# Partial orders generalizing the weak order on Coxeter groups

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**Abstract**

We define a new family of partial orders generalizing the weak order on Coxeter groups called $T$-orders, where $T$ is a set of reflections determining the covers in this order. We show that the Grassmann and Lagrange orders on the Coxeter groups of type $A_n$ and $B_n$ introduced by Bergeron and Sottile are in fact $T$-orders. These partial orders were used to compute certain products in the cohomology ring of the flag manifolds associated to the complex Chevalley groups of these types. We exhibit $T$-orders generalizing these orders to partial orders for the Coxeter groups of type $D_n$, $E_6$, and $E_7$.

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## 1 Introduction and statement of results

In an effort to further understand combinatorial properties of the Littlewood-Richardson coefficients, N. Bergeron and F. Sottile defined a new partial order on the symmetric group called the *Grassmannian order* in [1]. They further extended their work to a partial order (called the *Lagrangian order*) on the Coxeter group of type $B_n$ in [2]. These partial orders enabled Bergeron and Sottile to compute some products in the cohomology rings of flag manifolds and establish new relations among all structure constants.

One difficulty in working with these orders is that their definitions can be somewhat complicated and in any case lack easy generalizations to other Coxeter groups. In 2000 L. Evani, in her Ph.D. thesis ([7]), gave a permutation statistic based definition of the Grassmannian order. This work was continued in [3], where the authors simplified Evani's proof and derived many of the properties of the Grassmannian order from the permutation statistic definition.

In this article we continue the investigation of the orders of Bergeron and Sottile and provide a general framework for understanding such orders based on subsets of the full set of reflections of the Coxeter group in question. In particular, we provide a general framework for these orders that allows them to be viewed as generalizations of the weak order. Moreover, this generalization allows one to create many different partial orders, even allowing one to interpolate between the weak order and the Grassmannian and Lagrangian orders. Finally, we suggest what might be reasonable orders to consider for other Coxeter groups that might allow calculations like the ones that Bergeron and Sottile do in other systems.

We begin by setting some notation. A *partial order* on a set $S$ is a reflexive, anti-symmetric, transitive relation $\leq$ on $S$, and a partially ordered set $P = (S, \leq)$ is a set $S$ and a partial order $\leq$ on $S$. We say that $b$ is a *cover* of $a$ in $P$ if $a \leq b$ and $a \leq c \leq b$ implies $a = c$ or $a = b$. In this case, we write $a \prec b$. As in this paper we will have many different partial orders on a given set, we shall typically subscript the orders and use the $\prec$ symbol with the subscript to denote the cover in that particular order. We call a sequence $a_1 \prec a_2 \prec \cdots \prec a_k$ (where $a_i \in S$ for all $i$ and $a_i$ is covered by $a_{i+1}$) a *saturated chain in $P$*. Finally, we say that a partial order on $S$ is *rooted at $a$* if for all $b \in S$ we have $a \leq b$.

A *Coxeter system of type $M$* is a pair $(W, R)$ where $W$ is a group, $R = \{r_i\}_{i=1}^n$ is a set of generators for $W$, and $M = (m_{ij})_{i,j=1}^n$ is a symmetric matrix, called a *Coxeter matrix*, satisfying $m_{ij} \in \mathbb{N}$ and $m_{ii} = 1$ for all $i, j$. The group $W$ is defined by the relations $(r_i r_j)^{m_{ij}} = 1$.

The *length* of $v \in W$, denoted $l(v)$, with respect to $R$ equals $k$ if $k$ is the smallest integer such that $v = r_{i_1} r_{i_2} \cdots r_{i_k}$ for some $r_{i_j} \in R$. The (left) *weak order* on $W$ with respect to $R$, denoted $\leq_w$, is defined by its covers as follows: $u \prec_w v$ if and only if

(1) $v = ru$ for some $r \in R$,
(2) $l(v) = l(u) + 1$.

Thus, the length is the rank function of the weak order. A *reflection* of $W$ is a conjugate of $r_i$ for some $r_i \in R$, and the $r_i \in R$ are called *fundamental reflections.*

Next, let $T \subset W$ be a set of reflections containing $R$. A *$T$-gallery for $v \in W$* is a sequence

$$\Gamma := e \leq_w t_1 \leq_w t_2 t_1 \leq_w \cdots \leq_w t_k \cdots t_2 t_1 = v,$$

where $t_i \in T$ for $i = 1, 2, \ldots, k$. Such a gallery is said to have *length* $k$, denoted by $l_T(\Gamma) = k$, and we say that $\Gamma$ is a *minimal $T$-gallery* if there are no $T$-galleries from $e$ to $v$ of shorter length.

We define the *$T$-order* on $W$ as follows: For $u, v \in W$ we have $u \leq_T v$ if and only if there is a minimal $T$-gallery for $v$ passing through $u$. For example if $T = R$, then the $T$-order is (by definition) the weak order. Note that $\leq_T$ is necessarily a ranked order.

In Section 2 we provide the following method to determine if a given partial order on $W$ is a $T$-order.

**Theorem 1** *Suppose $\leq$ is a partial order on $W$ rooted at the identity $e$ with rank function $\rho$ and let $T \subset W$ be a set of reflections containing $R$. If*

*(W) for $u, v \in W$, $u \leq v$ implies $u \leq_w v$;*
*(C) the relation $u \prec v$ implies $v = tu$ for some $t \in T$; and*
*(R) for $u \in W$ and $t \in T$, $u \leq_w tu$ implies $\rho(tu) \leq \rho(u) + 1$, with equality if and only if $u \leq tu$,*

*then $\leq$ is the $T$-order on $W$.*

In Section 3 we briefly study the Grassmannian order on the symmetric group. Let $(W, R)$ be the Coxeter system of type $M = A_n$. Then $W = \mathrm{Sym}(n+1)$,

$r_i = (i, i+1)$ (for $i = 1, \ldots, n$), and for $i, j \in \{1, \ldots, n\}$,

$$m_{ij} = \begin{cases} 1 & \text{if } i = j \\ 3 & \text{if } |i-j| = 1 \\ 2 & \text{otherwise.} \end{cases}$$

In Bergeron & Sottile [1] the authors defined a partial order called the *Grassmann order* on the Coxeter group $A_n$ in terms of the action on $[n+1] = \{1, 2, \ldots, n+1\}$. This order is defined as follows: For $u, v \in A_n$ we have $u \leq_G v$ if and only if

(G1) For all $a \in [n+1]$, $a < u(a)$ implies $u(a) \leq v(a)$;
(G2) For all $a \in [n+1]$, $a > u(a)$ implies $u(a) \geq v(a)$,
(G3) For all $a, b \in [n+1]$, $a < v(a)$, $v < u(b)$ (or $a > v(a)$, $b > v(b)$), and $a < b$, $v(a) < v(b) \Longrightarrow u(a) < u(b)$.

Using Theorem 1 we obtain the following characterization of this partial order.

**Theorem 2** *Let* $(W, \{r_i\}_{i=1}^n)$ *be the Coxeter system of type* $A_n$*, i.e.* $W = \mathrm{Sym}(n+1)$ *and* $r_i = (i, i+1)$*, and let*

$$T_G = \{(i, j) \mid 1 \leq i < j \leq n\}.$$

*Then the Grassmann order is equivalent to the* $T_G$*-order.*

Note that $T_G$ is the set of all reflections of $W$. Consequently this theorem implies that $a \leq_G b$ if and only if there exists a $T_G$-gallery for $b$ that goes through $a$, or more precisely, that $b$ can be written as a product of reflections $t_k t_{k-1} \cdots t_1$ such that $t_m \cdots t_1 \leq_w t_{m+1} \cdots t_1$ for $m = 1, \ldots, k$ and $a = t_p \cdots t_1$ for some $p \leq k$.

In Section 4 we study the Lagrange order. Let $(W, R)$ be the Coxeter system of type $M = B_n$. Then $W$ is the group of all permutations on a set of size $2n$ preserving a given partition of this set into 2-subsets. More explicitly we can think of the set as $[\pm n] = \{-n, \ldots, -2, -1, 0, 1, 2, \ldots, n\}$ and have the group act on this set subject to the relation $w(-i) = -w(i)$, for all $i = 1, 2, \ldots, n$. For $k = 1, 2, \ldots, n$ we set $\overline{k} = -k$. The generators $r_i$ are then represented by the following permutations: for $1 \leq i < n$ we have $r_i = (n-i, n-(i-1))(\overline{n-i}, \overline{n-(i-1)})$, and $r_n = (1, \overline{1})$. (Note that every element of $W$ fixes 0 in this description, but later as we shall relate the groups of type $B_n$ and $A_{2n}$, it is easier to include 0 here.) The matrix $M$ is given by

$$m_{ij} = \begin{cases} 1 & \text{if } \ i = j \\ 3 & \text{if } \ |i-j| = 1, i, j < n \\ 4 & \text{if } \ \{i, j\} = \{n-1, n\} \\ 2 & \qquad \text{otherwise.} \end{cases}$$

In Bergeron & Sottile [2] the authors defined a partial order called the *Lagrange order* on the Coxeter group $B_n$ in terms of the action on $[\pm n]$ as follows [2]:

For $u, v \in B_n$ we have $u \leq_L v$ if and only if

(L1) Whenever $a \in [\pm n]$ with $a > u(a)$ we have $u(a) \geq v(a)$; and
(L2) Whenever $a, b \in [\pm n]$ with $a < b$, $v(a) > a$, $v(b) > b$, and $v(a) < v(b)$ then $u(a) < u(b)$.

The rank function of this order is given by (see [2])

$$\begin{aligned} \rho_L(u) &= \textstyle\sum_{a,0>u(a)} |u(a)| - \#\{(a,b) \mid 0 < a < b, a = u(a) > u(b)\} \\ &\quad - \#\{(a,b) \mid a < b, a > u(a), b > u(b), u(a) > u(b)\} - \textstyle\sum_{0>a>u(a)} |a|. \end{aligned}$$

Using Theorem 1 we obtain the following characterization of the Lagrange order.

**Theorem 3** *Let $(W, \{r_i\}_{i=1}^n)$ be the Coxeter system of type $B_n$ and let*

$$T_L = \{r_l^{r_l r_{l-1} \cdots r_k} \mid 1 \leq k \leq l \leq n\}.$$

*Then the Lagrange order is equivalent to the $T_L$-order.*

Here, for $u, v \in W$ we define $u^v = v^{-1}uv$. Note that $r_l^{r_l} = r_l$ so all fundamental reflections are in $T$. In terms of permutations of $[\pm n]$ the set $T$ in Theorem 3 consists of all reflections of the form $(i, \bar{\imath})$ and $(i, j)(\bar{\imath}, \bar{\jmath})$. Thus $T$ contains all reflections of $W$ except those of the form $(i, \bar{\jmath})(\bar{\imath}, j)$.

In Section 5 we generalize the Grassmannian and Lagrange orders to Coxeter groups of type $M = D_n$, $E_6$, and $E_7$. The resulting partial orders (defined on page 15) will be called rooted $\mu$-Bruhat-Chevalley orders, where $(M, \mu)$ $(\mu \neq \emptyset)$ is one of

(1) $M = A_n$, $\mu \subseteq \{1, 2, \ldots, n\}$,

[2] actually they defined it for a group $B_\infty$ acting on $\mathbb{Z} \setminus \{0\}$

(2) $M = D_n$, $\mu \subseteq \{1, n-1, n\}$,
(3) $M = E_6$, $\mu \subseteq \{1, 6\}$, or
(4) $M = E_7$, $\mu = \{7\}$.

Bearing the characterization of the Grassmann and Lagrange order as a $T_\mu$-order in mind we construct a $T_\mu$-order from each rooted $\mu$-Bruhat-Chevalley order $\leq_{\underline{\mu}}$ by taking

$$T_\mu = \{r \text{ a reflection of } W \mid u \prec_{\underline{\mu}} ur \text{ for some } u \in W\}.$$

**Theorem 4** *Let $(M, \mu)$ be one of the above. Then the $T_\mu$-order derived from the $\mu$-Bruhat-Chevalley order on the Coxeter group of type $M$ satisfies*

$$T_\mu = \{\textit{all reflections of } W\}.$$

**Problems** (1) Apparently the rooted $\mu$-Bruhat-Chevalley order is equivalent to its derived $T$-order if $(M, \mu) = (A_n, \{1, 2, \ldots, n\})$. It would be interesting to know if this is true also of the other rooted $\mu$-Bruhat-Chevalley orders.

(2) We would like to see if the partial orders here will be suitable for finding Pieri-type formulas for flag manifolds of type $D_n$, $E_6$, and $E_7$. In connection with this we note that in [2] the authors use a rather complicated labeled version of the Lagrange order on $B_n$ for deriving their Pieri-type formula for the flag manifold of type $C_n$. Could it be that the rooted $\mu$-Bruhat-Chevalley order for $C_n$ we define in Section 5 might also serve that purpose?

(3) We construct the rooted $\mu$-Bruhat-Chevalley orders from so-called $\mu$-Bruhat-Chevalley orders. It follows from our construction that we can only define these for Dynkin diagrams $M$ and subsets $\mu$ of the collection of indices labeling so-called minuscule weights. This restricts our generalization of the Grassmann and Lagrange orders to the list given in Table 1. We wonder if this connection with minuscule weights is merely a coincidence emerging from our method or that it has actual implications for the Schubert calculus of the associated flag manifolds.

## 2 $T$-orders

We begin this section by deriving an important property of $T$-orders, and then we will prove Theorem 1. Let $P$ be a set. A sequence $p_1, p_2, \ldots, p_k$, where $p_i \in P$ for $i = 1, 2, \ldots, k$, is called *coherent* with respect to some partial order $\leq$ on $P$ if $p_1 \leq p_2 \leq \cdots \leq p_k$. If $(P, \leq)$ is a partially ordered set, then for any $p, q \in P$ we define the *$p$-$q$ interval* to be the set

$$[p, q] = \{x, \in P \mid p \leq x \leq q\}.$$

**Proposition 5** *Let $(W, R)$ be a Coxeter system and let $T$ be a set of reflections containing $R$. Then for $u, v, w \in W$ with $u \leq_T w$, the map $v \mapsto vu^{-1}$ defines an isomorphism $[u, w]_T \to [e, wu^{-1}]_T$.*

**Proof:** Since $u \leq_T w$ we have $u \leq_w w$. If $u \leq_w v \leq_w w$, then also $vu^{-1} \leq_w wu^{-1}$, and therefore every $T$-gallery from $u$ to $w$ can be translated to give a $T$-gallery from $e$ to $wu^{-1}$ and vice versa. Thus left translation by $u$ (and $u^{-1}$) is a length-preserving bijection between $T$-galleries from $u$ to $w$ and from $e$ to $wu^{-1}$. Restricting this to those of minimal length shows that translation by $u$ (and $u^{-1}$) is a bijection of minimal $T$-galleries, and the result follows. □

We now prove Theorem 1.

**Proof** (of Theorem 1)**:** Suppose $\leq$ is a partial order on $W$ rooted at $e$ with rank function $\rho$ satisfying properties (W), (C), and (R). Let $T \supseteq R$ be the set of reflections given in property (C). Then we can write any sequence of covers in the $\leq$ order as

$$\Gamma := e \prec t_1 \prec t_2t_1 \prec \cdots \prec t_k \cdots t_2t_1,$$

where $t_i \in T$ for $i = 1, 2, \ldots, k$.

By property (W) the sequence $\Gamma$ is coherent for the weak order and hence is a $T$-gallery. We must prove that $\Gamma$ is a minimal $T$-gallery, i.e., that it has minimal length.

As $\Gamma$ consists of proper covers in $\leq$ we have $\rho(t_k \cdots t_2t_1) = k$. By property (R) however, for any reflections $s_i \in T$ $(i = 1, \ldots, l)$ we have $\rho(s_l \cdots s_1) \leq l$. Consequently, if

$$e, s_1, s_2s_1, \ldots, s_l \cdots s_1$$

is a $T$-gallery for $w = t_k \cdots t_1$ (and thus $s_l \cdots s_1 = w$), it must be the case that

$$l \geq \rho(s_l \cdots s_1) = \rho(w) = k.$$

Thus this $T$-gallery is minimal. This proves that $a \leq b$ implies $a \leq_T b$.

For the reverse direction, we need to establish that if $t_1, \ldots, t_k \in T$ are such that

$$e \prec_T t_1 \prec_T t_2t_1 \prec_T \cdots \prec_T t_k \cdots t_2t_1$$

is a saturated chain in the $T$-order for $w = t_k \cdots t_1$, then it is a saturated chain in the $\leq$-order. For this we again examine $\rho(t_k \cdots t_2t_1)$. Since $\alpha \leq_T \beta$ implies $\alpha \leq_w \beta$, property (R) implies $\rho(t_k \cdots t_2t_1) \leq k$. Consequently there exists a saturated chain of length at most $k$ in the $\leq$-order. But we already showed that saturated chains in the $\leq$-order are necessarily saturated chains in the $T$-order, implying that a minimal $T$-gallery for $w$ is of length $\rho(t_k \cdots t_1)$. By assumption, a saturated chain from $e$ to $w$ in the $T$-order has length $k$.

Consequently $\rho(t_k \cdots t_1) \geq k$, implying that $\rho(t_k \cdots t_1) = k$. To get equality, condition (R) gives by induction that $\rho(t_s \cdots t_1) = 1 + \rho(t_{s-1} \cdots t_1)$ so that $t_{s-1} \cdots t_1 \prec t_s \cdots t_1$ for all $s$. This implies that

$$e \prec t_1 \prec t_2 t_1 \prec \cdots \prec t_k \cdots t_1$$

as desired. □

## 3 The Grassmann order

In this section we prove Theorem 2. Let $(W, R)$ be a Coxeter system of type $M = A_n$. The Grassmannian order on $W$ was defined in Section 1. For $u \in W$ define

$$u_\uparrow = \{a \in [n+1] \mid u(a) > a\}$$
$$u_\downarrow = \{a \in [n+1] \mid u(a) < a\}$$

An *inversion* for $u \in W$ is a pair $(a, b)$ with $a, b \in [n+1]$ such that $a < b$ and $u(a) > u(b)$. The set of inversions of $u$ is denoted $\mathrm{Inv}(u)$. We can refine this notion as follows:

$$u_{\uparrow\uparrow} = \{(a, b) \in \mathrm{Inv}(u) \mid a \in u_\uparrow, b \in u_\uparrow\}$$
$$u_{\uparrow\downarrow} = \{(a, b) \in \mathrm{Inv}(u) \mid a \in u_\uparrow, b \in u_\downarrow\}$$
$$u_{\downarrow\downarrow} = \{(a, b) \in \mathrm{Inv}(u) \mid a \in u_\downarrow, b \in u_\downarrow\}$$

It turns out (see [3]) that the fixed point inversions play no effective role in the Grassmannian order.

Let $\rho_G(u)$ be the rank of $u$ in the Grassmannian order.

**Proposition 6** ([3] Corollary 2) *For $u \in W$ we have*

$$\rho_G(u) = |u_{\uparrow\downarrow}| - |u_{\uparrow\uparrow}| - |u_{\downarrow\downarrow}|.$$

In [3], it is shown that the rank function together with the weak order relation determines the Grassmannian order. The three conditions for the $T$-order arise naturally in the proof, and it was this that led to the idea of codifying orders by the reflection set $T$.

**Proposition 7** ([3] Proposition 5) *For $u, v \in W$, $u \leq_G v$ implies $u \leq_w v$.*

Let $T$ be the set of all reflections. Note that $T \supseteq R$.

**Proposition 8** ([3] Corollary 17) *The set $T$ has the property that $u \prec_G v$ implies $u = tv$ for some $t \in T$.*

**Proposition 9** ([1], [3] Proposition 13) *For $u \in W$ and $t \in T$, $u \leq_w tu$ implies $\rho_G(tu) \leq \rho_G(u)+1$, and moreover equality holds if and only if $u \leq_G tu$.*

**Proof** (of Theorem 2)**:** By Proposition 7, 8, and 9 above, the Grassmann order satisfies conditions (W), (C), and (R) respectively where $T$ is the set of all reflections. The result now follows from Theorem 1. □

## 4 The Lagrange order

We prove that the Lagrange order $\leq_L$ with rank function $\rho_L$ has properties (W, C, R) from Theorem 1. The proof proceeds along the lines of the proof for the Grassmannian order, and indeed, the proofs of each of the three conditions are similar to those for the Grassmannian order in [3]. In particular, we analyze the inversion sets on related permutations and use the Bergeron-Sottile characterization of the rank function.

Recall that we view $B_n$ as a subgroup of $\mathrm{Sym}([\pm n])$, and use the notation $\overline{a} = -a$. We will think of $B_n$ as acting on the set $[\pm n]$ from the left.

Then every reflection in $B_n$ is either of the form $(i, \overline{\imath})$ with $1 \leq i \leq n$, or of the form $(i, j)(\overline{\imath}, \overline{\jmath})$, or of the form $(i, \overline{\jmath})(\overline{\imath}, j)$ for some $1 \leq i < j \leq n$.

We will need the notion of a root as a subset of $B_n$ (see Ronan [8, Ch. 2]). We simultaneously view $B_n$ as a chamber system over the set $I = \{1, 2, \ldots, n\}$ where $x \sim_i y$ if and only if $y = r_i x$ (note that all actions are from reverse side in comparison to loc. cit.). The group $B_n$ acts as a group of automorphisms on this chamber system by multiplication from the right in addition to acting on the set $[\pm n]$

For every reflection $r$ we let $\alpha_r^+$ (resp. $\alpha_r^-$) be the set of elements $x$ such that there exists a gallery $\gamma = (e = c_0, c_1, \ldots, c_k = x)$ with the property that $c_{j-1} r = c_j$ for an even (resp. odd) number of $j$'s with $1 \leq j \leq k$. It can be shown that this is well-defined. This entails that $x$ and $y$ are in the same (opposite) root if and only if for any gallery $\gamma = (x = d_0, d_1, \ldots, d_l = y)$ we have $d_{j-1} r = d_j$ for an even (resp. odd) number of $j$'s with $1 \leq j \leq l$. The sets $\alpha_r^+$ and $\alpha_r^-$ are called respectively the *positive root* and *negative root* associated to $r$.

Note that $\alpha_r^\epsilon r = \alpha_r^{-\epsilon}$, for $\epsilon = +, -$, and $B_n = \alpha_r^+ \uplus \alpha_r^-$, where $\uplus$ denotes disjoint union.

We will first identify these roots in terms of inversions. For $1 \leq i < j \leq n$ and the indicated reflection $r$ we define two sets $\beta_r^+$ and $\beta_r^-$ as follows:

| $r$ | $\beta_r^+$ | $\beta_r^-$ |
|---|---|---|
| $(i,-i)$ | $\{w \mid (-i,i) \notin \mathrm{Inv}(w)\}$ | $\{w \mid (-i,i) \in \mathrm{Inv}(w)\}$ |
| $(i,j)(-i,-j)$ | $\{w \mid (i,j),(-j,-i) \notin \mathrm{Inv}(w)\}$ | $\{w \mid (i,j),(-j,-i) \in \mathrm{Inv}(w)\}$ |
| $(i,-j)(-i,j)$ | $\{w \mid (-j,i),(-i,j) \notin \mathrm{Inv}(w)\}$ | $\{w \mid (-j,i),(-i,j) \in \mathrm{Inv}(w)\}$ |

**Lemma 10** *For $\epsilon = +,-$ we have $\alpha_r^\epsilon = \beta_r^\epsilon$.*

**Proof:** Note that

$$(i,j) \in \mathrm{Inv}(w) \iff (-j,-i) \in \mathrm{Inv}(w) \quad \text{and}$$
$$(-i,j) \in \mathrm{Inv}(w) \iff (-j,i) \in \mathrm{Inv}(w)$$

so that $B_n = \beta_r^+ \uplus \beta_r^-$.

Now let $1 \leq i < j \leq n$ and $r = (i,j)(-j,-i)$. Suppose that $c$ and $d$ are chambers in $B_n$ with $c \sim_l d$ and $l < n$, that is $d = r_l c$. Let $c \in \beta_r^\epsilon$ and $d \in \beta_r^\eta$ for $\epsilon, \eta \in \{+,-\}$.

We claim that $cr = d$ if and only $\eta = -\epsilon$. By definition of a root and since $e$ is contained in the positive root, we then have $\alpha_r^+ = \beta_r^+$ and $\alpha_r^- = \beta_r^-$.

Let us now prove our claim. Assume first that $cr = d$. Then $d(i) = (cr)(i) = c(r(i)) = c(j)$ and $d(j) = (cr)(j) = c(r(j)) = c(i)$ so that $\eta = -\epsilon$, as desired.

Now assume that $\eta = -\epsilon$. Then $d(i) = (r_l c)(i) = r_l(c(i))$ and $d(j) = (r_l c)(j) = r_l(c(j))$. So if for instance $c(i) < c(j)$, then as $d(i) > d(j)$ and $r_l = (n-l, n-l-1)(-(n-l), -(n-l-1))$, we must have $c(i) = n-l-1$ and $c(j) = n-l$ or $c(i) = -(n-l)$ and $c(j) = -(n-l-1)$ so that $d = r_l c = cr$. The cases where $r = (-i,i)$ and/or $r_l = (-1,1)$ are handled similarly. □

**Lemma 11** *Let $z, y \in B_n$. Then, the following are equivalent:*

*(1)* $z \leq_w y$,
*(2) for any reflection $r$ we have* $z \in \alpha_r^- \Longrightarrow y \in \alpha_r^-$,
*(3)* $\mathrm{Inv}(z) \subseteq \mathrm{Inv}(y)$.

**Proof:** (1) $\iff$ (2): Since the weak order is the $T$-order where $T$ is the set of all fundamental reflections, we see that $z \leq_w y$ if and only if there is a minimal gallery in the chamber system $B_n$ from $e$ to $y$ through $z$. By Proposition 2.8 of Ronan [8] with $x = 1$ this happens precisely if $z$ is contained in every root that contains both $e$ and $y$, that is, if $z$ is contained in every positive root on

$y$. The result now follows since $\alpha_r^- = B_n \setminus \alpha_r^+$.

(2) $\iff$ (3): As Lemma 10 shows the negative roots are precisely those having the corresponding inversions (see chart) this follows immediately. $\square$

**Lemma 12** *For $u, v \in B_n$ we have $u \leq_L v$ implies $u \leq_w v$. In particular, $u_{\uparrow\downarrow} \subseteq v_{\uparrow\downarrow}$, $u_{\uparrow\uparrow} \subseteq v_{\uparrow\uparrow}$, and $u_{\downarrow\downarrow} \subseteq v_{\downarrow\downarrow}$.*

**Proof:** From (L1) it immediately follows that $u_\downarrow \subseteq v_\downarrow$. Moreover, it follows from (L1) that if $a \in [\pm n]$, then $-a > u(-a)$ implies $u(-a) \geq v(-a)$. That is, if $a < u(a)$ then $a < u(a) \leq v(a)$ so $u_\uparrow \subseteq v_\uparrow$. Consequently $u_{\uparrow\downarrow} \subseteq v_{\uparrow\downarrow}$.

From (L2) combined with the above, we claim that if $(a,b) \in u_{\uparrow\uparrow}$ then $(a,b) \in v_{\uparrow\uparrow}$. Indeed, if $u(a) > a$ and $u(b) > b$ then $v(a) > a$, $v(b) > b$ by (L1) and so, by (L2), if $u(a) > u(b)$, then $v(a) > v(b)$. Thus $u_{\uparrow\uparrow} \subseteq v_{\uparrow\uparrow}$. Clearly if $(a,b) \in u_{\downarrow\downarrow}$, then $(-b,-a) \in u_{\uparrow\uparrow}$ which implies $(-b,-a) \in v_{\uparrow\uparrow}$ and that in turn implies $(a,b) \in v_{\downarrow\downarrow}$. Thus $u_{\downarrow\downarrow} \subseteq v_{\downarrow\downarrow}$.

To finish the proof, it remains to show that inversions $(a,b)$ of $u$ with $u(a) = a$ or $u(b) = b$ are inversions of $v$. Let us assume that $u(a) = a$ and $(a,b)$ is an inversion of $u$. Then $b \in u_\downarrow$ implying $b \in v_\downarrow$. If $(a,b)$ is not an inversion of $v$, we would then have $v(a) < v(b) \leq u(b) < u(a) = a$. But then $v(-a) > v(-b) \geq u(-b) > u(-a) = -a > -b$. But $v(-b) < v(-a)$, $v(-b) > -b$, and $v(-a) > -a$ imply $u(-b) < u(-a)$, a contradiction. Thus $(a,b)$ is an inversion of $v$ in this case.

On the other hand, if $(a,b)$ is an inversion of $u$ with $u(b) = b$, we have $a \in u_\uparrow$, so that $a \in v_\uparrow$ and $a < b = u(b) < u(a) \leq v(a)$ by (L1). If $(a,b)$ were not an inversion of $v$, we would have $a < v(a)$, $b < v(b)$, and $v(a) < v(b)$ which would imply by (L2) that $u(a) < u(b)$ which it is not. Hence $(a,b)$ is again an inversion of $v$.

Consequently $\mathrm{Inv}(u) \subseteq \mathrm{Inv}(v)$, which, by Lemma 11, implies $u \leq_w v$. $\square$

We now examine $B_n$ as a subgroup of $A_{2n}$, viewed as the symmetric group acting on the set $[\pm n]$. Recall that we denote the Grassmann order on this group by $\leq_G$. By Proposition 6 its rank function can be expressed as $\rho_G(u) = |u_{\uparrow\downarrow}| - |u_{\uparrow\uparrow}| - |u_{\downarrow\downarrow}|$. The following is proved in Bergeron & Sottile [2].

**Theorem 13**

(1) *The Lagrange order is the order induced on $B_n$ as a subgroup of $A_{2n}$ by the Grassmann order.*

(2) *If $\psi(u) = |\{i \in [n] \mid i \cdot u(i) < 0\}|$, then*

$$\rho_L(u) = (\rho_G(u) + \psi(u))/2.$$

We will now look at covers in the Lagrange order. Let

$$T_L = \{(i,j)(\bar{\imath},\bar{\jmath}) \mid 1 \le i < j \le n\} \cup \{(i,\bar{\imath}) \mid 1 \le i \le n\}.$$

Notice that $T_L$ contains $R = \{(i,i+1)(\bar{\imath},\overline{\imath+1}) \mid 1 \le i \le n\} \cup \{(1,\bar{1})\}$.

**Lemma 14** *For $u, v \in B_n$, $u \prec_L v$ implies $v = tu$ for some $t \in T_L$.*

**Proof:** By Theorem 2.11 (4) in [2] $u \prec_L v$ if and only if $e \prec_L vu^{-1}$ and this is clearly equivalent to $\rho_L(vu^{-1}) = 1$. Set $t = vu^{-1}$. By Theorem 13 $\rho_L(t) = (\rho_G(t) + \psi(t))/2$ and as $\rho_G(t) = 1$, we either have $\rho_G(t) = 2$ and $\psi(t) = 0$ or $\rho_G(t) = 1$ and $\psi(t) = 1$. The latter case only occurs if $t$ is a reflection of the form $(i,\bar{\imath})$ for some $i \in [n]$. In the former case we do not have sign changes so we may look at $t$ restricted to $[n]$ and we get $\rho_G|_{[n]}(t) = 1$; so $t$ is a cover of $e$ in the Grassmann order and therefore by Theorem 2 is a reflection of the form $(i,j)(\bar{\imath},\bar{\jmath})$ for some $1 \le i < j \le n$. $\square$

Recalling that $r_i = (n-i, n-i+1)$ for $i < n$ and $r_n = (1,\bar{1})$, a straightforward check shows for $i < j$ that

$$(i,j)(\bar{\imath},\bar{\jmath}) = r_l^{r_l r_{l-1} \cdots r_k} \quad \text{and}$$
$$(i,\bar{\imath}) = r_n^{r_n r_{n-1} \cdots r_l},$$

where $l = n - i$ and $k = n - j + 1$. Consequently,

$$T_L = \{r_l^{r_l r_{l-1} \cdots r_k} \mid 1 \le k \le l \le n\}.$$

**Lemma 15** *For $u \in B_n$ and $t \in T_L$, $u \le_w tu$ implies $\rho_L(tu) \le \rho_L(u) + 1$, and equality holds if and only if $u \le_L tu$.*

**Proof:** If $t$ is of the form $(i,j)(\bar{\imath},\bar{\jmath})$, then $\psi(tu) = \psi(u)$, and by Theorem 13, it suffices to show that $\rho_G(tu) = \rho_G(u) + 2$. By Proposition 9 we know that for any reflection $t' = (a,b)$ of $[\pm n]$, we have $\rho_G(t'u) \le \rho_G(u) + 1$. Thus

$$\begin{aligned}\rho_G\left((i,j)(\bar{\imath},\bar{\jmath})u\right) &\le 1 + \rho_G((\bar{\imath},\bar{\jmath})u)\\ &\le 2 + \rho_G(u).\end{aligned}$$

so that $\rho_L(tu) \le \rho_L(u) + 1$, as desired. Moreover, we have equality if and only if we have equality in both steps of the above equation. But by Proposition 9, equality implies $u \le_G tu$. Theorem 13 implies that this holds if and only if $u \le_L tu$ as desired.

Now let $t = (i,\bar{\imath})$ $(i > 0)$. Recalling that $\psi(tu) = |\{j \in [n] \mid j \cdot tu(j) < 0\}|$ and noting that $j \cdot tu(j) = -j \cdot u(j)$ if and only if $u(j) = \pm i$, it follows that $\psi(tu) = \psi(u) \pm 1$, so that $\psi(tu) \le \psi(u) + 1$. Considered as an element of $A_{2n}$ however, $t \in T_G$, so that $\rho_G(tu) \le \rho_G(u) + 1$ by Proposition 9. Consequently,

$$\begin{aligned} 2\,\rho_L(tu) &= \rho_G(tu) + \psi(tu) \\ &\leq 1 + \rho_G(u) + 1 + \psi(u) \\ &= 2 + 2\,\rho_L(u). \end{aligned}$$

Thus $\rho_L(tu) \leq 1+\rho_L(u)$ as desired. Again, equality holds if and only if equality holds in the above equation, and again this holds if and only if $u \leq_L tu$ by Theorem 13. □

## 5 Generalizations to other Coxeter groups

We will now generalize the procedure for creating the Grassmann and Lagrange orders for the Coxeter groups of type $A_n$ and $B_n$ to Coxeter groups of type $D_n$, $E_6$, and $E_7$ and exhibit the corresponding $T$-orders.

We first consider certain sub-orders of the Bruhat-Chevalley order $\leq_{BC}$. Let $(W, \{r_i\}_{i\in I})$ be a Coxeter system with diagram $M$, and label the nodes of the diagram in the standard way by the elements of $I = \{1, 2, \ldots, n\}$ (see Bourbaki [5]). Recall that $u \prec_{BC} v$ in the Bruhat-Chevalley order on $W$ if

(1) $ur = v$ for some reflection $r$ and
(2) $l(u) + 1 = l(v)$,

where $l(w)$ denotes the length of a reduced expression in the generators $r_i$ for $w \in W$. This partial order plays a key role in Chevalley's formula for computing certain products in the Chow ring associated to the Chevalley group with diagram $M$ (see Proposition 10 in [6]). As we do not wish to go into the background of this formula, we will simply state it and indicate our interest in it. We refer the interested reader to Chevalley's paper [6].

We first introduce some notation needed only in the remainder of this section. Let $\Phi$ be the root system of type $M$ and let $\Delta$ be a $\mathbb{Z}$-basis of size $n$ for $\Phi$ (in fact there are many such bases). Let $\Phi^+$ denote the set of positive roots, that is, the roots whose coefficients on the basis $\Delta$ are all positive. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. For any root $\alpha$ let $r(\alpha)$ be the reflection sending $\alpha$ to $-\alpha$. Then for each $i = 1, 2, \ldots, n$, the fundamental reflection $r_i$ equals $r(\alpha_i)$.

Chevalley's formula explicitly describes multiplication of cohomology classes $X(w)$, $w \in W$, as follows (here $S_i = X(w_0 r_i)$ with $w_0$ the longest element of $W$):

$$[S_i][X(v)] = \sum_u c(\alpha_i, \beta) X(v).$$

| $M$ | $\mathrm{Min}(M)$ |
|---|---|
| $A_n$ | $\{1, 2, \ldots, n\}$ |
| $B_n$ | $\{n\}$ |
| $C_n$ | $\{1\}$ |
| $D_n$ | $\{1, n-1, n\}$ |
| $E_6$ | $\{1, 6\}$ |
| $E_7$ | $\{7\}$ |
| $E_8, F_4, G_2$ | $\emptyset$ |

Table 1
Dynkin diagrams and their Minuscule weights

Here $u$ runs over all elements in $W$ with $u \prec_{BC} v$ such that $v = ur(\beta)$. For $\alpha \in \Delta$ and $\beta \in \Phi$ the coefficient $c(\alpha, \beta)$ is defined as follows:

$$c(\alpha, \beta) = a(\alpha, \beta) \frac{\langle \alpha \mid \alpha \rangle}{\langle \beta \mid \beta \rangle},$$

where $a(\alpha, \beta)$ is the coefficient of $\alpha$ in the expression for $\beta$ as a linear combination of elements in $\Delta$ and $\langle \cdot \mid \cdot \rangle$ is the bilinear form on $\Phi$.

**Remark 16** *Chevalley's formula works for any spherical diagram $M$. If we take $M = A_n$ and $i = k$, then in the sum we get a coefficient of 1 exactly for those $u$ such that $u \prec_k v$ in the $k$-Bruhat order, and all the other coefficients are 0. Note that this is also true for Monk's formula.*

*If we take $M = B_n$ and $i = n$, then in the sum we get a coefficient of 1 exactly for those $u$ such that $u \prec_0 v$ in the 0-Bruhat order, and all the other coefficients are 0.*

We will will use Remark 16 as the characteristic property to generalize the $k$-Bruhat and 0-Bruhat orders.

Consider the sets

$$\mathrm{Min}(M) = \{i \in I \mid c(\alpha_i, \beta) \in \{0, 1\} \text{ for all } \beta \in \Phi^+\}.$$

One can then verify that we get one of the pairs in Table 1. Notice that $\mathrm{Min}(M)$ only depends on $M$ since $W$ acts as a linear isometry group on $\Phi$ that is transitive on the collection of $\mathbb{Z}$-bases for $\Phi$.

We now define a partial order on the Coxeter group with diagram $M$ that coincides with the $k$-Bruhat order for $A_n$ and the 0-Bruhat order for $B_n$.

**Definition 17** *For $\mu \subseteq \text{Min}(M)$, let $u \leq_\mu v$ if and only if*

*(1) $ur = v$, where $r \in \mathcal{R}$, and*
*(2) $l(u) + 1 = l(v)$*

*and take the transitive closure. Here*

$$\mathcal{R} = \begin{cases} \bigcup_{m\in\mu}\{r_m\}^{W_{I-\{m\}}} & \text{if } M = A_n, D_n, E_6, E_7, \\ \{r_n, r_{n-1}^{r_n}\}^{W_{I-\{n\}}} & \text{if } M = B_n, \\ \{r_1\}^{W_{I-\{1\}}} \cup \{r_n^{r_{n-1}\cdots r_1}\} & \text{if } M = C_n. \end{cases}$$

*We call this order the* $\mu$-Bruhat-Chevalley order *on $W$.*

For $M = A_n$ and $\mu = \{k\}$ we have the $k$-Bruhat order defined in [1], while for $M = B_n$ and $\mu = \{n\}$ we have the 0-Bruhat order defined in [2].

**Remark 18** *Note that for $M = A_n$, $D_n$, $E_6$, and $E_7$ the set $\mathcal{R}$ is the collection of all reflections $r_\beta$ such that $a(\alpha_m, \beta) = 1$. For these diagrams we have $a(\alpha_m, \beta) = c(\alpha_m, \beta)$.*

*For $M = B_n$ the set $\{r_n\}^{W_{I-\{n\}}}$ is the collection of reflections $r_\beta$ such that $a(\alpha_n, \beta) = c(\alpha_n, \beta) = 1$ and the reflections $r_\beta \in \{r_{n-1}\}^{r_n W_{I-\{n\}}}$ where $a(\alpha_n, \beta) = 2$ and $\langle \beta \mid \beta \rangle = 2$.*

*For $M = C_n$ the set $\{r_1\}^{W_{I-\{1\}}}$ is the collection of reflections $r_\beta$ such that $a(\alpha_1, \beta) = c(\alpha_n, \beta) = 1$ and the reflection $r_n^{r_{n-1}\cdots r_1}$ where $a(\alpha_1, \beta) = 2$ and $\langle \beta \mid \beta \rangle = 2$.*

We are now in a position to define the generalizations of the Grassmann and Lagrange order. Define the *rooted $\mu$-Bruhat-Chevalley order* on $W$ by setting $u \leq_{\underline{\mu}} v$ if and only if there exists $\zeta \in W$ such that

$$\zeta \leq_\mu u\zeta \leq_\mu v\zeta$$

and take the transitive closure. Note that the identity $e \in W$ is the unique minimal element for this partial order.

For $M = A_n$ and $\mu = \{1, 2, \ldots, n\}$ the rooted $\mu$-Bruhat-Chevalley order is the Grassmann order. For $M = B_n$ and $\mu = \{\alpha_n\}$ the rooted $\mu$-Bruhat-Chevalley order is the Lagrange order.

Bergeron and Sottile proved that in the Grassmann order all reflections cover the identity. This is true also of the other simply laced rooted $\mu$-Bruhat-Chevalley orders given $\mu$ as in Table 1.

**Proposition 19** *Let $(W, \{r_i\}_{i\in I})$ be a Coxeter group with diagram $M$ defined over $I$ and let $\mu \subseteq I$ be non-empty. Let $(M, \mu)$ be one of the following:*

*(1)* $M = A_n$, $\mu \subseteq \{1, 2, \ldots, n\}$,
*(2)* $M = D_n$, $\mu \subseteq \{1, n-1, n\}$,
*(3)* $M = E_6$, $\mu \subseteq \{1, 6\}$,
*(4)* $M = E_7$, $\mu = \{7\}$.

*Then in the rooted $\mu$-Bruhat-Chevalley order on $W$ all reflections cover the identity.*

In the proof we will use the following notation. For $J \subseteq I$ we let $M_J = (m_{kl})_{k,l\in J}$ and $W_J = \langle r_j \mid j \in J\rangle$. One can verify that $(W_J, \{r_j\}_{j\in J})$ is a Coxeter system with diagram $M_J$. For $u, v \in W$ we write the conjugate of $u$ under $v$ as $u^v = v^{-1}uv$.

**Proof:** We will prove the following: Given $m_1 \in \mu$ and a reflection $r \in W$ there exists a $K = \{m_1, m_2, \ldots, m_k\} \subseteq I$ and $w \in W_{I-K}$ with $r = {r_{m_1}}^{(r_{m_2}r_{m_1})(r_{m_3}r_{m_2})\cdots(r_{m_k}r_{m_{k-1}})w}$. Moreover, $M_K$ is of type $A_k$ and $m_1$ labels one of the end-nodes.

Let $r$ be any reflection. We proceed by induction on the pairs $(M, \mu)$ in the list. If $M = A_1$ and $\mu = \{1\}$ then $r$ can only be $r_1$ and we must take $K = \{1\}$ and $w = e$.

Now suppose $M \neq A_1$. Then since $M$ is simply-laced and connected any two reflections are conjugate and so $r \in {W_{\{m_1\}}}^W$. It was proved by Blok and Brouwer [4] that

$$ {W_{\{m_1\}}}^W = \bigcup_{j\in I} W_{\{j\}}^{W_{I-\{m_1\}}}, $$

and so $r \in W_{\{j\}}^{W_{I-\{m_1\}}}$ for some $j \in I$.

If $r \notin W_{I-\{m_1\}}$ then $r = {r_{m_1}}^w$ for $w \in W_{I-\{m_1\}}$ and we can take $K = \{m_1\}$.

Otherwise $r \in W_{I-\{m_1\}}$. In fact $r \in W_{I_2}$ for some connected component $M_{I_2}$ of $M_{I-\{m_1\}}$ for, if $M_{I-\{m_1\}}$ is disconnected then generators corresponding to distinct connected components of $M_{I-\{m_1\}}$ commute and $r$ is the conjugate of only one of these generators. Let $m_2$ be the unique neighbor of $m_1$ in $M_2 = M_{I_2}$. One verifies easily that if $\mu_2 = \{m_2\}$ then $(M_2, \mu_2)$ is again in the above list.

By induction we can now find $K_2 = \{m_2, m_3, \ldots, m_k\} \subseteq I_2$ and $w \in W_{I_2-K_2}$ with $r = {r_{m_2}}^{(r_{m_3}r_{m_2})(r_{m_4}r_{m_3})\cdots(r_{m_k}r_{m_{k-1}})w}$. In addition $M_{K_2}$ is of type $A_{k-1}$ and $m_2$ labels one of the end-nodes.

Substituting $r_{m_2} = {r_{m_1}}^{r_{m_2}r_{m_1}}$ we are done. It is clear that the diagram induced on $K = \{m_1, m_2, \ldots, m_k\}$ is of type $A_k$ and that $m_1$ labels one of the end-nodes. Notice that $r_{m_1}{}^{(r_{m_2}r_{m_1})(r_{m_3}r_{m_2})\cdots(r_{m_k}r_{m_{k-1}})} = r_{m_k}$.

Let $r$, $m_1$, $K = \{m_1, m_2, \ldots, m_k\}$, and $w \in W_{I-K}$ be as above. Let $\eta = (r_{m_2}r_{m_1})(r_{m_3}r_{m_2})\cdots(r_{m_k}r_{m_{k-1}})$ so that $r = r_{m_1}{}^{\eta w}$. Set $\zeta = (\eta w)^{-1}$. We now prove that $\zeta \leq_\mu e\zeta \prec_\mu r\zeta$.

It is clear that $\zeta \leq_\mu e\zeta = \zeta$. As for the second inequality we have $\zeta r_{m_1} = r\zeta$ where $m_1 \in \mu$. Thus in order to show $\zeta \prec_\mu r\zeta$ we only have to establish that $l(\zeta) + 1 = l(\zeta r_{m_1})$.

First of all, since $w \in W_{I-K}$ and $\eta, r_{m_1} \in W_K$ we have $l(\zeta) = l(w^{-1}) + l(\eta^{-1})$ and $l(\zeta r_{m_1}) = l(w^{-1}) + l(\eta^{-1} r_{m_1})$ so we may assume that $w = e$.

Now $\zeta$ and $\zeta r_{m_1}$ are elements of a Coxeter group of type $A_k$ where $m_1$ is the label of one of the end-nodes. Writing $i$ for $r_{m_i}$ we get $\zeta = \eta^{-1} = ((k-1)k)\cdots(23)(12)$ and $\zeta r_{m_1} = ((k-1)k)\cdots(23)(12)1$. Now using that non-neighboring labels commute we can rewrite these as $\zeta = ((k-1)(k-2)\cdots 21)(k(k-1)\cdots 2)$ and $\zeta r_{m_1} = ((k-1)(k-2)\cdots 21)(k(k-1)\cdots 21)$.

Keeping this notation one can easily check that $1(21)(321)\cdots(k(k-1)\cdots 21)$ is an expression of length $k(k+1)/2$ for the longest element $w_0$ in the Coxeter group of type $A_k$ and hence is the reduced expression for this element. Since the above expressions for $\zeta$ and $\zeta r_{m_1}$ are subwords of the above expression for $w_0$, they are reduced and so their length is simply the number of generators in these expressions. We find $l(\zeta) + 1 = l(r\zeta)$ so that $\zeta \leq_\mu e\zeta \prec_\mu r\zeta$.

This implies $e \prec_{\underline{\mu}} r$. Suppose that this is not the case and $s \in W$ satisfies $s \leq_{\underline{\mu}} r$. Then there is some element $\alpha$ with $\alpha \leq_\mu s\alpha \leq_\mu r\alpha$. As $r$ is a fundamental reflection we have $l(r\alpha) \leq l(\alpha) + 1$, so either $s = e$ or $s = r$. □

From the rooted $\mu$-Bruhat-Chevalley order we construct a $T$-order in the following way.

$$T = \{r \text{ a reflection of } W \mid u \prec_{\underline{\mu}} ur \text{ for some } u \in W\}.$$

It was proved by Bennett et al. ([3]) that this method applied to the Grassmann order yields a $T$-order which is in fact equivalent to the Grassmann order. Similarly, Theorem 3 shows that this method applied to the Lagrange order yields a $T$-order which is in fact equivalent to the Lagrange order. As the Grassmann order is the rooted $\mu$-Bruhat-Chevalley order for $M = A_n$ and $\mu = \{1, 2, \ldots, n\}$ and the Lagrange order is the rooted $\mu$-Bruhat-Chevalley order for $M = B_n$ and $\mu = \{n\}$, the above orders in some sense generalizes the Grassmann and Lagrange order for the Coxeter groups of type $D_n$, $E_6$, and $E_7$.

We can now prove Theorem 4

**Proof** (of Theorem 4)**:** In the construction of the set $T$ above take $u = e$. Then according to Proposition 19 we can take $v = r$ to be any reflection. $\square$

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