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On the regularity and uniformness conditions on quadrilateral grids

So-Hsiang Chou *, Songnian He

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403-0221, USA

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Abstract

In the literature there are various regularity conditions as well as many different so-called almost parallelogram conditions on the subdivisions of quadrilateral grids. In this paper we investigate and clarify their relations.

Keywords: Quadrilateral grids; Almost parallelogram; Quasi-regular; Regular

1. Introduction

In the discretization of partial differential equations using the finite element, finite difference, or finite volume method, one needs to choose a proper grid system for the problem domain. Next to rectangular grids, quadrilateral grids seem to have the simplest data structure and at the same are flexible enough to fit domains with complicated boundary geometry. It is not surprising that quadrilateral grids have been used in various application fields [6,4,8,7,11,14]. The rigorous analysis of mixed finite element or finite volume methods using quadrilateral grids is more recent. To get optimal order or superconvergence error estimates, one has to impose certain regularity and/or quasi-uniformness conditions on quadrilateral grids [1,2,4,5,9,18]. In the literature, one can find many different regularity and quasi-uniformness conditions, and confusion may arise as to how they are related. The purpose of this paper is to investigate and clarify their relations. In Section 2, we study various regularity conditions, carefully classifying them as regular and quasi-regular conditions. In Section 3, we look at several almost parallelogram conditions based either on angle conditions or the distance between midpoints of the diagonals. In Section 4 we investigate the equivalence relation between regularity condition and the min–max angle and edge ratio condition. Finally, a brief summary of the main results is given at the end of the paper.

* Corresponding author.
E-mail addresses: chou@bgnet.bgsu.edu (S.-H. Chou), songnianh@yahoo.com (S. He).

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2. Regularity conditions on quadrilateral grids

Let h_K denote the diameter of a quadrilateral K. A set of quadrilaterals $\mathcal{Q}_h = \{K_i\}$, $h = \max_i h_{K_i}$, is called a subdivision of a polygonal domain Ω if

1. $\overline{\Omega} = \cup K_i$,
2. The intersection of K_i and K_j, $i \neq j$ is either the empty set or a common edge, or a common node.

For each quadrilateral K with counterclockwise oriented vertices $\mathbf{x}_i$, $i = 1, 2, 3, 4$, let S_i be the subtriangle of K with vertices $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1} (\mathbf{x}_0 = \mathbf{x}_4)$. Let

$$\rho_K = \min_{1 \leqslant i \leqslant 4} \{\text{diameter of circle inscribed in } S_i\}. \tag{2.1}$$

We say that a family of subdivisions $\{\mathcal{Q}_h\}$ is *regular* [10] if there exists a positive constant C_1, independent of h, such that

$$\frac{h_K}{\rho_K} \leqslant C_1, \quad \forall K \in \mathcal{Q}_h. \tag{2.2}$$

Next we say that a family of subdivisions $\{\mathcal{Q}_h\}$ is *quasi-regular* if there exists a positive constant C_2 independent of h such that $|K|$, the area of the quadrilateral K, satisfies

$$|K| \geqslant C_2 h_K^2 \quad \forall K \in \mathcal{Q}_h. \tag{2.3}$$

We now show that condition (2.3) is indeed weaker and allows for possibility of some thin triangles S_i. Hence it is justified to call condition (2.3) quasi-regular.

Theorem 2.1. *Condition* (2.2) *implies condition* (2.3), *but not vice versa.*

Proof. Referring to the triangle ΔABC and its inscribed circle in Fig. 1, one has

$$|K| \geqslant \pi r^2 \geqslant \pi \left(\frac{\rho_K}{2}\right)^2 \geqslant \pi \left(\frac{h_K}{2C_1}\right)^2 = \frac{\pi}{4C_1^2} h_K^2.$$

Now we give a counter example to show that the converse is *not* true. Construct a quadrilateral ABCD (cf. the right half of Fig. 1) so that

$$|AB| = |BD| = |DA| = h_K, \quad AC \perp BD, \ |EC| = h_K^2.$$

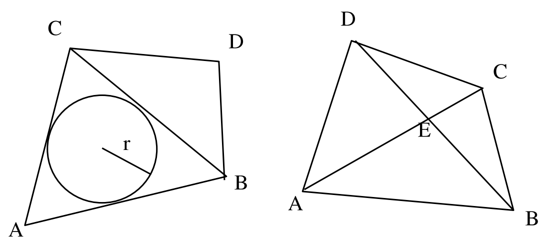

Fig. 1. Figures for Theorem 2.1.

Then

$$|K| = \frac{\sqrt{3}}{4}h_K^2 + \frac{1}{2}h_K^3 \geqslant \frac{\sqrt{3}}{4}h_K^2.$$

Since $\rho_K \leqslant h_K^2$ the partition is not regular.

This completes the proof. □

In view of the above proof, we now introduce a new *shape quasi-regular* condition: there exists a positive constant C_1 independent of h such that

$$\frac{h_K}{\rho_K^*} \leqslant C_1, \qquad \rho_K^* = \max_{1\leqslant i\leqslant 4}\{\text{diameter of circle inscribed in } S_i\}, \quad \forall K \in \mathcal{Q}_h. \tag{2.4}$$

In contrast to the quantity ρ_K in (2.1), the new parameter ρ_K^* is a maximum rather than a minimum to allow for possible thin triangles S_i. It is not surprising the following holds.

Theorem 2.2. *Condition* (2.3) *is equivalent to condition* (2.4).

Proof. Condition (2.4) implies condition (2.3): Similar to the case of condition (2.2) implying condition (2.3), one only has to take an inscribed circle with the diameter $2r = \rho_K^*$.

Condition (2.3) implies condition (2.4):

Let the quadrilateral K be $P_1P_2P_3P_4$ (cf. Fig. 2). Then

$$|K| = \tfrac{1}{2}(|P_1P_2| + |P_1P_4| + |P_2P_4|)r_1 + \tfrac{1}{2}(|P_2P_3| + |P_3P_4| + |P_2P_4|)r_2.$$

Assume $|K| \geqslant C_2h_K^2$. Then

$$\tfrac{1}{2}(|P_1P_2| + |P_1P_4| + |P_2P_4|)r_1 + \tfrac{1}{2}(|P_2P_3| + |P_3P_4| + |P_2P_4|)r_2 \geqslant C_2h_K^2$$

and hence

$$\tfrac{3}{2}(r_1 + r_2)h_K \geqslant C_2h_K^2.$$

Since $\frac{3}{2}\rho_K^* \geqslant \frac{3}{2}(r_1 + r_2)$ we have $(h_K/\rho_K^*) \leqslant 3/2C_2$. This completes the proof. □

We next introduce another quasi-regularity condition based on the maximum diameter of the circles contained in K, and show that it is equivalent to the previous two quasi-regularity conditions (2.3) and (2.4).

Theorem 2.3. *Let there be a constant $C > 0$ independent of h such that*

$$\frac{h_K}{\tilde{\rho}_K} \leqslant C, \tag{2.5}$$

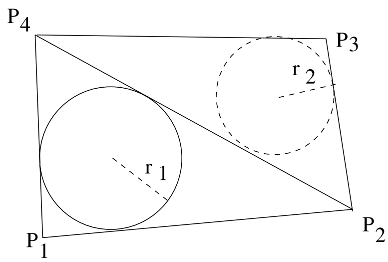

Fig. 2. Figure for Theorem 2.2.

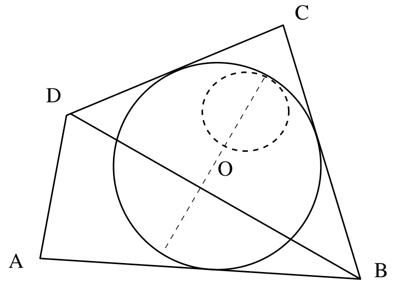

Fig. 3. Figure for Theorem 2.3.

where $\tilde{\rho}_K = \max\{$*the diameter of any circle contained in the quadrilateral K*$\}$. *Then conditions* (2.3), (2.4) *and* (2.5) *are equivalent*.

Proof. We have seen that conditions (2.3) and (2.4) are equivalent, and it remains to show that conditions (2.5) and (2.4) are equivalent.

Condition (2.4) implies condition (2.5):

Since $(h_K/\rho_K^*) \leqslant C$ and $\tilde{\rho}_K \geqslant \rho_K^*$, we have

$$\frac{h_K}{\tilde{\rho}_K} \leqslant \frac{h_K}{\rho_K^*} \leqslant C.$$

Condition (2.5) implies condition (2.4):

If (2.5) holds, then $(h_K/\tilde{\rho}_K) \leqslant C$. In reference to Fig. 3, by definition, there is a circle with diameter $\tilde{\rho}_K$ contained in the quadrilateral as shown in the figure. Without loss of generality, assume that the center O of this circle is either on the diagonal DB or falls in the triangle ΔBCD. One can then draw a smaller circle with the diameter $\tilde{\rho}_K/2$ in the triangle ΔBCD. Hence $\rho_K^* \geqslant (\tilde{\rho}_K/2)$. Consequently,

$$\frac{h_K}{2\rho_K^*} \leqslant \frac{h_K}{\tilde{\rho}_K} \leqslant C$$

or

$$\frac{h_K}{\rho_K^*} \leqslant 2C$$

and (2.4) holds. This completes the proof. □

3. Almost parallelogram conditions

We now turn to the issue of how to call a given quadrilateral almost parallelogram. First we can use the following two angles to measure it. Let K be a quadrilateral $ABCD$ (see Fig. 4), and denote by α the angle between the two normals to the opposite sides AB and DC. Similarly, we have an angle α' for the opposite sides AD and BC. Now let $APCD$ form a parallelogram, then the smallness of both $\beta := \pi - \alpha = \angle BAP$ and $\beta' := \pi - \alpha' = \angle PCB$ would be a good indicator of almost parallelogram [12]. On the other hand, the smallness of the quantity d_K, the distance between the two midpoints of the two diagonals of K, is also a good indicator of almost parallelogram [4,5,16]. Now we relate the two parameters.

Theorem 3.1. *Let* d_K, β, *and* β' *be defined as in the preceding paragraph. Under the regularity condition* (2.2), *the following are equivalent*:

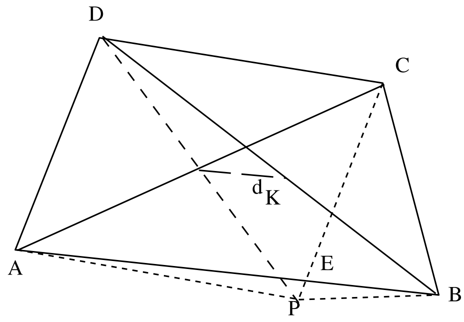

Fig. 4. Figure for Theorem 3.1.

The $o(h)$ *midpoints-distance condition:*

$$d_K = o(h_K), \quad \text{as } h \to 0. \tag{3.6}$$

The $o(1)$ *small angles condition: for the angles* α *and* α' *between the normals to the opposite sides of K*

$$\beta := \pi - \alpha = o(1), \qquad \beta' := \pi - \alpha' = o(1), \quad \text{as } h \to 0. \tag{3.7}$$

Here $q = o(1)$ means $\lim_{h\to 0} q = 0$.

Proof. First we show that condition (3.6) implies condition (3.7).

In reference to Fig. 4, let the quadrilateral K be $ABCD$. Draw the auxiliary line CP parallel to AD so that AP is parallel to DC. Assume for the time being that P falls outside of the quadrilateral K. Connecting PB we have $|PB| = 2d_K$. Let the circumscribed circle of the triangle ΔPBA have diameter d. Since $\beta = \angle PAB$, first by elementary geometry and then by regularity condition (2.3)

$$\frac{|PB|}{\sin\beta} = d \geqslant \rho_K \geqslant \frac{h_K}{C_1},$$

which upon combining $|PB| = 2d_K$ implies $d_K \geqslant (1/2C_1)h_K \sin\beta \geqslant 0$.

From $d_K = o(h_K)$ we know $h_K \sin\beta = o(h_K)$ as h tends to zero. Thus $\sin\beta = o(1)$ and equivalently $\beta = o(1)$. Similarly, $\beta' = o(1)$ as $h \to 0$.

Now we show that condition (3.7) implies condition (3.6).

Consider the triangle ΔAPE whose area

$$|\Delta APE| = \tfrac{1}{2}|AE||AP|\sin\beta \leqslant \tfrac{1}{2}h_K^2 \sin\beta.$$

and on the other hand

$$|\Delta APE| = \tfrac{1}{2}|PE|\,l$$

where l is the distance between the two parallel lines AD and PC.

Obviously $l \geqslant \rho_K$ and hence

$$\frac{h_K}{2C_1}|PE| \leqslant \frac{1}{2}|PE|\rho_K \leqslant |\Delta APE| \leqslant \frac{1}{2}h_K^2 \sin\beta.$$

Thus

$$|PE| \leqslant C_1 h_K \sin\beta. \tag{3.8}$$

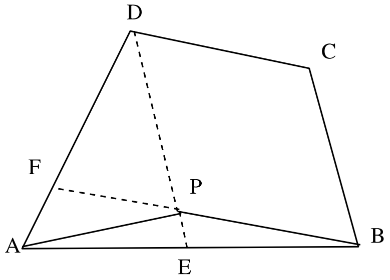

Fig. 5. Figure for Theorem 3.1.

Similarly, one has

$$|EB| \leqslant C_1 h_K \sin \beta'. \tag{3.9}$$

Since β, $\beta' = \mathrm{o}(1)$ we have from (3.8) and (3.9) that $|PE|$, $|EB| = \mathrm{o}(h_K)$, and hence $|PB| = \mathrm{o}(h_K)$. Thus $d_K = \mathrm{o}(h_K)$.

We note that if P falls inside the quadrilateral, we can show that all sides of the smaller quadrilateral $AEPF$ in Fig. 5 are $\mathrm{o}(h_K)$, and hence its diagonal $|PA| = 2d_K = \mathrm{o}(h_K)$ still holds. This completes the proof. □

From the above arguments, it is not hard to see the following theorem holds.

Theorem 3.2. *Under the regularity assumption* (2.2), *we have that the* $o(h^2)$ *midpoints distance condition*

$$d_K = o(h_K^2) \tag{3.10}$$

is equivalent to the $o(h)$ *angle conditions*

$$\beta = o(h_K), \qquad \beta' = o(h_K).$$

Here $q = o(h^m)$ *means there exists a constant* C *such that* $|q(h)| \leqslant Ch^m$ *for all* $0 \leqslant h \leqslant h_0$ *for some* h_0.

One can obtain a family of almost parallelograms satisfying (3.10) by successively refining each quadrilateral in the coarser grid into four smaller quadrilaterals by connecting the opposite midpoints. This is so, since one can show via elementary analytic geometry that $d_{h/2} = (1/4)d_h$, where d_h and $d_{h/2}$ are, respectively, the distances between midpoints for the two levels.

Now it is possible to have very narrow and skewed parallelograms in the subdivision, violating the regularity condition. In the literature, there appear the following two kinds of the so-called h^2-parallelogram conditions [3,4,6,17] combined with regularity conditions of some sort, and we show that they are more or less equivalent.

Given a quadrilateral K one can find two adjacent sides of the quadrilateral K to set up a parallelogram P_K that is contained in K. Let $\tilde{\rho} :=$ the largest diameter of any disk contained in the parallelogram P_K, and let $\tilde{h}$ be the diameter of P_K. (Cf. Fig. 6.)

Theorem 3.3. *The following two almost parallelogram-regularity conditions are equivalent*:

$$(I): \begin{cases} \dfrac{h_K}{\rho_K} \leqslant C \\ |BE| = o(h_K^2) \end{cases} \qquad (II): \begin{cases} \dfrac{\tilde{h}}{\tilde{\rho}} \leqslant \tilde{C} \\ |BE| = o(\tilde{h}^2). \end{cases}$$

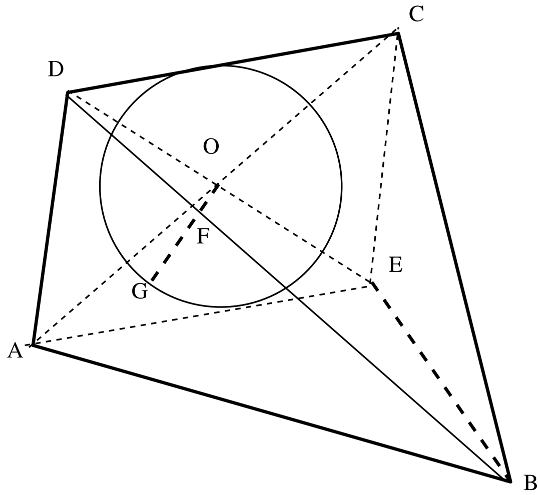

Fig. 6. Figure for Theorem 3.3.

Proof. We first show that (I) $\Rightarrow$ (II). Obviously $\rho_K \leqslant \tilde{\rho} \leqslant \tilde{h} \leqslant h_K$. Hence (see Fig. 6)

$$\frac{\tilde{h}}{\tilde{\rho}} \leqslant \frac{h_K}{\rho_K} \leqslant C, \qquad \frac{|BE|}{\tilde{h}^2} \leqslant \frac{|BE|}{\rho_K^2} \leqslant \frac{C^2|BE|}{h_K^2} \leqslant C^*,$$

and (II) holds.

Now we show that (II) $\Rightarrow$ (I), provided that $\tilde{h}$ is small enough. Let (II) hold: $(\tilde{h}/\tilde{\rho}) \leqslant \tilde{C}$, $(|BE|/\tilde{h}^2) \leqslant \tilde{C}'$. Assume $\tilde{h}$ is small enough so that $\tilde{h} \leqslant (1/4\tilde{C}\tilde{C}')$. First $|BE| = \mathrm{o}(h_K^2)$ since $\tilde{h} \leqslant h_K$ and $|BE| = \mathrm{o}(\tilde{h}^2)$. Next we show the regularity condition $(h_K/\rho_K) \leqslant C$ holds. Since the diameter $\tilde{\rho}$ must equal the distance between the two longer parallel sides, one can draw a circle with diameter $\tilde{\rho}$ and the center O at the intersection of the two diagonal lines of the parallelogram. Obviously, the diameters of the inscribed circles of ΔABC, ΔBCD, and ΔACD, respectively, are at least $\tilde{\rho}/2$, since all these triangles must contain at least half of the above mentioned circle.

Thus to prove the regularity of K, all we need to show is that the inscribed circle of the (remaining) triangle ΔABD is of moderate size. Let us drop a vertical line from O to BD with foot at F. Then $|OF| \leqslant |BE| \leqslant \tilde{C}'\tilde{h}^2$, and

$$|FG| = |OG| - |OF| \geqslant \frac{\tilde{\rho}}{2} - \tilde{C}'\tilde{h}^2 \geqslant \frac{\tilde{\rho}}{2} - \tilde{C}'\tilde{C}\tilde{h}\tilde{\rho} \geqslant \frac{\tilde{\rho}}{2} - \frac{1}{4}\tilde{\rho} = \frac{1}{4}\tilde{\rho},$$

which means that the inscribed circle of ΔABD cannot have diameter less than $(1/4)\tilde{\rho}$ and consequently $\rho_K \geqslant (1/4)\tilde{\rho}$.

Let d be the diameter of ΔABC. Then

$$d\tilde{h} + \tilde{C}'\tilde{h}^2 \leqslant \tilde{h} + \tilde{C}'\frac{1}{4\tilde{C}\tilde{C}'}\tilde{h} \leqslant \left(1 + \frac{1}{4\tilde{C}}\right)\tilde{h}.$$

Hence

$$h_K \leqslant \tilde{h} + \left(1 + \frac{1}{4\tilde{C}}\right)\tilde{h} \leqslant \left(2 + \frac{1}{4\tilde{C}}\right)\tilde{h}.$$

Consequently

$$\frac{1}{4}\tilde{\rho} \leqslant \rho_K \leqslant h_K \leqslant \left(2 + \frac{1}{4\tilde{C}}\right)\tilde{h},$$

which implies

$$\frac{h_K}{\rho_K} \leqslant \frac{\left(2 + \frac{1}{4\tilde{C}}\right)\tilde{h}}{\frac{1}{4}\tilde{\rho}} \leqslant \left(8 + \frac{1}{\tilde{C}}\right)\tilde{C} = (8\tilde{C} + 1).$$

This completes the proof. □

4. Edge ratio and maximum–minimum angle conditions for regularity

For triangular grids it is well known that regularity is equivalent to the minimum angle condition which says that the minimum angles of the triangles in the partitions are uniformly bounded below by a constant. It is thus natural to put forth the following condition for quadrilateral grids.

The edge ratio and min–max angle condition: Let there be positive constants C and C' independent of h such that for all $K \in \mathcal{Q}_h$

$$\begin{cases} \dfrac{h_K}{h'_K} \leqslant C, & h'_K = \text{the length of the shortest edge in } K, \\ |\cos\theta^i_K| \leqslant C' < 1, & \theta^K_i = \text{interior angles of } K, i = 1,2,3,4. \end{cases} \tag{4.11}$$

As in the triangular grid case, we have the following theorem.

Theorem 4.1. *The usual regularity condition* (2.2) *is equivalent to the edge ratio and min–max angle condition* (4.11).

Proof. We first show (2.2) implies (4.11). Since the shortest edge must be greater than the diameter of the smallest inscribed circle in the S_i's defined in (2.3), we have $h'_K \geqslant \rho_K$ and hence $(h_K/\rho_K) \leqslant C_1$ implies $(h_K/h'_K) \leqslant C_1$. It remains to show that the second condition of (4.11) holds. In reference to Fig. 7, consider the triangle ΔABD in the quadrilateral $K = ABCD$. Let $\tilde{h}$ = the diameter of this triangle, O = the center of its inscribed circle whose diameter is $\tilde{\rho}$. Since $h_K \geqslant \tilde{h}$ and $\rho_K \leqslant \tilde{\rho}$, we have $(\tilde{h}/\tilde{\rho}) \leqslant C$ from the fact $(h_K/\rho_K) \leqslant C$. As in the figure, since $DE \perp AB$ we have $\sin(\angle DAB) = (|DE|/|AD|)$. Obviously,

$$\sin(\angle DAB) = \frac{|DE|}{|AD|} \geqslant \frac{\tilde{\rho}}{\tilde{h}} \geqslant \frac{1}{C},$$

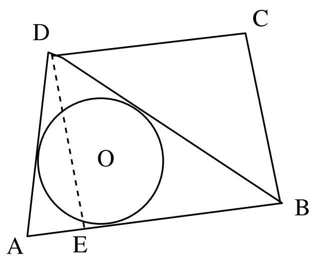

Fig. 7. Figure for Theorem 4.1.

which implies

$$|\cos(\angle DAB)| \leqslant \sqrt{1-\frac{1}{C^2}} = C'.$$

Similarly we can show that

$$|\cos(\angle ABC)|, |\cos(\angle BCD)|, |\cos(\angle CDA)| \leqslant C'.$$

This shows the validity of the second inequality of (4.11).

We now show that (4.11) implies (2.2). Without loss of generality, assume the circle in the figure is the inscribed circle in S_i that has the smallest diameter, i.e., $\rho_K = \tilde{\rho}$. From the area relation,

$$(|AB| + |BD| + |AD|)\frac{\rho_K}{2} = |AB||DE| = |AB||AD|\sin(\angle DAB),$$

and hence

$$\rho_K = \frac{2|AB||AD|\sin(\angle DAB)}{|AB| + |BD| + |AD|}.$$

Therefore

$$\frac{h_K}{\rho_K} = \frac{h_K(|AB| + |BD| + |AD|)}{2|AB||AD|\sin(\angle DAB)} = \frac{h_K}{2|AD|}\left(1 + \frac{|BD|}{|AB|} + \frac{|AD|}{|AB|}\right)\frac{1}{\sin(\angle DAB)}.$$

From condition (4.11) we deduce now

$$\frac{h_K}{\rho_K} \leqslant \frac{h_K}{2h'_K}\left(1 + \frac{h_K}{h'_K} + \frac{h_K}{h'_K}\right)\frac{1}{\sqrt{1-\cos^2(\angle DAB)}} \leqslant \frac{C}{2}(1+2C)\frac{1}{\sqrt{1-(C')^2}} = C''.$$

This completes the proof. □

Finally, we point out that it is shown in [17], Lemma 1, that quasi-regularity condition (2.5) plus the $o(h^2)$—almost parallelogram condition (3.10) imply condition (4.11).

5. Summary

In this paper we first investigated relations between several shape regularity conditions on quadrilateral subdivisions. Two are based on the ratios of the form $h_K/\tilde{\rho}_K$ (cf. (2.4) and (2.5)) and one on the areas (2.3), and we demonstrated in Theorem 2.3 that they are equivalent. These conditions are all quasi-regular, weaker than the regularity condition (2.2), where $\tilde{\rho}_K$ is related to the four subtriangles in the quadrilateral. We next looked at two types of almost parallelogram conditions, one based on the distance between the midpoints of the two diagonals of a quadrilateral and one based on on two angles formed by opposite normals of the sides of a quadrilateral. It is shown in Theorems 3.1 and 3.2 that they are equivalent provided that the subdivisions are regular. Finally in Theorem 4.1 we showed that the regularity condition (2.2) can be interpreted as an edge ratio and min–max angle condition.

6. Uncited references

[13,15]

Acknowledgement

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