it turns out to be satisfied whenever we need it. This condition is necessary in Theorem 3.16 as we will show now.

Let Δ be the building obtained from $\mathrm{Sp}_6(2)$. Label its Dynkin diagram M of type C_3 with the set $I = \{1, 2, 3\}$, where 3 corresponds to the long roots. Let R be a $\{1, 2\}$-residue. The panels of Σ meeting R in a chamber are $\{3\}$-residues. These panels determine three roots α_1, α_3 and α_5 of $\Phi^{(1)}(R)$. The corresponding root groups commute pairwise (e.g. because the sum of any two of these roots is not a root) so that we have $U^{(1)}(R) = U_{\alpha_1} \times U_{\alpha_3} \times U_{\alpha_5}$.

Let Y be a 2-panel on R containing c and let Z be the $\{2, 3\}$-residue on c. Then we may assume that the roots of $\Phi^0(Z)$ that contain c are (in cyclic order): $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, such that α_4 is a root of $\Phi^0(Y)$. Let u_i be the unique non-trivial element of U_{α_i} $(i = 1, 2, 3, 4)$. Then $[u_1, u_4] = u_2u_3$ (check with Chevalley commutator relations, or see Example 3.2.4). We have $u_1^{u_4} = u_1^{-1}u_2u_3$, which is not an element of $U^{(1)}(R)$ although $u_1 \in U^{(1)}(R)$ and $u_4 \in L(R)$.

Thus we see that $L(R)$ doesn't always normalize $U^{(1)}(R)$ if R meets a I_4 residue (Z in this example) in a panel (Y in this example).

The next theorem is analogous to items (i) and (iii) of Theorem 3.13 in combination with Corollary 3.14.

Theorem 3.17 *Let C be the connected component of $Far_\Delta(R)$ containing the residue $\widehat{R}$ on Σ opposite to R. Then,*

(i) $P^{(1)}(R)$ is the stabilizer of C in $P(R)$; this group acts transitively on the chambers of C,

(ii) $U^{(1)}(R)$ acts sharply 1-transitively on the residues of Δ opposite to R that lie in C.

Corollary 3.18 *The number of connected components of $Far_\Delta(R)$ equals the index $[P(R) : P^{(1)}(R)]$.*

Proof: (of Theorem 3.17) (i) We first note that C comprises those elements of $\mathrm{Far}_\Delta(R)$, that can be joined to $\hat{c}$ by a gallery (of finite length) in $\mathrm{Far}_\Delta(R)$. As $P(R)$ stabilizes $\mathrm{Far}_\Delta(R)$ and preserves adjacency, the stabilizer of C in $P(R)$ comprises precisely those elements $g \in P(R)$ for which there is a gallery in $\mathrm{Far}_\Delta(R)$ joining $\hat{c}$ to $g\hat{c}$. Since $P(R)$ is transitive on the chambers of $\mathrm{Far}_\Delta(R)$, it is immediate that the stabilizer of C in $P(R)$ is transitive on the chambers of C.

We prove that $P^{(1)}(R)$ is contained in the stabilizer of C in $P(R)$. More precisely, we show that if $g \in P^{(1)}(R)$, then $g\hat{c} \in C$. If $g \in L(R)$, then $g\hat{c} \in \widehat{R} \subset C$ and hence $L(R)$ preserves C. Now let $g \in U_\alpha$ for some $\alpha \in \Phi^+(R;T)$, where T is a rank 1 residue that intersects R in a chamber d (note that $d \in \Sigma$). Then the residue $\widehat{T}$ on Σ opposite T intersects $\widehat{R}$ in a unique chamber $\hat{d}$ on Σ opposite d. As $g \in U(R) \cap L(T)$ it stabilizes both R and $\widehat{T}$. Thus $\widehat{R}$ and $g\widehat{R}$ are residues opposite R that intersect the rank 1 residue $\widehat{T}$ in the chambers $\hat{d}$ and $g\hat{d}$ respectively. Hence $g\widehat{R} \subset C$.

For the converse inclusion, we show by induction on l that all elements in $P(R)$ that send $\hat{c}$ to a chamber of C that can be connected to $\hat{c}$ by a gallery in $\mathrm{Far}_\Delta(R)$ of length at most l, are in fact elements of $P^{(1)}(R)$.

Let $g \in P(R)$ be in the stabilizer of C. Suppose that there is a gallery $\hat{c} = x_0, x_1, \ldots, x_l = g\hat{c}$ of chambers in C. If $l = 0$ then g stabilizes R and $\widehat{R}$ so that $g \in L(R) \leq P^{(1)}(R)$, by Theorem 3.13. Suppose now that $l \geq 1$. Then by the induction hypothesis there is an element $f \in P^{(1)}(R)$ such that $f\hat{c} = x_{l-1}$. Let $\widehat{T}$ be the panel on $f^{-1}x_{l-1} = \hat{c}$ and $f^{-1}x_l = c'$. If $\widehat{T} \subseteq \widehat{R}$, then there is an element $e \in L(R)$ with $e\hat{c} = c'$. If not then let T be the residue on c opposite to $\widehat{T}$. Since $\hat{c} \in \widehat{R} \cap \widehat{T}$, we have $c \in R \cap T$. Now let α be the unique root in $\Phi^+(R;T)$. As $\alpha \supset R \cap \Sigma$, no chamber of α is opposite any chamber of R and therefore $\hat{c}, c' \in \widehat{T} \setminus \alpha$. The group $U_\alpha \leq U^{(1)}(R)$ acts simple transitively both on the chambers of $T \setminus \alpha$ and $\widehat{T} \setminus \alpha$. Thus there exists an element $e \in U_\alpha$ such that $e\hat{c} = c'$. It follows that $fe \in P^{(1)}(R)$ sends $\hat{c}$ to $g\hat{c}$. Hence $g(fe)^{-1}$ stabilizes both R and $\widehat{R}$, which implies (by Theorem 3.13) $(fe)^{-1}g = l \in L(R)$ so that $g = fel \in P^{(1)}(R)$.

(*ii*) Since $U(R)$ acts sharply 1-transitively on the collection of all residues opposite to R, the subgroup $U^{(1)}(R)$ acts sharply 1-transitively on the residues opposite to R in C. □

Products of buildings

Suppose that Δ is a building with diagram M and that M is the disjoint union $M_1 \sqcup M_2$. Then Δ is a direct product $\Delta_1 \times \Delta_2$, where Δ_i is a building with diagram M_i ($i = 1, 2$). The distance between two chambers (x_1, x_2) and (y_1, y_2) ($x_i, y_i \in \Delta_i$, $i = 1, 2$) is the product $\delta((x_1, x_2), (y_1, y_2)) = \delta_1(x_1, y_1)\delta_2(x_2, y_2)$, where δ_i is the distance function of Δ_i ($i = 1, 2$). The residue R is also a direct product $R_1 \times R_2$ where R_i is a residue of Δ_i ($i = 1, 2$) and it follows that $\mathrm{Far}_\Delta(R) = \mathrm{Far}_{R_1}(\Delta_1) \times \mathrm{Far}_{R_2}(\Delta_2)$. So if, for instance, $\mathrm{Far}_{R_i}(\Delta_i)$ has n_i connected components ($i = 1, 2$), then $\mathrm{Far}_\Delta(R)$ has $n_1 n_2$ connected components.

The apartment Σ, being a building with diagram M, is a product $\Sigma_1 \times \Sigma_2$, where Σ_i is an apartment of the building Δ_i and its root are of the form $\alpha_1 \times \Sigma_2$ or $\Sigma_1 \times \alpha_2$, where α_i is a root of Σ_i ($i = 1, 2$). The corresponding root groups U_{α_1} and U_{α_2} commute because the rank 2 residues that both α_1 and α_2 are roots of are of type $A_1 \times A_1$, by the Chevalley commutator relations. Hence the group G is of the form $G_1 \times G_2$, where G_i is the automorphism group of Δ_i generated by the root groups of the apartment Σ_i.

For the same reason all groups defined above, such as B, N, H, $L(R)$, $U(R)$ and $U^{(1)}(R)$, turn out to be direct products of their equivalents in the groups G_i. Clearly then we have

$$\begin{aligned} P(R) &= \langle U(R), L(R)\rangle \\ &= \langle U(R_1), U(R_2), H_1, H_2, G_1(R_1), G_2(R_2)\rangle \\ &= \langle U(R_1), H_1, G_1(R_1)\rangle \times \langle U(R_2), H_2, G_2(R_2)\rangle \\ &= P(R_1) \times P(R_2) \end{aligned},$$

and similarly for $P^{(1)}(R)$. Here we have written X_1 and X_2 instead of $X_1 \times e_2$ and $e_1 \times X_2$ for subgroups X_i and unit elements e_i of G_i $(i = 1, 2)$.

Examples

Except for the symbols Δ, c, $U(c)$ and $U^{(1)}(c)$, these examples are written in the language of Theorem 1.63.

Let M be a Dynkin diagram of type B_2 or C_2. Let Φ be a root system of type M with a system $\Pi = \{a, b\}$ of fundamental roots. Consider a Chevalley group G with Dynkin diagram M defined over a field $\mathbb{F}$. Let (B, N) be a Tits system for G and let Δ be the building corresponding to this (B, N)-pair. Let c be the chamber corresponding to B. We note that the roots a and b contain c and are determined by the two panels on c.

Denote the root group of the root $r \in \Phi$ by X_r. Then

$$\begin{aligned} U(c) &= \langle X_r \mid r \in \Phi^+\rangle \quad \text{and} \\ U^{(1)}(c) &= \langle X_a, X_b\rangle. \end{aligned}$$

Proposition 3.19 *We have $U(c) = U^{(1)}(c)$ unless $G \cong Sp_4(2)$ in which case we have $[U(c) : U^{(1)}(c)] = 2$. As a consequence, the chamber system $Far_\Delta(c)$ is connected unless $G \cong Sp_4(2)$, in which case it has two connected components.*

Proof: We may assume that $a, b \in \Phi$ are such that $2a+b \in \Phi$. From Theorem 1.63 it follows that

$$[x_b(u), x_a(t)] = x_{a+b}(C_{11ab}(-t)u)x_{2a+b}(C_{21ab}(-t)^2u),$$

where C_{11ab} and C_{21ab} are 1 or -1. This implies that $[X_b, X_a]$ contains all elements of the form $x_{a+b}(1)x_{2a+b}(s)$ with $s \in \mathbb{F} \setminus \{0\}$. Multiplying these elements by $(x_{a+b}(1)x_{2a+b}(1))^{-1}$ we find that all elements of the form $x_{2a+b}(s-1)$ with $s \in \mathbb{F} \setminus \{0\}$ are in $[X_b, X_a]$. These elements generate X_{2a+b} unless $\mathbb{F} = \mathbb{F}_2$. So, unless $\mathbb{F} = \mathbb{F}_2$, we have $\langle [X_b, X_a]\rangle = \langle X_{a+b}, X_{2a+b}\rangle$, which implies $U^{(1)}(c) = U(c)$. If $\mathbb{F} = \mathbb{F}_2$, we immediately see from the commutator formula that $(x_b(1)x_a(1))^2 = [x_b(1), x_a(1)] = x_{a+b}x_{2a+b}$ has order 2 which means that $U^{(1)}(c) = \langle x_a(1), x_b(1)\rangle$ has index 2 in the group $U(c)$ of order 2^4. This proves part one of the Proposition.

The second part is a consequence of part one and Corollary 3.18. □

Now let M be a Dynkin diagram of type G_2. Let Φ be a root system of type M and let $\Pi = \{a, b\}$ be a fundamental system of roots. Let G, B, N, Δ, c, $U(c)$ and $U^{(1)}(c)$ be defined in a similar fashion as above. Finally assume that $\mathbb{F} = \mathbb{F}_2$.

Proposition 3.20 *We have $[U(c) : U^{(1)}(c)] = 4$ and consequently $Far_\Delta(c)$ has four connected components.*

Proof: We may assume that $a, b \in \Phi$ are such that $3a + 2b \in \Phi$. Let us write x_r for $x_r(1)$ ($r \in \Phi$). From Theorem 1.63 it follows that

$$[x_b, x_a] = x_{a+b}x_{2a+b}x_{3a+b}x_{3a+2b}.$$

Most of the commutators $[x_s, x_r]$ ($r, s \in \Phi$) are seen to be trivial for the reason that $r + s$ is not a root.

The remaining non-trivial commutators are the following ones: $[x_{a+b}, x_a] = x_{3a+b}x_{3a+2b}$, $[x_{2a+b}, x_a] = x_{3a+b}$, $[x_{3a+b}, x_b] = x_{3a+2b}$ and $[x_{2a+b}, x_{a+b}] = x_{3a+2b}$. Now one computes that

$$\begin{aligned} (x_b x_a)^4 &= [x_b, x_a]^2 \\ &= (x_{a+b}x_{2a+b}x_{3a+b}x_{3a+2b})^2 \\ &= x_{a+b}x_{2a+b}x_{a+b}x_{2a+b}(x_{3a+b})^2(x_{3a+2b})^2 \\ &= x_{3a+2b}. \end{aligned}$$

Hence $(x_b x_a)$ has order 8, which means that $U^{(1)}(c)$ is a dihedral subgroup of order 2^4 in the group $U(c)$ of order 2^6 so that $[U(c) : U^{(1)}(c)] = 4$. The second statement follows from Corollary 3.18. □

The fundamental group of the geometry far from a point in this generalized hexagon is studied in Cohen and Tits [16]. These authors also indicate a connection to the local structure at the fixed flag.

3.2.5 Local structure of $\mathrm{Far}_\Delta(R)$

We study the relation between the local and the global structure of $\mathrm{Far}_\Delta(R)$. The main results are the following: $\mathrm{Far}_\Delta(R)$ is connected if and only if the intersection of $\mathrm{Far}_\Delta(R)$ with any rank 2 residue is connected or empty (Theorem 3.29). This enables us to determine exactly for which buildings Δ and residues R the chamber system $\mathrm{Far}_\Delta(R)$ is connected. Intermediate results show that if X is a rank 2 residue and x, y are chambers in $\mathrm{Far}_\Delta(R) \cap X$ that are connected by a gallery in $\mathrm{Far}_\Delta(R)$, then in fact there is a gallery in $\mathrm{Far}_\Delta(R) \cap X$ connecting x and y (Corollary 3.25). Moreover if $\mathrm{Far}_\Gamma(R)$ is disconnected, then the 'gaps' are not too large, they occur inside $\mathrm{Far}_\Delta(R) \cap X$ for certain rank 2 residues X (Corollary 3.28).

Let as before $\hat{c}$ be the chamber on Σ opposite to c. Let S denote a rank 2 residue on c and let $\widehat{S}$ denote the residue on $\hat{c}$ opposite to S.

Lemma 3.21 *The residues S and $\widehat{S}$ are isomorphic via the projection map and the action of $L(S)$ respects this isomorphism. The isomorphism maps $Far_S(R \cap S)$ to $Far_\Delta(R) \cap \widehat{S}$.*

Proof: The first statement is Theorem 2.40 of Tits [47]. By applying '$\mathrm{proj}_{S'}$' to both sides of the equality in Lemma 3.5, we see that the map restricts to an isomorphism between $\mathrm{Far}_{S \cap R}(S)$ and $\mathrm{Far}_\Delta(R) \cap \widehat{S}$.

The last statement is obvious: $L(S)$ is the stabilizer of S and $\widehat{S}$. Its action preserves distances, hence it respects the projection map. □

Thus we can study the structure of $\mathrm{Far}_\Delta(R)$ in the neighbourhood of $\hat{c}$ by studying the geometries $\mathrm{Far}_{R\cap S}(S)$ for all residues S (e.g. of rank 2) on c.

The suitable stabilizer of (a connected component of) $\mathrm{Far}_S(R\cap S)$ will be defined analogously to the stabilizer of (a connected component of) $\mathrm{Far}_\Delta(R)$. To this end we regard S as a building on its own embedded into Δ.

The chamber system $\Sigma(S) = \Sigma\cap S$ is an apartment for S that has root system $\Phi^0(S)$. Let $G(S)$ be the automorphism group of S generated by the root groups U_α ($\alpha \in \Phi^0(S)$). Then $(B(S) = \mathrm{Stab}_{G(S)}(c), \mathrm{Stab}_{G(S)}(\Sigma(S)))$ is a Tits system for $G(S)$. Put $H(S) = B(S) \cap N(S)$.

For two residues R, S that have non-empty intersection and $i = 1, 2$ put $\Phi^{(i)}(R; S) = \bigcup_{T\subseteq S} \Phi^+(R; T)$, where T ranges over all residues of rank i of S that intersect, but are not contained in, R. This can be viewed as an extension of the definition of $\Phi^{(i)}(R)$ by setting $\Phi^{(i)}(R) = \Phi^{(i)}(R; \Delta)$. Define (with R as above)

$$\begin{aligned}
L(R;S) &= \langle H(S), U_\alpha | \alpha \in \Phi^0(R;S)\rangle \\
U(R;S) &= \langle U_\alpha | \alpha \in \Phi^+(R;S)\rangle \\
P(R;S) &= \langle U(R;S), L(R;S)\rangle \\
U^{(1)}(R;S) &= \langle U_\alpha | \alpha \in \Phi^{(1)}(R;S)\rangle \\
P^{(1)}(R;S) &= \langle U^{(1)}(R;S), L(R;S)\rangle
\end{aligned}$$

One retrieves the groups $L(R)$, $U(R)$,... and so on by substituting Δ for S.

Corollary 3.22 *The subgroup $P(R;S)$ of $G(S)$ stabilizes $\mathrm{Far}_{R\cap S}(S)$, and it acts transitively on the chambers of $\mathrm{Far}_{R\cap S}(S)$. The subgroup $P^{(1)}(R;S)$ is the full stabilizer in $P(R;S)$ of a connected component of $\mathrm{Far}_{R\cap S}(S)$ and it acts transitively on the chambers of this component. In particular the number of connected components of $\mathrm{Far}_{R\cap S}(S)$ equals the index $[P(R;S) : P^{(1)}(R;S)]$.*

Proof: These statements follow from Theorems 3.13 and 3.17 applied to the Moufang building S and its automorphism group $G(S)$. □

The relation between the groups $L(R)$, $U(R)$, $P(R)$, $U^{(1)}(R)$, $P^{(1)}(R)$ and their equivalents in $G(S)$ are just as you expected.

Theorem 3.23 *For any two residues R and S on c, we have*

(i) $L(R;S) = L(R) \cap G(S)$,

(ii) $U(R;S) = U(R) \cap G(S)$,

(iii) $P(R;S) = P(R) \cap G(S)$,

(iv) $U^{(1)}(R;S) = U^{(1)}(R) \cap G(S)$,

and if $P^{(1)} = U^{(1)} \cdot L(R)$, *also*

(v) $P^{(1)}(R;S) = P^{(1)}(R) \cap G(S)$.

Before we prove this, we need a lemma in the vein of Lemma 3.15.

Lemma 3.24 *Let R, S be residues on Σ that have non-empty intersection. Then*

$$\Phi^{(1)}(R) \cap \Phi^0(S) = \Phi^{(1)}(R; S).$$

Proof: It is clear that the inclusion '$\supseteq$' holds. Now let T be a residue of rank 1 that meets R in one chamber x, let $\alpha \in \Phi^+(R;T)$ and let r be the reflection that interchanges α and $-\alpha$. Then T is a residue of rank 1 that meets $R' = rR$ in one chamber rx, and $\alpha \in \Phi^-(R';T)$. Suppose that $\alpha \in \Phi^0(S)$ as well. Then R' and S also have non-empty intersection.

Now α is a root of both $\Phi^+(R \cap S; S)$ and $\Phi^-(R' \cap S; S)$. Any such root is an element β of both $\Phi^+(R)$ and $\Phi^-(R')$. Hence every gallery from R to R' contains at least one pair (y, y') of consecutive chambers such that $y \in \beta$ and $y' \in -\beta$. Since x, rx is a gallery from R to R', the only such root is α.

Let $y_0, \dots y_k$ be a minimal gallery in S from R to R' and let $\beta_i \in \Phi^0(S)$ be the roots that satisfy $y_{i-1} \in \beta_i$, $y_i \notin \beta_i$ $(i = 1, \dots, k)$. We know that precisely one of these, β_j say, is α; the other roots are either in $\Phi^0(R;S)$ or $\Phi^0(R';S)$. Hence the rank 1 residue Y on y_{j-1} and y_j meets R in the single chamber y_{j-1}, and cleary $\alpha \in \Phi^+(R;Y) \subset \Phi^{(1)}(R;S)$. □

Proof: (of Theorem 3.23) We first consider the inclusions '$\subseteq$'.

(i): The group $L(R;S)$ is a subgroup of $G(S)$ because it is generated by certain root groups U_α for which $\alpha \in \Phi^0(S)$. As $G(S) \leq L(S)$, we know from Theorem 3.16 that $G(S)$ stabilizes both S and S'. The same proposition, applied to S and $R \cap S$, shows that $L(R;S)$ stabilizes $R \cap S$ and $\mathrm{op}_{\Sigma(S)}(R \cap S)$. Thus $L(R;S)$ stabilizes $R \cap S \subset R$ and $\mathrm{proj}_{S'}(\mathrm{op}_{\Sigma(S)}(r \cap S)) = \mathrm{op}_\Sigma(R \cap S) \subset \mathrm{op}_\Sigma(R)$ and it follows that $L(R;S) \leq L(R)$.

(ii), (iv): These follow from part (ii) of Lemma 3.15 with $\epsilon = +$ by taking for T the residue S itself, and a residue of rank 1 in S that meets but is not included in R, respectively.

(iii), (v) These follow from the cases above.

We now address the inclusions '$\supseteq$'. We will omit to explicitly invoke Theorem 3.16 each time it is used.

(i): The subgroup $L(R) \cap G(S)$ of $G(S)$ stabilizes S, $\mathrm{op}_\Sigma(S)$, R and $\mathrm{op}_\Sigma(R)$. Hence it stabilizes $S \cap R$ and $\mathrm{proj}_S(\mathrm{op}_\Sigma(S \cap R))$ which by Lemma 3.5 equals $\mathrm{op}_{\Sigma(S)}(S \cap R)$. This implies that $L(S) \cap G(S) \leq L(R;S)$.

(ii): Let $g \in U(R) \cap G(S)$. Then, g fixes every chamber of $S \cap R$. Choose $u \in U(R;S) \subset U(R)$ such that gu^{-1} stabilizes $\mathrm{op}_{\Sigma(S)}(S \cap R)$ as well. Then $gu^{-1} \in L(R;S) \cap U(R) \subset L(R) \cap U(R) = \{1\}$, and so $g = u$.

(iii): The subgroup $P(R) \cap G(S)$ of $G(S)$ stabilizes $S \cap R$. This means that $P(R) \cap G(S) \leq P(R;S)$.

(iv): Let $g \in G(S) \cap U^{(1)}(R)$ be a product $g = u_1 \cdots u_k$ with $u_i \in U_{\alpha_i}$ $(\alpha_i \in \Phi^{(1)}(R))$. Since for all i, we have $c \in S \cap R \subset \alpha_i$, we have $\alpha_i \in \Phi^0(S) \cup \Phi^+(S)$. Delete all u_i in the product $u_1 \cdots u_k$ for which $\alpha_i \in \Phi^+(S)$ to obtain $g' \in U^{(1)}(R;S)$ (see Lemma 3.24). Then, $g'g^{-1} \in H(S) \cap U^{(1)}(R) \subset H \cap U^{(1)}(R) = \{1\}$.

(v): Let $g \in P^{(1)}(R) \cap G(S)$ and let $l \in L_S(R) \subset P^{(1)}(R)$ be such that gl^{-1} fixes every chamber of $S \cap R$. Furthermore, choose $u \in U_S(R) \subset U(R)$ such that $h = gl^{-1}u^{-1}$ stabilizes $\mathrm{op}_{\Sigma \cap S}(S \cap R)$ as well. Then, $h \in H_S$ and $u = h^{-1}gl^{-1} \in P^{(1)}(R) \cap U_S(R)$. Now use the fact that $P^{(1)}(R)$ is the semidirect product $U^{(1)}(R) \cdot L(R)$ and (iv) to see that $u \in U_S(R, 1)$ It follows that $g = hul \in P_S(R, 1)$. □

Corollary 3.25 *(to Theorem 3.23) Suppose that R is a residue that meets no $I_2(m)$ residue with $m \geq 5$ in more than one chamber.*

Let x, y be chambers in $Far_R(\Delta) \cap X$ for some rank 2 residue X. If there is a gallery in $Far_\Delta(R)$ joining x and y, then there is a gallery in $Far_\Delta(R) \cap X$ joining x and y.

Proof: Since Δ is thick and its diagram is spherical, a connected component of M containing an $I_2(m)$ diagram with $m \geq 5$ is in fact equal to this $I_2(m)$ diagram. Hence M can be decomposed as $M = M_1 \sqcup M_2$, such that all $I_2(m)$ subdiagrams with $m = 3, 4$ are in M_1 and all subdiagrams $I_2(m)$ with $m \geq 5$ are in M_2. Accordingly, there are buildings Δ_i $(i = 1, 2)$ such that $\Delta = \Delta_1 \times \Delta_2$, and residues R_i of Δ_i such that $R = R_1 \times R_2$. Then $\mathrm{Far}_\Delta(R) = \mathrm{Far}_{\Delta_1}(R_1) \times \mathrm{Far}_{\Delta_2}(R_2)$. Further there are residues X_i of Δ_i $(i = 1, 2)$ such that $X = X_1 \times X_2$. Hence $\mathrm{Far}_\Delta(R) = \mathrm{Far}_{\Delta_1}(R_1) \cap X_1 \times \mathrm{Far}_{\Delta_2}(R_2) \cap X_2$.

First assume that both X_1 and X_2 have rank 1, i.e. X has diagram $A_1 \sqcup A_1$. Since $x = x_1 \times x_2$ with x_i $(i = 1, 2)$ a chamber of X_i, $\mathrm{Far}_{\Delta_i}(R_i) \cap X_i$ is non-empty and hence $\mathrm{Far}_\Delta(R) \cap X$ is connected and we are done.

Next, assume that X_j has rank 2 for some $j \in \{1, 2\}$. Then $\mathrm{Far}_\Delta(R) \cap X$ is connected if and only if $\mathrm{Far}_{\Delta_j}(R_j) \cap X_j$ is connected and so we will assume that $\Delta = \Delta_j$, $R = R_j$, ... and so on. If $j = 1$ and R meets a B_2 residue in more than one chamber, then by Lemma 3.21 and the rank 2 results from Subsection 3.2.2 the set $\mathrm{Far}_\Delta(R) \cap X$ is connected for all rank 2 residues X and we are done again.

In the remaining cases we know that R meets no $I_2(m)$ residue with $m \geq 4$ in more than one chamber. This enables us to use Theorems 3.16 and 3.23.

By applying some element of $P(R)$ we may assume that $x = \hat{c}$. Put $\widehat{S} = X$ and let S be the residue on c opposite to $\widehat{S}$. Further let $\widehat{R}$ be the residue on $\hat{c}$ opposite to R. In order to show that the chambers $x, y \in \mathrm{Far}_\Delta(R) \cap \widehat{S}$ are joined by a gallery in $\mathrm{Far}_\Delta(R) \cap \widehat{S}$ we find an element in $P^{(1)}(R; S)$ that maps x to y.

By assumption the chamber y lies in the connected component of $\mathrm{Far}_\Delta(R)$ on $\hat{c}$. By Theorem 3.17 there is an element $g \in P^{(1)}(R)$ such that $g\hat{c} = y$. The residue $\widehat{S}$ is opposite to both S and gS because it is stabilized by g. The residue R is opposite to $R' = g\widehat{R}$ because it is stabilized by g. Hence $\widehat{S} \cap R'$ is opposite to both $S \cap R$ and $g(S \cap R)$ (cf. Lemma 3.4). By Theorem 3.13 there exists an element $f \in U(\widehat{S} \cap R')$ that sends $g(S \cap R)$ back to $S \cap R$. This element also fixes every chamber of $\widehat{S} \cap \widehat{R}$. Then fg stabilizes $S \cap R$, and hence both S and R, and $\widehat{S}$ and sends $\hat{c}$ to y. By Theorem 3.13 we have $fg \in L(S)$ which implies that $fg = g'h$ with $g' \in G(S)$ and $h \in H$. Now g' stabilizes R and sends $\hat{c}$ to y which means that $g' \in P(R)$ stabilizes the connected component that contains $\hat{c}$. Thus $g' \in P^{(1)}(R) \cap G(S)$. Since we can apply Theorem 3.23 we find that $g' \in P^{(1)}(R; S)$. Then by Corollary 3.22 x and y are in a connected component of $\mathrm{Far}_\Delta(R) \cap X$. □

Theorem 3.23 says that if you have $L(R)$, $U(R)$, $P(R)$, $U^{(1)}(R)$ or $P^{(1)}(R)$, then you have their equivalents inside $G(S)$ for every rank 2 residue S on c. We shall now prove that the opposite almost holds. The 'almost' will turn out to be irrelevant for our purposes.

For any chamber $x \in R$, let

$$\begin{aligned} P_x(R) &= \langle P(R;S) \mid S \text{ a rank 2 residue on } x\rangle \qquad \text{and} \\ P_x^{(1)}(R) &= \langle P^{(1)}(R;S) \mid S \text{ a rank 2 residue on } x\rangle. \end{aligned}$$

Proposition 3.26 *We have $P(R) = P_x(R)H$ and $P^{(1)}(R) = P_x^{(1)}(R)H$.*

Recall that H fixes every chamber on Σ, in particular it fixes $\hat{c}$. Hence the orbits of a chamber in $\mathrm{Far}_\Delta(R)$ under the actions of $P(R)$ (resp. $P^{(1)}(R)$) and $P_x(R)$ (resp. $P_x^{(1)}(R)$) are the same. For the proof of this proposition we need the lemma below.

Recall that, for a residue R, the set $\Phi^{(2)}(R)$ is the union, taken over all residues T of rank 2 that intersect but are not contained in R, of the set of roots of T that contain $R \cap \Sigma$. Let us put $U^{(2)}(R) = \langle U_\alpha \mid \alpha \in \Phi^{(2)}(R)\rangle$.

Lemma 3.27 *Suppose that Δ is a thick Moufang building of spherical type. Let x be a chamber of the apartment Σ. Then, with the above notation, $U^{(2)}(x) = U(x)$.*

In the proof we will identify Σ with the Coxeter building W such that $x \in \Sigma$ is identified with $1 \in W$. For $w \in W$ and $J \subseteq I$ let $\Phi^+(w;J)$ be the set of roots of the J residue wW_J that contain w. Let us put $U(w;J) = \langle U_\alpha \mid \alpha \in \Phi^+(w;J)\rangle$. Notice that if w is shortest in wW_J, then $\Phi^+(w;J) \subseteq \Phi^+(x)$ and, accordingly, $U(w;J) \leq U(x)$.

Proof: If $\alpha \in \Phi^+(x)$, then $\alpha \in \Phi^+(w;J)$ for some $w \in W$ and J with $|J| = 2$ such that w is shortest in wW_J. Thus it suffices to show for each such w and J that $U(w;J) \leq U^{(2)}(x)$. We do this using induction on $l = l(w)$. For $l = 0$ this follows immediately from the definitions. Let $l \geq 1$ and assume $w'f$ ($f \in I$) is a reduced expression for w. Let w^{fi} and w^{fj} be the shortest elements of $w'W_{\{f,i\}}$ and $w'W_{\{f,j\}}$ respectively. Since $l(w^{fi}), l(w^{fj}) \leq l(w') \leq l-1$, we conclude that $U(w'f;\{i\}) = U(w;\{i\}), U(w'f;\{j\}) = U(w;\{j\}) \subset U^{(2)}(x)$. If $m_{ij} \leq 3$, then $\langle U(w;\{i\}), U(w;\{j\})\rangle = U(w;\{i,j\})$. Otherwise we have $m_{ij} = 4$ and we may assume that $m_{if} = 2$. Using the Chevalley commutator relations (see Example 3.2.4) one can show that $\langle U(w;\{i\}), U(w;\{j\}), U(wi;\{j\})\rangle = U(w;\{i,j\})$. Hence from the inclusion $U(wi;\{j\}) \leq U^{(2)}(x)$ it follows that $U(w;\{i,j\}) \leq U^{(2)}(x)$ for every $i,j \in I$ that occurs. It remains to show that $U(wi;\{j\}) = U(w'if;\{j\}) \subset U^{(2)}(x)$. Let v^{jf} be shortest in $w'iW_{\{j,f\}}$. Then $l(v^{jf}) \leq l$ and, as $m_{jf} = 2, 3$ and $l(w'i) = l(w'f) = l$, we may conclude that $U(v^{jf};\{jf\}) \subset U^{(2)}(x)$ and, in particular, $U(w'fi;\{j\}) \subset U^{(2)}(x)$. □

Remark: Analogous versions can be formulated for other classes of buildings; e.g. for (thick) affine Moufang buildings without residues of type $\widetilde{C}_2$ over $\mathbb{F}_2$.

Proof: (of Proposition 3.26) By Theorem 3.23 we have '$\supseteq$' in (i) and (*ii*).

We first show that $G(R) = \langle U_\alpha \mid \alpha \in \Phi^0(R)\rangle$ is contained in both $P_x(R)$ and $P_x^{(1)}(R)$. Apply Lemma 3.27 to the Moufang building R and the chamber $c \in \Sigma(R)$ to see that $U(c;R)$ is contained in both $P_x(R)$ and $P_x^{(1)}(R)$. Since, for all rank 2 residues S on c, the group $L(R;S)$ contains a subgroup of N that induces the action of the Coxeter group of type $\mathrm{type}(R \cap S)$ on Σ, the groups $P_x(R)$ and $P_x^{(1)}(R)$ both contain a subgroup of N that induces the action of the Coxeter group of type $\mathrm{type}(R)$ on Σ. As, for any element $n \in N$, we have $U_\alpha^n = U_{n\alpha}$, we conclude that $U_\alpha \leq P_x(R), P_x^{(1)}(R)$ for all $\alpha \in \Phi^0(R)$.

By Proposition 3.27 applied to Δ and $x = c$, we have $U \subset P_x(R)$ which implies $U(R) \leq P_x(R)$. For every root $\alpha \in \Phi^{(1)}(R)$ there is an element $n \in G(R)$ and a rank 2 residue S on c such that $\alpha = n\alpha'$ for some root $\alpha' \in \Phi^{(1)}(c; S)$. Then $U_\alpha = U_{\alpha'}^n \leq P_x^{(1)}(R)$ which implies $U^{(1)}(R) \leq P^{(1)}(R)$.

We have shown that $\langle U(R), G(R)\rangle \leq P_x(R) \leq P(R) = \langle U(R), L(R)\rangle$ and $\langle U^{(1)}(R), G(R)\rangle \leq P_x^{(1)}(R) \leq P^{(1)}(R) = \langle U^{(1)}(R), L(R)\rangle$. As $L(R) = \langle H, G(R)\rangle$ and H normalizes every root group, we have $P(R) = P_x(R)H$ and $P^{(1)}(R) = P_x^{(1)}(R)H$. □

The geometric interpretation of Proposition 3.26 is the following.

Corollary 3.28 *For every two chambers $x, y \in Far_\Delta(R)$ there is a sequence $X_0, X_1, \ldots, X_k$ of rank 2 residues such that*

(i) $x \in X_0$, $y \in X_k$ and

(ii) $X_{i-1} \cap X_i$ contains a chamber of $Far_\Delta(R)$.

Proof: We may assume that $c \in R$ and $x = \hat{c}$. Let $g \in P(R)$ be such that $gx = y$. Then, by Proposition 3.26 applied to the residue R and chamber $c \in \Sigma$ $g = g_0 g_1 \cdots g_k h$ where $h \in H$ and $g_i \in P(R; S_i)$ for certain rank 2 residues S_i on c. We may assume that $h = 1$ since it fixes x. Put $Y_i = \mathrm{op}_\Sigma(S_i)$. Then $X_0 = Y_0$ and $X_i := g_0 g_1 \cdots g_{i-1} Y_i$ for $1 \leq i \leq k$ satisfy (i) and (*ii*). We prove this by induction on k.

If $k = 0$, then apparently $x, y \in Y_0 = X_0$ and we are done. If $k > 0$, then put $x' = x$ and $y' = g_0^{-1} y = g_1 g_2 \cdots g_k x$. By induction the sequence $X_0', X_1', \ldots, X_{k-1}'$ with $X_i' = g_0' g_1' \cdots g_{i-1}' Y_i$, where $g_i' = g_{i+1}$ $(0 \leq i \leq k-1)$ satisfies (i) and (*ii*). Now let $X_0 = Y_0$ and $X_i = g_0 X_{i-1}'$ for $1 \leq i \leq k$. Then $x \in X_0$, $y \in X_k$ and $g_0 g_1 \cdots g_{i-1} x \in X_{i-1} \cap X_i$ $(1 \leq i \leq k)$ are chambers of $\mathrm{Far}_\Delta(R)$. □

Thus if $\mathrm{Far}_\Delta(R)$ is disconnected, we can find rank 2 residues that contain chambers of two or more connected components. We are now ready to prove one of the main results of this chapter. For a chamber x, the set $E_2(x)$ is the collection of chambers lying in some rank 2 residue on x.

Theorem 3.29 *Suppose that Δ is a thick Moufang building of spherical type and that R is a residue that meets no $I_2(m)$ residue with $m \geq 5$ in more than one chamber. Let c be a chamber of R and let $\hat{c}$ be a chamber opposite to c. Then, the following four statements are equivalent:*

(i) $Far_\Delta(R)$ is connected,

(ii) $Far_{S\cap R}(S)$ is connected for all rank 2 residues S on c,

(iii) $Far_\Delta(R) \cap \widehat{S}$ is connected for all rank 2 residues $\widehat{S}$ on $\hat{c}$,

(iv) $Far_\Delta(R) \cap E_2(\hat{c})$ is connected.

Proof: (of Theorem 3.29)

(i) $\Rightarrow$ (iii): This follows immediately from Corollary 3.25.

(ii) $\Rightarrow$ (i): This follows from Corollary 3.26: Condition (ii) means that $P^{(1)}(R;S) = P(R;S)$ for all rank 2 residues S on c. Hence $P_c^{(1)}(R) = P_c(R)$ and as the orbits of c' under the action of these groups and the groups $P^{(1)}(R)$ and $P(R)$ are the same, we see that $\mathrm{Far}_\Delta(R)$ is connected. (Note that, in case of connectedness, we can a posteriori conclude that $P^{(1)} = P(R)$ as well.)

(ii) $\Leftrightarrow$ (iii): This is Lemma 3.21.

(iii)$\Rightarrow$(iv): Clear.

(iv)$\Rightarrow$(iii): This follows immediately from Corollary 3.25. $\square$

It is now possible to determine for which spherical buildings Δ and residues R the set $\mathrm{Far}_\Delta(R)$ is connected. By Remark 3.2.4 following Theorem 3.17, it suffices to consider only those buildings that have has a connected diagram M. For spherical buildings with a connected diagram M of rank 2, the known results are given in Theorems 3.9, 3.11 and 3.12. For spherical buildings with a connected diagram of rank ≥ 3, the complete results are given in the next theorem.

Theorem 3.30 *Let Δ be a building of spherical type with a connected diagram M, naturally labeled by an index set I of size at least 3, and let R be a J-residue on a chamber c. Then, $Far_\Delta(R)$ is connected except when*

(i) $M = C_n$, Δ is defined over the field $\mathbb{F}_2$ and $J \cap \{n-1, n\} = \emptyset$

(ii) $M = F_4$, Δ is defined over the field $\mathbb{F}_2$ and $J \cap \{2, 3\} = \emptyset$.

Proof: (of Theorem 3.30) Since M is connected and has rank ≥ 3, it has no $I_2(m)$ subdiagrams with $m \geq 5$. Hence, by Theorem 3.29, $\mathrm{Far}_\Delta(R)$ is disconnected precisely when a rank 2 residue S on c exists such that $\mathrm{Far}_S(S \cap R)$ is disconnected. By Theorem 3.9 and the discussion preceding it, this happens only when S is a residue of type $\mathrm{Sp}(4,2)$ and $S \cap R$ is a chamber. The theorem lists all buildings Δ of spherical type with connected diagram of rank 3 and residues R for which such an S exists. $\square$

3.2.6 The number of connected components in Far$_\Delta(R)$

This subsection is devoted to the computation of the number of connected components of the chamber system $\mathrm{Far}_\Delta(R)$ in the cases we haven't covered yet. We will do this by computing the index of the stabilizer of a connected component of $\mathrm{Far}_\Delta(R)$ in the stabilizer of $\mathrm{Far}_\Delta(R)$ in the automorphism group G of Δ (cf. Corollary 3.14, Theorem 3.17 and Corollary 3.18).

Suppose that M can be decomposed as $M = M_1 \sqcup M_2$. Let Δ_i ($i = 1, 2$) be buildings with diagram M_i such that $\Delta = \Delta_1 \times \Delta_2$ and let R_i be residues of Δ_i such that $R = R_1 \times R_2$.

Lemma 3.31 *With the notation just defined, we have*

$$Far_{\Delta}(R) = Far_{\Delta_1}(R_1) \times Far_{\Delta_2}(R_2).$$

Hence, if $Far_{\Delta_i}(R_i)$ ($i = 1, 2$) has n_i connected components, then $Far_{\Delta}(R)$ has $n = n_1 \cdot n_2$ connected components.

The proof is straightforward. Therefore from now on we will assume that M is connected.

Theorem 3.32 (Brouwer [7], Abramenko [1]) *Let Δ be a thick Moufang generalized n-gon obtained from the group G. Suppose that it has a Dynkin diagram M defined over the set $I = \{1, 2\}$ such that 1 corresponds to a short root. Then if R has type $J \subseteq I$ the chamber system $Far_{\Delta}(R)$ has N connected components, where G, J and N are given in the following table.*

G	J	N
$Sp_4(2)$	$\{1, 2\}$	2
$G_2(2)$	$\{1\}$	2
	$\{1, 2\}$	4
$G_2(3)$	$\{1, 2\}$	3
${}^2F_4(2)$	$\{1\}$	2
	$\{1, 2\}$	2

From now on we will assume that M is spherical, connected and has rank at least 3. We know that then $\mathrm{Far}_{\Delta}(R)$ is disconnected only if Δ is $\mathrm{Sp}_{2n}(2)$ or $F_4(2)$ and R meets no $\mathrm{Sp}_4(2)$-residue in more than one chamber.

Theorem 3.33 *Let Δ be isomorphic to $Sp_{2n}(2)$ or $F_4(2)$ and let R be a residue such that $Far_{\Delta}(R)$ is disconnected. Then $Far_{\Delta}(R)$ has 2^{n-1} components if $\Delta = Sp_{2n}(2)$ and 16 components if $\Delta = F_4(2)$.*

One feature of this situation is expressed in the following lemma.

Lemma 3.34 *Let Δ be isomorphic to $Sp_{2n}(2)$ or $F_4(2)$ and let R be a residue such that $Far_{\Delta}(R)$ is disconnected. Then, the number of connected components of $Far_{\Delta}(R)$ only depends on Δ (i.e. not on R).*

Proof: We must show that $[P(R) : P^{(1)}(R)] = [P(c) : P^{(1)}(c)]$. Since we have $P^{(1)}(c) \leq P(c), P^{(1)}(R) \leq P(R)$, it sufices to show that $[P(R) : P(c)] = [P^{(1)}(R) : P^{(1)}(c)]$ instead.

In order to do this we use the decomposition $P(R) = U(R) \cdot L(R)$ and $P^{(1)}(R) = U^{(1)}(R) \cdot L(R)$ (R satisfies the condition of Theorem 3.16). Note that $P(c) = U(c)$, $P^{(1)}(c) = U^{(1)}(c)$ and $L(R) = G(R)$ since $H = \{1\}$. As $U(R) \subset U(c)$, we find $[P(R) : U(c)] = [G(R) : U(c; R)]$ (cf. Theorem 3.23). Similarly, we have $[P^{(1)}(R) : U^{(1)}(c)]$ because $U^{(1)}(R) \subset U^{(1)}(c)$. This inclusion is a little less obvious. Let $\alpha \in \Phi^+(R; T)$ for some rank 1 residue T on Σ that meets R in precisely one chamber. Clearly α is a root of the smallest residue X that contains R and T. Since R meets no $I_2(m)$ residue with $m \geq 4$ in more than one chamber and $\mathrm{rank}(X) = \mathrm{rank}(R) + 1$, the residue X is a building without $I_2(m)$ residues with $m \geq 4$. Therefore $\mathrm{Far}_c(X)$ is connected which means that $U(c; X) = U^{(1)}(c; X)$. Thus $U_{\alpha} \subset U^{(1)}(c; X) \subset U^{(1)}(c)$ and so we see that $U^{(1)}(R) \leq U^{(1)}(c)$.

Finally we have $U(c; R) = U^{(1)}(c; R)$, because R is a building without $I_2(m)$ residues with $m \geq 4$, and we are done. □

Remark: Note that the conclusion of this lemma does not apply to all spherical buildings. The set of chambers far from a fixed chamber in the generalized hexagon $G_2(2)$ has four components, far from a point (short root geometry) we find two components, and far from a line there is only one connected component.

Proof: (of Theorem 3.33) By Lemma 3.34, we may assume that R is the chamber c. The number of connected components in $\mathrm{Far}_\Delta(c)$ equals the index $[P(c) : P^{(1)}(c)]$ (by Corollary 3.18). Since G is defined over $\mathbb{F}_2$, we have $H = \{1\}$ so that $P(c) = U(c)$ and $P^{(1)}(c) = U^{(1)}(c)$.

We compute the index $[U(c) : U^{(1)}(c)]$ by hand for $M = C_n$ and by computer for $M = F_4$.

Generators for $U^{(1)}(c)$. Let M be the diagram of Δ and let $(W, \{r_i\}_{i\in I})$ be the Coxeter system with diagram M. We may identify W with N since $W \cong N/(B\cap N)$ and $B\cap N = \{1\}$, and r_i becomes the reflection determined by the i-panel on c. Let α_i $(i \in I)$ be the positive roots corresponding to r_i $(i \in I)$. By u_i and u_i^- we denote the non-zero elements of the root groups U_{α_i} and $U_{-\alpha_i}$ respectively. Then $U^{(1)}(c) = \langle u_i \mid i \in I\rangle$.

Computations in $\mathbf{Sp}_{2n}(2)$. We will need to do some computations involving the elements u_i and u_i^-. One may do these computations using the faithfulness of the $\mathrm{Sp}_{2n}(2)$-action on an $\mathbb{F}_2$-vector space endowed with a symplectic form $(\cdot,\cdot)$ given by

$$(e_i, e_j) = \begin{cases} 1 & \text{if } i \equiv j \bmod n \\ 0 & \text{otherwise} \end{cases}$$

with respect to a basis $\mathcal{E} = \{e_i | i = 1, \dots, 2n\}$. If our apartment Σ comprises all totally isotropic subspaces spanned by subsets of $\mathcal{E}$ and c is the chamber $\{\langle e_1\rangle, \langle e_1, e_2\rangle, \dots, \langle e_i \mid i = 1, 2, \dots, n\rangle\}$, then the action of u_i and u_i^- $(i \in I)$ is given by

$$u_i\colon \begin{cases} e_{i+1} & \mapsto e_i + e_{i+1} \\ e_{n+i} & \mapsto e_{n+i} + e_{n+i+1} \end{cases}, \quad u_i^-\colon \begin{cases} e_i & \mapsto e_i + e_{i+1} \\ e_{n+i+1} & \mapsto e_{n+i} + e_{n+i+1} \end{cases}$$

for $1 \le i \le n-1$ and $u_n\colon e_{2n} \mapsto e_n + e_{2n}$, $u_n^-\colon e_n \mapsto e_n + e_{2n}$.

If the computations can be done inside a rank two residue one may also use the Chevalley commutator relations directly (Theorem 1.63).

The case $M = C_n$

We put $M = C_n$. We first indicate a subbuilding of type D_n. Let $s_i = r_i$ for $i \in I \setminus \{n\}$ and $s_n = {r_{n-1}}^{r_n}$. Then $(W_\circ, \{s_i\}_{i\in I})$ is a Coxeter system with diagram $M_\circ = D_n$. Thus the collection of roots $\Phi_\circ$ determined by the reflections in $W_\circ$ form a root system with Dynkin diagram $M_\circ$. The group $G_\circ = \langle U_\beta \mid \beta \in \Phi_\circ\rangle$ is the Chevalley group $O^+_{2n}(2)$. Let $\Delta_\circ$ be the building associated with $G_\circ$.

Define β_i, $-\beta_i$ and v_i, ${v_i}^-$ $(i \in I)$ in the same way as α_i, $-\alpha_i$ and u_i, u_i^- respectively. Then $\Pi_\circ = \{\beta_i \mid i = 1, \dots, n\}$ is a fundamental set of roots for $\Phi_\circ$. Thus if d is the chamber of $\Delta_\circ$ corresponding to $\Pi_\circ$, then $U_\circ^{(1)}(d) = \langle U_{\beta_i} \mid i = 1, \dots, n\rangle$ is the stabilizer of a connected component of $\mathrm{Far}_{\Delta_\circ}(d)$. However, since $M_\circ$ has no I_4 subdiagrams, there is just one connected component and this group is in fact $U_\circ(d) = \langle U_\beta \mid \beta \in \Phi_\circ^+\rangle$.

An 'orthogonal' subgroup of $U^{(1)}(c)$. Let $D \subset U^{(1)}(c)$ be the group generated by $d_i = u_i$ for $i \in I \setminus \{n\}$ and $d_n = u_{n-1}^{u_n}$.

We will now show that $v_i^{u_n^-} = d_i$ for $i \in I$. First note that $r_i = u_i u_i^- u_i = u_i^- u_i u_i^-$. If $m_{ij} = 2$, then $[u_i, u_j] = [u_i, u_j^-] = 1$ and hence $d_i = v_i^{u_n^-}$ if $i < n-1$. Further, one can verify that $[u_{n-1}, u_n^-] = 1$ either by checking the action of these elements on the $\mathrm{Sp}_{2n}(2)$-module or by looking at the Chevalley commutator relations and observing that '$\alpha_{n-1} - \alpha_n$ is not a root'. It then follows that $v_{n-1}^{u_n^-} = u_{n-1} = d_{n-1}$. Finally

$$\begin{aligned} v_n^{u_n^-} &= u_{n-1}^{u_n^- u_n u_n^- u_n^-} \\ &= u_{n-1}^{u_n^- u_n} \\ &= u_{n-1}^{u_n} = d_n \end{aligned}$$

as required.

The sizes of $U(c)$ and D. Since $D \cong U_\circ(d)$, we can compute the sizes of $U(c)$ and D as 2^k and 2^l, where k and l are the diameters of the Coxeter groups W and $W_\circ$ respectively. This yields $|U(c)| = 2^{n^2}$ and $|D| = 2^{n(n-1)}$.

The indices $[U^{(1)}(c) : D]$ and $[U(c) : U^{(1)}(c)]$. We have $U^{(1)}(c) = \langle D, u_n \rangle$ and $D^{u_n} = D$. Hence $U^{(1)}(c) = D \cdot \langle u_n \rangle$ and $[U^{(1)}(c) : D] = 2$. Finally, $[U(c) : U^{(1)}(c)] = [U(c) : D]/[U^{(1)}(c) : D] = 2^{n-1}$ and we are done.

The case $M = F_4$

Here we have asked GAP [29] to compute the index $[U : U^{(1)}(c)]$. Starting out with the root system Φ of type F_4, we have chosen a fundamental system Π and we have determined the set of positive roots Φ^+ with respect to Π. Then we introduced an element u_r of order 2 for each root $\mathsf{r} \in \Phi^+$ and implemented the Chevalley commutator relations (see Theorem 1.63), under the condition that for any two positive roots r and s we have $i\mathsf{r} + j\mathsf{s}$ only if $i \leq 1$ or $j \leq 1$; this condition holds for root systems that do not have subsystems of type G_2. By imposing these relations on the elements introduced above, we obtain the group U. Then $U^{(1)}(c)$ is the subgroup of U generated by $\{u_\mathsf{r} \mid \mathsf{r} \in \Pi\}$. We find $[U : U^{(1)}(c)] = 16$.

The GAP-code is included below. We first list (in two columns to save space) the full root system of type F_4 (`AllRootsF4`), the first four roots form the fundamental system and the roots in the left-hand column are the positive roots (`PositiveRootsF4`) with respect to this fundamental system.

```
AllRootsF4:=
[
 [   1,  -1,   0,   0],   [  -1,   1,   0,   0],
 [   0,   1,  -1,   0],   [   0,  -1,   1,   0],
 [   0,   0,   1,   0],   [   0,   0,  -1,   0],
 [-1/2,-1/2,-1/2, 1/2],   [ 1/2, 1/2, 1/2,-1/2],
 [   1,   0,  -1,   0],   [  -1,   0,   1,   0],
 [   1,   0,   0,   0],   [  -1,   0,   0,   0],
 [   0,   1,   0,   0],   [   0,  -1,   0,   0],
 [   0,   0,   0,   1],   [   0,   0,   0,  -1],
 [   1,   1,   0,   0],   [  -1,  -1,   0,   0],
 [   0,   1,   1,   0],   [   0,  -1,  -1,   0],
 [   0,   0,   1,   1],   [   0,   0,  -1,  -1],
 [   1,   0,   1,   0],   [  -1,   0,  -1,   0],
 [   0,   1,   0,   1],   [   0,  -1,   0,  -1],
 [   1,   0,   0,   1],   [  -1,   0,   0,  -1],
 [  -1,   0,   0,   1],   [   1,   0,   0,  -1],
 [   0,  -1,   0,   1],   [   0,   1,   0,  -1],
 [   0,   0,  -1,   1],   [   0,   0,   1,  -1],
 [-1/2,-1/2, 1/2, 1/2],   [ 1/2, 1/2,-1/2,-1/2],
 [ 1/2,-1/2,-1/2, 1/2],   [-1/2, 1/2, 1/2,-1/2],
 [-1/2, 1/2,-1/2, 1/2],   [ 1/2,-1/2, 1/2,-1/2],
 [ 1/2, 1/2,-1/2, 1/2],   [-1/2,-1/2, 1/2,-1/2],
 [-1/2, 1/2, 1/2, 1/2],   [ 1/2,-1/2,-1/2,-1/2],
 [ 1/2,-1/2, 1/2, 1/2],   [-1/2, 1/2,-1/2,-1/2],
 [ 1/2, 1/2, 1/2, 1/2],   [-1/2,-1/2,-1/2,-1/2]
];
```

Similarly, we define `PositiveRootsF4`. We have a function generating the group U:

```
UofBinChevGroup:= function(PositiveRoots,AllRoots)
local Diffroot,i,j,NewRelation,positiveroots,pr,ps,qr,qs,r,
      rChain,Relations,s,sChain,SizePositiveRoots,Sumroot,U;

positiveroots:= Set(PositiveRoots);
SizePositiveRoots :=  Size(positiveroots);
U:= FreeGroup(SizePositiveRoots,"u");
Relations:= List([ ]);
for r in [1..SizePositiveRoots] do
  Append(Relations,[(U.(r))^2]);
od;
```

```
for r in [1..SizePositiveRoots] do
  for s in [(r+1)..SizePositiveRoots] do


    pr:= 0;
    Diffroot:= PositiveRoots[s] - PositiveRoots[r];
    while Diffroot in AllRoots
    do
      pr:= pr + 1;
      Diffroot:= Diffroot - PositiveRoots[r];
    od;


    qr:= 0;
    rChain:= List([ ]);
    Sumroot:= PositiveRoots[s] + PositiveRoots[r];
    while Sumroot in PositiveRoots
    do
      Append(rChain,[Position(PositiveRoots, Sumroot)]);
      qr:= qr + 1;
      Sumroot:= Sumroot + PositiveRoots[r];
    od;


    ps:= 0;
    Diffroot:= PositiveRoots[r] - PositiveRoots[s];
    while Diffroot in AllRoots
    do
      ps:= ps + 1;
      Diffroot:= Diffroot - PositiveRoots[s];
    od;


    qs:= 0;
    sChain:= List([ ]);
    Sumroot:= PositiveRoots[r] + PositiveRoots[s];
    while Sumroot in PositiveRoots
    do
      Append(sChain,[Position(PositiveRoots, Sumroot)]);
      qs:= qs + 1;
      Sumroot:= Sumroot + PositiveRoots[s];
    od;


    if  qr = 0  then
      NewRelation:= U.(r)^-1 * U.(s)^-1 * U.(r) * U.(s);
```

```
    elif  qr >= qs  then
      NewRelation:= U.(r)^-1 * U.(s)^-1 * U.(r) * U.(s);
      for i in [1..qr]
      do
        if  EuclideanRemainder(Binomial(pr+i,i),2) <> 0  then
          NewRelation:= NewRelation * U.(rChain[i]);
        fi;
      od;

    elif  qr < qs  then
      NewRelation:= U.(s)^-1 * U.(r)^-1 * U.(s) * U.(r);
      for j in [1..qs]
      do
        if  EuclideanRemainder(Binomial(ps+j,j),2) <> 0  then
          NewRelation:= NewRelation * U.(sChain[j]);
        fi;
      od;
    fi;

    Append(Relations,[NewRelation]);
  od;
od;

U:= U / Relations;
return(U);
end;;
```

Now type

```
gap>U:=UofBinChevGroup(PositiveRootsF4,AllRootsF4);;
gap>U1:=Subgroup(U,[U.1,U.2,U.3,U.4]);;
gap>Index(U,U1);
```

and we get

```
gap>16
```

. □

3.2.7 Proving (dis-) connectedness without groups

In this subsection we prove the following theorem in a purely combinatorial (geometric) way. Furthermore we will give a geometric description of the connected components of the geometry far from a chamber in the building $\mathrm{Sp}_{2n}(2)$.

Theorem 3.35 *Let Δ be a spherical building wih a connected diagram of rank at least 3 and let R be a residue. Then $Far_\Delta(R)$ is connected except possibly if*

(i) *$M = C_n$, Δ is defined over the field $\mathbb{F}_2$ and $J \cap \{n-1, n\} = \emptyset$, or*

(ii) *$M = F_4$, Δ is defined over the field $\mathbb{F}_2$ and $J \cap \{2, 3\} = \emptyset$.*

The difference with Theorem 3.30 is that we can only indicate in which cases $\mathrm{Far}_\Delta(R)$ is connected and not in which cases it isn't. Note however that we do obtain the same list.

Remark: We note that the (dis-)connectedness of $\mathrm{Far}_\Delta(R)$ for buildings with diagram $A_1 \times A_1$, A_2 and B_2 was proved by purely combinatorial means (see Proposition 3.8 and Theorem 3.9).

Moreover the result on Moufang polygons given in Theorem 3.12 has also been proved in a geometric way by Van Maldeghem and others.

First we will consider buildings without $\mathrm{Sp}_4(2)$ residues.

Proposition 3.36 *Let Δ be a building of spherical type with a connected diagram of rank at least 3 without residues of type $Sp_4(2)$. Then $Far_\Delta(R)$ is connected for every residue R.*

This will follow from a direct application of the next theorem.

Define the distance between two sets of chambers X and Y as $\mathrm{d}(X, Y) = \min\{\mathrm{d}(x, y) \mid x \in X, y \in Y\}$. In this context, a gallery is regarded as a set of chambers.

For any residue X let $\mathcal{B}^d_\Delta(X)$ be the set of chambers of Δ at distance $\geq d$ from R. A chamber of Δ at maximal distance from R has distance $d = \mathrm{diam}(\Delta) - \mathrm{diam}(R)$ to R, and one has $\mathcal{B}^d_\Delta(R) = \mathrm{Far}_\Delta(R)$. So we have $\mathcal{B}^0_\Delta(R) \supset \mathcal{B}^1_\Delta(R) \supset \cdots \supset \mathrm{Far}_\Delta(R)$. Note that if $\mathcal{B}^d_\Delta(R)$ is connected for some d, then $\mathcal{B}^i_\Delta(R)$ is connected for all $i \leq d$: every chamber of $\mathcal{B}^i_\Delta(R)$ can be joined by a gallery in $\mathcal{B}^i_\Delta(R)$ to a chamber in $\mathcal{B}^d_\Delta \subset \mathcal{B}^i_\Delta(R)$.

Theorem 3.37 (cf. Mühlherr and Ronan [34], Theorem 1.5.) *Let Δ be a spherical building and let R be a residue. Suppose that for some $d \in \mathbb{N}$ the set $\mathcal{B}^d_\Delta(R)$ is connected and that for all rank 2 residues X incident with $\mathcal{B}^d_\Delta(R)$ the set $Far_X(r)$ (with $r = \mathrm{proj}_X(R)$) is connected. Then $Far_\Delta(R)$ is connected.*

Proof: (of the theorem) We show by induction on $k \geq d$ that every two chambers x and y at distance k from R can be joined by a gallery at distance k from R; the case $k = \mathrm{diam}(\Delta) - \mathrm{diam}(R)$ is the desired result. For $k = d$ this is true by assumption. Now let $k > d$. Choose chambers x' and y' adjacent to x and y respectively at distance $k-1$ from R. By the induction hypothesis there exists a gallery between x' and y' at distance $k-1$ from R. Extend this gallery in the obvious way to a gallery γ between x and y.

We show by induction on the number $n(\gamma)$ of chambers in γ at distance $k-1$ from R that we can replace γ by a gallery between x and y at distance k from R. If $n(\gamma)=0$, we are done. Let $n(\gamma)=n>0$. Then there is a subgallery a,b,c of γ such that $\mathrm{d}(a,R)=k$ and $\mathrm{d}(b,R)=k-1$. Let X be a rank 2 residue on $\{a,b,c\}$ and put $r=\mathrm{proj}_X(R)$. Now there is a number l such that $\mathrm{d}(p,R)=l$ for all chambers $p\in r$. Hence, as a is further from R than b is, $X\setminus r$ is non-empty. We can choose a chamber $c'\in X\setminus r$ adjacent to c at distance k from R because $\mathrm{d}(c,R)\geq k-1$. Then $\mathrm{d}(a,r)=\mathrm{d}(c',r)$. By the assumption on the rank 2 residues, inside X we can find a gallery between a and c' at distance at least $\mathrm{d}(r,a)$ from r (see the remarks made just before Theorem 3.37). The chambers of this gallery lie at distance at least $\mathrm{d}(R,a)=k$ from R. Extend this gallery in the obvious way to a gallery δ between a and c. If in γ we replace a,b,c by δ, then $n(\gamma)=n-1$. By the induction hypothesis (on $n(\gamma)$) we can now replace γ by a gallery at distance k from R. $\square$

Proof: (of Proposition 3.36) Since the diagram of Δ is connected of rank at least 3, the rank 2 residues have diagram $A_1\times A_1$, A_2 or B_2. It follows from Theorem 3.9 and the discussion preceding it, that $\mathrm{Far}_X(r)$ (with $r=\mathrm{proj}_X(R)$) is connected for all rank 2 residues and arbitrary R. Clearly, for $d=0$, the set $\mathcal{B}^d_\Delta(R)=\Delta$ is connected. Now apply Theorem 3.37. $\square$

The following step is to consider the buildings $\mathrm{Sp}_{2n}(2)$ and $F_4(2)$ and restrict to residues R such that the projection of R onto no $\mathrm{Sp}_4(2)$ residue is a chamber.

Proposition 3.38 *If $\Delta=Sp_{2n}(2)$ and R is an i-object ($1\leq i\leq n$) or $\Delta=F_4(2)$ and R is an i-object ($i=1,4$), then $Far_\Delta(R)$ is connected.*

We prove this by simply checking that with this assumption on R the projection of R onto X is not a chamber, for any $\mathrm{Sp}_4(2)$ residue X.

For the moment let R and X be any two residues. One can always find an apartment incident with X and R and this apartment then contains $\mathrm{proj}_X(R)$ too. Therefore, when determining $\mathrm{proj}_X(R)$, we may confine our attention to residues R and X on a fixed apartment Σ.

We will now describe the apartments of type C_n and F_4 and show how one computes $\mathrm{proj}_X(R)$ for given residues X and R. We will describe these apartments by presenting the Coxeter graphs of type $C_{n,1}$ and $F_{4,1}$ (see 1.1.5). The vertices and some of the edges of these graphs are depicted in Figure 3.1. One constructs a Coxeter graph of type $C_{n,1}$ from the complete graph K_{2n} on $2n$ vertices by removing the n edges of a perfect matching of the vertices. In the picture we chose to remove the edges of the matching (i,i') ($1\leq i\leq n$). The i-cliques ($1\leq i\leq n$) of this graph are the objects of type i of the apartment. Two objects are incident whenever one of them is contained in the other. One constructs a Coxeter graph of type $F_{4,1}$ from the picture above as follows. Insert edges in the following places: between ∞ and every vertex of the cube C and likewise for ∞' and C', between the vertices i and i' for $1\leq i\leq 8$ and, finally, for all $x\in\{a,b,c,d,e,f\}$, between the vertex x and all vertices that lie on the face of C or C' with label x. The 1-cliques, 2-cliques, 3-cliques and octahedra in this graph are the objects of type 1, 2, 3 and 4 respectively of the apartment. Two objects are incident whenever one of them is contained in the other.

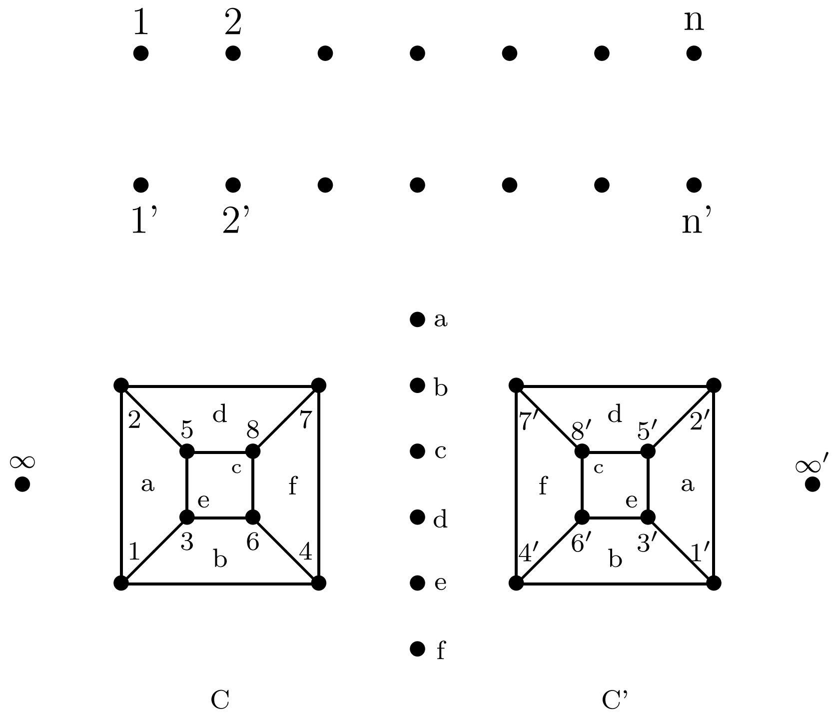


Figure 3.1: The Coxeter graphs of type $C_{n,1}$ (top) and $F_{4,1}$ (bottom)

When working with these pictures, it is best to view a residue (of type $J \subset I$) as a flag (of type $I \setminus J$). For given flags X and R, the i object of the flag $\text{proj}_X(R)$ is the unique object closest to R among all objects of type i that are incident with X, if it exists.

EXAMPLE: Consider the apartment of type C_4. We describe an object by the set of its vertices. Let X and R be the flags $\{2', 3', 4'\}$ and $\{1, 2\}$. Both $3'$ and $4'$ are closest 1-objects to R on X. The 2-object $\{3', 4'\}$ is the unique 2-object on X closest to R because its 1-objects are closest to R. Clearly the only 3-object incident with X is X itself. The 4-object $\{1, 2', 3', 4'\}$ is the unique 4-object on X closest to R. Hence $\text{proj}_X(R) = (\{3', 4'\}, \{2', 3', 4'\}, \{1, 2', 3', 4'\})$.

Proof: (of Proposition 3.38) Using the description of the apartment of Δ given above, one can easily verify that under the assumption of the proposition the projection of R onto an arbitrary $\text{Sp}_4(2)$ residue is not a chamber. It follows from Lemma 3.8 and Theorem 3.9 that $\text{Far}_X(r)$ (with $r = \text{proj}_X(R)$) is connected for all rank 2 residues X.

Clearly, for $d = 0$, the set $\mathcal{B}^d_\Delta(R) = \Delta$ is connected. Now apply Theorem 3.37. □

Finally we have to consider the following cases: (*i*) $\Delta = \text{Sp}_{2n}(2)$ and R is a flag of rank at least 2 but not of type $\{n-1, n\}$ and (*ii*) $\Delta = F_4(2)$ and R is a flag that contains a flag either of type $\{1, 4\}$ or of type $\{2\}$.

The following examples show that in these cases there exist $\text{Sp}_4(2)$ residues X such that the projection of R onto X is a chamber. Hence we cannot apply Theorem 3.37 as we did before.

Example: To find such 'obstructing' residues, first notice that either $\Delta = \mathrm{Sp}_{2n}(2)$ and R is a residue of corank at least 2, or $\Delta = F_4(2)$ and R is contained in a residue of cotype 2, 3 or $\{1,4\}$. Now look at the cases where R is such a residue.

Let A be the Coxeter graph of type $C_{n,1}$ or $F_{4,1}$ as described above. For the duration of this example, we use the following ad hoc notation. For a subset X of the vertex set of A and a flag R, let $\langle X\rangle$ be the smallest object containing X, and let $\langle R, X\rangle$ be the flag on R and the smallest object containing R and X.

Assume that A is the Coxeter graph of type $C_{n,1}$ and suppose that R is the flag $\langle\langle 1',\dots,i'\rangle, (i+1)',\dots,j'\rangle$ of type $\{i,j\}$ $(1 \le i < j \le n)$. Let X be a flag of type $\{1,\dots,n-2\}$ on the object $\langle 1,\dots,\widehat{\imath},\dots,\widehat{\jmath},\dots,n\rangle$ of type $(n-2)$. Then $\mathrm{proj}_X(R)$ is the flag that is incident with X and $\langle\langle 1,\dots,i',\dots,\widehat{\jmath},\dots,n\rangle, j'\rangle$ and this is a chamber.

Assume now that A is the Coxeter graph of type $F_{4,1}$. Let X be the flag $\langle\langle a\rangle, b\rangle$ of type $\{1,4\}$. If R is the flag $\langle\infty, 1\rangle$ of type 2 or the flag $\langle\langle\infty\rangle, c\rangle$ of type $\{1,4\}$, then $\mathrm{proj}_X(R)$ is the flag $\langle\langle\langle\langle a\rangle,1\rangle,3\rangle,b\rangle$ and this is a chamber.

If R' is a flag containing one of these 'extremal' flags R, then the projection $\mathrm{proj}_X(R')$ is a flag that contains $\mathrm{proj}_X(R)$ and hence again this will be a chamber.

Proof: (of Theorem 3.35)
In view of Proposition 3.36 and Proposition 3.38 we only have to consider the following cases: (*i*) $\Delta = \mathrm{Sp}_{2n}(2)$ $(n \ge 2)$ and (*ii*) $\Delta = F_4(2)$, and R is a panel of a $\mathrm{Sp}_4(2)$ residue. It suffices to show that for these buildings, $\mathrm{Far}_\Delta(R)$ is connected if R is such a panel. In case (*i*) we do this by induction on $n \ge 2$. Note that for $n = 2$ this is true (by Theorem 3.9). The remainder of the proof applies to both case (*i*) and (*ii*). Notice however that in the treatment of case (*i*) certain lower rank results follow from the induction hypothesis, whereas in the treatment of case (*ii*) these follow from the case (*i*) result.

Let us call the objects of type 1, 2 and 3 points, lines and planes respectively. Let x, y be two chambers in $\mathrm{Far}_\Delta(R)$. We first find a path $P_0, L_0, \dots, L_k, P_{k+1}$ with $x \in P_0$, $y \in P_{k+1}$ and $P_i \cap L_i \ne \emptyset \ne L_k \cap P_{k+1}$, where (P) P_i are points far from R and (L) L_i are lines far from R. Such a path yields a gallery in $\mathrm{Far}_\Delta(R)$ between x and y: If P is a point far from R, then $r = \mathrm{proj}_P(R)$ is a panel of the same type as R and we may assume ((*i*): induction/(*ii*): case (*i*) result) that $\mathrm{Far}_P(r)$ is connected. If L is a line far from R, then either we have the same situation as for P or L has no $\mathrm{Sp}_4(2)$ residues (or both), so that again $\mathrm{Far}_L(r)$ (with $r = \mathrm{proj}_L(R)$) is connected.

Let Q be the point on R. Then, by Lemma 3.38, $\mathrm{Far}_\Delta(Q)$ is connected and so there is a path as above that satisfies (P) but possibly not (L). Suppose for instance that line $L = L_i$ $(i \in \{0,\dots,k\})$ is not far from R. Choose a plane π on L far from $\mathrm{proj}_L(R)$. One can check that $\mathrm{proj}_{(P,\pi)}(R)$ (with $P = P_i$) contains L (*). Clearly then, the same holds for $P = P_{i+1}$. This means that we can join P_i to P_{i+1} by a path on π whose points are far from R and whose lines are further from R than L is. Choose any line $L' \ne L$ on P_i. This line contains at least one point $P' \ne P_i$ that is far from R. Since the points and lines of π form a projective plane, there is also a line $L'' \ne L$ that contains P' and P_{i+1}. By the preceding L' and L'' are further from R than L is.

By an easy induction on the number of lines of the path that do not lie far from R, we can replace this path by one that satisfies both (P) and (L). □

It is a somewhat tedious job to check statement (*) explicitly. However, for completeness sake, we will do it here. We use the notation of the previous example once more. Consider the Coxeter graph of type $C_{n,1}$. We may assume that R is the flag $\langle \cdots \langle 1\rangle, 2\rangle, \ldots\rangle, n-2\rangle, n-\epsilon\rangle$ of type $\{1, \ldots, n-2, n-\epsilon\}$ ($\epsilon \in \{0,1\}$) and that $P = \langle 1'\rangle$. There are essentially two cases. The line L is $\langle 1', l\rangle$, where $l = 2$ or $l = i, i'$ with $3 \leq i \leq n$ respectively. The projection $\mathrm{proj}_L(R)$ then contains the plane $\langle 1', l, r\rangle$ where $r = 3$ or $r = 2$ respectively. In either case the plane $\pi = \langle 1', l, r'\rangle$ is the plane on L far from this projection. Since the line $\langle 1', r'\rangle$ is further from R than L, the projection $\mathrm{proj}_{(P,\pi)}(R)$ is contained in L.

Consider the Coxeter graph of type $F_{4,1}$. We may assume that R is the flag $\langle\langle\langle\infty\rangle, 1, 2\rangle, a\rangle$ and $P = \infty'$. Then, up to an isomorphism fixing R and P, the possibilities are listed in the table below.

L	$\{\infty', 1'\}$
$\mathrm{proj}_L(R)$	$(1', \{\infty', 1'\}, \{\infty', 1', 2'\}, \langle\infty', a\rangle)$
π	$\{\infty', 1', 4'\}$
$\mathrm{proj}_{(P,\pi)}$	$(\infty', \{\infty', 1'\}, \{\infty', 1', 4'\}, \langle\infty', c\rangle)$
L	$\{\infty', 3'\}$
$\mathrm{proj}_L(R)$	$(3', \{\infty', 3'\}, \{\infty', 3', 1'\}, \langle\infty', a\rangle)$
π	$\{\infty', 3', 6'\}$
$\mathrm{proj}_{(P,\pi)}$	$(\infty', \{\infty', 3'\}, \{\infty', 3', 6'\}, \langle\infty', b\rangle)$

The symplectic case

We will now determine the number of connected components of the set of chambers of the $\mathrm{Sp}_{2n}(2)$ building far away from a fixed chamber c in a combinatorial way. This approach will gain us some insight in the geometry of these components as well.

Theorem 3.39 *Fix a chamber c in the $Sp_{2n}(2)$ building Δ, where $n > 0$, and consider the geometry Δ' of elements far from c. Then Δ' has 2^{n-1} connected components.*

There are various ways to understand the occurrence of components here. Maybe the best way to describe these components is as corresponding to the $O^+_{2n}(2)$ subgeometries of the $\mathrm{Sp}_{2n}(2)$ geometry.

Fix an $\mathrm{Sp}_{2n}(2)$ geometry and an $O^+_{2n}(2)$ subgeometry. In the $\mathrm{Sp}_{2n}(2)$ chamber system $\mathcal{C}$, an $O^+_{2n}(2)$ chamber corresponds to a pair of n-adjacent chambers (c', c'') (for the usual labeling of the diagram), two chambers (c', c'') and (d', d'') being $(n-1)$- (or n-) adjacent, when $c' \neq d'$ but $c'' = d''$ (or $c'' \neq d''$ but $c' = d'$). Let D^1 be the collection of chambers in $\mathcal{C}$ occurring in such pairs. Let $\{c, c', c''\}$ be an n-panel in $\mathcal{C}$, with $c \in \mathcal{C} \setminus D^1$, $c', c'' \in D^1$. Then a chamber $d' \in D^1$ is far from c if and only if it occurs in an $O^+_{2n}(2)$ chamber (d', d'') opposite to (c', c''). Since the projection of c into the n-panel $\{d, d', d''\}$ is not far from c, we see that n-edges in $\mathrm{Far}_c(\Delta)$ remain in D^1. Since it is clear that D^1 is closed for j-edges, $j < n$, we see that any connected component of $\mathrm{Far}_c(\Delta)$ meeting D^1 is contained in D^1. But $D^1 \cap \mathrm{Far}_c(\Delta)$ is connected (for example because the D_n diagram has no B_2 subdiagrams), so is a connected component.

Let us redo the above in geometric terms. Let 'i-object' stand for 'totally isotropic (t.i.) i-space'. Two chambers c and c' in an $Sp_{2n}(q)$ geometry are far (opposite) if and only if whenever A and A' are i-objects in c and c', respectively, we have $A^\perp \cap A' = 0$.

Proposition 3.40 *Fix an n-object U with hyperplane ($(n-1)$-object) H in the $Sp_{2n}(2)$ geometry. Let F be the flag (H, U). The connected component of the geometry far from F containing a given n-object V is the geometry far from H in the $O^{+}_{2n}(2)$ geometry defined by the quadric Q that defines the given symplectic form, for which H and V are totally singular, but U is not. (That is, $H = Q \cap U$.)*

The conditions that Q defines the given symplectic form, and vanishes identically on H and V but not on U, and that Q defines a hyperbolic quadric, indeed determine Q uniquely. If we choose a basis, then Q is determined by the symplectic form, except for its diagonal elements. Requiring that Q vanishes on H determines $n-1$ diagonal elements; one more is fixed since Q must not vanish on U; one more is fixed since Q must be hyperbolic, so that $H^{\perp}/H$ is a hyperbolic line; finally there are $n-1$ diagonal elements to choose freely, giving 2^{n-1} connected components.

Remark: Note that we used the subbuilding of type $O^{+}_{2n}(2)$ in the computation of the number of components on page 3.2.6 as well (it was called '$\Delta_{\circ}$' there).

Chapter 4

The subgeometry of $F_{4,1}(q)$ far from a point

In this chapter Δ is a building with diagram F_4, defined over a finite field $\mathbb{F}_q$. Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the 1-shadow space of Δ. This is the geometry of type $F_{4,1}(q)$ we will be interested in. We will study the subgeometry $\text{Far}(\infty)$ of Γ consisting of the points and lines far away from a given point ∞ (see Chapter 3). The main result is Theorem 4.15 which describes a 13-class association scheme obtained from the collinearity graph of $\text{Far}(\infty)$ and lists all its intersection numbers.

4.1 Introduction to the geometries involved

4.1.1 The F_4 geometry

An F_4 *geometry* is the geometry of some building with diagram F_4. An $F_{4,i}$ *geometry* ($i = 1, 2, 3, 4$) is the i-shadow space of an F_4 geometry. If the building is defined over a finite field $\mathbb{F}_q$ we use the terms $F_4(q)$ geometry and $F_{4,i}(q)$ geometry, respectively.

In the F_4 geometry an object of type 1, 2, 3 and 4 will be called a *point*, *line*, *plane* and *symplecton* respectively.

In an $F_{4,i}$ geometry the *points* are the objects of type i and the *lines* are the sets of points incident to an i-panel of the building (see Subsection 1.2.4).

We first address the thin case. Let M denote the Dynkin diagram of type F_4, defined over the set $I = \{1, 2, 3, 4\}$. Let $(W, \{r_i\}_{i=1}^4)$ be the Coxeter system with diagram M. Then the set

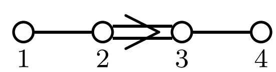


Figure 4.1: The Dynkin diagram of type F_4

W together with i-adjacency relations given by $x \sim_i y$ if and only if $x^{-1}y = r_i$ $(i = 1, 2, 3, 4)$ forms the chamber system of a thin building with diagram M.

The thin F_4 geometry is the geometry of this building; it is the geometry whose objects of type i are the left cosets of the subgroups $W_{I-\{i\}}$ $(i = 1, 2, 3, 4)$ and in which two objects are incident whenever they intersect non-trivially.

The thin $F_{4,1}$ geometry is the 1-shadow space of the thin F_4 geometry: it is the point-line geometry whose points are the objects of type 1 and in which two points are collinear whenever they are incident to a common object of type 2 (lie on a common 1-panel). We will now present the collinearity graph of the thin $F_{4,1}$ geometry. Complete the graph in Figure

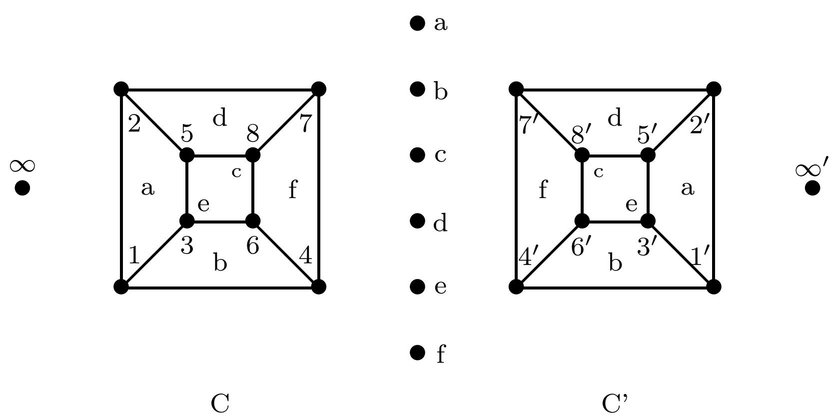


Figure 4.2: The thin $F_{4,1}$ geometry

4.2 by inserting edges in the following places: between ∞ and every vertex of the cube C and likewise for ∞' and C', between the vertices i and i' for $1 \leq i \leq 8$ and, finally, for all $x \in \{a, b, c, d, e, f\}$, between the vertex x and all vertices that lie on the face of C or C' with label x. The graph thus obtained is called the *Coxeter graph of type $F_{4,1}$*.

The thin F_4 geometry can be recovered from this graph as follows. The 1-cliques, 2-cliques, 3-cliques and octahedra in this graph are the objects of type 1, 2, 3 and 4 respectively and two objects are incident whenever one of them is contained in the other.

4.1.2 A B_n and a C_n geometry

The notions of B_n geometry, C_n geometry, $B_{n,i}$ geometry $C_{n,i}$ geometry $(i = 1, 2, \ldots, n)$, $B_n(q)$, $C_n(q)$, $B_{n,i}(q)$ and $C_{n,i}(q)$ geometry are defined as for the F_4 diagram.

We are interested in the $B_n(q)$ buildings and $C_n(q)$ buildings $\mathrm{O}_{2n+1}(q)$ and $\mathrm{Sp}_{2n}(q)$ because (for $n = 3$) they are contained in the $F_4(q)$ building in such a way that their geometries are subgeometries of the $F_4(q)$ geometry. The geometries of type $B_{n,1}(q)$, $B_{n,n}(q)$, $C_{n,1}(q)$ and $C_{n,n}(q)$ obtained from these buildings are often denoted as $\mathrm{O}_{2n+1}(q)$, $\mathrm{DO}_{2n+1}(q)$, $\mathrm{Sp}_{2n}(q)$ and $\mathrm{DSp}_{2n}(q)$ respectively and we will do so here as well.

We first address the thin case. Let M denote the Dynkin diagram of type B_n or C_n, defined over the set $I = \{1, 2, \ldots, n\}$. Let $(W, \{r_i\}_{i=1}^n)$ be the Coxeter system with diagram M.

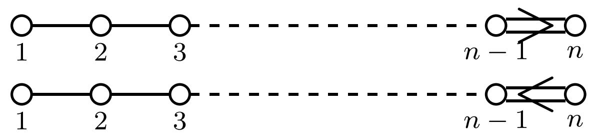


Figure 4.3: The Dynkin diagrams of type B_n (top) and C_n (bottom)

Then the set W together with i-adjacency relations given by $x \sim_i y$ if and only if $x^{-1}y = r_i$ ($i = 1, 2, \ldots, n$) forms the chamber system of a thin building with diagram M. Clearly the Coxeter diagrams underlying the Dynkin diagrams of type B_n and C_n are the same, hence the thin buildings are the same.

From this building we obtain the thin B_n geometry and the thin $B_{n,1}$ geometry as we did for the F_4 diagram above. We will now present the collinearity graph of the thin $B_{n,1}$ geometry. Complete the graph in Figure 4.4 by inserting edges between any pair of vertices

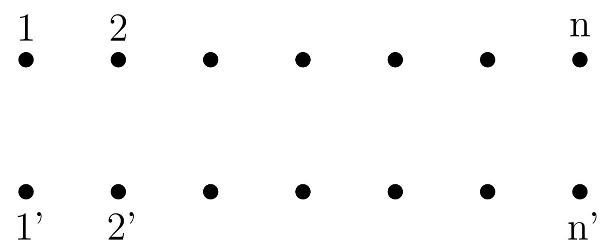


Figure 4.4: The thin $B_{n,1}$ geometry

except those from the matching $\{1, 1'\}, \{2, 2'\}, \ldots, \{n, n'\}$. The graph thus obtained is called the *Coxeter graph of type* $B_{n,1}$ (or $C_{n,1}$).

The thin B_n geometry can be recovered from this graph as follows. The i-cliques ($i = 1, 2, \ldots, n$) in this graph are the objects of type i and two objects are incident whenever one of them is contained in the other.

4.2 Properties of the geometries involved

In this subsection we recover some properties of the geometries of type $F_{4,1}(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{O}_{2n+1}(q)$ and $\mathrm{DO}_{2n+1}(q)$ directly from the diagram and the apartment.

4.2.1 Properties recovered from the apartment

Let M be any spherical diagram defined over an index set I and let $(W, \{r_i\}_{i \in I})$ be the Coxeter system with diagram M. Let Σ be the Coxeter building obtained from this Coxeter system and let Δ be a building with diagram M.

Recall that in Σ the distance between two chambers x and y is just $\delta(x, y) = x^{-1}y$; the distance between two residues (or flags) $R = xW_J$ and $S = yW_K$ ($J, K \subset I$) is $\{\delta(r, s) \mid r \in R, s \in S\}$ which is the double coset $W_J x^{-1} y W_K$. Similarly, the distance between two residues (or flags) R and S of Δ is $\delta(R, S) = \{\delta(r, s) \mid r \in R, s \in S\}$, where δ now denotes the distance in Δ.

Now suppose that Σ is an apartment of Δ incident to R and S, then since apartments are convex, we have $\delta(R \cap \Sigma, S \cap \Sigma) \subseteq \delta(R, S)$. Since for any other apartment Σ' incident to R and S there is an isomorphism between Σ and Σ' stabilizing R and S, for any such apartment we get the same set at the left hand side of the inclusion. As for any pair of chambers in R and S we can find such an apartment the inclusion is an equality.

The *projection* of S onto R is the residue $\mathrm{proj}_R(S) \cup_{s \in S} (\mathrm{proj}_R(s)$. It follows from the convexity of apartments that any apartment incident to R and S is incident to $\mathrm{proj}_R(S)$ too. In view of the previous paragraphs we see that we can measure the distances involving R, S and $\mathrm{proj}_R(S)$ inside any apartment incident to R and S.

Thus if we want to know for given residues R and S of given types and at given distance from each other what is the type of $\mathrm{proj}_R(S)$ and its distance to S, we only have to consider this for a thin building.

Lemma 4.1 *Let Δ be a building with a spherical diagram defined over an index set I. Let R and S be residues and let $P = \mathrm{proj}_R(S)$. Then for any $J \subset I$ all J-residues incident with P are at the same distance from S. Moreover, this distance is minimal among all J-residues incident with R.*

Lemma 4.2 Brouwer et al. [8]) *Let Γ be the i-shadow space of Δ, let p be a point and let l be a line. Then either all points on l have equal distance to p or there is a unique point on l at minimal distance from p and all others have equal (and larger) distance to p.*

Proof: Let π be the i-panel representing l. Let Σ be an apartment on p and π. We have $\delta(p \cap \Sigma, \pi \cap \Sigma) = \delta(p, \pi)$. The two chambers of $\Sigma \cap \pi$ are either at equal distance from p or not. In the first case apparently all chambers of π have equal distance to p. In the second case the chamber on $\pi \cap \Sigma$ closer to p is the projection $\mathrm{proj}_\pi(p)$ which is contained in all apartments on p and π. Hence all other chambers on π are at equal (larger) distance from p.

Using the previous lemma, it is not difficult to show that if x, y are chambers on π contained in points X and Y, then x is closer to π than y is if and only if X is closer to p than Y is. □

4.2.2 Properties of the $F_{4,1}$ geometry

We keep the notation from Subsection 4.2.1 and set $M = F_4$. The distances between two points of the thin $F_{4,1}$ geometry are the double cosets $W_{I-\{1\}} w W_{I-\{1\}}$ ($w \in W$). For each of these double cosets we can find representatives of minimal length. The names of the distances and a corresponding minimal double coset representative is given in the table below.

Distance	Double coset representative
0	1
1	r_1
2s	$r_1r_2r_3r_2r_1$
2ns	$r_1r_2r_3r_4r_2r_3r_2r_1$
3	$r_1r_2r_3r_4r_2r_3r_4r_1r_2r_3r_2r_1$

The numbers in the 'Distance' column refer to the distance in the collinearity graph; two points at distance 2 in the collinearity graph are said to be at distance 2s (resp. 2ns) from one another if they are (not) incident to a common symplecton.

By looking at Figure 4.2 and applying Lemmas 4.1 and 4.2 we easily derive the following properties of the $F_{4,1}$ geometry.

For $\lambda = 0, 1, 2s, 2ns, 3$ and any point p let $C_\lambda(p)$ be the collection of points at distance λ from p.

Lemma 4.3 *Given a point p, a line l, a plane V and a symplecton S,*

(i) if $C_3(p) \cap l \neq \emptyset$ then $C_{2ns}(p) \cap l$ consists of a point and $l \setminus C_{2ns}(p) \subset C_3(p)$.

(ii) if $C_3(p) \cap V \neq \emptyset$ then $C_{2ns}(p) \cap V$ consists of a line and $V \setminus C_{2ns}(p) \subset C_3(p)$.

(iii) if $C_3(p) \cap S \neq \emptyset$ then $C_0(p) \cap S = C_1(p) \cap S = \emptyset$, $C_{2s}(p) \cap S$ consists of a point s, $C_{2ns}(p) \cap S = C_1(s) \cap S$ and $C_3(p) \cap S = C_{2s}(s) \cap S$.

(iv) if $C_0(p) \cap S = C_3(p) \cap S = \emptyset$, then $C_1(p) \cap S$ is a line m, $C_{2s}(p) \cap S = \cap_{q \in m} C_1(q)$.

† The properties...were used by Cohen [18] as part of an axiom system for the metasymplectic spaces.

4.2.3 Properties of the $B_{n,1}$ and $C_{n,1}$ geometries

We keep the notation from Subsection 4.2.1 and set $M = B_n$.

In the table below we list for each double coset of $W_{I-\{1\}}$ in W a representative of minimal length and the name of the corresponding distance in the $B_{n,1}$ (and $C_{n,1}$) geometry.

Distance	Double coset representative
0	1
1	r_1
2	$r_1r_2r_3 \cdots r_{n-1}r_nr_{n-1} \cdots r_2r_1$

The numbers in the 'Distance' column are the distance in the collinearity graph. We see that if two points have distance 2 from each other, then they are not incident to any common object.

By looking at figure 4.4 and applying Lemmas 4.1 and 4.2 we easily derive the following properties of the $B_{n,1}$ and $C_{n,1}$ geometries.

For $\lambda = 0, 1, 2$ and any point p let $C_\lambda(p)$ be the collection of points at distance λ from p.

Lemma 4.4 *Given a point p and an i-object O ($i \in \{1, \ldots, n\}$) such that $p \notin O$, then $C_1(p) \cap O$ is the set of points of an $(i-1)$-object incident with O.*

4.2.4 Properties of the $B_{n,n}$ and $C_{n,n}$ geometries

In the table below we list for each double coset of $W_{I-\{n\}}$ in W a representative of minimal length and the name of the corresponding distance in the $B_{n,n}$ (and $C_{n,n}$) geometry.

Distance	Double coset representative
0	1
1	r_n
2	$r_n r_{n-1} r_n$
$\vdots$	$\vdots$
n	$r_n r_{n-1} \ldots r_2 r_1 r_n r_{n-1} \cdots r_3 r_2 r_n \cdots r_{n-1} r_n$

The numbers in the 'Distance' column are the distance in the collinearity graph. We see that if two points have distance i from each other, then they are incident to a common j-object ($0 < j \leq n-i$) but not to a common k-object ($k > n-i$).

By looking at figure 4.4 and applying Lemmas 4.1 and 4.2 we easily derive the following properties of the $B_{n,n}$ and $C_{n,n}$ geometries.

For $\lambda = 0, 1, \ldots, n$ and any point p let $C_\lambda(p)$ be the collection of points at distance λ from p.

Lemma 4.5 *Given a point p and a line l, then for some $i \in 1, \ldots, n$ we have $|l \cap C_{i-1}(p)| = 1$ and $l \setminus C_{i-1}(p) \subset C_i(p)$.*

4.3 Association schemes

Let M be any spherical diagram defined over an index set I and let $(W, \{r_i\}_{i\in I})$ be the Coxeter system with diagram M. Let Σ be the Coxeter building obtained from this Coxeter system and let Δ be a *finite* building with diagram M obtained from a group G with a Tits system (B, N).

Suppose that Γ is the i-shadow space of Δ for some $i \in I$. Then G acting on Γ not only preserves distances but it actually acts transitively on the set of pairs of points at any given distance. Let Λ be an ordered set whose elements represent the possible distances between two points of Γ. Then the distance relations R_λ ($\lambda \in \Lambda$) form the set of relations of a group scheme whose vertices are the points of Γ. Many of these schemes are distance regular graphs (see Brouwer et al. [8] Ch. 10). By convention we will assume that the first two relations are $\lambda = 0, 1$ representing identity and collinearity respectively.

For any triple $i, j, k \in \Lambda$ let p^i_{jk} denote, as customary, for any pair x, y in relation i, the number of z in relation j with x and in relation k with y. For various geometries we will compute the numbers p^0_{jj} and the numbers p^i_{j1}. For readability we put $k_i = p^0_{ii}$ and $r_{ij} = p^i_{j1}$ $(i, j \in \Lambda)$.

Lemma 4.6

(i) We have $\sum_{j\in\Lambda} r_{ij} = \sum_{j\in\Lambda} r_{i'j}$ *for any two elements* $i, i' \in \Lambda$.

(ii) We have $k_i r_{ij} = k_j r_{ji}$ *for any two elements* $i, j \in \Lambda$.

Proof: Any two points have an equal number of neighbours, hence (i). Relation 1 is symmetric. Thus, given a point x, (ii) follows by double counting the pairs of points (y, z) where x is in relation i with y and in relation j with z and where y and z are in relation 1. □

We note that if a parameter is non-zero in case Δ is thin (i.e. is the Coxeter building) then it will be non-zero in all other cases; the converse is often false. We note that all distance relations for Γ are already visible in the thin case i.e. k_i is non-zero for all $i \in \Lambda$ and all buildings (see Subsection 4.2.1).

4.3.1 An association scheme for the $F_{4,1}(q)$ geometry

We keep the notation of Subsection 4.3 and let M be the diagram F_4. We consider the 1-shadow space Γ of the $F_4(q)$ geometry. Thus, for $\lambda \in \Lambda = \{0, 1, 2s, 2ns, 3\}$, the relation R_λ comprises the pairs of points that are (0) identical, (1) collinear, (2s) at distance 2 and contained in a common symplecton, (2ns) at distance 2 but not contained in a common symplecton and (3) at distance 3.

Proposition 4.7 (cf. Cohen [19]) *There exists a 5-class group scheme induced by the group* $F_4(q)$ *on the collinearity graph of the* $F_{4,1}(q)$ *geometry. Its relations are* R_λ $(\lambda \in \Lambda)$ *defined above and its parameters are as displayed in Figure 4.5.*

The computation of the parameters of the scheme has three kinds of ingredients.

(i) Lemma 4.6.

(ii) Using the properties of diagrams for geometries and the fact that the geometries $A_{n,1}(q)$ and $C_{n,1}(q)$ have $(q^{n+1} - 1)/(q - 1)$ and $(q^{2n} - 1)/(q - 1)$ points respectively, we can compute the number of flags of any given type incident with a given flag of any other type.

(iii) Specific geometric properties as given in Lemmas 4.3, 4.4 and 4.5.

Proof: The fact that the distance relations of the geometry form the relations of a group scheme was explained in Subsection 4.3. Thus we only have to calculate the parameters of the scheme.

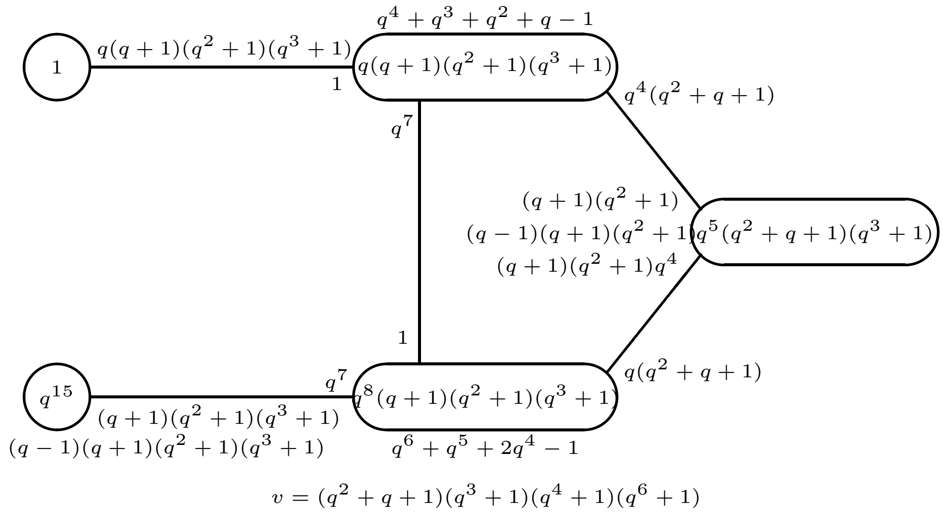


Figure 4.5: A 5-class group-scheme on the points of $F_{4,1}$.

Fix a point p and, for $\lambda \in \Lambda$ let $R_\lambda(p)$ denote the set of points in relation R_λ with p. Then $k_\lambda = |R_\lambda(p)|$ and, for any $x \in R_i(p)$ the number of points in $R_j(p)$ that are collinear to x is r_{ij}.

It is trivial that $k_0 = r_{10} = 1$ and r_{00}, r_{02s}, r_{02ns}, r_{03}, r_{2s0}, r_{2ns0}, r_{30} are all equal to 0. Further, by looking at an apartment we also see that $r_{2s3} = r_{32s} = 0$. Using (ii) we compute that the number of lines on a point is $(q+1)(q^2+1)(q^3+1)$ and hence we also find r_{01} and k_1. If a line is contained in $R_1(p)$, then it lies on a plane containing p. Thus using (ii) we find r_{11}. A pair of points in R_{2s} is contained in a unique symplecton. Then using (ii) we find k_{2s}. A line L on a point in $R_{2s}(p)$ either has 1 point in $R_1(p)$ and q points in $R_{2s}(p)$ or it has 1 point in $R_{2s}(p)$ and q points in $R_{2ns}(p)$. In the former case L is contained in a unique symplecton on p. Thus computing the number of lines on a point in a symplecton we find r_{2s1}, r_{2s2s}. Then $r_{2s2ns} = r_{00} + r_{01} - r_{2s1} - r_{2s2s}$. Now with (1) we find $r_{12s} = k_{2s}r_{21}/k_1$ and $r_{12ns} = r_{00} + c01 - r_{11} - r_{12s}$. A point in $R_{2ns}(p)$ has a unique neighbour in $R_1(p)$, that is $r_{2ns1} = 1$. Now with (1) we find k_{2ns} and r_{2ns2s}. Given a point $x \in R_{2ns}(p)$ and the unique point $z \in R_1(p) \cap R_1(x)$, then all neighbours of x in $R_{2ns}(p)$ are either on the line zx but different from z and x, or on a plane V on zx but not on the lines zx and $V \cap R_1(p) \cup R_{2s}(p)$ or in a unique symplecton on zx but not coplanar to zx. Now using (ii) we can compute r_{2ns2ns} and then by (1) we find r_{2ns3}. Any line on a point of $R_3(p)$ contains 1 point in $R_{2ns}(p)$ and q points in $R_3(p)$. Knowing the number of lines on a point we find r_{32ns} and r_{33}. Finally with (ii) we find k_3. □

4.3.2 Association schemes for the $B_{n,1}$, $C_{n,1}$, $B_{n,n}$ and $C_{n,n}$ geometries

We keep the notation of of Subsection 4.3. The diagram M is of type B_n or C_n and Δ is the building obtained from the group $\mathrm{O}_{2n+1}(q)$ and $\mathrm{Sp}_{2n}(q)$ in the respective cases.

We shall compute the parameters of the association scheme of the 1- and n-shadow space of the geometry of Δ. Here, we use the same kinds of ingredients we used in the proof of

Proposition 4.7. Since the number of points of the geometries $O_{2n+1}(q)$ and $\mathrm{Sp}_{2n}(q)$ is the same and the apartments of the buildings of type B_n and C_n are the same, we get the same results for the schemes derived from the B_n and C_n geometries.

We first consider the 1-shadow space Γ of the geometry of Δ. Thus, for $\lambda \in \Lambda = \{0, 1, 2\}$ the relation R_λ comprises the pairs of points that are at distance λ from each other in the collinearity graph.

Proposition 4.8 *There exists a 3-class group scheme induced by the group* $O_{2n+1}(q)$ *(resp.* $Sp_{2n}(q)$*) on the collinearity graph of the* $O_{2n+1}(q)$ *geometry (resp. the* $Sp_{2n}(q)$ *geometry). Its relations are* R_λ $(\lambda \in \Lambda)$ *defined above and its parameters are as displayed in Figure 4.6.*

The proof is straightforward.

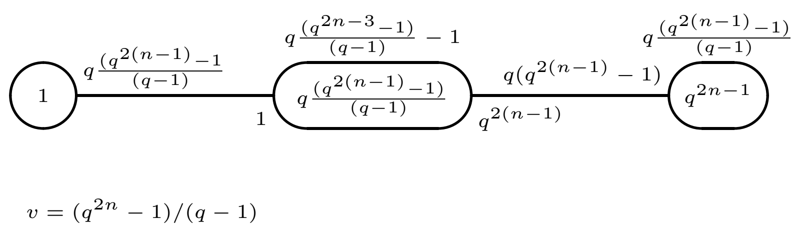


Figure 4.6: The distance distribution diagram for $O_{2n+1}(q)$ and $\mathrm{Sp}_{2n}(q)$.

We consider the n-shadow space Γ of the geometry of Δ. Thus, for $\lambda \in \Lambda = \{0, 1, \ldots, n\}$ the relation R_λ comprises the pairs of points that are at distance λ from each other in the collinearity graph.

Proposition 4.9 *There exists an* $(n+1)$*-class group scheme induced by the group* $O_{2n+1}(q)$ *(resp.* $Sp_{2n}(q)$*) on the collinearity graph of the* $DO_{2n+1}(q)$ *geometry (resp. the* $DSp_{2n}(q)$ *geometry). Its relations are* R_λ $(\lambda \in \Lambda)$ *defined above and, for* $n = 3$*, its parameters are as displayed in Figure 4.7.*

Proof: The fact that the distance relations of the geometry form the relations of a group scheme was explained in Subsection 4.3. Thus we only have to calculate the parameters of the scheme. Let us assume, without loss of generality, that $M = C_n$.

Then, for $i \leq n - 1$ a residue of type $\{i, i+1, \ldots, n\}$ is (isomorphic to) the building obtained from $\mathrm{Sp}_{2(n-i)}(q)$ and a residue of type $\{1, \ldots, n-1\}$ is the building with diagram

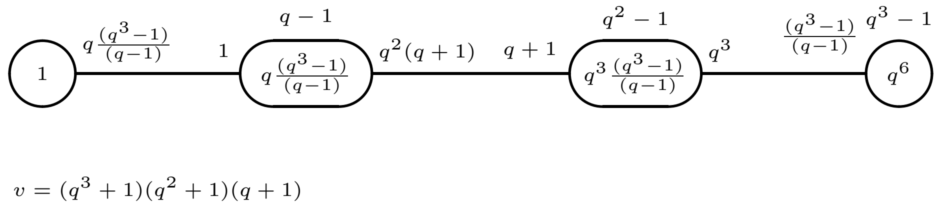


Figure 4.7: The distance distribution diagram for $\mathrm{DO}_7(q)$ and $\mathrm{DSp}_6(q)$.

A_{n-1} defined over $\mathbb{F}_q$ (i.e. the projective $(n-1)$-space $\mathrm{PG}(n-1,q)$). Hence there are $N_n^q = \prod_{i=1}^n \frac{q^{2i}-1}{q-1} / \prod_{i=1}^n \frac{q^i-1}{q-1} = \prod_{i=1}^n (q^i+1)$ n-objects.

Let M_i^q be the number of i-objects at distance i from a fixed i-object in a $\mathrm{DSp}_{2i}(q)$ geometry. Then M_i^q $(i=1,2,\ldots,n)$ can inductively be computed using $k_i = \left[{n \atop n-i}\right] M_i^q$ and $N_n^q = \sum_{i=0}^n k_i$ where $\left[{k \atop l}\right] = \prod_{i=1}^{l-1} \frac{q^{k-i}-1}{q^{l-i}-1}$ is the number of l-objects in a building of type $A_{(k-1)}(q)$.

For any two n-objects X and Y at distance i from each other, we can compute the number of n-objects in $R_1(X) \cap R_j(Y)$ $(j=0,1,\ldots,n)$ using the following observation.

Let X and Y be n-objects whose largest common object is an $(n-i)$-object U. In the folowing table we have listed the possible positions of an n-object at distance 1 from Y, indicating the largest object it has in common with X and U.

X	U	
$n-(i-1)$	$n-i$	$(i>1)$
$n-i$	$n-i$	
$n-(i+1)$	$n-(i+1)$	$(i<n)$

There are $\left[{i \atop i-1}\right]$ $(n-1)$-objects incident with Y that contain U and the remaining $\left[{n \atop n-1}\right] - \left[{i \atop i-1}\right]$ meet U in an $(n-(i+1))$-object. Each of the former $(n-1)$-objects is contained in precisely 1 n-object that meets X in an $(n-(i-1))$-object and in q n-objects that meet X in an $(n-i)$-object (one of which is Y). Each of the latter $(n-1)$-objects is contained in precisely 1 n-object that meets X in an $(n-i)$-object, namely Y, and in q n-objects that meet X in an $(n-(i+1))$-object. Thus we find $r_{i,i-1} = \left[{i \atop i-1}\right]$, $r_{i,i} = (q-1)\left[{i \atop i-1}\right]$ and $r_{i,i+1} = q(\left[{n \atop n-1}\right] - \left[{i \atop i-1}\right])$. □

4.4 Association schemes associated to the subgeometry far from a point

Let M be a spherical diagram defined over a set I and let Δ be a finite building with diagram M obtained from a group G with a Tits system (B,N). Let Γ be the i-shadow space of the geometry of Δ for some $i \in I$. The ordered set of possible distances between points of Γ is denoted by Λ. Given $\lambda \in \Lambda$, R_λ is the the collection of pairs of points at distance λ and for any point p, $R_\lambda(p)$ is the set of points at distance λ from p. Suppose that μ is the largest distance.

In subsequent sections we will do the following for each of the pairs $(G, i) = (F_4(q), 1)$, $(\mathrm{O}_7(q), 1)$, $(\mathrm{Sp}_6(q), 1)$, $(\mathrm{O}_7(q), 3)$, $(\mathrm{Sp}_6(q), 3)$. Fix a point ∞ and consider the subgeometry of Γ far from ∞; this is the point-line geometry whose point set is $R_\mu(\infty)$ and in which two points are collinear if they are collinear in Γ. Suppose that P_∞ is the parabolic subgroup (of type $I - \{i\}$) of G stabilizing ∞. Then of course P_∞ stabilizes $R_\mu(\infty)$ and in fact it acts transitively on it because G acts strongly transitively on Δ. Hence the orbitals under the action of P_∞ on $R_\mu(\infty)$ form the classes of an association scheme.

For $(\mathrm{O}_7(q), 1)$ and $(\mathrm{Sp}_6(q), 1)$ we will compute the parameters of this group scheme. The result is presented in Theorem 4.10.

For $(F_4(q), 1)$ the number of classes of this group scheme turns out to depend on q and we consider a scheme, some of whose classes are the union of several classes of the group scheme, for which this is not the case. By passing from the former to the latter scheme we donot seem to lose much information about the geometry. The result is presented in Theorem 4.15.

4.4.1 Long root geometry

Let M be a spherical Dynkin diagram defined over an index set I. Let G be a Chevalley group with diagram M. Let Φ be a root system for G and let $\Pi = \{\alpha_i\}_{i \in I}$ be a fundamental system for Φ. For every root α (not just those in Φ) let X_α be the corresponding root group.

Let (B, N) be the Tits system for G described on page 28.

Consider the positive root α_* of maximal height with respect to Π; this is the unique positive root with the property that for all roots α in Π, and indeed in Φ^+, $\alpha + \alpha_*$ is not a root.

Let $\mathcal{X}$ be the conjugation class of X_{α_*}. Consider the action of G by conjugation on $\mathcal{X}$ and let P be the stabilizer of X_{α_*} under this action.

Then it follows from the Chevalley commutator relations that P contains all root groups belonging to positive roots. It also contains H because this subgroup normalizes every root subgroup X_α with $\alpha \in \Phi$. Hence B stabilizes X_{α_*} and we see that P is a parabolic subgroup.

In order to determine the type of P we first consider the Bruhat decomposition of P. If P contains $X_{-\alpha_i}$ then it contains $BW_{\{i\}}B$ and, more generally, if P contains $X_{-\alpha_i}$ for all $i \in J \subset I$, then it contains BW_JB. It follows at once from the Chevalley commutator relations that P contains X_{α_i} if α_i and $(\pm)\alpha_*$ have innerproduct 0. To see for which α_i this holds we consider the *extended Dynkin diagram*: this is the diagram obtained in the same way as the Dynkin diagram itself, but starting out from the union of Π and $\{-\alpha_*\}$. In figure 4.8 we have drawn the extended Dynkin diagrams $\tilde{C}_n$ and $\tilde{F}_4$. We see that in both cases P contains the standard parabolic subgroup $P_{I-\{1\}}$. Also we see that the parabolic subgroup P doesn't contain X_{α_1} and hence must be equal to $P_{I-\{1\}}$.

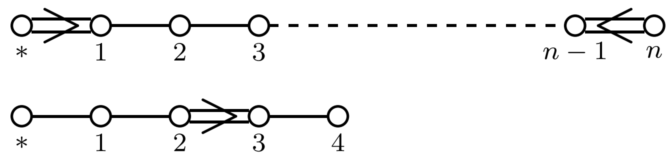


Figure 4.8: The extended Dynkin diagrams of type C_n and F_4.

Let Γ be the 1-shadow space of Δ. The set of conjugates of $P_{I-\{1\}}$ can be viewed as the point set of this geometry (see the end of Subsection 1.2.3). Thus we see that there is a 1-1 correspondence between the conjugates of X_{α_*} and the points of Γ and that the action of G on both sets by conjugation respects this correspondence.

Our next aim is to indicate how the distance relations R_λ ($\lambda \in \Lambda$) between pairs of points in Γ translate to relations between the root subgroups.

Let x and y be two points of Γ. First of all, since G acts transitively on the collection of pairs of points at any given distance, we may assume that x and y both correspond to a root in Φ.

Angle between roots	Root-group relation	Distance in $F_{4,1}$	Distance in $\mathrm{Sp}_{2n}(q)$
0	0	0	0
$\pi/3$	1	1	-
$\pi/2$	2	2s	1
$2\pi/3$	3	2ns	-
π	4	3	2

Let X and Y be the root groups corresponding to x and y. We have one of the following

1 $[X,Y] = \{1\}$ $\langle X,Y\rangle$ contains $q+1$ elements of $\mathcal{X}$.

2 $[X,Y] = \{1\}$ the only elements of $\mathcal{X}$ in $\langle X,Y\rangle$ are X and Y.

3 $|\langle X,Y\rangle| = q^3$ $[X,Y] = \langle X,Y\rangle' = Z(\langle X,Y\rangle) \in \mathcal{X}$ and $[X,Y]$ is in relation 1 with both X and Y.

4 $\langle X,Y\rangle \cong \mathrm{SL}_2(q)$

Note that in case (4) the points X and Y are opposite. The $q+1$ elements of $\mathcal{X}$ in $\langle X,Y\rangle$ form what we will call the *hyperbolic line* spanned by X and Y.

These properties are invariant under the action of G. A great advantage of this characterization of distances is that if we want to study collections of a (not too large) number of points, then we can forget about the group G and look at the subgroup these points generate between them. Often this subgroup is contained in a small classical group which is easier to handle.

4.5 The geometry far from a point in $\mathbf{Sp}_{2n}(q)$

Let M be the Dynkin diagram of type C_n and let Δ be the building with diagram C_n obtained from the Chevalley group $G = \mathrm{Sp}_{2n}(q)$. Let Γ be the 1-shadow space of the geometry of this building. Recall that the set R_λ ($\lambda \in \Lambda = \{0, 1, 2\}$) comprises the pairs of points of Γ at distance λ.

Given a hyperbolic line θ and a point $t \in \theta$ we define the following sets of points (See Proposition 4.11).

$$\begin{array}{ll} X_{0,t} & = R_0(t) \\ X_{1,t} & = R_1(t) \setminus \cup_{t' \in \theta \setminus \{t\}} R_1(t') \\ X_{1'} & = \cap_{t \in \theta} R_1(t) \end{array}$$

Now fix a point ∞. Recall that $R_2(\infty)$ is the point set of $\mathrm{Far}(\infty)$, the subgeometry of Γ far from ∞ ($\mu = 2$, in the terminology of Subsection 4.3). For any point o in $R_2(\infty)$ (or $\mathrm{Far}(\infty)$) there is a hyperbolic line θ on ∞ and o. We define relations C_i, $\bar{C}_i$ ($i = 0, 1$) between points of $R_2(\infty)$ as follows.

$$\begin{array}{lll} C_i(o) = & X_{i,o} & (i = 0, 1) \\ \bar{C}_i(o) = & \bigcup_{t \in \theta - \{o, \infty\}} X_{i,t} & (i = 0, 1) \end{array}$$

Theorem 4.10 *Let Γ be an $Sp_{2n}(q)$ geometry with q even. Then there exists a 4-class group scheme on the q^{2n-1} points far from any given point ∞ whose classes are C_i and $\bar{C}_i$ ($i = 0, 1$) and whose parameters are as depicted in Figure 4.9.*

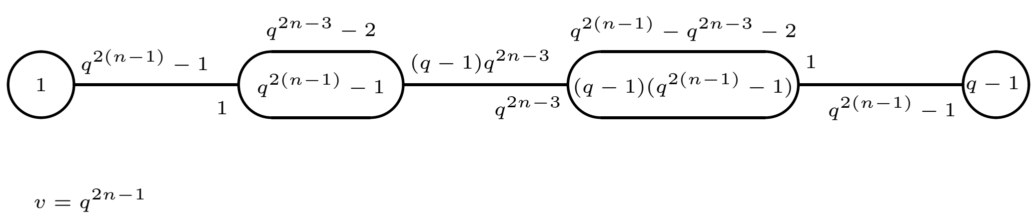


Figure 4.9: An association scheme on the points far from a given point in the $\mathrm{Sp}_{2n}(q)$ geometry. From left to right: C_0, C_1, $\bar{C}_1$, $\bar{C}_0$.

The proof of the theorem will be given at the end of this section and will be prepared during the next few subsections.

4.5.1 Definition of the scheme

Consider the natural polar geometry for $\mathrm{Sp}_{2n}(q)$. Given two non-collinear points x and y we can define a *hyperbolic line* xy on them in three ways:

(i) by its natural embedding into $\mathrm{PG}(2n-1,q)$: it is the set of $q+1$ points on the projective line $x+y$.

(ii) geometrically: let $A^{\perp}$ be the collection of points collinear to every element of the point set A. Then $xy = \{x,y\}^{\perp\perp}$.

(iii) group-theoretically: Given a point x we have a group T_x of transvections $t_x(\lambda) : y \mapsto y + \lambda(y,x)x$ ($\lambda \in \mathbb{F}_q$). Two groups T_x, T_y generate a group $L = \langle T_x, T_y\rangle$ that is isomorphic to $\mathrm{SL}_2(q)$ if and only if x and y are non-collinear; in that case L contains $q+1$ groups T_z (one for each point z on $x+y$) and these points z form xy.

We note that the groups of transvections mentioned in (iii) are precisely the long-root subgroups of the Chevalley group $\mathrm{Sp}_{2n}(q)$. The definition of a hyperbolic line given here is the same as given on page 108.

Fix a hyperbolic line θ. The stabilizer in G of θ has three orbits on the points. We call them X_0, X_1 and $X_{1'}$. In the next table we give the size of each orbit X and for a point $x \in X$ we give the number of points in θ at any given distance from it.

X	$\lvert R_0(x)\cap\theta\rvert$	$\lvert R_1(x)\cap\theta\rvert$	$\lvert R_2(x)\cap\theta\rvert$	$\lvert X\rvert$
X_0	1	0	q	$q+1$
X_1	0	1	q	$(q+1)(q^{2n-2}-1)$
$X_{1'}$	0	$q+1$	0	$(q^{2n-2}-1)/(q-1)$

Using that $y \in R_1(x)$, $y \in R_2(x)$ means (i) (in $\mathrm{PG}(2n-1,q)$, with $(\cdot,\cdot)$ being the symplectic form) $(x,y)=0$ resp. $(x,y)\neq 0$, (ii) (geometrically) $y \in x^{\perp}$ resp. $y \notin x^{\perp}$ and (iii) (group-theoretically) $[T_y,T_x]=\{1\}$ resp. $[T_x,T_y] > \{1\}$, we can interpret this table accordingly.

For $i=0,1$ and $t\in\theta$ let $X_{i,t}$ be the set of points in X_i whose unique closest point on θ is t; a point in $X_{i,t}$ is said to have *position* $[i,t]$ with respect to θ. Then $X_{0,t}=\{t\}$ so that $|X_{0,t}| = |X_0|/|\theta| = 1$ and also, by transitivity of G_θ on θ, we have $|X_{1,t}| = |X_1|/|\theta| = q^{2n-2}-1$.

Proposition 4.11 *Let Γ be an $Sp_{2n}(q)$ geometry and let ∞ be any given point. There is a group scheme on the set of points far from ∞ which has four classes if q is even and six otherwise. For even q the classes containing a point o far from ∞ are:*

$C_0(o) = X_{0,o}$ *(*$=\{o\}$*),*

$C_1(o) = X_{1,o}$*,*

$\bar{C}_1(o) = \cup_{t\in\theta\setminus\{\infty,o\}} X_{1,t}$*,*

$\bar{C}_0(o) = \cup_{t\in\theta\setminus\{\infty,o\}} X_{0,t}$ *(*$=\theta\setminus\{\infty,o\}$*).*

If q is odd, then the class $\bar{C}_i(o)$ $(i = 0, 1)$ splits into two sets $\bar{C}_{i,\mathrm{sq}}$ and $\bar{C}_{i,\mathrm{nsq}}$ of the form $\cup_t X_{i,t}$, where t runs over all squares and non-squares in $\theta \setminus \{\infty, o\}$ respectively.

Proof: We clearly have $R_2(\infty) = \cup_{t\in\theta\setminus\{\infty\}}(X_{0,t} \cup X_{1,t}) = \cup_{i=0,1}(C_i(o) \cup \bar{C}_i(o))$. Let G be the group $\mathrm{Sp}_{2n}(q)$. We have a G_∞-scheme because G_∞ is transitive on $R_2(\infty)$ and the relations given above are (the) orbits of the point stabilizer $G_{\infty,o}$ of o in G_∞, as we will see now.

The group G_θ acts as $\mathrm{SL}_2(q)$ on the points of θ. This action is sharply 3-transitive if q is even and so in that case $G_{\infty,o} \le G_\theta$ has two orbits on $X_0 \cap \theta = \theta \setminus \{\infty\}$, namely $C_0(o)$ and $\bar{C}_0(o)$. If q is odd the group $G_{\infty,o}$ has two orbits on $\theta \setminus \{\infty, o\}$; when viewing $\theta \setminus \{\infty\}$ as an affine line these are the collections of squares and non-squares.

For non-collinear points x and y let C_{xy} be the subgroup $\langle T_u \mid (u, x) = (u, y) = 0\rangle$ i.e. the subgroup generated by the root groups that centralize T_x and T_y. This group, which is naturally isomorphic to $\mathrm{Sp}_{2n-2}(q)$, fixes every point of xy and acts transitively on the points of $\{x, y\}^\perp$. Thus it acts transitively on the collection of lines containing any given point $t \in xy$.

For such a point $t \in xy$ and a line l on t, let $u, v \in \{x, y\}^\perp$ be such that $u \in l$ and $(u, v) = 1$. It can be calculated that the group $\langle T_u, T_v\rangle \le C_\theta$ acts transitively on the points of $l \setminus \{t, u\}$.

Taking $xy = \theta$ and $L = \langle T_x, T_y\rangle$ we find that $C_\theta \le G_{\infty,o}$ is transitive on $X_{1,t}$ for any $t \in \theta$, so in particular on $C_1(o)$, and that $L_{x,y} \cdot C_\theta \le G_{\infty,o}$ is transitive on $\bar{C}_1(o)$ if q is even and has the two indicated orbits if q is odd. □

Proposition 4.12 *The parameters of the group scheme in Proposition 4.11 are as displayed in Figures 4.9 and 4.10 in case q is even or odd, respectively.*

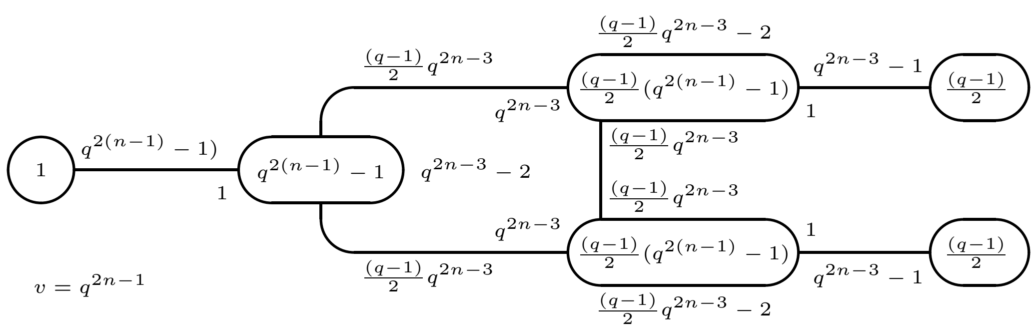


Figure 4.10: A group scheme on the points far from a given point in the $\mathrm{Sp}_{2n}(q)$ geometry for q odd. From left to right: C_0, C_1, and $\bar{C}_{1,\mathrm{sq}}$, $\bar{C}_{0,\mathrm{sq}}$ (top), and $\bar{C}_{0,\mathrm{nsq}}$, $\bar{C}_{0,\mathrm{nsq}}$ (bottom).

The proof is easily constructed from the results of the next subsection.

De association schemes referred to in Proposition 4.11 can be coordinatized as follows. Consider the $2n$-dimensional embedding V of the $\mathrm{Sp}_{2n}(q)$ geometry. Take a basis $\{e_i\}_{i=1}^{2n}$ for V such that the symplectic form $(\cdot, \cdot)$ is defined by $(e_j, e_k) = 1$ if and only if $j = 2i - 1$ and $k = 2i$ for some $i = 1, 2, \ldots, n$. Choose $\infty = (1 : 0 : \cdots : 0)$ and $o = (0 : 1 : 0 : \cdots : 0)$.

Then,

$$\begin{array}{ll} C_0(o) & = \{(0:1:0:\cdots:0)\} \\ C_1(o) & = \{(0:1:u_1:\cdots:u_{2n-2}) \mid u_i \in \mathbb{F}_q^*\} \\ \widehat{C}_1(o) & = \{(a:1:u_1:\cdots:u_{2n-2}) \mid a, u_i \in \mathbb{F}_q^*\} \\ \widehat{C}_0(o) & = \{(a:1:0:\cdots:0) \mid a \in \mathbb{F}_q^*\}. \end{array}$$

In case q is odd, both $\widehat{C}_1(o)$ and $\widehat{C}_0(o)$ split into two classes; such a class consists only of those elements in which a is square (or non-square).

This presentation can be used to prove Theorem 4.10 and Propositions 4.11 and 4.12. However we will take a different approach in order to deal with all geometries involved in a uniform way. One of the goals of this chapter is to examine all lines containing at least two points far from a point inside the metasymplectic geometry $F_{4,1}(q)$, and from this point of view this section may be regarded as a preparation, or indeed as part of that programme.

4.5.2 Calculation of the parameters

Keeping the notation introduced in the previous subsection, we will for $i, i' \in \{0,1\}$ and $t, t' \in \theta \setminus \{\infty\}$ determine how many neighbours a point of $X_{i,t}$ has in $X_{i',t'}$. We do this in the following way.

For every line L of the geometry that contains a point x in $R_2(\infty)$ we determine which distance relations occur between the points of L and the points of θ, and how many times they occur. These data comprise what we will call the *distribution* of L with respect to θ. It is the distribution of the points of L over the sets $X_{i,t}$ for all $i = 0, 1$ and $t \in \theta$. Also for every point x in $R_2(\infty)$ we determine the possible distributions of the lines on x, and for every such distribution, how many lines on x have that distribution.

To put it differently, we study configurations of three and four points in which one point has prescribed distance 2 to the other points and in which two of those other points are at distance 1 from each other. In this way we gather a little more information than strictly needed to compute the parameters of the scheme.

We label the distributions of lines with respect to the hyperbolic line θ with a roman numeral, preceded by the letters '**Sp**' and followed by a sequence of zero or more variables representing points of θ that have a special position with respect to the points of L. Each column of the table contains the position with respect to θ of a point on L and the number of points on L having that position. So for instance there are lines that intersect θ; the distribution of such lines is given the label $[\mathbf{Sp} : \mathbf{i}, t]$, where t represents the intersection point on θ.

If $n = 1$, then there are no points outside the hyperbolic line θ so we will assume that $n \geq 2$.

Let $t \in \theta$ (in position $X_{0,t}$). Let L be a line on t. Then L has a unique point of $X_{1'}$ and the remaining $q-1$ points are clearly in $X_{1,t}$. Thus we find a line with the following distribution.

[$\mathbf{Sp}:\mathbf{i},t$]

0	$[1,t]$	$1'$
1	$q-1$	1

All $(q^{2(n-1)}-1)/(q-1)$ lines on t have distribution [$\mathbf{Sp}:\mathbf{i},t$].

Let $x \in X_{1,t}$. Then clearly there is a unique line with distribution [$\mathbf{Sp}:\mathbf{i},t$] on x.

Let L be a line, not with distribution [$\mathbf{Sp}:\mathbf{i},t$], on x that contains another point $y \in X_{1,t}$. Then there is a unique plane V on t and L. The set $M = d_1(t') \cap V$, for $t' \in \theta \setminus \{t\}$, is a line whose points are all in $X_{1'}$. The points of $V \setminus (\{t\} \cup M)$ are all in $X_{1,t}$. Hence L has the following distribution.

[$\mathbf{Sp}:\mathbf{ii},t$]

$[1,t]$	$1'$
q	1

All q lines of V on x different from the line xt have this distribution. Each such line lies in a unique plane on the line xt. There are $(q^{2(n-2)}-1)/(q-1)$ planes on the line xt. Hence on a point $x \in X_{1,t}$, in total, there are $q(q^{2(n-2)}-1)/(q-1)$ lines with distribution [$\mathbf{Sp}:\mathbf{ii},t$]. We note that if L is a line on x that contains a point $y \in X_{1'}$, then L has distribution [$\mathbf{Sp}:\mathbf{i},t$] or [$\mathbf{Sp}:\mathbf{ii},t$] because L lies on a plane that contains t.

Let L be a line on x, not with distribution [$\mathbf{Sp}:\mathbf{i},t$] or [$\mathbf{Sp}:\mathbf{ii},t$]. Since L has a point in $d_2(t')$ for every $t' \in \theta \setminus \{t\}$, the hyperplane $\{t'\} \cup d_1(t')$ meets L in a single point. The points $d_1(t') \cap L$ ($t' \in \theta$) are all distinct, because L has no point in $X_{1'}$. As θ and L both have $q+1$ points, L has the following distribution.

[$\mathbf{Sp}:\mathbf{iii}$]

1
$q+1$

We have considered all lines on $x \in X_{1,t}$. Hence on a point $x \in X_{1,t}$, in total, there are $q^{2(n-1)}-1)/(q-1)-(1+q(q^{2(n-2)}-1)/(q-1)) = q^{2n-3}$ lines with distribution [$\mathbf{Sp}:\mathbf{iii}$].

Let $x \in X_{1'}$. Then on x, for each $t \in \theta$, there is a unique line xt with distribution [$\mathbf{Sp}:\mathbf{i},t$].

Suppose that V is a plane on xt for some $t \in \theta$. Let $M = d_1(t') \cap V$ for some $t' \in \theta \setminus \{t\}$. We encountered such a line above. It has the following distribution.

[$\mathbf{Sp}:\mathbf{iv}$]

$1'$
$q+1$

There is just one such line on V. The remaining $q-1$ lines of V on x contain a point of $X_{1,t}$ but none in X_0, and hence have distribution [$\mathbf{Sp}:\mathbf{ii},t$]. There are $q^{2(n-2)}-1)/(q-1)$ planes on xt. Hence on a point of $X_{1'}$, in total, we find $q^{2(n-2)}-1$ lines with distribution [$\mathbf{Sp}:\mathbf{ii},t$] and $(q^{2(n-1)}-1)/(q-1)$ lines with distribution [$\mathbf{Sp}:\mathbf{iv}$].

For $i, i' \in \{0,1\}$ and $t, t' \in \theta \setminus \{\infty\}$ we will determine how many neighbours a point of $X_{i,t}$ has in $X_{i',t'}$. Let $t \in \theta$. Let $x \in X_{0,t}$, that is $x = t$. Then all $(q^{(2(n-1)} - 1)/(q-1)$ lines on x have distribution $[\mathbf{Sp} : \mathbf{i}, t]$. Each of these lines contains $q-1$ points of $X_{1,t}$. So x has $q^{2(n-1)} - 1$ neighbours in $X_{1,t}$.

Let $x \in X_{1,t}$ for some $t \in \theta$. Then there is one line on x with distribution $[\mathbf{Sp} : \mathbf{i}, t]$ and it has one point in $X_{0,t}$ and $q-2$ other points in $X_{1,t}$. Moreover there are $q(q^{2(n-2)} - 1)/(q-1)$ lines on x with distribution $[\mathbf{Sp} : \mathbf{ii}, t]$ and each of these lines has $q-1$ other points in $X_{1,t}$. Finally there are q^{2n-3} lines on x with distribution $[\mathbf{Sp} : \mathbf{iii}]$ and each of these lines has one point in $X_{1,t'}$, for each $t' \in \theta \setminus \{\infty\}$.

In case q is even this easily produces the desired parameters. In case q is odd we just make a remark on the neighbours of a point $x \in \bar{C}_{1,\mathrm{sq}}(o)$. A line on x only has points in $\bar{C}_{1,\mathrm{nsq}}$ if it has distribution $[\mathbf{Sp} : \mathbf{iii}]$; in that case it has $(q-1)/2$ other points in $\bar{C}_{1,\mathrm{sq}}$ and $(q+1)/2$ points in $\bar{C}_{1,\mathrm{nsq}}$.

4.6 The geometry far from a point in $\mathrm{DO}_7(q)$

Let M be the Dynkin diagram of type B_n and let Δ be the building with diagram B_n obtained from the Chevalley group $G = \mathrm{O}_7(q)$. Let Γ be the 3-shadow space of the geometry of this building; this we called the $\mathrm{DO}_7(q)$ geometry. Recall that the set C_λ ($\lambda \in \Lambda = \{0,1,2,3\}$) comprises the pairs of points of Γ at distance λ.

Theorem 4.13 *Let Γ be a $\mathrm{DO}_7(q)$ geometry. Then there exists a 5-class association scheme on the q^6 points far from any given point ∞ whose classes are C_i ($i = 0, 1, 2\mathrm{g}, 2\mathrm{h}, 3$) and whose parameters are as depicted in Figure 4.11.*

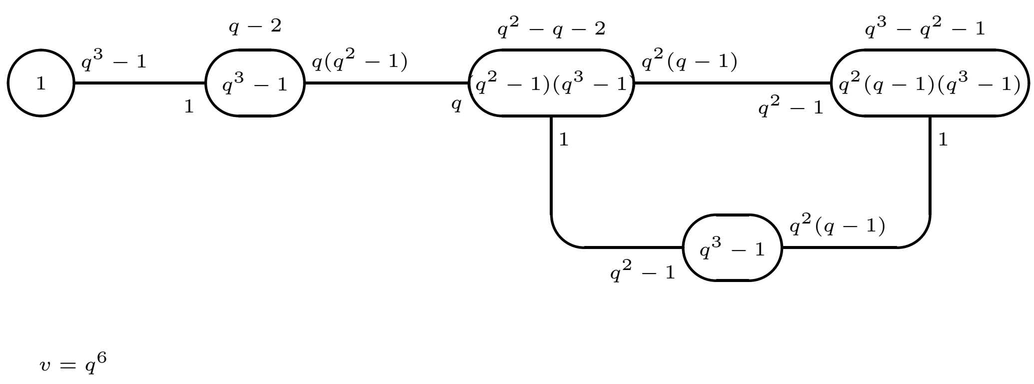


Figure 4.11: The distance distribution diagram of $\mathrm{Far}(\infty)$ in the $\mathrm{DO}_7(q)$ geometry. From left ro right: C_0, C_1, $C_{2\mathrm{g}}$, $C_{2\mathrm{h}}$, C_3.

The proof of the theorem will be given at the end of this section and will be prepared during the next few subsections.

4.6.1 Configurations of a point and a pair of far points

In this geometry two points that are far away from each other donot span a 'hyperbolic line' as was the case in the $\mathrm{Sp}_{2n}(q)$ geometry, but they do determine a unique subgeometry, that we will call a 'hyperbolic hyperplane'. These hyperbolic hyperplanes will play a role much the same as the hyperbolic lines did in the previous section.

A subspace $\bar{\mathsf{U}} \subset \bar{\mathsf{V}}$ of dimension $2n$ such that the quadratic form restricted to $\bar{\mathsf{U}}$ is non-degenerate, and that contains t.s. n-spaces is a hyperbolic quadric; here we will often refer to it as a hyperbolic hyperplane. The geometry of t.s. i-spaces ($i = 1, 2, \ldots, n$) on $\bar{\mathsf{U}}$ with the induced incidence relation form a geometry of type O^+_{2n}.

If X and Y are disjoint t.s. n-spaces, then they determine a unique subgeometry of type O^+_{2n}; it is the subspace spanned by X and Y. The most important property of this geometry is that there are two classes of t.s. n-spaces; two n-spaces belong to the same *class* if and only if the codimension of their intersection in either n-space is even.

We set $n = 3$. Fix two disjoint planes O and ∞. Let X be a plane. We say that X has position $[i, j]$, $(i, j = 0, 1, 2, 3)$ with respect to O and ∞ if $O \in d_i(X)$ and $\infty \in d_j(X)$. Since O and ∞ are disjoint and X is a 3-space, we have $i + j \geq 3$.

Let Θ be the hyperbolic hyperplane on O and ∞. Then either $X \subset \Theta$ or $X \cap \Theta$ is a line l_X. If X has type $[i, j]$ with $i + j = 3$, then clearly $X \subset \Theta$. Note that since $n = 3$, any two planes of Θ in the same class are equal or intersect in a point; in particular O and ∞ are in different classes. Hence if X has type $[i, j]$ with $i + j = 4, 6$, then $X \not\subset \Theta$. A plane X with position $[i, j]$ with $i + j = 5$, that satisfies $X \subset \Theta$ (resp. $X \not\subset \Theta$) will be said to have type $[i, j]_h$ (resp.$[i, j]_g$).

We denote the set of points in position *pos* by X_{pos}. The sizes of these sets will be determined in the next subsection.

4.6.2 Configurations of a line and a pair of far points

Let l be a line on O. We know that the geometry of planes on l has type $\mathrm{O}_3(q)$, and has $q+1$ planes. One of these is O (in position $[0, 3]$). Since $l \not\subset \infty$ we have $\mathrm{codim}(\infty, l^\perp \cap \infty) = 2$, so there is a unique point on ∞ that is contained in a plane ∞_l on l. This plane has type $[1, 2]$ and hence the remaining $q - 1$ planes on l have type $[1, 3]$. Such lines are said to have distribution $[\mathbf{DO} : \mathbf{i}, O]$.

There are $(q^3-1)/(q-1)$ lines on O. Thus, in total there are $(q-1)(q^3-1)/(q-1) = q^3-1$ planes in position $[1, 3]$ and $(q^3 - 1)/(q - 1)$ planes in position $[1, 2]$.

Let p be a point on O. We know that the geometry of lines and planes on p has type $\mathrm{O}_5(q)$, and has $(q^4 - 1)/(q - 1)$ planes (and lines).

Since $p \not\subset \infty$ we have $\mathrm{codim}(\infty, p^\perp \cap \infty) = 2$, so that there is a unique plane ∞_p on p that meets ∞ in a line. The plane ∞_p has type $[2,1]$.

We use the isomorphism $\mathrm{DO}_5(q) \cong \mathrm{Sp}_4(q)$ to apply the results of Section 4.5 to the present situation. Let the non-collinear planes O and ∞_p correspond to the non-collinear points o and ∞ of the $\mathrm{Sp}_4(q)$ geometry, respectively. Let h be the set of planes on p that corresponds to θ. Then the sets of planes corresponding to $X_1(o)$, $X_{1'}$ and $X_1(\infty)$ comprise the planes on p in position $[1,3]$, $[1,2]$ and $[2,2]$ respectively. A plane on p in position $[2,3]_g$ or $[2,3]_h$ corresponds to a point in $\bar{C}_1(o)$ or $\bar{C}_0(o)$, respectively.

We define the lines correponding to the lines with distribution $[\mathbf{Sp}:\mathbf{ii}, o]$, and $[\mathbf{Sp}:\mathbf{iii}]$ with respect to θ and $o \in \theta$ to be lines with distribution $[\mathbf{DO}:\mathbf{ii}, O]$ and $[\mathbf{DO}:\mathbf{iii}]$.

A plane in position $[2,j]$ $(j = 1,2,3)$ lies on a unique point of O. There are $(q^3-1)/(q-1)$ points on O and on each line of O there are $(q+1)$ points. Using this and the results of Subsection 4.5.2 we can compute the entries of the following table, whose AB-entry contains, for each plane of the set A, the number of planes in B it is collinear to.

	$X_{0,3}(O)$	$X_{1,3}(O)$	$X_{2,3g}(O)$	$X_{2,3h}(O)$	$X_{3,3}$
$X_{0,3}(O)$	0	q^3-1	0	0	0
$X_{1,3}(O)$	1	$q-2$	$(q^2-1)q$	0	0
$X_{2,3g}(O)$	0	q	q^2-q-2	1	$q^2(q-1)$
$X_{2,3h}(O)$	0	0	q^2-1	0	$q^2(q-1)$

The last column is explained as follows. Consider a plane X that meets O in a point x and is disjoint from ∞. Suppose l is a line on X. If l lies on x, then no plane on l is in $d_3(O)$. On the other hand, if l is not on x, then all planes different from X are in $d_3(O)$ and precisely one of them meets ∞ and the remaining $q-1$ are in position $[3,3]$. There are $(q^3-1)/(q-1) - (q^2-1)/(q-1) = q^2$ such lines on X, each contained in $q-1$ planes in position $[3,3]$. Thus we find lines in the following distributions.

$[\mathbf{DO}:\mathbf{iv}, O]$

$[2,3]_g$	$[3,3]$	$[3,2]_g$
1	$q-1$	1

$[\mathbf{DO}:\mathbf{v}, O]$

$[2,3]_h$	$[3,3]$	$[3,2]_h$
1	$q-1$	1

The positions $[3,2]_g$ and $[3,2]_h$ are defined by interchanging the role of O and ∞ in the definition of $[2,3]_g$ and $[2,3]_h$. The occurrence of such points in these lines is explained below.

We have seen that $|X_{1,3}| = q^3-1$. Further $|X_{2,3g}| = |\bar{C}_1(o)|(q^3-1)/(q-1) = (q^2-1)(q^3-1)$ and $|X_{2,3h}| = |\bar{C}_0(o)|(q^3-1)/(q-1) = (q^3-1)$.

Let X be a plane in position $[3,3]$. As for ∞, there is a unique plane X_p on p that meets X in a line. Since X is disjoint from O and ∞, X_p meets O only in the point p and it meets ∞ in at most one point; in other words, X_p has type $[2,2]$, $[2,3]_g$ or $[2,3]_h$. In the first case there is a line on p that meets both X and ∞. Accordingly, we say that p has type (a), (b) or (c) (with respect to X).

Proposition 4.14 *Let O, ∞ and X be pairwise disjoint planes in a geometry of type $\mathrm{O}_7(q)$. Then there is a unique line on O, each point of which lies on a line that meets both ∞ and X in a point.*

Proof: Let Θ be the hyperbolic hyperplane containing O and ∞. We recall that, as $n = 3$, two planes in the same class of planes in the $\mathrm{O}_6^+(q)$-geometry are either equal or have a point in common. Hence O and ∞ are in different classes and X, being disjoint from both planes, cannot be contained in Θ. Since Θ is a hyperplane $\Theta \cap X$ is a line m. Let x be a point on m. Suppose that $Y_O, Y_\infty \subset \Theta$ are planes containing p that meet O and ∞ respectively in a line. Then Y_O and Y_∞ are in different classes and as $p \in Y_O \cap Y_\infty$, this intersection is a line, and this line meets both $Y_O \cap O$ and $Y_\infty \cap \infty$. Note immediately that this means that this is the only such line on x. All lines that meet each of the three planes lie in Θ, hence no point of $X \setminus m$ lies on such a line. Clearly, the argument is symmetric in O, ∞ and X. □

Let l and l' be lines that meet O, ∞ and X and let q, q' be the intersection points of these lines with O, respectively. Let V and V' be the planes on l and l' that meet O in lines m and m' and let $p \in O$ be the intersection point of m and m'.

Since the geometry of lines and planes on p is of type $\mathrm{O}_5(q)$, hence has no triangles, and the planes V and V' meet O in distinct lines, these planes only meet in p. Let ∞_p (resp. X_p) be the projection of ∞ (resp. X) onto p; this is the plane on p that meets ∞ (resp. X) in a line. Clearly then the planes V and V' meet the plane ∞_p (and X_p) in distinct lines. In fact p must have type (c) because there are two planes that meet O, ∞_p and X_p in a line.

Conversely, suppose that p has type (c). Then every line on p and O lies in a unique plane that meets both ∞ and X in a point. On each point in position (a) there is exactly one such line, hence there is at most one point contained in all such lines. Thus p is the unique point in position (c) on O. The remaining $q^2 - 1$ points of O apparently have type (b).

From the preceding paragraphs, we can see that on X there are three kinds of lines l. First, if l is one of the $q + 1$ lines on the unique point in position (c), then there is a unique plane on l that meets both O and ∞ in a point. The remaining $q - 1$ planes on l then are in position $[3, 3]$. Such lines are said to have distribution $[\mathbf{DO} : \mathbf{vi}]$.

Second, if l is the line that is contained in any hyperbolic hyperplane Ξ that contains two of the planes O, ∞ and X, then, apart from X, there are $q - 2$ planes in position $[3, 3]$ and there are two planes on l that meet O and ∞ respectively in a point. The plane on l that meets O (resp. ∞) in a point has position $[2, 3]_h$ (resp. $[3, 2]_h$). This follows from the fact that p is collinear to all lines that meet O, ∞ and X. These lines have distribution $[\mathbf{DO} : \mathbf{v}, O]$.

Third, if l is any of the $q^2 - 1$ other lines, then, apart from X, there are $q - 2$ planes that have position $[3, 3]$ and there are two planes that meet O and ∞ respectively in a point. The unique plane on l that meets O (resp. ∞) in a point must have position $[2, 3]_g$ (resp. $[3, 2]_g$). These are lines with distribution $[\mathbf{DO} : \mathbf{iv}, O]$.

Thus X is collinear to $(q+1)(q-1) + q^2(q-2) = q^3 - q^2 - 1$ planes in position $[3, 3]$, to 1 plane in position $[2, 3]_h$ and to $q^2 - 1$ planes in position $[2, 3]_g$.

4.7 Far from a point in the $F_{4,1}(q)$ geometry

We will consider a building Δ with diagram F_4 defined over the field $\mathbb{F}_q$. Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the $F_{4,1}(q)$ geometry that is the 1-shadow space of (the geometry of) Δ. We fix a point ∞ and consider the subgeometry of Γ whose points are the points of Γ at maximal distance (i.e. distance 3) from ∞ and in which two points are collinear if they are collinear in the original geometry Γ. Our aim is to describe this geometry in some detail. More precisely we will prove the following result.

Theorem 4.15 *Let Γ be an $F_{4,1}(q)$ geometry with q even. Then there exists a 13-class association scheme on the q^{15} points far from any given point ∞ whose parameters are as depicted in on 131.*

The classes are denoted by R_i and $\bar{R}_i$ ($i = 0, 1, 2\mathrm{g}, 2\mathrm{h}, 3, 3'$) and R_4 and are defined on page 120. Also the terms 'position' and 'distribution' will be explained below.

In his dissertation [36], Riebeek determined the parameters of this scheme in the special case where $q = 2$ by computer. One of the motivations behind the present chapter was to verify his result in a purely theoretic way.

Proof: We fix a point $o \in \mathrm{Far}(\infty)$ and let θ denote the hyperbolic line spanned by ∞ and o. Given a point $x \in \mathrm{Far}(\infty)$ in a given position with respect to θ and a line on x, we determine the possible distribution of the points of this line among the sets $R_i(o)$. Further we determine on how many lines on x a given distribution occurs. This will be done in the subsequent subsections. After that computing the parameters of the scheme is a matter of elaborate bookkeeping. □

Notation

For a point x, we denote the set of points at distance 0,1,2s,2ns,3 from x by $d_0(x)$, $d_1(x)$, $d_2(x)$, $d_3(x)$, $d_4(x)$. Whenever we think of points as long-root subgroups we can use the relations f_i ($i = 0, 1, 2, 3, 4$) from Cooperstein [24] as well, and then $d_i(x) = f_i(x)$. However, we will sometimes use the isomorphism $O_7(q) \cong Sp_6(q)$, and then the points will no longer be long-root subgroups so that we cannot use the notation from Cooperstein [24].

4.7.1 Configurations of a point and a hyperbolic line

We now fix a point ∞. Let o be a point opposite to ∞. Then o and ∞ determine a *hyperbolic line* θ (which depends on the choice of o). Let $N = N(\langle\theta\rangle)$ and $C = C(\langle\theta\rangle)$.

The relations of the scheme in Theorem 4.15 will be constructed from the orbits of points under the action of N. These orbits have been studied in detail by Cooperstein [24]. In the table below we have listed some information about these orbits

orbit	0	1	2	3	4	size of orbit
X_0	1	–	–	–	q	$q+1$
X_1	–	1	–	–	q	$(q-1)(q+1)^2(q^2+1)(q^3+1)$
$X_{1'}$	–	1	–	q	–	$(q+1)^2(q^2+1)^2(q^3+1)$
$X_{2\mathrm{h}}$	–	–	1	–	q	$(q+1)(q^6-1)$
$X_{2\mathrm{g}}$	–	–	1	–	q	$(q+1)(q^4-1)(q^6-1)$
$X_{2'}$	–	–	1	q	–	$(q+1)^2(q^2+1)^2(q^6-1)$
$X_{2''}$	–	–	$q+1$	–	–	$(q^6-1)/(q-1)$
X_3	–	–	–	1	q	$q^3(q+1)(q^4-1)(q^6-1)$
$X_{3'}$	–	–	–	2	$q-1$	$q^7(q+1)(q^4-1)(q^3+1)/2$
$X_{3''}$	–	–	–	$q+1$	–	$q^3(q^6-1)(q^2+1)(q+1)$
X_4	–	–	–	–	$q+1$	$q^7(q-1)(q^4-1)(q^3-1)/2$

where the i-entry $(i = 0, 1, 2, 3, 4)$ of a row is the number of points in $d_i(y) \cap \theta$ for any point y in the orbit corresponding to that row.

We say that a point of X_i $(i \in \{0, 1, 1', 2\mathrm{h}, 2\mathrm{g}, 2', 2'', 3, 3', 3'', 4\})$ is *in position* $[i]$ (with respect to θ). We often refine this by specifying (some of) the points on θ that are closest to it. We denote the set of points in position *pos* by X_{pos}.

Thus, for instance, a point x is in position $[1, t]$ if $d_1(x) \cap \theta = \{t\}$ and $\theta \setminus d_1(x) \subset d_4(x)$ and it is in position $[3', t_1, t_2]$ if $d_3(x) \cap \theta = \{t_1, t_2\}$ and $\theta \setminus d_3(x) \subset d_4(x)$. Further a point x is in position $[3', t]$ if it is in position $[3']$ and $t \in d_3(x) \cap \theta$. Clearly we have the following partitions:

$$X_i = \bigcup_{t\in\theta} X_{i,t} \qquad (i = 0, 1, 1', 2, 2', 3)$$
$$X_{3'} = \bigcup_{t_1,t_2} X_{3',t_1,t_2}$$

The sizes of these sets are as may be expected: $(q + 1)|X_i(t)| = |X_i|$ $(t \in \theta,\ i = 0, 1, 1', 2, 2', 3)$ and $\binom{q+1}{2}|X_{3'}(t_1, t_2)| = |X_{3'}|$ $(t_1, t_2 \in \theta)$. This is a consequence of the fact that the group $\langle\theta\rangle$ $(\cong SL_2(q))$ is 2-transitive on the $q+1$ points of the hyperbolic line θ and hence permutes the elements of the partition transitively in each N-orbit.

Let us focus on the sets $X_{2\mathrm{h},t}$ and $X_{2\mathrm{g},t}$ $(t \in \theta)$. Fix $t \in \theta$ and let $y \in d_2(t)$. Then there is a unique symplecton S containing t and y. Let $s = \mathrm{proj}_S(t')$ for some $t' \in \theta$. Then apparently s is in position $[2'']$, that is $\theta \subset d_2(s)$. Further, the points in $(d_1(s) \setminus d_1(t)) \cap S$ are in position $[2', t]$, the points in $d_1(s) \cap d_1(t)$ $(\subset S)$ are in position $[1', t]$ and the points in $(d_1(t) \setminus d_1(s)) \cap S$ are in position $[1, t]$.

Recall that S is a geometry of type $O_7(q)$ with q even and that hence S can also be viewed as a geometry of type $\mathrm{Sp}_6(q)$. The points s and t determine a symplectic hyperbolic line st.

For $y \in X_{2,t} \cap S$, we consider the set $(d_1(y) \cap d_1(t)) \cap d_1(s)$ of points in position $[1', t]$. This set is contained in S and either has $(q^4-1)/(q-1)$ or $(q^3-1)/(q-1)$ element(s), depending on whether y lies on the hyperbolic line st or not. We say that y is a point in position $[2\mathrm{h}, t]$ ('h' for 'hyperbolic') or $[2\mathrm{g}, t]$ ('g' for 'generic') in the respective cases. Accordingly, we define the sets $X_{2\mathrm{g},t}$, $X_{2\mathrm{h},t}$ for every $t \in \theta$.

The number of points in position $[2, t]$ in S is $|S| - |d_1(s) \cap d_1(t)| - 2|\{s\} \cup d_1(s) \setminus (d_1(s) \cap d_1(t))| = q^4(q-1)$. Of these there are $q-1$ points in position $[2\mathrm{h}, t]$ and $(q^4-1)(q-1)$ in position $[2\mathrm{g}, t]$. The number of symplecta S on t equals $(q^6-1)/(q-1)$. Hence $|X_{2\mathrm{g},t}| = (q^4-1)(q^6-1)$ and $|X_{2\mathrm{h},t}| = (q^6-1)$. Indeed, the sizes of $X_{2\mathrm{h}}$ and $X_{2\mathrm{g}}$ match those displayed in the table above.

The relations that will give the distance distribution diagram of Theorem 4.15 for $\mathcal{X}$ endowed with the collinearity relation are denoted by R_i, $\bar{R}_i$ ($i = 0$, 1, 2g, 2h, 3, 3′, 4) and are defined by

$$\begin{aligned}
R_i(o) =\ & X_{i,o} && (i = 0, 1, 2\mathrm{g}, 2\mathrm{h}, 3)\\
\bar{R}_i(o) =\ & \textstyle\bigcup_{t \in \theta - \{o,\infty\}} X_{i,t} && (i = 0, 1, 2\mathrm{g}, 2\mathrm{h}, 3)\\
R_{3'}(o) =\ & \textstyle\bigcup_{t \in \theta - \{o,\infty\}} X_{3',o,t} &&\\
\bar{R}_{3'}(o) =\ & \textstyle\bigcup_{t_1,t_2 \in \theta - \{o,\infty\}} X_{3',t_1,t_2} &&\\
R_4(o) =\ & X_4. &&
\end{aligned}$$

Note that in case $q = 2$, the set $\bar{R}_{3'}$ is empty.

Lemma 4.16 *The relations R_i and $\bar{R}_i$ ($i = 0, 1, 2\mathrm{g}, 2\mathrm{h}, 3, 3', 4$) are invariant under the action of G_∞.*

Proof: Let $o, o' \in \mathcal{X}$ and let θ, θ' be the hyperbolic lines on ∞ determined by these points respectively. Then any element from G_∞ that sends o to o', and such an element exists, automatically maps θ to θ'. The lemma now follows from the fact that the sets $\mathcal{X}$, $R_i(x)$ and $\bar{R}_i(x)$ ($i = 0, 1, 2\mathrm{g}, 2\mathrm{h}, 3, 3', 4$, $x = o, o'$) are defined in terms of distances that are preserved by G. □

4.7.2 Configurations of a (geometric) line and hyperbolic line

In order to prove Theorem 4.15, We will follow the same strategy as in Section 4.5. Fix a hyperbolic line θ. We study the relations between (the points of) a line L of the geometry and (the points of) the hyperbolic line θ. We will use a similar labeling system for lines as we had for points. More precisely, for each point on the line L we determine its position with respect to θ, and these positions determine what we will call the *distribution* of L with respect to θ. We label a distribution with a roman numeral A. We sometimes refine this labeling by specifying certain points $t_1, \ldots, t_k \in \theta$ ($k \in \mathbb{N}$) that are in a special position with respect to L. We then say that L has *distribution* $[A, t_1, \ldots, t_k]$ with respect to θ and the points $t_1, \ldots, t_k$. The set of lines with distribution *distr* is denoted by L_{distr}.

So for instance, a line has distribution $[\mathbf{i}, t]$ (see below) if it has a point $t \in \theta$. In the tables we display the number of points on L in each occurring position (with respect to θ).

We donot consider all lines; we are mainly interested in those lines that contain a point x such that $d_4(x) \cap \theta \neq \emptyset$, that is a point of $\mathrm{Far}(\infty)$.

4.7.3 Lines inside a symplecton meeting θ

First we consider lines that lie inside a symplecton on $t \in \theta$. If S is a symplecton and a is a point with $d_4(a) \cap S \neq \emptyset$, then the projection $a_S = \mathrm{proj}_S(a)$ is a (unique) point. Moreover, we have $d_2(a) \cap S = \{a_S\}$, $d_3(a) \cap S = d_1(a_S) \cap S$ and $d_4(a) \cap S = d_2(a_S) \cap S$.

For a symplecton S on t, the points $\mathrm{proj}_S(t')$ ($t' \in \theta$) all coincide, for instance because each of these points is contained in $d_2(t) \cap d_2(t')$ and hence has position $[2'']$. Call this point s.

The position of a point in S with respect to the hyperbolic line θ is completely determined by its position with respect to the symplectic hyperbolic line st. For instance, we have $H \setminus \{s, t\} = X_{2\mathrm{h},t} \cap S$.

With a little knowledge of the geometry of type $\mathrm{Sp}_6(q)$ (or $\mathrm{O}_7(q)$) it is easy to determine all possible distributions of lines in S and the number of lines with any given distribution on any given point of S (see also Section 4.5).

Distribution	$[0, t]$	$[1, t]$	$[1', t]$	$[2\mathrm{g}, t]$	$[2\mathrm{h}, t]$	$[2', t]$	$[2'']$
$[\mathbf{i}, t]$	1	$q-1$	1				
$[\mathbf{ii}, t]$			$q+1$				
$[\mathbf{iii}, t]$		q	1				
$[\mathbf{iv}, t]$			1	$q-1$	1		
$[\mathbf{v}, t]$			1	q			
$[\mathbf{vi}, t]$		1		$q-1$		1	
$[\mathbf{vii}, t]$			1			$q-1$	1
$[\mathbf{viii}, t]$			1			q	

All $(q^3+1)(q^2+1)(q+1)$ lines on a (the) point of $X_{0,t}$ have distribution $[\mathbf{i}, t]$. In the following table the LP-entry is the number of lines with distribution L on a point in position P.

Distribution	$[1, t]$	$[2\mathrm{g}, t]$	$[2\mathrm{h}, t]$
$[\mathbf{i}, t]$	1	0	0
$[\mathbf{iii}, t]$	$q(q^3-1)/(q-1)$	0	0
$[\mathbf{iv}, t]$	0	1	$(q^4-1)/(q-1)$
$[\mathbf{v}, t]$	0	$q(q+1)$	0
$[\mathbf{vi}, t]$	1	q^3	0

4.7.4 Lines with a point in X_1

Let $x \in d_1(t)$ for some $t \in \theta$. We consider lines L on x that are not contained in a symplecton on t.

Lemma 4.17 *Let L be a line on $x \in d_1(t)$ for some $t \in \theta$ that is not contained in a symplecton on t. Then $L \setminus \{x\}$ is contained in one of $X_{1'}$, $X_{2'}$, $X_{3''}$, $X_{3,t}$ or $X_{3',t}$ and the latter holds if and only if $x \in X_{1,t}$.*

Proof: The group $\langle t \rangle$ fixes t and x and acts sharply 1-transitively on the points of both $\theta \setminus \{t\}$ and $L \setminus \{x\}$ (see Cooperstein [24]). This proves the first part. If $x \in X_{1,t}$, then $x \in d_4(t')$ for every $t' \in \theta$. This means that we can only have $L \setminus \{x\} \subset X_{3,t} \cup X_{3',t}$. Let $y \in L \setminus \{x\}$ such that $\theta \subset d_3(y) \cap d_4(y)$. Clearly $\{t\} \subseteq d_3(y) \cap \theta$, but if we have equality, then the action of $\langle t \rangle$ shows that $L \subset d_4(t')$ for all $t' \in \theta \setminus \{t\}$. This contradiction implies $y \in X_{3',t}$.

Conversely, if $y \in X_{3',t}$, then there exists $t' \in \theta$ with $y \in d_4(t') \cap L$. By transitivity the unique point of $d_3(t') \cap L$ is in the $\langle t \rangle$ orbit of y. Hence $x \in d_4(t')$ and we are done. □

We find the following (possible) distributions for lines:

[**ix**]

$[1', t]$
$q+1$

[**x**, t]

$[1', t]$	$2'$
1	q

[**xi**, t]

$[1', t]$	$3''$
1	q

The line with distribution [**ix**] (resp. [**x**, t]) has one point in $X_{1',t'}$ (resp. $X_{2',t'}$) for each $t' \in \theta \setminus \{t\}$.

Moreover we find lines with the following distributions:

[**xii**, t]

$[1', t]$	$[3, t]$
1	q

[**xiii**, t]

$[1, t]$	$[3', t]$
1	q

The line with distribution [**xii**, t] (resp. [**xiii**, t]) has one point in $X_{3,t'}$ (resp. $X_{3,t,t'}$) for each $t' \in \theta \setminus \{t\}$.

A point $y \in d_3(t)$ is collinear to precisely one point of $d_1(t)$. Hence if $y \in X_{3,t}$, then it lies on precisely one line with distribution [**xii**, t] and if $y \in X_{3',t,t'}$, then it lies on precisely one line with distribution [**xiii**, t] and one line with distribution [**xiii**, t']. On a point of $X_{1,t}$, there are q^6 lines that are not contained in a symplecton on t and all these have distribution [**xiii**, t].

The remaining part of this subsubsection will be needed in Subsection 4.7.7. Let E be a line with distribution [**xi**] and let V be a plane on E. Then, by looking at an apartment, we see that, for every $t \in \theta$, there is a unique point $p_t = d_1(t) \cap V$, the set $(d_1(t) \cup d_2(t)) \cap V$ is a line, L_t say, and $V \setminus L_t \subset d_3(t)$. For every $t \in \theta$ the point p_t has position $[1', t]$ and together they fill E. The lines L_t meet in $V \setminus E$ and intersect in a point u that has position $[2'']$. Thus, on a point p_t and in V there is one line that has distribution [**xi**] (namely E), there is one line that has distribution [**vii**, t] (namely L_t) and the remaining lines have distribution [**x**, t].

4.7.5 Lines with a point in X_2

Let $x \in d_2(t)$ for some $t \in \theta$. Let S be the symplecton on t and x and let L be a line on x not contained in S. Let V be the unique plane on L that meets S in a line and call this line M. Let $z \in M$ be the unique point of $d_1(t) \cap V$. In view of Lemma 4.17 we distinguish the cases $z \in X_{1',t}$ and $z \in X_{1,t}$. First suppose that $z \in X_{1',t}$. We restrict to the case where L has at least one point in $d_4(t')$ for some $t' \in \theta$ so that M has distribution $[\mathbf{iv}, t]$, $[\mathbf{v}, t]$ or $[\mathbf{viii}, t]$ (and not $[\mathbf{vii}, t]$ which wouldn't interest us). Now the unique point of $d_3(t') \cap L$ equals x if and only if M has distribution $[\mathbf{viii}, t]$. Since $d_4(t') \cap L \neq \emptyset$ we have at least one line with distribution $[\mathbf{xii}, t]$ on z; hence we have $d_4(t') \cap V \neq \emptyset$ for all $t' \in \theta \setminus \{t\}$. Then $L_{t'} = d_3(t') \cap V$ is a line for every $t' \in \theta \setminus \{t\}$. These lines all coincide, either they equal M or they have distribution $[\mathbf{xi}, t]$. Thus, we find that either M has distribution $[\mathbf{viii}, t]$ and $V \setminus S \subset X_{3,t}$ or M has one of the distributions $[\mathbf{iv}, t]$ and $[\mathbf{v}, t]$, there is a single line M' with distribution $[\mathbf{xi}, t]$ in V and $V \setminus (S \cup M') \subset X_{3,t}$. Thus if $x \in X_{2g,t} \cup X_{2h,t}$, then L has one of the following two distributions:

$[\mathbf{xiv}, t]$

$[2h, t]$	$3''$	$[3, t]$
1	1	$q-1$

$[\mathbf{xv}, t]$

$[2g, t]$	$3''$	$[3, t]$
1	1	$q-1$

.

If $x \in X_{2',t}$ and $z \in X_{1',t}$, then L has the following distribution:

$[\mathbf{xvi}, t]$

$[2', t]$	$[3, t]$
1	q

Suppose $z \in X_{1,t}$. Again we restrict to the case where L contains at least one point in $d_4(t')$ for some $t' \in \theta$ so that M has distribution $[\mathbf{vi}, t]$. Then by Lemma 4.17 every other line of V on z has distribution $[\mathbf{xiii}, t]$. No two of the lines $M_{t'} = d_3(t') \cap V$ with $t' \in \theta \setminus \{t\}$, coincide because V has no point in $X_{3''}$, but they all contain the point $p \in X_{2',t}$ on M.

$[\mathbf{xvii}, t_1, t_2]$

$[2', t_1]$	$[3', t_1, t_2]$
1	q

$[\mathbf{xviii}, t]$

$[2g, t]$	$[3', t]$
1	q

The lines $M_{t'}$ have distribution $[\mathbf{xvii}, t_1, t_2]$ (with $t_1 = t$) and the lines meeting M outside $\{z, p\}$ have distribution $[\mathbf{xviii}, t]$; these contain one point from $X_{3',t,t'}$ for every $t' \in \theta \setminus \{t'\}$.

As for the number of lines L with a certain distribution on $x \in d_2(t)$, we note that L uniquely determines M and the distribution of M. On the other hand M lies on q^2 planes V outside S and we know how many lines on x and V have a certain distribution. Thus by double counting lines M and L on x with the appropriate distribution we can compute that a point of $X_{2h,t}$ lies on $q^3(q^4-1)/(q-1)$ lines with distribution $[\mathbf{xiv}, t]$ and that a point of $X_{2g,t}$ lies on $q^3(q^3-1)/(q-1)$ lines with distribution $[\mathbf{xv}, t]$ and on q^6 lines with distribution $[\mathbf{xviii}, t]$.

The end of this subsection will be needed when dealing with somewhat far symplectons in Subsection 4.7.9. Let $x \in X_{2''}$. For each $t \in \theta$ there is a unique symplecton S_t on x and t. Consider the geometry of type $\mathrm{Sp}_6(q)$ of symplecta, planes and lines on x and call these objects Points, Lines and Planes respectively. Then the Points S_t ($t \in \theta$) form a symplectic hyperbolic line.

Let us prove this. Suppose S is a Point that is colLinear to S_{t_i} for certain $t_i \in \theta$ ($i = 1, 2$). Let V_i be the Line $S_{t_i} \cap S$ ($i = 1, 2$) and let $L_i \subset V_i$ be the line contained in $d_1(t_i)$. Then the points of L_i ($i = 1, 2$) have position $[1', t_i]$. The lines L_1 and L_2 are disjoined and contained in S. Let M_i ($i = 1, 2$) be two lines that meet both L_1 and L_2. These lines have distribution [**ix**]. For every $t \in \theta$ let $y_{i,t}$ ($i = 1, 2$) be the point on $M_i \cap d_1(t)$. Since the lines M_1 and M_2 are disjoint, we find that, for every $t \in \theta$, the symplecta S and S_t have three distinct points in common and as $S \neq S_t$, we conclude that the Points S and S_t lie on a common Line. Thus every Point is colLinear to one or all Points of the set $\{S_t \mid t \in \theta\}$, which therefore forms a hyperbolic line (this is the geometric definition of a symplectic hyperbolic line).

It follows that any symplecton on x has a plane in common with S_t for at least one $t \in \theta$. In particular, every symplecton S on x satisfies $d_4(t) \cap S = \emptyset$ for one or all $t \in \theta$.

4.7.6 Lines with a point in $X_{3'}$

Let $x \in X_{3',t_1,t_2}$ for certain $t_1, t_2 \in \theta$. Let z_i ($i = 1, 2$) be the unique point in $d_1(x) \cap X_{1,t_i}$ (see Lemma 4.17). We consider all lines L on x that do not contain a point of X_4.

Lemma 4.18 *Given a point $t \in$ theta let $x \in d_3(t)$, let z be the point in $d_1(t) \cap d_1(x)$ and let L be a line on x. Then $L \cap d_4(t)$ is non-empty if and only if L is not contained in a symplecton on z; in that case it equals $L \setminus \{x\}$.*

Proof: Let A be an apartment on t and the flag (x, L, S) for some symplecton S. Then the apartment also contains the unique point $z \in d_1(t) \cap d_1(x)$, because the line zx is the projection of t onto x and z is the projection of t onto zx. We see that we could have chosen S to contain z if and only if L has no point in $d_4(t)$. Moreover, in that case x is the unique point of $L \cap d_3(t)$. □

We claim that L is contained in a symplecton S on x and z_1 or z_2. By Lemma 4.18, if L has a point in X_4, then L is not contained in a symplecton on z_i ($i = 1, 2$). Conversely, if L is not contained in a symplecton on z_1 or z_2, we find that $L \setminus \{x\}$ has a unique point in $d_3(t)$ for every $t \in \theta \setminus \{t_1, t_2\}$, hence it has a point in X_4.

By symmetry assume that S is a symplecton on L and z_1. In view of Lemma 4.3, we first want to locate the projections of $t \in \theta$ on S. Since xz_1 has distribution $[\mathbf{xiii}, t_1]$ we have $\emptyset \neq d_4(t) \cap L \subset S$ for all $t \in \theta \setminus \{t_1\}$ and $\emptyset \neq d_1(t_1) \cap S$ ($t_1 \notin S$). Hence $\pi(t) = d_2(t) \cap S$ is a point for every $t \in \theta \setminus \{t_1\}$ and $M = d_1(t_1) \cap S$ is a line with distribution $[\mathbf{iii}, t_1]$. The unique point p on M with position $[1', t_1]$ is collinear with the point $\pi(t)$ for every $t \in \theta$. The line N on p and $\pi(t)$ has distribution $[\mathbf{x}, t]$ and contains the point $\pi(t)$ for all $t \in \theta$.

Now using Lemma 4.3 we can easily determine the distribution of any line in S. We will simply list the results in the next subsubsection.

On a plane containing xz_1 we find a line with distribution $[\mathbf{vi}, t_1]$, hence the plane was described when we found lines with this distribution in Subsection 4.7.5. On x there is one line with distribution $[\mathbf{xvii}, t_1, t_2]$, namely $d_3(t_2) \cap V$ and the remaining $q-1$ lines are with distribution $[\mathbf{xviii}, t_1]$.

A line with distribution $[\mathbf{xviii}, t_1]$ or $[\mathbf{xvii}, t_1, t]$ $(t \in \theta \setminus \{t_1\})$ is contained in a unique plane on xz_1. Hence on a point in position X_{3', t_1, t_2} in total there are $(q^3-1)/(q-1)$ lines with distribution $[\mathbf{xvii}, t_1, t_2]$ and $q^3 - 1$ lines with distribution $[\mathbf{xviii}, t_1]$.

Let V be a plane of S on N. The line $d_1(z_1) \cap V$ is contained in $d_2(t_1)$ and has distribution $[\mathbf{viii}, t_1]$. We have $V \setminus (d_2(t_1) \cup N) \subset X_{3''}$. Hence we find the following distribution for lines on V.

$[\mathbf{xix}, t, t']$

$[2', t]$	$[2', t']$	$3''$
1	1	$q-1$

Let V be a plane on $\pi(t_2)$ and x. The line on $\pi(t_2)$ and x has distribution $[\mathbf{xvii}, t_2, t_1]$. The line $d_1(p) \cap V$ lies on $\pi(t_2)$ and has distribution $[\mathbf{xix}, t_1, t_2]$. We have $V \setminus d_1(p) \subset X_{3', t_1, t_2}$. Thus in V and on x we find $q-1$ lines with the following distribution.

$[\mathbf{xx}, t_1, t_2]$

$3''$	$[3', t_1, t_2]$
1	q

If L is a line on x not coplanar to z_1 or $\pi(t_2)$, then it is opposite N (in S) and we see that it has the following distribution:

$[\mathbf{xxi}, t]$

$[3, t]$	$[3', t]$
1	q

A line L with distribution $[\mathbf{xx}, t]$ or $[\mathbf{xxi}, t]$ lies in a unique symplecton on xz_1. Inside such a symplecton we know how many lines on x have that distribution. Using this and a double count we find that on a point of X_{3', t_1, t_2} there are $(q^3-1)(q+1)$ lines with distribution $[\mathbf{xx}, t_1, t_2]$ and $q^2(q^3-1)$ lines with distribution $[\mathbf{xxi}, t_1]$.

The lines on x containing a point of X_4 will be treated in Subsections 4.7.8 and 4.7.9.

4.7.7 Lines with a point in X_3

Let $x \in X_{3,t}$ for some $t \in \theta$. We consider all lines L on x that do not contain a point from X_4.

There is precisely one line on x containing a point of $d_1(t)$ and it has distribution $[\mathbf{xii}, t]$.

We consider lines L on x that contain a point $y \in d_2(t)$. In Subsection 4.7.5 we have seen that L has distribution $[\mathbf{xiv}, t]$, $[\mathbf{xv}, t]$ or $[\mathbf{xvi}, t]$. We determine how often each distribution occurs among the lines on x.

Let z be the unique point in $d_1(x) \cap d_1(t)$. Then these lines L lie in a plane V on the line xz. Consider the unique line M that is contained in a (unique) symplecton S on zt. Then the distribution of M (which is $[\mathbf{iv}, t]$, $[\mathbf{v}, t]$ or $[\mathbf{viii}, t]$) determines the distribution of the lines in V as described in Subsections 4.7.4 and 4.7.5.

Let E be a line with distribution $[\mathbf{ix}]$ on z (in fact it is unique). Let $M_x = M$ and let M_E be the line on S that is coplanar to E. Then the distribution of M_x is determined by the distribution of M_E.

Look at the geometry of symplecta, planes and lines on z as a geometry of type $Sp_6(q)$ and call these objects Points, Lines and Planes respectively. The Planes E, zt and zx are pairwise disjoint. Then we know from Section 4.6 that the Point S has one of three possible positions: (a) The Planes M_x and M_E share a Line that meets zt (in S), zx and E in a Point. (b) The Plane M_x is not colLinear to M_E nor is it colLinear to all Planes that are colLinear to zt and M_E. (c) The Plane M_x is not colLinear to M_E but is colLinear to all Planes that are colLinear to zt and M_E.

Now we note that M_E being coplanar to E contains the unique point s in position $[2'']$ of S (see the end of Subsection 4.7.4). It follows that M_E has distribution $[\mathbf{vii}, t]$. We note that we can characterize case (c) (resp. (b)) by the fact that they occur precisely when M_x has (no) point on the hyperbolic line ts.

Thus in case (a) M_x has distribution $[\mathbf{viii}, t]$; in case (b) M_x has distribution $[\mathbf{v}, t]$; in case (c) M_x has distribution $[\mathbf{iv}, t]$.

In Section 4.6 we have seen that case (a), (b) and (c) occur $q+1$, q^2-1 and 1 time(s) respectively. Hence on x, in total, there are $(q+1)q$ lines with distribution $[\mathbf{xvi}, t]$, there are $(q^2-1)q + (q-1) = q^3 - 1$ lines with distribution $[\mathbf{xv}, t]$ and there is 1 line with distribution $[\mathbf{xiv}, t]$ with respect to θ and t.

Let L be a line on x that has a point in $d_3(t)$ but none in $d_2(t)$. Then L is contained in $d_3(t)$. By considering an apartment on t and (x, L) we see that L is contained in a unique symplecton S on zx (remember z is the unique point in $d_1(t) \cap d_1(x)$), but not in a plane on zx (cf. Lemma 4.18).

Let M_t and M_E be the lines on S that are coplanar to zt and E respectively. At the end of Subsection 4.7.4 we saw that M_E has distribution $[\mathbf{vii}, t]$ or $[\mathbf{x}, t]$.

Now $\emptyset \neq d_4(t') \cap L \subset S$ for all $t' \in \theta \setminus \{t\}$ and $t \notin S$ but $d_1(t) \cap S \neq \emptyset$. Since L is opposite M_E (in S) the distribution of L is determined by the distribution of M_E by the fact that to each point on M_E there is a unique point on L closest to it. It follows that if M_E has distribution $[\mathbf{vii}, t]$, then L has the following distribution:

$[\mathbf{xxii}, t]$

$3''$	$[3, t]$
1	q

If M_E has distribution $[\mathbf{x}, t]$, then L has distribution $[\mathbf{xxi}, t]$.

There are three cases. Either (a) M_E and M_t lie in a common plane V of S or M_E is not coplanar to M_t and (c) (resp. (b)) M_E is (not) coplanar to all lines of S that are coplanar to xz and zt.

In case (a) we see that M_E is contained in a symplecton on zt and has distribution $[\mathbf{vii}, t]$. In cases (b) and (c), M_E is not contained in a symplecton on zt and hence must have distribution $[\mathbf{x}, t]$.

In Section 4.6 we have seen that case (a), (b) and (c) occur $q+1$, q^2-1 and 1 time(s) respectively. In each symplecton on zx there are q^3 lines on x that are not coplanar to zt. Hence on x, in total, there are $(q+1)q^3$ lines with distribution $[\mathbf{xxii}, t]$, and there are q^5 lines with distribution $[\mathbf{xxi}, t]$ with respect to θ and t.

4.7.8 Somewhat far planes

We say that a plane is *somewhat far* with respect to θ if, for every $t \in \theta$ the plane contains a point of $d_4(t)$ and has no points in $\cap_{t\in\theta} d_3(t)$ (that is, with position $[3'']$).

Let us look somewhat more closely at these planes. Suppose V is a somewhat far plane. Then the sets $L_t = V \cap d_3(t)$ $(t \in \theta)$ are lines no three of which are on a common point, because such a common point would have position $[3'']$. Hence the lines L_t $(t \in \theta)$ have q points with position $[3', t]$ and the remaining point has position $[3, t]$. Thus L_t $(t \in \theta)$ has distribution $[\mathbf{xxi}, t]$.

We find that the somewhat far plane V contains precisely one point with position $[3, t_1]$ and $[3', t_1, t_2]$ for every $t_1, t_2 \in \theta$. The $\binom{q}{2}$ points of V that are not on any of the lines L_t have distribution $[4]$.

We will see that somewhat far planes exist and are quite useful to us. Fix $t \in \theta$ and let $L = L_t$. Let $x \in L_t$ be the point with position $[3, t]$ and let z be the unique point in $d_1(x) \cap d_1(t)$. Then L and zx are contained in a unique symplecton S. A plane on L has a point in $d_4(t)$ if and only if it is not contained in S. Moreover, for $t' \in \theta \setminus \{t\}$ we have $x \in d_4(t')$. Finally, if a plane has a point in $d_4(t)$, then it has no points with position $[3'']$ for these would have to lie on L and they don't, so that the plane is somewhat far from θ.

Thus a line with distribution $[\mathbf{xxi}, t]$ is contained in q^2 planes that are somewhat far from θ. We have seen in Subsection 4.7.7 that x lies on q^5 lines with distribution $[\mathbf{xxi}, t]$ and so x lies on q^7 somewhat far planes.

Now let y be a point on $V \setminus L_t$. We determine how many somewhat far planes lie on the line xy. Then any symplecton S' on zx contains a unique line L' that lies on a plane V' with y. The line L' has distribution $[\mathbf{xxi}, t]$ or $[\mathbf{xxii}, t]$ (see Subsection 4.7.7) and as V already contains $y \in d_4(t)$, this plane is somewhat far precisely in the former case.

Since L' is a line of S' that is not coplanar to xz we conclude from Subsection 4.7.7 that there are q^2 symplecta S' such that L' has distribution $[\mathbf{xxi}, t]$. Hence there are q^2 planes on x and y that are somewhat far.

It follows that $x \in X_{3,t}$ $(t \in \theta)$ is collinear to $q^7/q^2 = q^5$ points with position $[3, t_1]$ or $[3', t_1, t_2]$ for given $t_1, t_2 \in \theta \setminus \{t\}$, and is collinear to $q^7 \cdot (q(q-1)/2)/q^2 = q^5(q(q-1)/2)$ points with position $[4]$.

Now let $t_1, t_2 \in \theta$ and let $x \in X_{3',t_1,t_2}$. We have seen in Subsection 4.7.6 that x lies on $q^2(q^3-1)$ lines with distribution $[\mathbf{xxi}, t_i]$ $(i = 1, 2)$. Hence x lies on $q^4(q^3-1)$ planes that are somewhat far from θ (see above).

Now let y be a point of $V \setminus (d_3(t_1) \cup d_3(t_2))$. The line L on x and y satisfies $d_4(t) \cap L \neq \emptyset$ for all $t \in \theta$ so that $d_3(t) \cap L$ is a single point.

We determine how many planes V' on L are somewhat far from θ. For this it remains to determine how many planes V' on L contain a point in position $[3'']$. As L contains a single point in $d_3(t)$ for every $t \in \theta$, the set $L'_t = d_3(t) \cap V'$ is a line of V for every $t \in \theta$. If V' contains a point with position $[3'']$, then the lines L_{t_1} and L_{t_2} must coincide. Conversely, if these lines do coincide, then the intersection point of L_{t_1} and $L_{t'}$, for arbitrary $t' \in \theta \setminus \{t\}$, has position $[3'']$.

Let L_i $(i = 1, 2)$ be the line on x and z_i. The line L'_{t_i} $(i = 1, 2)$ is the unique line of V' that is contained in a symplecton on L_i. By considering again the geometry of type $\mathrm{Sp}_6(q)$ of symplecta, planes and lines on x, and using the results of Section 4.6, we see that there are $q + 1$ planes (Lines) on (the Plane) L that contain a line that is contained in a symplecton (Point) both on L_1 and L_2, that is where $L_{t_1} = L_{t_2}$; these were the $q + 1$ Lines on the unique Point of type (c) (see the end of Section 4.6). The remaining q^2 planes on x and y are apparently somewhat far from θ.

Hence the point $x \in X_{3',t_1,t_2}$ is collinear to $q^4(q^3 - 1) \cdot 1/q^2 = q^2(q^3 - 1)$ points in position *pos*, where *pos* is one of $[3, t_3]$, $[3', t_3, t_4]$, with $t_3, t_4 \in \theta \setminus \{t_1, t_2\}$. Moreover $x \in X_{3',t_1,t_2}$ is collinear to $q^4(q^3 - 1)(q(q - 1)/2)/q^2 = q^2(q^3 - 1)(q(q - 1)/2)$ points with position [4].

4.7.9 Somewhat far symplecta

We say that a symplecton is *somewhat far* from the hyperbolic line if it contains a point with position [4]. We note that this condition doesn't define the position of the symplecton uniquely.

Let S be a symplecton somewhat far from θ and let $x \in S$ be a point with position [4]. Then, for every $t \in \theta$, there is a unique point s_t in $d_2(t) \cap S$. For every $t \in \theta$, the points in $d_1(s_t) \cap S$ (resp. $S \setminus (\{s_t\} \cup d_1(s_t))$ are precisely the points in $d_3(t) \cap S$ (resp. $d_4(t) \cap S$). Since for every $t \in \theta$ a point s_t is in position $[2'']$, $[2', t]$ or $[2, t]$ and for every $t' \in \theta \setminus \{t\}$ and $i \in \{0, 1, 2\}$ we have $s_t \in d_i(s_{t'})$ if and only if $s_{t'} \in d_i(s_t)$ it follows that a priori there are three cases:

(i) All s_t's coincide.

(ii) The s_t's are distinct but pairwise collinear.

(iii) The s_t's are pairwise non-collinear.

From the end of Subsection 4.7.5 we may conclude that case (i) never occurs.

Using the isomorphism $\mathrm{O}_7(q) \cong \mathrm{Sp}_6(q)$ we think of S as the polar geometry of totally isotropic 1-, 2- and 3-spaces of a vector space V of dimension 6 over $\mathbb{F}_q$ endowed with a non-degenerate symplectic form. We use the symbol $\perp$ for the orthogonality relation on V with respect to this symplectic form.

Let $\mathcal{O} = \{s_t \mid t \in \theta\}$ and let $\mathsf{O} \subset \mathsf{V}$ be the subspace spanned by $\mathcal{O}$.

The points of $\mathcal{O}$ do not form a single projective line because then we would have $s_t \in d_1(x)$ for some $t \in \theta$ contradicting $x \in d_4(t)$. Then, using that a point of S that is collinear to three points of $\mathcal{O}$ must be collinear to all points of $\mathcal{O}$, we find that O has dimension 3.

Let x be a point in $x \in S \cap X_4$. Then $x \notin \mathsf{O}$ so that $\dim(x+\mathsf{O}) = 4$, and hence the number of points in $X_{3''} \cap S$ that are collinear to x (i.e. in $(x+\mathsf{O})^\perp$) is $q+1$. Then double counting the pairs of lines on x containing a point from $X_{3''}$ and symplecta containing these lines, we find that a point in X_4 is collinear to $(q+1)(q^3+1)$ points in $X_{3''}$. Here we use that a line on x contains precisely one point from $d_3(t)$ for every $t \in \theta$. Similarly one can compute that a point in X_4 is collinear to $q^2(q^3+1)$ points of $X_{3',t,t'}$ and of $X_{3,t}$ for any given $t, t' \in \theta$.

Now we focus on lines L that contain a point in $d_4(t)$ for every $t \in \theta$. For any $t \in \theta$ the set $d_3(t) \cap L$ is a single point and the remaining points of L are in $d_4(t)$.

If $d_3(t) \cap L$ is the same point for all $t \in \theta$, then L has the following distribution.

[**xxiii**]

$3''$	4
1	q

Above we saw that a point $x \in X_4$ lies on $(q+1)(q^3+1)$ lines with distribution [**xxiii**].

We now restrict to considering lines L that have no point with position $[3'']$. From Subsection 4.7.8 we learn that we are looking at those lines of a somewhat far plane V that do not have distribution $[\mathbf{xxi}, t]$ for some $t \in \theta$. We will study somewhat far planes more closely.

Let S be a symplecton containing the somewhat far plane V. Since V is singular and has no points in $X_{3''} \cap S = \mathsf{O}^\perp$, we have $V \cap \mathsf{O} = \{0\}$. Now $\perp$ (or $d_1(\cdot)$) defines an isomorphism between the geometry of points and lines on O and the geometry of lines and points on V.

Let y be a point of $V \cap X_{3',t_1,t_2}$ for certain $t_1, t_2 \in \theta$. Let z_i $(i = 1, 2)$ be the unique point in $d_1(y) \cap d_1(t_i)$. Let L_i $(i = 1, 2)$ be the line on y and z_i. Consider the geometry of type $\mathrm{Sp}_6(q)$ of symplecta, planes and points on y and call these Points, Lines and Planes respectively. Note that a symplecton is somewhat far if and only if it contains a somewhat far plane. The symplecta and planes on y that are somewhat far are precisely those not meeting L_1 and L_2. We claim that the geometry of Points and Lines not meeting L_1 and L_2 is connected. It then follows that $\mathcal{O}$ is embedded into O in the same way in every somewhat far symplecton S on y.

Let S_i $(i = 1, 2)$ be Points not on L_1 and L_2 and let M_i be a Plane on S_i not meeting L_1 and L_2. Then for the pair (L_1, M_2) there a Line on M_1 such that a Plane meeting M_1 outside this Line meets at most one of L_1 and M_2 in a Line and the same holds for the pair (L_2, M_2). Thus M_1 has $q^2 - q \geq 2$ points that are contained in a Plane that meets M_2 in a Line but is disjoint from L_1 and L_2. Such a plane contains a Line which does not meet L_1 and L_2 and joins a Point of M_1 to a Point of M_2.

Thus the structure of a somewhat far plane on y is determined by the way $\mathcal{O}$ is embedded into O inside a symplecton S that we may assume to be in case (iii). The fact that $\dim(\mathsf{O}) = 3$ and that no two (resp. three) points of $\mathcal{O}$ are on a geometric (resp. projective) line shows that O has $q+1$ geometric lines meeting in a common point n and that $\mathcal{O}$ is an oval not containing n.

The $q^2 - 1$ lines on a somewhat far plane V corresponding to points in $\mathsf{O} \setminus (\mathcal{O} \cup \{n\})$ have the following distribution:

[$\mathbf{xxiv}, t$]

$[3,t]$	$3'$	4
1	$q/2$	$q/2$

The points in position $3'$ have position $[3', t_{i_1,i_2}]$, where $\{t_{i_1}, t_{i_2}\}$ $(i = 1, \ldots, q/2)$ is a partition of $\theta \setminus \{t\}$.

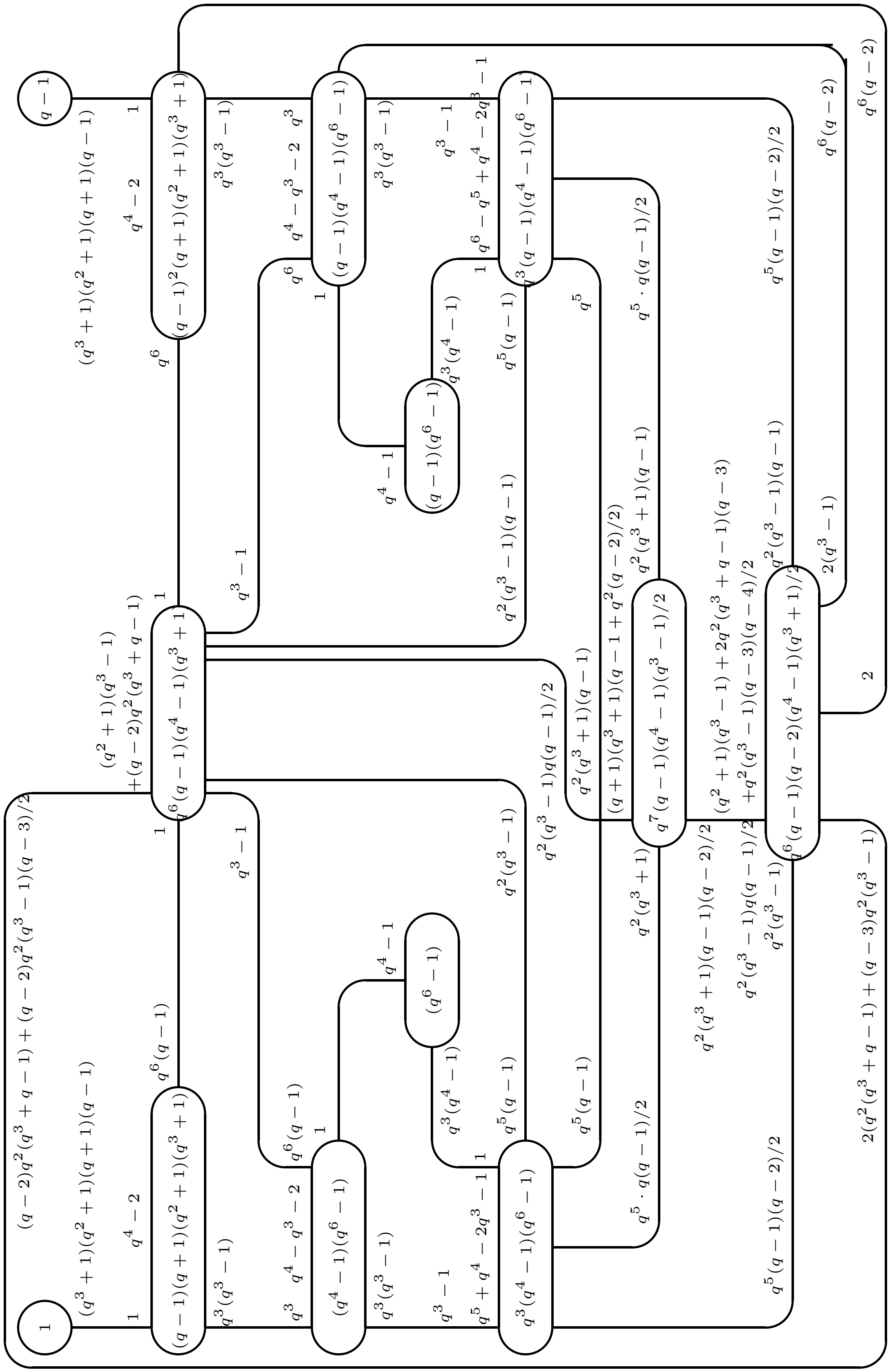
Note that a line containing a point of X_4 has distribution [$\mathbf{xxii}$] or [$\mathbf{xxiv}, t$] for some $t \in \theta$.

The unique line on a somewhat far plane V corresponding to $n \in \mathsf{O}$ has the following distribution:

[$\mathbf{xxv}$]

3
$q+1$

For every $t \in \theta$ the unique point of V with position $[3,t]$ lies on L. Hence a somewhat far plane contains a unique line in position [$\mathbf{xxv}$], containing all its points in position [3]. Using the results on somewhat far symplectons (page 128) we can compute the number of lines with distribution [$\mathbf{xxiv}, t$] and [$\mathbf{xxv}$] on points in X_4, $X_{3,t}$ and $X_{3,t,t'}$.



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Nederlandse samenvatting

Over Meetkundes afgeleid van Gebouwen
Rieuwert Blok

Dit proefschrift behandelt enkele meetkundige aspecten van meetkundes die afgeleid zijn van gebouwen.

Hoofdstuk 1 dient ter inleiding. Na een korte introductie van het begrip diagrammeetkunde, wordt enige aandacht besteed aan het kamersysteem, de meetkunde en schaduwruimten van een (sferisch) gebouw. Vervolgens wordt aangegeven hoe deze objecten verkregen kunnen worden met behulp van groepen met een (B, N)-paar. In het laatste deel van dit hoofdstuk wordt beschreven hoe men via het begrip Chevalleygroep de standaard inbedding van een schaduwruimte in een Liealgebramoduul construeert.

In Hoofdstuk 2 worden genererende puntverzamelingen voor bepaalde schaduwruimten van sferische gebouwen bestudeerd. Aangetoond wordt dat voor schaduwruimten behorende bij een knoop van het diagram dat correspondeert met een zogenaamd minuscuul gewicht de verzameling punten op een appartement de meetkunde genereert. Deze eigenschap blijkt vrijwel rechtstreeks afgeleid te kunnen worden van de diameter van de bijbehorende Coxetergrafen. Bovendien wordt er, op grond van de dimensie van de standaardinbedding van schaduwruimten in een Lie algebra moduul, geconcludeerd dat er geen andere schaduwruimtes zijn met deze eigenschap. Het hoofdstuk wordt besloten met een studie naar genererende verzamelingen voor polaire ruimten behorend bij een hermitische of pseudo-kwadratische vorm.

Het onderwerp van hoofdstuk 3 is de deelmeetkunde van een schaduwruimte van een sferisch gebouw bestaande uit vlaggen op grote afstand van een gegeven vlag. Aangetoond wordt dat deze deelmeetkunde in bijna alle gevallen samenhangend is; de uitzonderingen ontstaan in meetkundes met 'korte' lijnen ($q = 2, 3$) en in sommige 'vrije constructies'. Na een overzicht van de, reeds bekende, resultaten voor gebouwen van rang 2, worden nu ook gebouwen van hogere rang bestudeerd. Het blijkt dat de deelmeetkunde ver van een vlag onsamenhangend is precies dan als hij een onsamenhangende deelmeetkunde van rang 2 bevat. Ten slotte wordt het aantal samenhangscomponenten voor alle behandelde gevallen bepaald door de index van de stabilisator van een samenhangscomponent binnen de stabilisator van de hele deelmeetkunde te berekenen.

In hoofdstuk 4 worden de parameters van enkele associatieschemas berekend. We zijn hier voornamelijk geïnteresseerd in het schema op de verzameling punten $\mathcal{X}$ ver van een vast gekozen punt ∞ binnen de metasymplectische meetkunde $F_4(q)$ (q even), waarbij de klassen gedefinieerd worden door afstandsrelaties. De berekening loopt globaal gezien als volgt. We kiezen een punt o in $\mathcal{X}$. Met behulp van de presentatie van de meetkunde als lange-wortel meetkunde kunnen de banen in de puntverzameling $\mathcal{X}$ onder de werking van de stabilisator van het paar (∞, o) worden bepaald; de bijbehorende orbitals vormen de klassen van het schema. Nu wordt op meetkundige wijze de doorsnede van iedere lijn met deze banen bepaald. Ter voorbereiding op, en als onderdeel van, de behandeling van de $F_4(q)$ meetkunde, worden de deelmeetkundes van type $\mathrm{Sp}_6(q)$ en $\mathrm{DO}_7(q)$ apart bestudeerd. Bij de bestudering van de $\mathrm{Sp}_6(q)$ meetkunde wordt naast een puur meetkundige beschrijving en de bekende 6-dimensionale inbedding gebruik gemaakt van de presentatie als lange-wortel meetkunde.

Curriculum Vitae

Rieuwert Blok werd op 13 augustus 1969 geboren te Amsterdam. Hij bezocht de Van Reenen school te Bergen (N.H.) en behaalde in 1987 zijn gymnasium-β diploma aan het Murmellius gymnasium te Alkmaar. In datzelfde jaar begon hij zijn studie wiskunde aan de Universiteit van Amsterdam en slaagde in juli 1988 voor zijn propaedeutisch examen. Naast logica en grondslagen van de wiskunde verdiepte Rieuwert Blok zich in algebra en meetkunde en in juni 1993 besloot hij zijn studie met de doctoraalscriptie getiteld 'Zelf-duale Goppa-codes op Supersinguliere Krommen' (cum laude, Prof. dr. G. van der Geer). In september 1993 trad hij als AIO in dienst bij de TU Delft om zich voor te bereiden op zijn promotie. Zijn promotie onderzoek dat heeft geleid tot de totstandkoming van dit proefschrift heeft hij verricht onder supervisie van prof. dr. A.E. Brouwer en Prof. dr. T.H.M. Smits. In het kader van dit onderzoek bezocht hij diverse internationale conferenties, colloquia en cursussen. Naast zijn onderzoek verzorgde Rieuwert Blok wiskunde onderwijs voor verscheidene studierichtingen van de TU Delft.