On the Cohomology of the Finite Chevalley Groups

Corneliu Hoffman

March 28, 2005

Abstract

Let $G$ is a group of Lie type, $K$ an algebraically closed field and $V$ a faithful irreducible $KG$ module. The aim of this paper is to provide bounds for $\dim H^1(G, V)$ as a function of $\dim V$.

1 Introduction and statement of results

Let $G$ be a finite group and $k$ a field of characteristic $p > 0$. Let $V$ a faithful $kG$ module. In the case when $G$ is a finite Chevalley group, explicit results for $H^1(G, V)$ have been obtained for specific modules (see [JP]) or for small rank groups (see [S1], [S2], [S4]). We will be concerned with bounding the dimension of $H^1(G, V)$ as a function of the dimension of $V$. The following result (see [AS]) shows that one can restrict ones attentions to simple groups and irreducible modules.

Theorem 1 If $G$ acts faithfully on $V$, then either $H^1(G, V) = 0$ or there exists a subnormal simple subgroup $L$ of $G$ such that

$$\dim H^1(G, V) \leq \dim H^1(L, W)$$

for any nontrivial irreducible $KL$ submodule of $W$ of $V$.

In [AG] it was shown that $\dim H^1(G, V) < \dim V$, and later in [G] the result was improved to $\dim H^1(G, V) \leq \frac{2}{3} \dim V$ and it was conjectured in that a better bound can be obtained (1/2 instead of 2/3). On the other hand concrete results as in [JP] or [S1] [S2] [S4] suggest that the universal bound is very crude and that better bounds could be obtained for particular groups. (In particular there is no example of a finite simple group $G$ and an irreducible $kG$ module $V$ with $\dim H^1(G, V) > 2$).

This paper will be concerned only with the case when $G$ is a finite Chevalley group. The main result is:

---

*This paper is part of my Ph.D. dissertation written at the University of Southern California. I will take this opportunity to thank my advisor, Robert Guralnick, for the encouragements and countless suggestions he gave me while I was writing this paper. I would like to thank the referee of the paper for pointing out the errors in the first version and for some useful references that simplified some of the arguments.
**Theorem 2** Let $G = G(q)$ be a finite simple group of Lie type. If $V$ is an irreducible $kG$ module where $char k = p$, then:

$$\dim H^1(G, V) \leq c \dim V.$$ 

where $c$ is as in Table 1 if $p$ does not divide $q$ and as in Table 2 if $p/q$.

The case of $G = A_n$ was considered in [GK] and the case of sporadic groups in [GH]. In particular in [GH] these results were combined to obtain:

**Theorem 3** If $G$ is a finite group, $k$ a field with $char k = p$ and $V$ an irreducible $kG$ module which is faithful as a $G$-module, then:

$$\dim H^1(G, V) \leq \frac{1}{2} \dim V$$

and the inequality is strict unless $F^*(G) = L_2(2^n), n > 1$ with $V$ the natural module (or a Frobenius twist of it) or $F^*(G) = A_6$ and $V$ the 4-dimensional heart of the natural permutation module in characteristic 3.

Since we have a duality between cohomology and homology for finite groups (see [Br]), we obtain the following which has recently been used by Segal and Shalev[SS].

**Corollary 1.1** If $G$ is a finite group, $k$ a field and $V$ an irreducible $kG$ module which is faithful as a $G$-module, then:

$$\dim H_1(G, V) \leq \frac{1}{2} \dim V$$

and the inequality is strict unless $G$ is of the type $L_2(2^n)$ with $V$ the natural module (or a twist of it) or $F^*(G) = A_6$ and $V$ the heart of the natural permutation module in characteristic 3 (in particular $V$ has dimension 4).

In the next two sections we will describe the techniques used in the proof of the main theorem and in the final one we’ll proceed with the case by case analysis.

## 2 Cross Characteristic

Consider $G$ a finite group and $V$ an irreducible $G$ module. Let:

$$[G, V] := \{gx - x \mid g \in G, x \in V\},$$

$$C_V(G) := V^G,$$

$$Der(G, V) := \{\text{set of derivations from } G \to V\}$$

$$IDer(G, V) := \{\delta \in Der(G, V) \mid \exists m \in V \text{ with } \delta(g) = gm - m\}$$
Note that $H^1(G, V) = \text{Der}(G, V)/I\text{Der}(G, V)$ and that in the case when $V$ is faithful as a $G$ module $\dim I\text{Der}(G, V) = \dim V$.

Also if $B \leq G$ then

$$\text{Der}_B(G, V) := \{\tau \in \text{Der}(G, V) \mid \tau \text{ is trivial on } B\}.$$ 

One of our main tools will be the following lemma (cf. [GK]):

**Lemma 2.1** Let $G$ and $V$ as before and suppose $T_1 \leq G$ and $T_2 \leq G$ so that $G = \langle T_1, T_2 \rangle$, $B \subset T_1 \cap T_2$, and $T_i = \langle B, S_i \rangle$ where $S_i \triangleleft T_i$ and both $S_i$ are $p'$ groups. Then:

a) $\dim \text{Der}(G, V) \leq \dim \text{Der}(T_1, V) + \dim \text{Der}_B(T_2, V)$.

b) $\text{Der}(T_i, V) \cong \text{Der}(B, V^{S_i}) \oplus [S_i, V]$. In particular if $B$ is cyclic, $\dim \text{Der}(T_i, V) \leq \dim V$.

c) $\text{Der}_B(T_i, V) \cong [S_i, V]^B$.

In most of the cases when we will need to apply Lemma 2.1, the corresponding $S_i$’s will either be abelian or will admit a filtration with $B$ invariant subgroups such that the factors are abelian. In this special case we will use the following:

**Corollary 2.1** With the notation of Lemma 2.1, suppose that there exists a $B$ invariant filtration

$$1 = R_k \triangleleft R_{k-1} \triangleleft \cdots \triangleleft R_0 = S_2$$

such that $\forall i, R_i/R_{i-1}$ is abelian, $\dim V^{S_1} \leq \dim V^{S_2}$, and there is $C \geq 2$ such that $\dim H^1(B, V) \leq (1/C)\dim V$ and all orbits of the action of $B$ on the nontrivial characters of $R_{i+1}/R_i$ have lengths $\geq C$ (In fact most of the time $C$ will be the bound we already have for the subgroup $B$). Then:

$$\dim H^1(G, V) \leq \frac{1}{C} \dim V \forall i.$$ 

**Proof.** Using Lemma 2.1 we get:

$$\dim \text{Der}(G, V) \leq \dim \text{Der}(T_1, V) + \dim \text{Der}_B(T_2, V) = \dim \text{Der}(B, V^{S_1}) + \dim [S_1, V] + \dim [S_2, V]^B.$$ 

From our hypothesis we can deduce that:

$$\dim \text{Der}(B, V^{S_1}) \leq \frac{1}{C} \dim V^{S_1} + \dim V^{S_1}.$$ 

Therefore we should consider the action of $B$ on $[S_2, V]$. We can decompose $V$ as a sum of irreducible $S_2$ modules ($S_2$ is a $p'$ group). The action of $B$ will permute those modules.
Consider $W_0 = [S_2, V]$ and note that $W_0 = [R_1, W_0] \oplus C_{W_0}(R_1)$ as a $B$ module and that $\dim C_{W_0}(R_1)^B \leq \frac{1}{C} \dim C_{W_0}(R_1)$. Inductively let $W_i = [R_i, W_{i-1}] \oplus C_{W_i}(R_{i+1})$ as $B$ modules and in consequence $\dim C_{W_i}(R_{i+1})^B \leq \frac{1}{C} \dim C_{W_i}(R_{i+1})$. Repeating the procedure until $i = k - 1$, and collecting the results we obtain:

$$\dim [S_2, V]^B = \sum_{i=0}^{k-1} \dim C_{W_i}(R_{i+1})^B \leq \frac{1}{C} \sum_{i=0}^{k-1} \dim C_{W_i}(R_{i+1}) = \frac{1}{C} \dim [S_2, V]$$

Therefore

$$\dim H^1(G, V) \leq \frac{1}{C} \dim V^{S_2} + \frac{1}{C} \dim [S_2, V] \leq \frac{1}{C} \dim V.$$

Next let $G$ a Chevalley group over $\mathbf{F}_q$ and let $V$ an irreducible $KG$ module where $K$ is a field (we can assume algebraically closed) of characteristic $p$ not dividing $q$. Then take $P_J = L_JHQ_J$ a maximal parabolic subgroup of $G$, $P_J'$ the opposite parabolic, $L_JH$ their common Levi factor ($L_J = O^{p'}(L_JH)$). Choose the set $J$ so that $B \cong L_J$ is a Chevalley group for which we already have a bound, say $\dim H^1(B, V) \leq \frac{1}{C_0} \dim V$ for some $C_0$. Then we can take $T_1 = BQ_J$, $T_2 = BQ_J'$ in Lemma 2.1.

For the construction of the filtration we can use the results in [ABS] concerning the action of the Levi factor of a parabolic subgroup on the corresponding unipotent radical. If $(G, \text{char} (\mathbf{F}_q))$ is one of $(B_n, 2)$, $(C_n, 2)$, $(F_4, 2)$, $(G_2, 2$ or $3)$, we say that $G$ is of special type and the results in [ABS] will not hold as described. However we will see that even in the special cases we can still apply similar results. Note that for classical groups in mixed characteristic we do not really need these results since we can deduce directly how $B$ acts.

To fix the notation, let $G$ be a Chevalley group with root system $\Phi$, $\Delta$ a fundamental set of roots, $J$ a subsystem of $\Delta$, and, $P_J$ the parabolic subgroup of $G$ corresponding with the subsystem $J$. Then $P_J = Q_JL_JH$ where:

$$Q_J := \langle X_\alpha : \alpha \in \Phi^+ \setminus ZJ \rangle,$$

$$L_J := \langle X_\alpha : \alpha \in \Phi \cap ZJ \rangle,$$

and $H$ is a Cartan subgroup normalizing $U := \langle X_\alpha : \alpha \in \Phi^+ \rangle$. Let also $L := L_JH$. Then if $\beta \in \Phi^+ \setminus (\Phi^+ \cap ZJ)$ write $\beta = \beta_J + \beta_{J'}$ where $\beta_J = \sum c_j \alpha_j$ the sum ranging over all $\alpha_j \in J$ and $\beta_{J'} = \sum d_i \alpha_i$ with the sum ranging over all $\alpha_i \in \Delta \setminus J$. Let:

$$\text{height}(\beta) := \sum c_j + \sum d_i,$$

$$\text{level}(\beta) := \sum d_i,$$

$$\text{shape}(\beta) := \beta_{J'}.$$
\[ Q(i) := \prod_{\text{level}(\beta) \geq i} X_{-\beta} \]
\[ V_S := \prod_{\text{shape}(\beta) = S} X_{-\beta} \]

Also let \( Q^{(i)} \) be the \( i \)th term in the descending central series for \( Q \). Then (cf. Theorem 2 in [ABS]):

**Lemma 2.2**

a) In the above notation, we have:

i) For each \( i \geq 1 \), \( Q^{(i)}/Q^{(i+1)} = \prod V_S \) the product ranging over all shapes of level \( i \).

ii) If \( S \) is a shape of level \( i \) then \( V_S \) is an irreducible \( \mathbb{F}_q \) module of highest weight \(-\beta\) where \( \beta \) is the unique root of minimal height and shape \( S \).

b) Also if \( G \) is a twisted Chevalley group corresponding to the graph automorphism \( \tau \) and if we consider the construction above at the level of the algebraic group, we get that:

i) If \( V^\tau_S = V_S \) then \( V_S \) is an absolutely irreducible \( \mathbb{F}_q \) module.

ii) If \( V^\tau_S \neq V_S \), then \((V_S + V^\tau_S)_{\tau}(\text{respectively } (V_S + V^\tau_S + V^{\tau^2}S)_{\tau})\) is an absolutely irreducible \( \mathbb{F}_{q^2} \) (respectively \( \mathbb{F}_{q^3} \)) module.

c) In the case when \( G \) is a group of special type, and if there are roots of different length having the same shape \( S \), then the decomposition will still hold but \( V_S \) will be indecomposable of length \( \leq 2 \).

Unfortunately in some cases we will have some one dimensional factors and that will not give too much information. However in those cases we will use the following form of Clifford’s theorem (cf 11.20 of [CR81]):

**Theorem 4** Let \( Q \) be a normal subgroup of a group \( R \) and \( k \) an algebraically closed field. Let \( M \) be a simple \( kR \)-module, and \( L \) a simple \( kQ \) submodule of \( M_Q \) such that \( L \) is stable relative to \( R \) (that is \( L \) is isomorphic to all its conjugates). Let \( G = R/Q \). Then

\[ M \cong L \otimes_K I \]

for a left ideal \( I \) in the \( G \) graded algebra \( E = \text{End}_{kR}L^R \). The \( R \) action on \( L \otimes_K I \) is given by:

\[ x \rightarrow U(x) \otimes V(x), x \in R \]

where \( U : R \rightarrow GL(L) \) is a projective representation of \( R \) on \( L \) and \( V \) is a projective representation of \( G \). The factor sets associated with \( U \) and \( V \) are inverses of each other.
Corollary 2.2 Let $G \cong SL_2(p^n), p^n = q > 4$, and $Q$ an extraspecial $p$ group of order $p^{2n+1}$ such that $G$ acts trivially on $Z(Q)$ and as the natural module on $Q/Z(Q)$. If $M$ is an irreducible representation of the semidirect product $R = QG$ such that $M_Q$ is not linear, then $M = W \otimes W'$ where $W$ is an nonlinear irreducible representation of $Q$ and $W'$ is a projective representation of $G$.

Proof. Using Theorem 4 we only need to prove that one can extend any nonlinear representation $\Psi$ of $Q$ to one of $R$. Let $W$ an irreducible character of $Q$ that is not linear. Then since this character vanishes outside of the center and since $G$ centralizes the center, $W^g = W$ where by $W^g$ we mean the twist by $g$ of the action of $Q$. Therefore there is some $x_g \in GL(W)$ such that $\Psi(h^g) = (\Psi(h))^{x_g}$ for all $h \in Q$. The map $g \rightarrow x_g$ gives an isomorphism $G \cong \langle x_g \rangle$ and $C_{GL(W)}(\Psi(Q)) = Z(GL(W))$ and therefore $\langle x_g \rangle$ is a central extension of $G$. However unless $q = 9$, $G$ is its own universal central extension hence this gives an extension of $\Psi$.

In the remaining case ($q = 9$) we note that up to isomorphism there is a unique extraspecial group of exponent 3 and order $3^{4+1}$. We also know that $C_{Aut(Q)}(Z(Q)) = 3^4Sp_4(3)$

$G$ is isomorphic to a subgroup of a copy of $Sp_4(3)$ in $C_{Aut(Q)}(Z(Q))$ and after repeating the argument above replacing $G$ with $Sp_4(3) \subset C_{Aut(Q)}(Z(Q))$, we see that we can extend $W$ to $Sp_4(3)$ (because of $Sp_4(3)$ has trivial Schur multiplier). Therefore we can extend the character to the semidirect product. \qed

In particular since by [LS] we have lower bounds on the dimension of the smallest irreducible projective representations of a group of Lie type, we will be able to bound below the orbit of a given character of $Q$.

2.1 Classical Groups

2.1.1 Linear Groups

Let $G \cong PSL_n(q)$ where $p$ does not divide $q$ and assume as before that $V$ is an irreducible $KG$ module where $char(K) = p$. Also for simplicity we can assume that $K$ is algebraically closed.

We will construct the $T_i = \langle S_i, B \rangle$ where:

$$S_1 := \langle \begin{pmatrix} 1 & v \\ 0 & I_{n-1} \end{pmatrix} \mid v \in F_q^{n-1} \rangle$$

$$S_2 := \langle \begin{pmatrix} 1 & 0 \\ v & I_{n-1} \end{pmatrix} \mid v \in F_q^{n-1} \rangle$$

$$B := \langle \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \rangle$$
where $A \in GL(n-1, q)$ has order $q^{n-1} - 1$ and $a = (\det A)^{-1}$.

Now we can easily see that both $S_i$'s have order $q^{n-1}$ which is prime to $p$ so we are in the settings of the Lemma 2.1. In particular $B$ is cyclic so:

$$\dim H^1(G, V) \leq \dim [S_2, V]^B.$$ 

Also we can see that both $S_i$ are abelian and that the action of $B$ on them is actually the natural representation for $a^{-1}A$ as an element of $GL_{n-1}(q)$. The action of $B$ on the characters is the dual representation for $B$.

Let $x$ be the generator of $B$. Then $y := x^{\frac{q^{n-1} - 1}{q-1}} = \text{diag}(b, \lambda, \ldots \lambda)$ where $b = a^{\frac{q^{n-1} - 1}{q-1}}$. Clearly $y^k$ is a scalar iff $\lambda^k$. Since $y \in SL_n(q)$, $b^k(\lambda^k)^{n-1} = I_n$. This means that $y^k$ is a scalar iff $(\lambda^k)^n = 1$. The smallest $k$ with this property is $(q-1)/\gcd(n, q-1)$ and so the order of $x$ is $(q^{n-1} - 1)/\gcd(n, q-1)$. In conclusion $B$ will act freely on the characters of $S_i$ and

$$\dim [S_i, V]^B = \frac{\gcd(n, q-1)}{q^{n-1} - 1} \dim [S_i, V] \leq \frac{\gcd(n, q-1)}{q^{n-1} - 1} \dim V.$$  

We proved:

**Proposition 2.1.1** If $G \cong PSL_n(q)$, $K$ is a field of characteristic $p | q$ and $V$ is an irreducible $KG$ module, then

$$\dim H^1(G, V) \leq \frac{\gcd(n, q-1)}{q^{n-1} - 1} \dim V.$$  

There are some problems for $n = 2$ and $q = 2, 3, 5$, because in these cases the bound will not be good enough. However we know that:

$$PSL_2(2) \cong S_3, PSL_2(3) \cong A_4, PSL_2(5) \cong A_5.$$ 

### 2.1.2 Symplectic Groups

Let $G \cong PSp_{2n}(q)$ and assume the same conditions for $K$ and $V$ as in 2.1.1. Then $G$ is of type $C_n(q)$. Then we have a fundamental system as follows:

$$e_1 - e_2 \quad e_2 - e_3 \quad \ldots \ldots \quad e_{n-1} - e_n \quad 2e_1$$

The full system of roots is:

$$\Phi = \left\{ \begin{array}{ll} \pm e_i \pm e_j; & i \neq j \\
\pm 2e_i; & i = 1, \ldots, n \end{array} \right. \quad i, j = 1, \ldots, n$$

We will consider a parabolic of type $A_{n-1}$ so the corresponding subsystem is:

$$J = \{ e_1 - e_2, \ldots, e_{n-1} - e_n \}.$$
$Q_J$ is an irreducible $n(n+1)/2$-dimensional $L_J$ module so it is the symmetric square of the natural representation of $L_{n-1}$. If $q$ is even, this module will be reducible but indecomposable of length 2, with factors isomorphic to $\Lambda^2 N$ and a Frobenius twist of $N$. If $q$ is odd the module is already irreducible. In particular the action of the Levi complement is nontrivial and we can use Corollary 2.1. The index of a proper subgroup of $L_n$ is (cf. [KL]) at least $(q^n - 1)/(q - 1)$ if $(n, q) \neq (2, 5), (2, 7), (2, 9), (2, 11), (4, 2)$ and respectively 5, 7, 6, 11, 8 in those last cases.

These bounds are better than $(q^n - 1 - 1)/\gcd(n, q - 1)$ (the bound for $L_n$) except for the case $(2, 9)$. This last case can be solved directly, as in 2.1.1 and still get a bound of $\frac{1}{q - 1} \dim V$. So we get:

**Proposition 2.1.2** If $G \cong PSp_{2n}(q)$, $K$ is a field of characteristic $p \mid q$ and $V$ is an irreducible $KG$ module, then

$$\dim H^1(G, V) \leq \frac{\gcd(n, q - 1)}{q^n - 1 - 1} \dim V.$$ 

Again if $n = 2$ and $q = 2, 3, 5$ the bounds are not satisfactory. However in this case we can use an argument similar with that one in 2.1.1. For that consider

$$S_1 := \left\langle \left( \begin{array}{cc} I_2 & S \\ 0 & I_2 \end{array} \right) \mid S = tS \right\rangle$$

$$S_2 := \left\langle \left( \begin{array}{cc} I_2 & 0 \\ S & I_2 \end{array} \right) \mid S = tS \right\rangle$$

$$B := \left\langle \left( \begin{array}{cc} A & 0 \\ 0 & tA^{-1} \end{array} \right) \right\rangle$$

where $A \in GL_2(q)$ has order $q^2 - 1$. We can then apply Lemma 2.1 b) in the special case when $B$ is cyclic. Let $a$ be the generator of $B$. We have to bound the length of the shortest orbit of the action of $a$ on $S_1$. Now we can see that in fact if $a^k$ fixes an element of $S_1$, this is equivalent to saying that $a$ is in $O_2(q)$. However $O_2^+(q) \cong \mathbb{Z}_{(q^2 - 1)/\gcd(2, q - 1)}$. Therefore the maximum order of $a^k$ is $(q \pm 1)/\gcd(2, q - 1)$. In other words $k \geq (q^2 - 1)/\gcd(2, q - 1) = (q - 1)\gcd(2, q - 1)$. This means that all orbits will have length $\geq (q - 1)\gcd(2, q - 1)$. This is still not effective if $q = 2$. We know that $Sp_4(2) \cong S_6$ and $p \neq 2$ so $A_6$ already contains the $p$ Sylow so we can apply Corollary 3 of [GK] and get a bound of 1/3. Therefore:

**Proposition 2.1.3** If $G \cong PSp_4(q), q \leq 5$, $K$ is a field of characteristic $p \mid q$ and $V$ is an irreducible $KG$ module, then

$$\dim H^1(G, V) \leq \frac{1}{\max\{3, (q - 1)\gcd(2, q - 1)\}} \dim V.$$ 

8
2.1.3 Orthogonal Groups

2.1.4 $\Omega_{2n+1}(q), n \geq 3, q \text{ odd}$

Let $G \cong \Omega_{2n+1}(q)$ and assume the same conditions for $K$ and $V$ as in 2.1.1. Then $G$ is of type $B_n(q)$. We have a fundamental system as follows:

\[
\begin{array}{cccccc}
\epsilon_1 - \epsilon_2 & \epsilon_2 - \epsilon_3 & \cdots & \epsilon_{n-1} - \epsilon_n & \epsilon_n
\end{array}
\]

The full system of roots is:

\[
\Phi = \{ \pm \epsilon_i \pm \epsilon_j; \ i \neq j, \ i, j = 1, \ldots, n \}
\]

Again take $J = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$ so the parabolic is of type $A_{n-1}$. In this case the unipotent radical is not going to be abelian. We can see that the order of $Q_J$ is also $q^{n(n+1)/2}$ but there are two shapes involved $\beta_1 = e_n$ and $\beta_2 = 2e_n$. We have that $\dim V_{\beta_1} = n$ and $\dim V_{\beta_2} = n(n-1)/2$ so $Q/Q^{(2)}$ is the natural representation of $L_n$ and $Q^{(2)}$ is the exterior square of the natural representation. In particular none of them is trivial and we can apply Corollary 2.1. A similar computation as in 2.1.2 yields

**Proposition 2.1.4** If $G \cong \Omega_{2n+1}(q), n \geq 3, q \text{ odd}, K$ is a field of characteristic $p \nmid q$ and $V$ is an irreducible $KG$ module, then

\[
\dim H^1(G, V) \leq \frac{\gcd(n, q-1)}{q^{n-1} - 1} \dim V.
\]

2.1.5 $\Omega_{2n}^+(q), n \geq 4$

Let $G \cong \Omega_{2n}^+(q)$ and assume the same conditions for $K$ and $V$ as in 2.1.1. Then $G$ is of type $D_n(q)$ and so we have a fundamental system as follows:

\[
\begin{array}{cccccc}
\epsilon_1 - \epsilon_2 & \epsilon_2 - \epsilon_3 & \cdots & \epsilon_{n-2} - \epsilon_{n-1} & \epsilon_{n-1} - \epsilon_n & \epsilon_{n-1} + \epsilon_n
\end{array}
\]

The full system of roots is $\Phi = \{ \pm \epsilon_i \pm \epsilon_j; i \neq j, i, j = 1, \ldots, n\}$. We will consider once again a parabolic corresponding to $J = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$ so the order of $Q_J$ will be $q^{n(n-1)/2}$ and the only shape is $\beta = \epsilon_{n-1} + \epsilon_n$. Thus $Q_J$ is irreducible so it is isomorphic with the exterior square of the natural representation. In particular all characters are linear and the action of $L_n$ on them is nontrivial so we can apply the same method as in 2.1.2 and get:

**Proposition 2.1.5** If $G \cong \Omega_{2n}^+(q), n \geq 4, K$ is a field of characteristic $p \nmid q$ and $V$ is an irreducible $KG$ module, then

\[
\dim H^1(G, V) \leq \frac{\gcd(n, q-1)}{q^{n-1} - 1} \dim V.
\]
2.1.6 \( \Omega_{2n}^{-}(q), n \geq 4 \)

The group is a twisted Chevalley group of type \( 2D_n \). We take the parabolic to be of type \( A_{n-2} \) and so the order of the unipotent radical will be \( q^{(n-1)(n+2)/2} \). There are 2 shapes involved. The first one, \( Q^{(2)} \) (the one corresponding to the “root” \( 2e_{n-1} \)) is invariant under the twisting so it is absolutely irreducible as \( F_q[L_n] \)-module, and the second, \( Q/Q^{(2)} \) (corresponding to the “root” \( \frac{1}{2}(2e_{n-1}) \)) is not invariant so it is absolutely irreducible as \( F_q[L_{n-1}] \)-module (using Lemma 2.2 b). Their dimensions are respectively \( (n-1)(n-2)/2 \) and \( n-1 \). In particular the action of \( L_{n-1} \) is nontrivial and we can apply the estimates in 2.1.1 and obtain:

**Proposition 2.1.6** If \( G \cong \Omega_{2n}^{-}(q), n \geq 4, K \) is a field of characteristic \( p \nmid q \) and \( V \) is an irreducible \( KG \) module, then

\[
\dim H^1(G, V) \leq \frac{\gcd(n-1, q-1)}{q^{n-2} - 1} \dim V.
\]

2.1.7 Unitary Groups

2.1.8 \( PSU_{2n}(q), n \geq 2 \)

In this case we are dealing with a group of type \( ^2A_{2n+1}(q) \). Consider a parabolic \( P = LQ \) so that the corresponding Levi complement \( L \) will be of type \( A_{n-1}(q^2) \). Then there is only one shape and the unipotent radical \( Q \) will be an absolutely irreducible \( F_q[L] \) module of dimension \( n^2 \) and so it is not trivial (it can even be described as a tensor product of the natural module with a twist of itself). In consequence we can apply the results regarding the maximal subgroups of \( L_n \) and our estimate for \( L_n \) and get:

**Proposition 2.1.7** If \( G \cong PSU_{2n}(q), K \) is a field of characteristic \( p \nmid q \) and \( V \) is an irreducible \( KG \) module, then

\[
\dim H^1(G, V) \leq \frac{\gcd(n, q^2 - 1)}{q^{2(n-1)} - 1} \dim V.
\]

2.1.9 \( PSU_{2n+1}(q) \)

We are in the case \( ^2A_{2n}(q) \) and we can consider \( P = LQ \) the parabolic corresponding to the subsystem of type \( A_{n-1}(q^2) \). There are two shapes one corresponding with the "root" \( \beta_1 = (e_{n-1} + e_{n+1})/2 \) and the other with the root \( \beta_2 = 2(e_{n-1} + e_{n+1})/2 \). The corresponding modules \( V_{\beta_1} \) (respectively \( V_{\beta_2} \)) are absolutely irreducible \( F_{q^2} \) (respectively \( F_q \)) modules of dimensions \( n \) (respectively \( n^2 \)). In particular they are nontrivial so we can apply Corollary 2.1 and get:

**Proposition 2.1.8** If \( G \cong PSU_{2n+1}(q), n \geq 2, K \) is a field of characteristic \( p \nmid q \) and \( V \) is an irreducible \( KG \) module, then

\[
\dim H^1(G, V) \leq \frac{\gcd(n, q^2 - 1)}{q^{2(n-1)} - 1} \dim V.
\]
Next let $G \cong \mathbf{2A}_2(q)$. In this case if we use two opposite Borel subgroups as $T_1, T_2$ and the Cartan subgroup as $B$, then $B$ is cyclic of order $(q^2 - 1)$ and the $S_i$’s are special groups of order $q^3$. If we use the matrix representation in Theorem 13.7.2 of [C1], we can see that in fact $|Z(S_i)| = q$ and $t \in B$ acts on this group as multiplication with $t^{q+1}$ and on $S_i/Z(S_i)$ as multiplication by $t^{q-1}$. In particular it follows that the orbits of $B$ on $S_i/Z(S_i)$ will have length $q^2 - 1$ unless $3 \mid q+1$ in which case their length is $(q^2 - 1)/3$.

We can improve the bound on the orbits of the nonlinear characters. To do that assume $q+1$ is not a power of $p$. In this case let $\chi$ be an irreducible nonlinear (Brauer) character of $S_2$ so that $C_B(x)$ is maximal $p'$ subgroup of $C_B(x)$ (where $C_B(x)$ is cyclic with $q+1$ elements). One can regard $\chi$ as a $S_1C$ module. It has dimension $q$ and following Hall-Higman Theorem (See 17.13 Satz of [Hu]) $\chi_{C} = \frac{q+1}{|C|} \rho - \lambda$ where $\rho$ is the regular representation of $C$ and $\lambda$ is linear. It then follows that the length of the orbits of the action of $C$ on the representations associated to $\chi$ are at least as large as the smallest index of a subgroup of $C$ and that is at least 2 if $q$ odd and at least 3 if $q$ even. The orbits of $B$ on the nonlinear characters will then be at least $d(q-1)$ where $d$ is the smallest divisor of $q+1$ which is relatively prime to $p$ ($d = 1$ if $q+1$ is a power of $p$). In any case if $q > 2$, this is smaller than $(q^2 - 1)/3$ hence using Lemma 2.1 and Corollary 2.1 we obtain:

**Proposition 2.1.9** If $G \cong \mathbf{2A}_2(q)$, $K$ is a field of characteristic $p$ such that $p$ does not divide $q$ and $V$ is an irreducible $KG$ module, then

$$\dim H^1(G, V) \leq \frac{1}{d(q-1)} \dim V$$

Where $d$ is the smallest divisor of $q+1$ such that $(d, p) = 1$ ($d = 1$ if $q+1$ is a power of $p$).

Note that if $q = 3$ and $p = 2$ the bound is still inefficient. We’ll deal with this case in 2.3.

### 2.2 Exceptional Chevalley Groups

#### 2.2.1 Exceptional Groups of Chevalley Rank One

This is the case when $G$ is of type $\mathbf{2B}_2$ or $\mathbf{3G}_2$. In order to deal with this cases, we will use the following (see also Proposition 2.5 of [G]).

**Proposition 2.2.1** Let $V$ be an indecomposable $KG$-module with $K$ a field of characteristic $p$ and $G$ a finite group. If the $V$ has a cyclic vertex (eq. Sylow $p$-subgroup of $G$ is cyclic) then $\dim H^1(G, V) \leq 1$.

**Proof.** Let $N := N_G(P)$ the normalizer of the Sylow subgroup. Using Green correspondence, if $V$ is a direct summand of the induced $U^G_N$ ($U$ is an indecomposable $N$ module, the Green correspondent of $V$) then $H^1(G, M) \leq H^1(G, U^G_N) = H^1(N, U)$. 

---

11
Moreover since the block of \( V \) is a uniserial module \( \dim H^1(G, U) \leq 1 \) (see 62.29 of [CR81]).

Assume \( p \) odd and \( |G| = \prod \Phi_{m_0}^r \) and \( m_0 \) is the smallest \( m \) so that \( p|\Phi_m(q) \). Using §10 of [GL] we get that if either \( p \) does not divide \( |W| \) the order of the Weyl group or \( pm_0 \) does not divide \( m \) for all \( m \) then the \( p \)-Sylow subgroup of \( G \) is homo-cyclic abelian of rank \( r_m \). But from Table 10:2 of [GL] we see that if \( G = 2B_2 \) then \( |G| = \Phi_1 \Phi_4 \) and if \( G = 2G_2 \) then \( |G| = \Phi_1 \Phi_2 \Phi_6 \). Also in both cases \( W \sim \mathbb{Z}/2\mathbb{Z} \). So if \( G \) is one of the above groups we have that the \( p \)-Sylow subgroup of \( G \) is cyclic. Now we can use Prop 2.2.1 and some results about the dimension of the minimal representations of \( G \) (cf [LS]), we get:

**Proposition 2.2.2**

i) If \( G \cong 2B_2(q) \), \( K \) is a field of characteristic \( p \neq 2 \) and \( V \) is an irreducible \( KG \) module, then

\[
\dim H^1(G, V) \leq \frac{1}{(q/2)^2(q - 1)} \dim V.
\]

ii) If \( G \cong 2G_2(q) \), \( K \) is a field of characteristic \( p > 3 \) and \( V \) is an irreducible \( KG \) module, then

\[
\dim H^1(G, V) \leq \frac{1}{q(q - 1)} \dim V.
\]

Let \( G = 2G_2(q) \) and \( p = 2 \), the results in [LM] give:

**Proposition 2.2.3**

If \( G \cong 2G_2(q), q > 3, K \) is a field of characteristic 2 and \( V \) is an irreducible \( KG \) module, then

\[
\dim H^1(G, V) \leq \frac{1}{q(q - 1)} \dim V.
\]

### 2.2.2 Exceptional Groups of Chevalley rank 2

This is the case when \( G \) is one of \( G_2(q), D_4(q^3) \) or \( E_8(2k+1) \). In the first two cases we will take a maximal parabolic of type \( A_1 \). Unfortunately the unipotent radicals will have one dimensional factors so we will need to use Corollary 2.2.

Consider \( SL_2(q) \) acting on a group \( Q \) so that the order of \( Q \) is \( pq^2 \), \( Q' = Z(Q) = \Phi(Q) \cong \mathbb{Z}/p\mathbb{Z} \) and \( Q/Z(Q) \cong N \) the natural module of \( L_2 \). Then using Corollary 2.2 we get that the orbit of the action of \( SL_2(q) \) has length at least equal to the number of elements of the minimal irreducible representation of \( SL_2(q) \) which is \((q-1)/2\).

First we will consider the case \( G \cong G_2(q) \) where \( q \) is not a power of 2 or of 3. Then the fundamental system is of the form:

\[
a \bullet \overline{\bullet} b
\]

where \( b \) is longer than \( a \). The full root system is:

\[
\Phi = \{ \pm 3a + 2b, \pm 3a + b, \pm 2a + b, \pm a + b, \pm a, \pm b \}
\]
If we take $J := \{b\}$, then we obviously have 3 shapes: $a, 2a, 3a$. The corresponding factors are: 

\[ V_a = X_a X_{a+b}, \quad V_{2a} = X_{2a+b}, \quad V_{3a} = X_{3a+2b} X_{3a+b} \]

Note that all the roots of a given shape have the same length therefore the same decomposition holds when $q$ is a power of 2 or 3.

The first and the third modules are isomorphic to the natural $L_2$-module but the second one is one dimensional so it is a trivial $L_2$-module. So we divide the characters of $Q$ in three categories:

i) characters that are nontrivial on $Q^{(3)}$.

ii) characters that are trivial on $Q^{(3)}$ but nontrivial on $Q^{(2)}$.

iii) Characters that are trivial on $Q^{(2)}$.

If the character $\psi$ is of type i) or iii) then we can identify it with a character of the natural $L_2$ module thus it is nontrivial so the orbits are of length at least $(q + 1)$ if $q \not\in \{5, 7, 11\}$ and at least $q$ if $q$ is one of the above.

The characters of type ii), are nonlinear characters of a special group of order $q^3$. If we factor by the kernel of such a character we see that we can assume that it is a character of an extra-special group of order $pq^2$. This extraspecial group will also have minimal exponent because 

\[(x_{a+b} t) x_a(s)^k = x_{a+b} x_a(k) x_{2a+b}(k(k-1)/2)\]

in $Q/Q^{(3)}$ so if $q$ is odd this gives that the exponent is $p$ and if $q$ is even the exponent is 4. Therefore we can apply Corollary 2.2 and get that the length of the orbits on the characters is at least $(q-1)/2$. Now if we put together all this information we get that the bound will be $\frac{\gcd(2,q-1)}{(q-1)}$.

Note that if $q = 2, 3, 4, 5$ the bound is ineffective. However the group $G_2(2)$ is not simple so we will not discuss this. We must consider separately the groups $G_2(3), G_2(4), G_2(5)$ and we’ll do that in 2.3.

Proposition 2.2.4 If $G \cong G_2(q), q > 5$, $K$ is a field of characteristic $p$ not dividing $q$ and $V$ is an irreducible $KG$ module, then

\[ \dim H^1(G,V) \leq \frac{\gcd(2,q-1)}{(q-1)} \dim V. \]

Next assume $G \cong 3D_4(q)$. Then the fundamental system consists (if we use the notation in Chapter 13 of [C1]) of the two vectors $p_1$ and $\alpha = \frac{1}{3}(p_2 + p_3 + p_4)$. Then consider the parabolic corresponding to $\{p_1\}$. There will be three shapes, $(\alpha, 2\alpha, 3\alpha)$ and the factors will be:

\[ V_\alpha = X_\alpha X_{\alpha+p_1}, \quad V_{2\alpha} = X_{2\alpha+p_1}, \quad V_{3\alpha} = X_{3\alpha+p_1} X_{3\alpha+2p_1} \]
Again we use the notation in [C1]. We can see that the situation is exactly the one in case $G_2$ but the Levi complement is $A_1(q)$ so we can repeat the argument and get:

**Proposition 2.2.5** If $G \cong^3 D_4(q)$, $q \neq 2, 3, 4, 5$, $K$ is a field of characteristic $p$ not dividing $q$ and $V$ is an irreducible $KG$ module, then

$$\dim H^1(G, V) \leq \frac{\gcd(2, q - 1)}{(q - 1)} \dim V.$$

The cases $G \cong^3 D_4(q)$ and $q = 2, 3, 4, 5$ will be considered in 2.3.

The only case left is $^2F_4(q)$. For this case we will use the results in §10 of [FS]. We will take a parabolic subgroup of type $^2B_2(q)$, $(R_2 = < U_2, U_2^q >$ in the notations of [FS]). Then the unipotent subgroup $U_{s_2}$ will have order $q^{10}$ and will be generated by:

$$U_{s_2}^2, U_{2s_2}^1, U_{2s_2}^2, U_{2s_2}^{1s_2}, U_{s_2}^{s_2}, U_{s_2}^{s_1}, U_{s_2}^{s_1}, U_{s_2}^{s_1}, U_1$$

we also have (cf (10. 3) of [FS]) that:

$$[U_{s_2}, U_{s_2}] = \Phi(U_{s_2})$$

$$\Phi([U_{s_2}, U_{s_2}]) = \Omega_1(U_{s_2}^{s_2}) \Omega_1(U_{s_2}^{s_2}) \Omega_1(U_{s_2}^{s_2}) U_1^{s_2} \Omega_1(U_{s_2}^{s_2}),$$

$$\Phi([U_{s_2}, U_{s_2}]) = 1$$

$$[\Phi(U_{s_2}), U_{s_2}] = \Omega_1(U_{s_2}) = Z(U_{s_2})$$

Now if $T_2 = [U_{s_2}, U_{s_2}] U_{s_2}^{s_2}$ then $R_2$ will act on $U_{s_2}/T_2$ as the symplectic representation of $Sz(q)$. The same will be true for the action of $R_2$ on $\Phi(U_{s_2})/Z(U_{s_2})$.

Let us define $V_1 := \Phi(U_{s_2})$ and $V_2 := U_{s_2}/\Phi(U_{s_2})$. Then both $V_i$’s are 5 dimensional $F_q$ -vector spaces and also indecomposable $Sz(q)$ modules. However they are not irreducible since they have a 1 dimensional submodule. So now as usual we will consider characters of $Q$ that are trivial on $V_1$ and regard them as characters of $V_2$ and also characters that are not trivial on $V_1$ and restrict them to $V_1$. Since both groups are abelian, the irreducible characters will be linear so they are elements of the dual modules. But the dual modules are indecomposable modules that have a submodule of dimension 4. Now $Sz(q)$ will act non trivially on this 4 dimensional module and also cannot fix an element outside it (because then the module would be decomposable). So we can see that the orbits under this action will be at least as large as the index of the largest subgroup of $Sz(q)$.

Then we can apply the results concerning subgroups of exceptional Chevalley groups (cf. [LiS] or [Sz]) and get that there are no subgroups of $Sz(q)$ of index smaller than $q^2 + 1$. Now the bound we have for $Sz(q)$ is $(q/2)^{\frac{1}{2}}(q - 1)$ which is bigger than $q^2 + 1$. So if we apply the usual argument:

**Proposition 2.2.6** If $G \cong^2 F_4(q), q > 2$, $K$ is a field of characteristic $p$ or $q \dim H^1(G, V) \leq \frac{1}{(q/2)^{\frac{1}{2}}(q - 1)} \dim V.$
Note that once again the case $q = 2$ will give no bound so we’ll consider the case $^{2}F_{4}(2)$ in 2.3.

2.2.3 $F_{4}(q)$

If $\{e_{i}\}_{i=1}^{4}$ is an orthonormal basis for $V$, then in the notations of 1, $\Delta = \{e_{1} - e_{2}, e_{2} - e_{3}, e_{3}, e_{4}, e_{3}, e_{4}\}$ and

$$\Phi = \left\{ \begin{array}{ll}
\pm e_{i} \pm e_{j} & i \neq j \\
\pm e_{i} & i = 1, 2, 3, 4 \\
\frac{1}{2}(\pm e_{1} \pm e_{2} \pm e_{3} \pm e_{4}) &
\end{array} \right. $n$

Pick $J = \{e_{1} - e_{2}, e_{2} - e_{3}, e_{3}\}$. Then $L_{J}$ is of the type $B_{3}$. Since $P_{J}$ is maximal, all roots of a given level will have the same shape so in particular $Q^{(i)} / Q^{(i+1)}$ is irreducible. We can easily see that

$$Q^{(2)} \cong V_{2\frac{1}{2}(-e_{1} - e_{2} - e_{3} + e_{4})} \cong X_{e_{4}} \prod X_{\pm e_{i} \pm e_{4}}(\dim Q^{(2)} = 7)$$

$$Q^{(1)} / Q^{(2)} \cong V_{2\frac{1}{2}(-e_{1} - e_{2} - e_{3} + e_{4})} \cong \prod X_{\frac{1}{2}(\pm e_{1} \pm e_{2} \pm e_{3} \pm e_{4})} \dim Q^{(1)} / Q^{(2)} = 8)$$

A 7-dimensional irreducible $\Omega_{7}(q)$-module has to be the natural one and an 8-dimensional irreducible $\Omega_{7}(q)$-module is the spin module. Note that the results of [ABS] will work in this case even if $q$ is even because in $V_{2\frac{1}{2}(-e_{1} - e_{2} - e_{3} + e_{4})}$ all roots have length 2 so this is still irreducible and an indecomposable module of length 2 cannot have dimension 7 (all modules of dimension $< 7$ are trivial).

In particular none of these modules are trivial, so we can use Corollary 2.1 and the fact that the length any orbit is at least the rank of the minimal permutation representation of $B_{3}$. The index of the largest subgroup of $\Omega_{7}(q)$ is $(q^{6} - 1)/(q - 1)$ if $q \geq 5$ and $q$ odd and $(\frac{1}{2})3^{3}(3^{3} - 1)$ if $q = 3$(cf Table 5. 2A in [KL]). Since from 2.1.3, the corresponding bound for $\Omega_{7}(q)$ is $C_{0} = (q^{2} - 1)/\gcd(3, q - 1)$, we get that:

**Proposition 2.2.7** If $G \cong F_{4}(q)$, $K$ is a field of characteristic $p|q$ and $V$ is an irreducible $KG$ module, then

$$\dim H^{1}(G, V) \leq \frac{\gcd(3, q - 1)}{(q^{2} - 1)} \dim V.$$

2.2.4 $E_{6}(q)$

In this case we have the following diagram:

```
  e_1 -- e_2 -- e_3 -- e_4 -- e_5 -- e_6 
/                   /                   / 
|                   |                   |
e_4 + e_5 
```

15
The full root system is:

$$\Phi = \begin{cases} 
\pm e_i \pm e_j & i \neq j, i, j = 1, \ldots, 5 \\
\frac{1}{2} \sum_{i=1}^{5} \varepsilon_i e_i + e_6 & \varepsilon_i = \pm 1 \prod_{i=1}^{5} \varepsilon_i = 1 \\
-\frac{1}{2} \sum_{i=1}^{5} \varepsilon_i e_i + e_6 & \varepsilon_i = \pm 1 \prod_{i=1}^{5} \varepsilon_i = 1 
\end{cases}$$

We will take a maximal parabolic of type $A_5(q)$. Then

$$J = \{ e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 + e_5, -\frac{1}{2} \sum_{i=1}^{6} e_i \}.$$ 

In this case there is only one level and respectively only one shape $\beta = e_4 - e_5$. Therefore $Q$ is an irreducible $L_6$ module of dimension 21 which means that it is the symmetric square of the natural module. In particular it is nontrivial so the length of the orbit of any point will be at least equal to the index of the maximal subgroup of $L_6$. This is (cf. Table 5, 2 A [KL]) $(q^6 - 1)/(q - 1)$. If we combine this with the fact that for $L_6$ we have the bound $C = (q^5 - 1)/(\gcd(6, q - 1))$ we conclude:

**Proposition 2.2.8** If $G \cong E_6(q)$, $K$ is a field of characteristic $p\mid q$ and $V$ is an irreducible $KG$ module, then

$$\dim H^1(G, V) \leq \frac{\gcd(6, q - 1)}{(q^6 - 1)} \dim V.$$  

2.2.5 $2E_6(q)$

In this case we will look to a parabolic group of type $2A_5(q)$. If we apply the same computation as in 2.2.4, we can see that actually there is only one shape $\beta = (e_4 - e_5)$. The dimension of $V_{\beta}$ is also 21 and $V_{\beta}^{\sigma} = V_{\beta}$ ($\sigma$ is the graph automorphism that gives the twisting) so using once again Lemma 2.2, we see that the action of the Levi complement on the unipotent radical is irreducible and nontrivial so we can look for the minimal length of an orbit which has to be at least $\frac{(q^6 - 1)(q^2 + 1)}{(q^2 - 1)}$ (that is because $q$ is a square and so it cannot be 2). Now this is a better bound than the one for $U_4(q)$ so if we apply Corollary 2.1 we get:

**Proposition 2.2.9** If $G \cong 2E_6(q)$, $K$ is a field of characteristic $p\mid q$ and $V$ is an irreducible $KG$ module, then

$$\dim H^1(G, V) \leq \frac{\gcd(2, q^2 - 1)}{(q^2 - 1)} \dim V.$$ 

2.2.6 $E_7(q)$

The fundamental system looks like:

$$\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & -\frac{1}{2} \sum_{i=1}^{7} e_i \\
\rightarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
e_5 & e_6 & & & & & & 
\end{array}$$

16
The full root system is
\[ \Phi = \begin{cases} 
\pm e_i \pm e_j & i \neq j \\
\pm e_7 & \\
\frac{1}{2} \sum_{i=1}^{6} \varepsilon_i e_i + e_7 & \varepsilon_i = \pm 1 \\
\frac{1}{2} \sum_{i=1}^{6} \varepsilon_i e_i + e_7 & \varepsilon_i = \pm 1 \\
-\frac{1}{2} \sum_{i=1}^{8} \varepsilon_i e_i & \varepsilon = \pm 1 \\
\prod_{i=1}^{6} \varepsilon_i = 1 \\
\prod_{i=1}^{8} \varepsilon_i = 1 
\end{cases} \]

We will take a parabolic of type $A_6$. Then
\[ J = \{ e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 + e_6, -\frac{1}{2} \sum_{i=1}^{7} e_i \} \]

Thus the order of $Q$ is $q^{42}$. In $\Phi^+ \setminus \mathbb{Z}J$ there are two levels (shapes). These are $\beta := e_5 - e_6$ and $\beta' := 2(e_5 - e_6)$. Then $Q$ will have 2 factors, $Q^{(2)} \cong V_{\beta'}$ of dimension 7 and $Q/Q^{(2)} \cong V_{\beta'}$ of dimension 35. But the irreducible representations of $L_7$ of dimension $\leq 42$ are ($M$ is the natural module (cf. Table 3 in [Lie]):

\[ M(dim = 7), \Lambda^2 M(dim = 21), \Lambda^3 M(dim = 35), S^2 M(dim = 28) \]

So the factors are the natural module, respectively the exterior cube of the natural module. In both cases the modules will be nontrivial and we can use the results about permutation representations. We have that in $L_7$ there are no groups of index smaller than $(q^7 - 1)/(q - 1)$. Therefore using Corollary 2.2, we get:

**Proposition 2.2.10** If $G \cong E_7(q)$, $K$ is a field of characteristic $p \not| q$ and $V$ is an irreducible $KG$ module, then
\[ \dim H^1(G, V) \leq \frac{\gcd(7, q - 1)}{(q^6 - 1)} \dim V. \]

2.2.7 $E_8(q)$

The fundamental system looks like:

\[ e_1 - e_2 \quad e_2 - e_3 \quad e_3 - e_4 \quad e_4 - e_5 \quad e_5 - e_6 \quad e_6 + e_7 \quad -\frac{1}{2} \sum_{i=1}^{8} e_i \]

and the full root system is
\[ \Phi = \begin{cases} 
\pm e_i \pm e_j & i \neq j \\
\pm e_7 & \\
\frac{1}{2} \sum_{i=1}^{8} \varepsilon_i e_i & \varepsilon_i = \pm 1 \\
\frac{1}{2} \sum_{i=1}^{8} \varepsilon_i e_i & \varepsilon = \pm 1 \\
\prod_{i=1}^{8} \varepsilon_i = 1 \\
\prod_{i=1}^{8} \varepsilon_i = 1 
\end{cases} \]
We will take a maximal parabolic of type $A_7$. Then

$$J := \{e_i - e_{i+1}, \frac{1}{2} \sum_{i=1}^{8} e_i \} (i = 1, \ldots, 8)$$

There will be three shapes $\beta_1 = e_6 - e_7$, $\beta_2 = 2(e_6 - e_7)$ and $\beta_3 = 3(e_6 - e_7)$. The order of the unipotent radical is $q^{92}$. We can also compute the dimensions of the modules $V_{\beta_i}$. They are:

$$\dim V_{\beta_1} = 56; \dim V_{\beta_2} = 28; \dim V_{\beta_3} = 8$$

So the factors of the group $Q_J$ as a $L_8$ module are isomorphic with the natural module, the exterior square of the natural module and respectively the exterior cube of the natural module.

In particular none of them is trivial so the length of the orbits will be at least the rank of the minimal permutation representation of $L_8(q)$ and that is $(q^8 - 1)/(q - 1)$. Also from the previous computations we have that the bound for $L_8$ is $C_0 = (q^7 - 1)/\gcd(n,q - 1)$ so:

**Proposition 2.2.11** If $G \cong E_8(q)$, $K$ is a field of characteristic $p \not| q$ and $V$ is an irreducible $KG$ module, then

$$\dim H^1(G,V) \leq \frac{\gcd(8,q - 1)}{(q^7 - 1)} \dim V.$$ 

### 2.3 The problematic cases

The first case we encountered is $G = U_3(3)$ and $p = 2$. The results in Tables A1 and A2 of [S3] solve this case and the case $p = 3$. In particular one gets a bound of $1/6$ for both these cases\(^1\).

The next cases we need to consider are $G_2(q)$ for $q = 3, 4, 5$. If $p^2$ does not divide the order of $G$ then the cohomology will have dimension at most 1 because the Sylow subgroup is cyclic or trivial. Therefore we get bounds of $1/14, 1/12, 1/120$ respectively (using table 5.3A of [KL] for the dimension of the smallest irreducible module).

If $G = G_2(3)$ and $p = 2$, then $G$ contains a cyclic subgroup $S_1$ of order 13 and there are three maximal subgroups of $G$ containing this subgroup. The three groups are isomorphic to $L_2(13), L_3(3) : 2, L_3(3) : 2$. Note that in each of these three groups the normalizer of the element of order 13 is isomorphic to $13 : 6$. Consider $x$ an element of order 6 that normalizes $S_1$ and $B = \langle x \rangle$. Then note that in each of the three subgroups the centralizer of an element of order 6 is trivial. Nevertheless the centralizer of an element of order 6 in $G$ has size at least 18. Take $g$ an element of order 3 that centralizes $x$ but does not belong to any of the three subgroups, then as before consider $S_2 = S_1^g$. The group generated by $S_1, S_2$ and $B$ cannot be included in

---

\(^1\)I would like to thank the referee for this references
any of the three maximal subgroups that contain $S_1$ hence it’s the whole group $G$ and we can apply Lemma 2.1 to get a bound of $1/6$ for $G_2(3)$.

If $G = G_2(4)$ and $p = 3, 5$ consider $S_1$ a cyclic subgroup of order 13. The maximal subgroups of $G_2(4)$ containing this group are isomorphic to $L_2(13)$ or to $U_3(4) : 2$. In each of these groups there is an element $x$ of order 6 normalizing $S_1$, let $B = \langle x \rangle$ and note that in the maximal groups containing $S_1$ the centralizer of an element of order 6 is trivial. Nevertheless in $G_2(4)$ the centralizer of an element of order 6 is at least of order 12 hence we can pick $g$ that centralizes $x$ but is not contained in any of the two groups. The group generated by $S_1, B, S_2 = S_1^g$ is the whole $G$ hence we can apply Lemma 2.1 and get a bound of $1/6$ for $G$.

We will consider the cases $^3D_4(q), q = 2, 3, 4, 5$ together.

Note that the $G(q)$ conjugacy classes of maximal tori in the algebraic group $G$ correspond to the $F$ conjugacy classes in the Weyl group of $G$. Since the Frobenius $F$ acts on $W$ via the symmetry of the Dynkin diagram, the size of that base field will not affect the number of $G(q)$ conjugacy classes. Moreover if $T = ^gT_0$ is an $F$ stable maximal torus corresponding to $w \in W$ via $\pi(g^{-1}F(g)) = w$, and $N$ is the normalizer of $T$ in $G$ then:

$$|N^F/T^F| = \{|x \in W |x^{-1}wF(x) = w\}|.$$

In particular $N^F/T^F$ does not depend on $q$ but on the action of $F$ on $W$ hence will be the same for all three groups.

If we take $T$ the maximal torus such that $T^F = q^4 - q^2 + 1$, then in each of the three groups $T$ will correspond to the same $w$ (up to conjugation in the finite group). To prove this note that by Proposition 3.3.5 of [C2], $|T^F| = P(q)$ implies that $P$ is the characteristic polynomial of $F_0^w$ in its action on $Y \otimes \mathbb{R}$.

The orders of the maximal tori $T^F$ are 13, 73, 241, 601 respectively and one can check using the ATLAS that there are no simple subgroups of $G$ containing those tori hence the normalizers are maximal. Moreover in $^3D_4(2)$ the normalizer is of type $13:4$ so in each of the other two there is an element $x$ of order 4 normalizing the torus.

Since in any of the four groups $^3D_4(q)$ an element of order 4 will have nontrivial centralizer, we can take $S_1$ the torus, $B = \langle x \rangle$, and $S_2 = S_1^g$ for $g \in C_G(x)$. It follows that $G = \langle S_1, B, S_2 \rangle$ and we can apply Lemma 2.1 and get a bound of $1/4$ for these groups.

Finally assume $G = ^2F_4(2)'$. We will treat both the natural characteristic and the mixed characteristic together. Assume first that the characteristic of the field is 13. Then since the 13 Sylow subgroup is cyclic we have that $\dim H^1(G, V) \leq 1$ so we need to find the dimension of the smallest representation of $G$ in characteristic 13. By [Hi]
this is 26.

If the characteristic of the field is not 13 then we can consider $S_1$ the cyclic group of order 13, $B = \langle x \rangle$ the group of order 6 that normalizes $S_1$. Then there is a unique maximal subgroup $H$ of $G$ containing $S_1$ and $H = L_3(3) : 2$. Also the centralizer of $x$ in $H$ is trivial and the centralizer of $x$ in $G$ has order at least 12. We can then take an element $g \in G$ that centralizes $x$ but does not belong to $H$ and construct $S_2 = S_1^g$. It will follow that $S_1, B, S_2$ generate the whole group $G$ and so by Lemma 2.1 there’s a bound of 1/6 for this group.

3 Natural Characteristic

3.1 The case $q \geq 5$

To fix the notations let $G$ a Chevalley group over the field of $q = p^n$ elements, $V$ an irreducible $F_p[G]$ module. Let $B$ a Borel subgroup of $G$ and consider $B = TQ$ where $T$ is a maximal torus and $Q$ the corresponding unipotent radical. Note that $B$ contains the $p$-Sylow subgroup of $G$ so the map $res: H^1(G, V) \rightarrow H^1(B, V)$ is injective for any $G$-module $V$.

If we obtain a common bound for all $B$ irreducible factors of $V$ and use the long exact sequence in cohomology we will get the bound for the composite module $V$. Therefore we can restrict our attention to irreducible $B$ modules. By Lemma 72.11 of [CR81], we have that if $M$ is such a module then $M$ is one dimensional and $Q$ acts trivially on it. Also $T$ is a $p'$ group so it has trivial cohomology hence:

$$H^1(B, M) = \text{Hom}_T(Q, M) = \text{Hom}_T(Q/Q', M)$$

One can decompose $Q/Q'$ as a direct sum of weight spaces of $T$. One should note that these weights are in fact roots. We have the following (cf [Se3] Theorem (5.5)):

**Lemma 3.1** In the above notations assume further that $q \geq 5$ and let $V_i, V_j$ two weight spaces in $Q/Q'$. Then $V_i$ and $V_j$ are inequivalent irreducible $F_p[T]$ modules unless $G = Sz(8)$, $2F_4(8)$, or $q = 5, 9$ and $G$ is of type $C_n$.

If $\lambda$ is a weight, denote by $n_\lambda$ the sum of the multiplicities in $V$ of all weights that are Galois conjugate to $\lambda$. In the following I will loosely refer to $n_\lambda$ as the multiplicity of $\lambda$. We then have:

**Lemma 3.2** Assume $q \geq 5$ and $G \neq Sz(8), 2F_4(8), C_n(5), C_n(9)$. If $S$ is the set of roots that will appear in $Q/Q'$ then

$$\dim H^1(G, V) \leq \sum_{\lambda \in S} n_\lambda.$$
Proof. We proved that $H^1(G,V)$ embeds in $\text{Hom}_T(Q/Q', V)$ and if we use Lemma 3.1 we get that the the dimension of $H^1(G,V)$ is less than or equals the number of $T$ composition factors factors of $V$ that are Galois conjugate to one of the factors of $Q/Q'$.

We know that the Weyl group of $G$ will act on the weight spaces and so if two roots $\lambda_1$ and $\lambda_2$ are conjugate under $W$ then $n_{\lambda_1} = n_{\lambda_2}$.

3.2 $G$ of type $A_m$

In this case the roots in $Q/Q'$ are just the ones in the fundamental system. Also all the roots have the same length thus are conjugate and have the same multiplicity $n$ in $V$. So $\dim V \geq nm(m + 1)$ and :

Proposition 3.2.1 If $G$ is of type $A_m(q)$, $q \geq 5$ and $V$ is an irreducible $kG$-module where $\text{char } k|q$ then:

$$\dim H^1(G,V) \leq \frac{1}{m+1} \dim V.$$ 

Note that in the case $m = 1$ the bound is quite inefficient. However using the results of [AJL], we have that if $G = L_2(p^k)$, then $\dim H^1(G,V) \leq 1$ hence the only case when the bound is $1/2$ is when $\dim V = 2$ and $\dim H^1(G,V) = 1$. In this case it follows that $p = 2$ or $p = 3, q = 9$ by [AJL]. In all other cases we get a bound of at least $1/3$ for $L_2(q)$.

3.3 $G$ of type $B_m$ or $C_m$

In the case $B_m$ the roots have two different lengths hence there are two conjugacy classes. Let $n_1$ the multiplicity of the short roots and $n_2$ the multiplicity of the long ones. Then we have that $\dim V \geq 2m n_1 + 2m(m - 1)n_2$ and $\dim H^1(G,V) \leq n_1 + (m - 1)n_2$.

Note that the same computation will work for $C_m$ if we call $n_1$ the multiplicity of the long roots and $n_2$ the multiplicity of the short ones.

Proposition 3.3.1 If $G$ is of type $B_m(q)$ or $C_m(q)$, $q \geq 5$ ($q \neq 9$ when $G = C_n$) and $V$ is an irreducible $kG$-module where $\text{char } k|q$ then:

$$\dim H^1(G,V) \leq \frac{1}{2m} \dim V.$$ 

3.4 $G$ of type $D_m$, $m \geq 4$

As in the case $A_n$, all the roots are conjugate so they have the same multiplicity $n$. Then $\dim V \geq 2m(m - 1)n$ and $\dim H^1(G,V) \leq nm$ so:
Proposition 3.4.1 If $G$ is of type $D_m(q)$, $q \geq 5$, $m \geq 4$ and $V$ is an irreducible $kG$-module where char $k | q$ then:

$$\dim H^1(G,V) \leq \frac{1}{2(m-1)} \dim V.$$

3.5 $G$ of type $E_i$ for $i = 6 \ldots 8$

In any of these three cases we have that all roots have the same length so there will be only one conjugacy class. If $n$ is the multiplicity of these roots, in $V$ we have that:

- In the case $E_6$, $\dim V \geq 72n$ and $\dim H^1(G,V) \leq 6n$
- In the case $E_7$, $\dim V \geq 126n$ and $\dim H^1(G,V) \leq 7n$
- In the case $E_8$, $\dim V \geq 240n$ and $\dim H^1(G,V) \leq 8n$

Proposition 3.5.1 If $G$ is of type $E_i(q)$, $i = 6,7,8 \ q \geq 5$ and $V$ is an irreducible $kG$-module where char $k | q$ then:

$$\dim H^1(G,V) \leq \frac{1}{c_i} \dim V.$$

where $c_6 = 12, c_7 = 18$, and $c_8 = 30$.

3.6 $G$ of type $F_4$

In this case there are two classes. Let $n_1$ (respectively $n_2$) be the multiplicity of the long (respectively short) root. Then we have that

$$\dim V \geq 24n_1 + 24n_2 \text{ and } \dim H^1(G,V) \leq 2n_1 + 2n_2$$

We can see then that :

Proposition 3.6.1 If $G$ is of type $F_4(q)$, $q \geq 5$ and $V$ is an irreducible $kG$-module where char $k | q$ then:

$$\dim H^1(G,V) \leq \frac{1}{12} \dim V.$$

3.7 $G$ of type $G_2$

Again in this case we have two conjugacy classes. Let as before $n_1$ (respectively $n_2$) be the multiplicity of the long (respectively short) root. Then:

$$\dim V \geq 6n_1 + 6n_2 \text{ and } \dim H^1(G,V) \leq n_1 + n_2$$

Proposition 3.7.1 If $G$ is of type $G_2(q)$, $q \geq 5$ and $V$ is an irreducible $kG$-module where char $k | q$ then:

$$\dim H^1(G,V) \leq \frac{1}{6} \dim V.$$
3.8 \textbf{G of type $^2A_n$}

We can see (using §13.3 of [C1]) that the fundamental system in this case is of type $C_{(n+1)/2}$ if $n$ odd and $B_{n/2}$ if $n$ even. In particular

\textbf{Proposition 3.8.1} If $G$ is of type $^2A_n(q)$, $q \geq 5$ and $V$ is an irreducible $kG$-module where $\text{char } k|q$. Let $t = n + 1$ if $n$ odd, $t = n$ if $n$ even. Then

$$\dim H^1(G, V) \leq \frac{1}{t} \dim V.$$  

3.9 \textbf{G of type $^2D_n$}

We can see (using §13.3 of [C1]) that the fundamental system in this case is of type $B_{n-1}$. So in arguing as above

\textbf{Proposition 3.9.1} If $G$ is of type $^2D_n(q)$, $q \geq 5$ and $V$ is an irreducible $kG$-module where $\text{char } k|q$ then:

$$\dim H^1(G, V) \leq \frac{1}{2(n-1)} \dim V.$$  

3.10 \textbf{G of type $^2E_6$}

We can see (using §13.3 of [C1]) that the fundamental system in this case is of type $F_4$.

\textbf{Proposition 3.10.1} If $G$ is of type $^2E_6(q), q \geq 5$ and $V$ is an irreducible $kG$-module where $\text{char } k|q$ then:

$$\dim H^1(G, V) \leq \frac{1}{12} \dim V.$$  

3.11 \textbf{G of type $^3D_4$}

We can see (using §13.3 of [C1]) that the fundamental system in this case is of type $G_2$.

\textbf{Proposition 3.11.1} If $G$ is of type $^3D_4(q)$, $q \geq 5$ and $V$ is an irreducible $kG$-module where $\text{char } k|q$ then:

$$\dim H^1(G, V) \leq \frac{1}{6} \dim V.$$
3.12 G of type $^2F_4(2^{2n+1})$, $n \geq 2$

Using §13. 3 of [Cl], we get that the roots form two concentric octagons and that the Weyl group will permute transitively their vertices. Therefore if $n_1, n_2$ are the multiplicity of the long respectively short root in $V$ then $\dim V \geq 8n_1 + 8n_2$ and $\dim H^1(G, V) \leq n_1 + n_2$.

**Proposition 3.12.1** If $G$ is of the $^2F_4(2^{2n+1})$, $n \geq 2$ and $V$ is an irreducible $kG$-module where $\text{char } k = 2$ then:

$$\dim H^1(G, V) \leq \frac{1}{8} \dim V.$$

3.13 Rank 1 Groups

These are the groups of type $^2B(2^{2n+1})$ and $^2G(3^{2n+1})$. For those the result is already known from [S2] and [S4]. In particular we have:

**Proposition 3.13.1** If $G$ is of type $^2B_2(2^n)$, (respectively $^2G_2(3^n)$), $n > 1$, $V$ is an irreducible $kG$-module char $k = 2$ (respectively $3$), then let $t = 4$ (respectively $t = 49$)

$$\dim H^1(G, V) \leq \frac{1}{t} \dim V.$$

Note that this will exhaust these cases since $^2G_2(3)$ and $^2B_2(2)$ are not simple.

4 The remaining cases

We still have to work out the cases $q < 5$ and the cases $C_n(5), C_n(9)$ and $^2F_4(8)$. Note that the group $^2B_2(8)$ was treated above. The strategy in this cases will be somehow similar with the one we used in the non natural characteristic case. Assume $G$ is one of the "remaining cases". Consider $P = LQ$ a parabolic subgroup such that we know an estimate for the cohomology of $L$. Since $P$ contains the $p$ Sylow subgroup of $G$, we have that:

$$\dim H^1(G, V) \leq \dim H^1(P, V) \leq \dim H^1(L, V) + \dim H^1(Q, V)^L$$

Once again we can restrict our consideration to $P$ irreducible factors of $V$ and an inductive argument will give us the general case.

Note that if $M$ is a $P$ irreducible factor of $V$, $Q$ acts trivially on $M$ hence the first cohomology group $H^1(Q, M)$ is in fact the group of homomorphisms $\text{Hom}(Q, M)$ so

$$\dim H^1(Q, M)^L \leq \dim \text{Hom}_L(Q, M)$$

Therefore we can use Lemma 2.2 to find the structure of the factors of $Q$ as $L$ modules. However we will only care about the top factor $Q/Q'$ because we are interested
in $\dim \text{Hom}_L(Q, M)$. We will also use the results of [JP] regarding the cohomology of minimal modules.

If $V_1$ is a factor of $V$ (as $kP$ modules) and it is not Galois conjugate to one of the factors of $Q/Q'$, then

$$\dim \text{Hom}_L(Q, V_1) = 0 \Rightarrow \dim H^1(P, V_1) \leq \dim H^1(L, V_1)$$

and we can use the estimates for $L$.

If $V_1$ is Galois conjugate to one of the factors of $Q/Q'$ then we know exactly its structure and using [JP] we can usually prove that $k = \dim H^1(L, V_1)$ is very small (usually $k = 0$ or 1) and $\dim V_1$ is usually large. We also have that $\dim \text{Hom}_L(Q, V_1) = 1$ therefore

$$H^1(P, V_1) \leq k + 1 \leq \frac{k + 1}{\dim V_1} \dim V_1$$

we can then pick $C$ to be the best bound among the factors to get

$$H^1(G, V) \leq H^1(P, V) \leq k + 1 \leq C \dim V$$

4.1 Linear Groups

At first we will consider $G \cong PSL_n(q)$ for $n \geq 3$ prime and $q < 5$. In this case consider $T$ a non-split maximal torus of $G$ corresponding with the conjugacy class of $n$ cycles in the Weyl group. This will be a cyclic group of order $\frac{q^n - 1}{q - 1}$ in $SL_n(q)$. Also $T$ will be normalized by an element $x$ of order $n$ and in fact (see for example Prop 4.3.6 in [KL]) the group generated by $T$ and $x$ will be maximal in $G$. A straightforward calculation shows that $N_G(x) \not\subset N_G(T)$ hence we can find an element $y \in N_G(x)$ such that $y \not\in N_G(T)$ and consider

$$S := \langle x, T \rangle; R := \langle x, T^n \rangle$$

We must note that since $n$ is prime, the orbits of the action of $x$ on $T$ can only be of length 1 or $n$.

Moreover $x$ acts nontrivially on $T$. Note also that as a group $T$ can be identified to $\mathbb{F}_{q^n}^*/\mathbb{F}_q^*$. The Frobenius acts on this group as an automorphism of order $n$ and it does not fix any element. Therefore the action of $x$ can be identified with some power of the Frobenius hence the orbits have all length $n$.

We can apply Lemma 2.1 and get the following:

**Proposition 4.1.1** If $G \cong PSL_n(q)$ for $n$ prime, $n \not| q$ and $q < 5$ and $V$ is an irreducible $kG$-module with $k$ algebraically closed and char($k$)/$q$ then:

$$\dim H^1(G, V) \leq \frac{1}{n} \dim V.$$
Note that the result will be true for \( n=2 \) as well because in this case the only simple group we need to worry about is \( PSL_2(4) \cong A_5 \) and for this group we have the result from [GK].

We will use this result to obtain one for general arbitrary \( m \). Let \( n \) the largest prime such that \( n \leq m \). We will prove inductively that

\[
\dim H^1(PSL_m(q), V) \leq \frac{1}{n} \dim V.
\]

If \( m = n \) then we already have the bound. Otherwise let \( P = LQ \) the parabolic whose Levi subgroup \( L \) is of type \( m-1 \). Arguing as above,

\[
\dim H^1(G, V) \leq \dim(G, V) \leq \dim H^1(L, V) + \dim \text{Hom}_{L}(Q, V).
\]

Note that the \( L \)-module structure of \( Q \) is in fact that of the natural module for \( PSL_n(q) \). But using [JP] we get that if \( N \) is the natural \( L \) module, \( H^1(L, N) = 0 \) unless \( L = PSL_2(2^n) \) or \( PSL_4(2) \). Nevertheless we do not use these cases in the induction process. Therefore if \( V' \) is a factor of \( V \) which is isomorphic to \( N \), we have \( \dim H^1(P, V') = 1 = (1/n) \dim V' \). If \( V'' \) is a factor not isomorphic to \( N \), we have \( \dim H^1(P, V'') = 0 \) and from the induction hypothesis, \( \dim H^1(P, V'') \leq \frac{1}{n} \dim V'' \).

Hence:

\[
\dim H^1(G, V) \leq \frac{1}{n} \dim V.
\]

**Proposition 4.1.2** If \( G \cong PSL_n(q) \), \( n \) is the largest prime so that \( n \leq m \), \( q < 5 \) and \( V \) is an irreducible \( kG \)-module with \( k \) algebraically closed and \( \text{char}(k) \mid q \) then:

\[
\dim H^1(G, V) \leq \frac{1}{n} \dim V.
\]

**Corollary 4.1** If \( G, V \) as in Prop 4.1.2, we have that:

\[
\dim H^1(G, V) < \frac{2}{m} \dim V.
\]

Proof: \( n \) in the Proposition can be taken > \( m/2 \).

### 4.2 Unitary Groups

The only cases we need to worry are \( q = 2, 3, 4 \).

For these cases, \( G \cong PSU_{2m}(q) \), we will consider the parabolic \( P \) of the type \( L_k(q^2) \). Then if \( P = LQ \), \( P \) will contain the \( p \) Sylow of \( G \) hence \( \dim H^1(G, V) \leq \dim H^1(P, V) \).

As above, \( Q \) will act trivially on an irreducible \( P \) module \( M \) and so after considering the \( P \) irreducible factors of \( V \) we can see that

\[
\dim H^1(P, M) \leq \dim H^1(L, M) + \dim \text{Hom}_{L}(Q, M).
\]

In fact one can easily see that \( Q \) as a \( F_{q^2} \) module is isomorphic to \( N \otimes N^{(2)} \). We will use the following:
Proposition 4.2.1 For any $n \geq 2, q < 5$

1. If $q = 3, 4$
   \[ \dim H^1(GL_n(q^2), N \otimes N^{(2)}) = 0. \]

2. If $q = 2$
   \[ \dim H^1(GL_n(q^2), N \otimes N^{(2)}) \leq 2. \]

Proof. Note that if $q \neq 2$ then the center of $GL_n(q)$ acts nontrivially on $N \otimes N^{(2)}$, hence the cohomology is trivial proving 1.

If $q = 2$ we use induction on $n$. If $n = 2$, then we can use 4.1 and get

\[ \dim H^1(L_2(q^2), N \otimes N^{(2)}) \leq 2/4 = 2. \]

If $n \geq 3$, consider a parabolic subgroup $P = LQ$ such that the Levi subgroup will be of type $L_{n-1}$ and the unipotent radical will be isomorphic with $N''$ the dual of the natural $L$ module. Then we can see that $N \cong D \oplus N'$ as $L$ modules. Here $D$ is one dimensional. Therefore:

\[ N \otimes N^{(2)} \cong D \oplus N' \oplus N''(2) \oplus (N' \otimes N''(2)). \]

and

\[
\begin{align*}
\dim H^1(L_n(q^2), N \otimes N^{(2)}) & \leq \dim H^1(P, N \otimes N^{(2)}) \\
& \leq \dim H^1(L, N \otimes N^{(2)}) + \dim \text{Hom}_L(Q, N \otimes N^{(2)}) \\
& \leq \dim H^1(L, k \oplus N' \oplus N''(2) \oplus (N' \otimes N''(2)) + \\
& + \dim \text{Hom}_L(N'', k \oplus N' \oplus N''(2) \oplus (N' \otimes N''(2))) \\
& \leq \dim H^1(L, k) + \dim H^1(L, N') + \dim H^1(L, N' \otimes N''(2)) \\
& + \dim H^1(L, N' \otimes N'(2)) \\
& \leq 2.
\end{align*}
\]

In the worst case (that is when $q = 2$ and all factors of $V$ are isomorphic to $N \otimes N^{(2)}$) we obtain that if $m \geq 3$, 

\[ \dim H^1(P, M) \leq \frac{3}{m^2} \dim V \leq \frac{1}{n} \dim V \]

where $n$ is the largest prime so that $n \leq m$. Note also that if $q = 3, 4$ then the corresponding $GL_m(q^2)$ give the better bound of $1/m$. 

Proposition 4.2.2

- If $G \cong PSU_{2m}(2), m \geq 3$ is the largest prime so that $n \leq m$, and $V$ is an irreducible $kG$-module with $k$ algebraically closed and $\text{char}(k) = 2$ then:

\[ \dim H^1(G, V) \leq \frac{1}{n} \dim V. \]
• If $G \cong PSU_{2m}(q), q = 3, 4,$ and $V$ is an irreducible $kG$-module with $k$ algebraically closed and $\text{char } k | q$ then:

$$\dim H^1(G, V) \leq \frac{1}{m} \dim V.$$ 

Note that $PSU_4(2) \cong PSp_4(3)$ so the restriction in the proposition is not very strong. Next consider $PSU_{2m+1}(q)$. Assume first that $m = 1$, then we only need to discuss two groups, $U_3(3)$ and $U_5(4)$. the first one was already consider in 2.3 and a bound of $1/3$ was obtained. the second one will be considered in 4.12.

Assume $m \geq 2$, in this case if we use the decomposition from 2.1.9, we get that $P = LQ$ with $P$ of type $A_{m-1}$ and such that $Q/Q'$ is the natural $L$ module. Also $\dim H^1(L, N) = 0$ if $m - 1 > 1$ or $q$ odd and $\dim H^1(L, N) = 1$ if $m = 2, q = 2^i$. We will consider the cases $U_5(2), U_5(4)$ in 4.12. As above note that if $q > 2$ then the groups $GL_m(q^2)$ will give better bounds. Therefore

**Proposition 4.2.3**

• If $G \cong PSU_{2m+1}(2), m \geq 3$, $n$ is the largest prime so that $n \leq m$ and, and $V$ is an irreducible $kG$-module with $k$ algebraically closed and $\text{char } k | q$ then:

$$\dim H^1(G, V) \leq \frac{1}{n} \dim V.$$ 

• If $G \cong PSU_{2m+1}(q), m \geq 2, q = 3, 4$, $(m, q) \neq (2, 4)$, and $V$ is an irreducible $kG$-module with $k$ algebraically closed and $\text{char } k | q$ then:

$$\dim H^1(G, V) \leq \frac{1}{m} \dim V.$$ 

4.3 $PSp_n(q)$

For symplectic groups in even characteristic we have that $PSp_{2n}(2^i) \cong P\Omega_{2n+1}(2^i)$ so we’ll treat them in 4.4 For the other symplectic groups we will use a similar idea with the one we used for $A_n$ in general. To fix the notations let $G$ of type $C_n(q)$, $M$ an irreducible $kG$ module, $P$ a parabolic subgroup of $G$ of type $A_{n-1}$. Also let $P = LQ$ the Levi decomposition. As we noted in 2.1.2, $Q$ is isomorphic as a $kL$ module with the symmetric square of the natural representation. We will prove that in general if $N$ is the natural representation of $G = GL_n(q)$ with q small, then $\dim H^1(G, S^2N) \leq 1$. If $q \neq 3$ note that the center of $G$ will act without fixed points on $S^2N$. But if we apply the inflation restriction sequence to this we get that

$$H^1(G, S^2N) \leq H^1(Z(G), S^2N) = 0$$

So the only case that we need to consider is $q = 3$. We will need the following result (cf Lemma 1.8 of [LiST] or [K]^2):
Proposition 4.3.1

\[ \dim H^1(GL_n(3), S^2N) \leq 1 \]

Using this and our estimate for \( GL_n \) to prove (note that for b) we will use the result in Proposition 3.2.1):

**Proposition 4.3.2**

a) If \( G \cong PSp_{2n}(q) \), \( q < 5 \), \( V \) is an irreducible \( G \)-module, and \( t \) is the largest prime \( t \leq n \) or \( t = 3 \) if \( n = 2 \) then:

\[ \dim H^1(G, V) \leq \frac{1}{t} \dim V. \]

b) If \( G \cong PSp_{2n}(q) \), \( q = 5, 9 \) then

\[ \dim H^1(G, V) \leq \frac{1}{\max\{3, n\}} \dim V. \]

**4.4** \( \Omega_{2n+1}(q) \), \( q < 5 \)

This is the case \( B_n \) and as in the non-natural characteristic, we will consider a parabolic subgroup \( P_J \) of type \( A_{n-1} \). If \( P = LQ \) is the Levi decomposition, then \( Q \) has order \( q^n(n+1)/2 \) and it is, as a \( GL_n \) module, indecomposable of length 2 having the factor \( Q/Q^{(2)} \cong N \) where \( N \) is the natural \( GL_n \) module. Again since \( P \) contains the \( p \) Sylow subgroup of \( G \), we have:

\[ \dim H^1(G, V) \leq \dim H^1(P, V) \]

and after passing to irreducible \( P \) factors of \( V \):

\[ \dim H^1(P, M) \leq \dim H^1(L, M) + \dim \text{Hom}_L(Q, M) \]

Now \( \dim \text{Hom}_L(Q, M) = 0 \) unless \( M \) is Galois conjugated to \( N \). But in this case we have to deal with \( H^1(L, N) \) and, using the results in [JP], this is zero unless \( n=2 \) and \( q = 2^k \) or \( n=3 \) and \( q = 2 \). Note that in the case \( G = Sp_4(4) = PO_5(4) \), [S2] gives a bound of 1/4. Also if \( n = 2, q = 2 \) the group is in fact \( S_6 \) and we obtain a bound of 1/4 from [GK]. We will consider the case \( PSp_6(2) \) in 4.12.

If we exclude the previous cases, we can use the bounds for \( A_n \) and obtain

\[ \dim H^1(G, V) \leq \frac{1}{t} \dim V \]

where \( t \) is the largest prime \( \leq n \).

Therefore we proved:

**Proposition 4.4.1** If \( G \cong P\Omega_{2n+1}(q), n \geq 2, q < 5, (n, q) \neq (3, 2) \), \( V \) is an irreducible \( G \)-module, and \( t \) is the largest prime \( t \leq n \) and \( t = 3 \) if \( (n, q) = (2, 3) \), \( t = 4 \) if \( (n, q) = (2, 2), (2, 4) \)

\[ \dim H^1(G, V) \leq \frac{1}{t} \dim V. \]

Note that the results of [ABS] might fail in characteristic 2, however if \( n \) is larger than 3, the \( n \) dimensional module cannot have length 2 so it has to be irreducible.
4.5 $\Omega^+_n(q), n \geq 3$

We are in the case $D_n$, we will consider once again $P = LQ$ the $A_{n-1}$ parabolic. Then if the characteristic is odd, $Q$ is isomorphic as an $L$ module with the exterior square of the natural representation.

The cohomology of this module is 0 by [JP]. Then we can use the same argument as above and get the same type of bound.

If we are in characteristic 2, we can encounter two types of problems. First, it is possible that the cohomology of $\Lambda^2 N$ is not trivial. This is the case when $(n-1, q) = (1, 2^n), (2, 2)$ or $(3, 2)$. The first two cases are not possible and so the only possibility is $\Omega_8(2)$. Still in this case the dimension of $Q$ is in fact 6 and thus we can see that even if $M \cong \Lambda^2 N$, thus $\dim H^1(G, M) \leq 2$, we still get the estimate of $\frac{1}{3}$ that we need for the argument.

The next problem we could encounter is that $Q$ might not be irreducible but indecomposable of length 2. This might happen only if $n=5$ and in this case $Q$ will have two factors, both isomorphic to the natural module. However in this case since $H^1(L_n, N) = 0$ we can modify the argument slightly and still get:

**Proposition 4.5.1** If $G \cong \Omega^+_n(q), q < 5$, $V$ is an irreducible $G$-module, and $t$ is the largest prime $t \leq n$ then:

$$\dim H^1(G, V) \leq \frac{1}{t} \dim V.$$ 

4.6 $\Omega^-_n(q), n > 3$

This time we are in the case $2D_n(q)$. Using the decomposition in 2.1.6 we get a parabolic $P$ in which the Levi complement is of the type $A_{n-2}$ and the unipotent radical will have $Q/Q'$ isomorphic with the natural $n-1$ module. Once again if $V'$ is a $L$ factor of $V$ that is isomorphic with the natural module, $H^1(L, V') = 0$. Therefore:

**Proposition 4.6.1** If $G \cong \Omega^-_n(q), n > 3, q < 5$, $V$ is an irreducible $G$-module, and $t$ is the largest prime $t \leq n-1$ then:

$$\dim H^1(G, V) \leq \frac{1}{t} \dim V.$$ 

4.7 $F_4(q)$

In this case we will take $P$ to be the parabolic subgroup that corresponds with the Levi complement of type $B_3$ if $q = 3$ and $C_3$ if $q$ is odd. There are two factors in the decomposition of the unipotent radical. $Q^{(2)} \cong N$ the natural module and $Q/Q^{(2)} \cong S$ the spin module. From [JP] $H^1(L, S) = 0$. So if $M$ is an irreducible $P$ module we have:

$$\dim H^1(P, M) \leq \dim H^1(L, M) + \dim \text{Hom}_L(Q, M) \leq \frac{1}{3} \dim M$$

In conclusion after passing to $P$ irreducible factors of $V$ we get:
Proposition 4.7.1 If $G$ is of type $F_4(q)$, $q < 5$, $V$ is an irreducible $G$-module, then
\[ \dim H^1(G, V) \leq \frac{1}{3} \dim V. \]

4.8 $E_6(q)$
In this case we will consider the $P = LQ$ the parabolic subgroup of type $A_5$. Then the corresponding unipotent subgroup $Q$ will be in fact isomorphic with $S^2N$ where $N$ is the natural module. We have proved that $\dim H^1(L, S^2N) = 0$ unless $q = 3$ and in this last case the cohomology is $1$ dimensional. Therefore we can see that:

Proposition 4.8.1 If $G$ is of type $E_6(q)$, $q < 5$, $V$ is an irreducible $G$-module, then
\[ \dim H^1(G, V) \leq \frac{1}{5} \dim V. \]

4.9 $^2E_6(q), q < 5$
We can use the same decomposition as in the proof of Proposition 2.2.5 and get a parabolic subgroup for which the Levi complement of type $^2A_5(q^2)$ and the unipotent radical is isomorphic with $S^2N$ where $N$ is the natural $6$ dimensional module. If we are in characteristic $2$, we have that this module is indecomposable of length $2$ with the top factor isomorphic with $N$ and $H^1(L, N) = 0$. If the characteristic is not $2$, the module $S^2N$ will be irreducible and $H^1(L, S^2N) = 0$. Since there is a bound of $1/3$ for $^2A_5$, then:

Proposition 4.9.1 If $G$ is of type $^2E_6(q), q < 5$, $V$ is an irreducible $G$-module, then
\[ \dim H^1(G, V) \leq \frac{1}{3} \dim V. \]

4.10 $E_7(q)$
Let $P = LQ$ be the parabolic of type $A_6$. Then then $Q$ will have two shapes, one of dimension $7$ ($V_{2(c_3-e_6)}$) and one of dimension $35$ ($V_{(c_3-e_6)}$). Then $Q/Q'$ is isomorphic with $\Lambda^2N$ where $N$ is the natural module and the cohomology of this module is trivial. If we use the same argument as before we get:

Proposition 4.10.1 If $G$ is of type $E_7(q), q < 5$, $V$ is an irreducible $G$-module, then
\[ \dim H^1(G, V) \leq \frac{1}{7} \dim V. \]
4.11 $E_8(q)$

Let $P = LQ$ be the parabolic of type $A_7$. In this case we will have three shapes the dimensions of which are 8, 28, 56. We have that $Q/Q'$ is $\Lambda^3 N$ where $N$ is the natural module. Also $H^1(\text{PSL}_8, \Lambda^3 N) = 0$. Thus:

**Proposition 4.11.1** If $G$ is of type $E_8(q)$, $q < 5$, $V$ is an irreducible $G$-module, then

$$\dim H^1(G, V) \leq \frac{1}{7} \dim V.$$  

4.12 The exceptions

Consider first $G = G_2(q)$, $2 < q < 5$. Note that if $q = 3^n$, the problem is already solved by [S4]. In particular we have that:

**Proposition 4.12.1** If $G$ is of type $G_2(3)$, $V$ is an irreducible $G$-module, then:

$$\dim H^1(G, V) \leq \frac{1}{49} \dim V.$$  

If $q = 4$ we can see that $G$ has a cyclic subgroup $S_1$ of order 13 and that there is an element $x \in G$ of order 6 that normalizes $S_1$. Let $B = \langle x \rangle$ and $T_1 = S_1B$. There are two maximal subgroups $H_1, H_2$ of $G$ that contain $T_1$. They are of type $L_2(13)$ and respectively $U_3(4) : 2$. In each of these two subgroups the centralizer of an element of order 6 is trivial. The centralizer of $x$ in $G$ has order at least 12 hence there is an element $g \in G$ of order 2 that centralizes $x$ but is not contained in any of the two subgroups. Let $S_2 = S_1^g$. If $S_2$ is included in $H_i$ one of the two subgroups, then there is a $g' \in H_1$ such that $g' g^{-1} \in N_G(T_1) = T_1$ hence $g \in H_i$ and that is a contradiction. It follows that $S_2$ does not belong to any of the $H_i$ hence $G = \langle S_1, B, S_2 \rangle$ and we can apply Lemma 2.1 to get a bound of $1/6$ for this group.

Consider next the case $2F_4(q)$. If $G = 2F_4(8)$ then by $[M]$, the normalizer in $G$ of an element of order 37 is maximal and it is isomorphic to the semidirect product of the cyclic group of order 37 with a cyclic group of order 12. Also using GAP the element of order 12 has nontrivial centralizer in $G$. Let $s$ an element of order 37, $a$ an element of order 12 in its normalizer and $g$ an element that centralizes $a$ but does not belong to the normalizer of $s$. Let $S_1 = \langle s \rangle$, $B = \langle a \rangle$, $S_2 = S_1^g$ and apply Lemma 2.1. Since the orbits of $B$ on $S_1$ have all length 12 we get a bound of $1/12$ for this group.

The case $G = 2F_4(2)'$ has been treated in 2.3.

If $G = U_3(4)$ then there is a maximal subgroup of type $13 : 3$ and the centralizer in $G$ of an element of order 3 has order 15. We can take $S_1$ a cyclic group of order 13, $B = \langle x \rangle$ a group of order 3 that normalizes $S_1$, $g \in C_G(x) \setminus B$ and $T_2 = T_1^g$ in Lemma 2.1 to get a bound of $1/3$ for this group.

In the case $G = PSU_5(2)$ consider $S_1$ the cyclic subgroup of order 11, let $B = \langle x \rangle$ a subgroup of order 5 that normalizes $S_1$ and $T_1 = S_1 : B$. There is a unique maximal subgroup $H$ of $G$ that contains $T_1$ and $H = L_2(11)$. In $H$ the centralizer of an element

32
of order 5 is trivial and in $G$ the order of the centralizer of $x$ is at least 15. We can take then $g$ an element of order 3 that centralizes $x$ but is not contained in $H$ and repeat the argument above to obtain a bound of $1/5$ for this group.

If $G = PSU_5(4)$ consider $P = LQ$ a parabolic of type $U_3(4)$, $P$ will contain the $p$ Sylow of $G$ therefore $\dim H^1(G, V) \leq \dim H^1(P, V)$. As above, $Q$ will act trivially on an irreducible $P$ module $M$ and so after considering the $P$ irreducible factors of $V$ we can see that for each factor $M$,

$$\dim H^1(P, M) \leq \dim H^1(L, M) + \dim \text{Hom}_L(Q, M).$$

Note that in this case, $Q/Q'$ is the natural module for $U_3(4)$ and we know that $H^1(U_3(4), N) = 0$ so we get a bound of $1/3$ for this group.

If $G = PSp_6(2)$, consider $S_1$ a cyclic subgroup of order 7, then $N_G(S_1) = 7 : 6$ will be included in 4 maximal subgroups, $H_1 = S_8$, $H_2 = U_3 : 2$, $H_3 = 2^6 : L_3(20)$ (parabolic), $H_4 = L_2(8) : 3$. Note that in each of these subgroups with the exception of $S_8$ the centralizer of an element of order 6 is trivial and that in the case of $S_8$ the centralizer can have order 18. Nevertheless in $G$ any element $x$ of order 6 is centralized by an involution that is not a power of $x$. We can then take $B = \langle x \rangle$ a subgroup of order 6 normalizing $S_1$, $g \in C_G(x) \setminus B$ an involution and $S_2 = S_2^g$. It will follow that $G = \langle S_1, B, S_2 \rangle$ hence applying Lemma 2.1 we get a bound of $1/6$ for this group.

References


[C1] R. Carter “Simple Groups of Lie Type”, Willey-Interscience


D. Segal and A. Shalev, On groups with bounded conjugacy classes, preprint.


<table>
<thead>
<tr>
<th>Name</th>
<th>Bound</th>
<th>Exceptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(PSL_n(q))</td>
<td>(\gcd(n,q-1)/q^{n-1}-1)</td>
<td>(n = 2, q = 2, 3, 5) (G \cong S_3, A_4, A_5)</td>
</tr>
<tr>
<td>(PSp_{2n}(q))</td>
<td>(\gcd(n,q-1)/q^{n-1}-1)</td>
<td>((n, q) = (2, 2) \Rightarrow c = 1/3)</td>
</tr>
<tr>
<td>(\Omega_{2n+1}(q), q \text{ odd}, n \geq 3)</td>
<td>(\gcd(n,q-1)/q^{n-1}-1)</td>
<td>none</td>
</tr>
<tr>
<td>(\Omega_{2n}(q), n \geq 4)</td>
<td>(\gcd(n,q-1)/q^{n-1}-1)</td>
<td>none</td>
</tr>
<tr>
<td>(\Omega_{2n}(q), n \geq 4)</td>
<td>(\gcd(n-1,q-1)/q^{n-1}-1)</td>
<td>none</td>
</tr>
<tr>
<td>(PSU_{2n}(q), n \geq 2)</td>
<td>(\gcd(n,q^2-1)/q^{n-1}-1)</td>
<td>none</td>
</tr>
<tr>
<td>(\Omega_{2n+1}(q))</td>
<td>(\gcd(n,q^2-1)/q^{n-1}-1)</td>
<td>(n = 1, q &gt; 3, c = \frac{1}{d(q-1)}, d</td>
</tr>
<tr>
<td>(^2B_2(q)/q \geq 8)</td>
<td>(\sqrt{(q/2)(q-1)})</td>
<td>none</td>
</tr>
<tr>
<td>(^2G_2(q), q &gt; 3)</td>
<td>(q(q-1))</td>
<td>(q = 3, 5 \Rightarrow c = 1/6)</td>
</tr>
<tr>
<td>(G_2(q)/q &gt; 2)</td>
<td>(\gcd(2q-1)/q^{n-1}-1)</td>
<td>(q = 2, 3, 5 \Rightarrow c = 1/4)</td>
</tr>
<tr>
<td>(^3D_4(q))</td>
<td>(\gcd(2q-1)/q^{n-1}-1)</td>
<td>(\frac{1}{\sqrt{(q/2)(q-1)}}) (G = ^2F_4(2)^\prime, c = 1/6)</td>
</tr>
<tr>
<td>(^2F_4(q))</td>
<td>(\gcd(3q-1)/q^{n-1}-1)</td>
<td>none</td>
</tr>
<tr>
<td>(F_4(q))</td>
<td>(\gcd(3q-1)/q^{n-1}-1)</td>
<td>none</td>
</tr>
<tr>
<td>(E_i(q), i = 6, 7, 8)</td>
<td>(\gcd(i,q-1)/q^{n-1}-1)</td>
<td>none</td>
</tr>
<tr>
<td>(^2E_6(q))</td>
<td>(\gcd(2q^2-1)/q^{n-1}-1)</td>
<td>none</td>
</tr>
</tbody>
</table>

Table 1: The mixed characteristic case
<table>
<thead>
<tr>
<th>Name</th>
<th>$q \geq 5$</th>
<th>$q &lt; 5$</th>
<th>Exceptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_n(q)$</td>
<td>$1/n$</td>
<td>$\frac{1}{\max_t {t \text{ prime}, t \leq n, t \mid q}}$</td>
<td>$L_2(q), q \text{ odd, } c = 1/3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$L_2(2^k), c = 1/2$</td>
</tr>
<tr>
<td>$U_{n+1}(q)$</td>
<td>$\frac{1}{n+1}$ if $n$ odd</td>
<td>$\frac{1}{\max_t {t \text{ prime}, t \leq n}}$ if $q = 2$</td>
<td>$U_3(3), c = 1/6$</td>
</tr>
<tr>
<td></td>
<td>$1/n$ if $n$ even</td>
<td>$\frac{1}{\lfloor \frac{n+1}{2} \rfloor}, q = 3, 4$</td>
<td>$U_3(4), c = 1/3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$U_5(2), c = 1/5$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$U_5(4), c = 1/3$</td>
</tr>
<tr>
<td>$PSp_{2n}(q), q$ odd</td>
<td>$1/2n$ if $q \neq 5, 9$</td>
<td>$\frac{1}{\max_t {t \text{ prime}, t \leq n}}$ if $q = 5, 9$</td>
<td>$PSp_6(3), c = 1/3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$PSp_4(q), c = 1/3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>for $q = 3, 5, 9$</td>
</tr>
<tr>
<td>$\Omega_{2n+1}(q)$</td>
<td>$1/2n$</td>
<td>$\frac{1}{\max_t {t \text{ prime}, t \leq n}}$ if $q = 5, 9$</td>
<td>$\Omega_5(3), c = 1/3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Omega_5(2), \Omega_5(4), c = 1/4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Omega_7(2), c = 1/6$</td>
</tr>
<tr>
<td>$\Omega_{2n}^+(q), n \geq 3$</td>
<td>$1/(2(n-1))$</td>
<td>$\frac{1}{\max_t {t \text{ prime}, t \leq n}}$</td>
<td>none</td>
</tr>
<tr>
<td>$\Omega_{2n}^-(q), n \geq 3$</td>
<td>$1/(2(n-1))$</td>
<td>$\frac{1}{\max_t {t \text{ prime}, t \leq n-1}}$</td>
<td>none</td>
</tr>
<tr>
<td>$E_6(q)$</td>
<td>$1/12$</td>
<td>$1/5$</td>
<td>none</td>
</tr>
<tr>
<td>$E_7(q)$</td>
<td>$1/18$</td>
<td>$1/7$</td>
<td>none</td>
</tr>
<tr>
<td>$E_8(q)$</td>
<td>$1/30$</td>
<td>$1/7$</td>
<td>none</td>
</tr>
<tr>
<td>$F_4(q)$</td>
<td>$1/12$</td>
<td>$1/3$</td>
<td>none</td>
</tr>
<tr>
<td>$G_2(q)$</td>
<td>$1/6$</td>
<td>$1/49$ if $q = 3$</td>
<td>none</td>
</tr>
<tr>
<td>$G_2(q)$</td>
<td>$1/49$</td>
<td>$13/42$ if $q = 4$</td>
<td>none</td>
</tr>
<tr>
<td>$E_6(q)$</td>
<td>$1/12$</td>
<td>$1/3$</td>
<td>none</td>
</tr>
<tr>
<td>$D_4(q)$</td>
<td>$1/6$</td>
<td>—</td>
<td>none</td>
</tr>
<tr>
<td>$B_2(q)$</td>
<td>$1/4$</td>
<td>$1/4$</td>
<td>none</td>
</tr>
<tr>
<td>$G_2(q)$</td>
<td>$1/49$</td>
<td>$1/49$</td>
<td>none</td>
</tr>
<tr>
<td>$F_4(q)$</td>
<td>$1/8$</td>
<td>—</td>
<td>$^{2}F_4(8), c = 1/12$</td>
</tr>
<tr>
<td>$F_4(q)$</td>
<td>$1/8$</td>
<td>—</td>
<td>$^{2}F_4(2), c = 1/13$</td>
</tr>
</tbody>
</table>

Table 2: The natural characteristic case